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EULER ANGLES, QUATERNIONS, AND TRANSFORMATION MATRICES FOR SPACE SHUTTLE ANALYSIS

MISSION PLANNING, MISSION ANALYSIS, AND SOFTWARE FORMULATION

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1.0 INTRODUCTION

Due to the extensive use of the quaternion in the onboard Space Shuttle Computer System, considerable analysis is being performed using relationships between the quaternion, the transformation matrix, and the Euler angles. This Design Note offers a brief mathematical development of the relationships between the Euler angles and the transformation matrix, the quaternion and the transformation matrix, and the Euler angles and the quaternion. The analysis and equations presented here apply directly to current Space Shuttle problems. Appendix A presents the twelve three-axis Euler transformation matrices as functions of the Euler angles, the equations for the quaternion as a function of the Euler angles, and the Euler angles as a function of the transformation matrix elements.

The equations of Appendix A are a valuable reference in Shuttle analysis work and this Design Note is the only known document where each of the twelve Euler angle to quaternion relationships are given. Appendix B presents a group of utility subroutines to accomplish the Euler-matrix, quaternion-matrix, and Euler-quaternion relationships of Appendix A.

2.0 DISCUSSION

2.1 Euler Angle Transformation Matrices

The following analysis and utility subroutines are offered to simplify computer programs when working with coordinate transformation matrices and their relationships with the Euler Angles and the Quaternions. The coordinate transformation matrices discussed here are defined using the following figure,

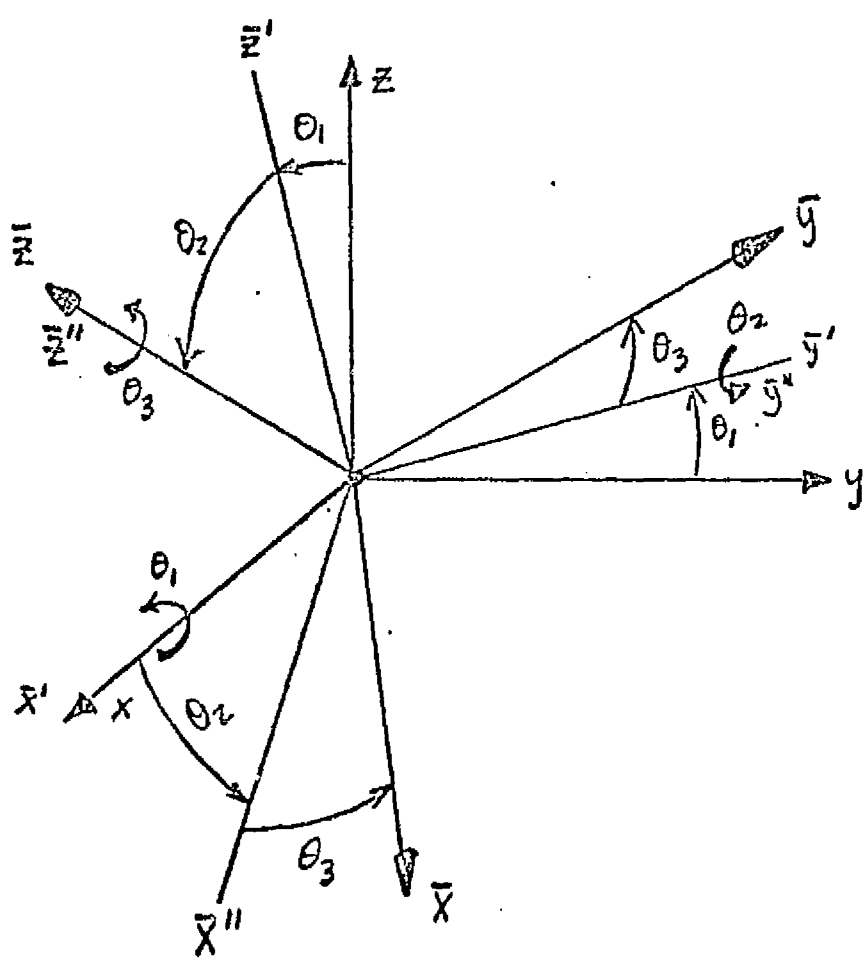

FIGURE 1

The transformation matrix M, is defined to transform vectors in the $\overline{x}$- system $(\overline{x}, \overline{y}, \overline{z})$ into the original x-system (x, y, z) and is given by the equation,

$$x = M\overline{x}$$

where (1)

$$x = (x, y, z) \text{ and } \overline{x} = (\overline{x}, \overline{y}, \overline{z}).$$

Using the right-hand rule for positive rotations, the M matrix in (1) above is constructed by the following analysis. The first rotation in Figure 1 above is about the x-axis by the amount θ_1. The single rotation about the x-axis results in the following transformation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \overline{x}' \\ \overline{y}' \\ \overline{z}' \end{pmatrix} \qquad (2)$$

or $x = X\overline{x}'$ in matrix form. Rotation about the $\overline{y}'$-axis by the amount θ_2 yields the intermediate transformation matrix:

$$\begin{pmatrix} \overline{x}' \\ \overline{y}' \\ \overline{z}' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \overline{x}'' \\ \overline{y}'' \\ \overline{z}'' \end{pmatrix} \qquad (3)$$

or $\overline{x}' = Y\overline{x}'$ in matrix form. Finally rotation about the $\overline{z}''$-axis by the amount θ_3 yields the intermediate transformation matrix,

$$\begin{pmatrix} \bar{x}'' \\ \bar{y}'' \\ \bar{z}'' \end{pmatrix} = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \qquad (4)$$

and in matrix form $\bar{x}'' = Z\bar{x}_1$. Now using the three equations,

$$x = X\bar{x}'$$
$$\bar{x}' = Y\bar{x}'' \qquad (5)$$
$$\bar{x}'' = Z\bar{x}$$

by substitution

$$x = (X\ Y\ Z)\ \bar{x}. \qquad (6)$$

Then from equation 1,

$$M = (X\ Y\ Z) \qquad (7)$$

Computation for the M matrix from the indicated matrix multiplication in equation (7) yields,

$$M = \begin{pmatrix} (\cos\theta_2\ \cos\theta_3) & (-\cos\theta_2\ \sin\theta_3) & (\sin\theta_2) \\ (\cos\theta_1\ \sin\theta_3 + \sin\theta_1\ \sin\theta_2\ \cos\theta_3) & (\cos\theta_1\ \cos\theta_3 - \sin\theta_1\ \sin\theta_2\ \sin\theta_3) & (-\sin\theta_1\ \cos\theta_2) \\ (\sin\theta_1\ \sin\theta_3 - \cos\theta_1\ \sin\theta_2\ \cos\theta_3) & (\sin\theta_1\ \cos\theta_3 + \cos\theta_1\ \sin\theta_2\ \sin\theta_3) & (\cos\theta_1\ \cos\theta_2) \end{pmatrix}. \qquad (8)$$

The matrix M in equation (8) is a function of;

(1) The three Euler angles θ_1 θ_2 and θ_3 and

(2) The sequence of rotations used to generate the matrix.

By examination of equation (7) it is possible to show that there are twelve possible Euler rotational sequences. If the (X Y Z) notation in equation (7) represents a rotation about the X axis, then the Y axis and finally the Z axis, then the following permutations of the rotational order represents the twelve possible Euler angle sets using three rotations,

$$\begin{array}{ccc} X\,Y\,Z & Y\,X\,Z & Z\,X\,Y \\ X\,Z\,Y & Y\,Z\,X & Z\,Y\,X \\ X\,Y\,X & Y\,X\,Y & Z\,X\,Z \\ X\,Z\,X & Y\,Z\,Y & Z\,Y\,Z \end{array} \tag{9}$$

Any repeated axis rotation such as XXY does not represent a three axis rotation but reduces to the two axis rotation XY. Hence the rotations described in (9) above represent all twelve possible sets of Euler angle defining sequences. Conversely then, for a given transformation matrix there are twelve Euler angle sets which can be extracted from the matrix. The Euler matrices corresponding to all possible rotational sequences of (9) above are presented in Appendix A.

The utility subroutines "EULMAT" generates the transformation matrix from a given Euler sequence and the Euler angles. The utility subroutine "MATEUL" extracts the Euler angles from a given Euler rotational sequence. The convention is established that the Euler angles occur in the same sequences as the axis rotations. Using this concept and functional notation, equation (7) could be expressed as

$$M = X\,Y\,Z = M(\theta_x, \theta_y, \theta_z) \tag{10}$$

and from (9)

$$M = X\ Z\ X = M(\theta_x, \theta_z, \theta_x') \text{ etc.} \tag{11}$$

This convention is assumed in both subroutines and also used throughout this design note when Euler angles are used. A brief explaination of the use of these two utility subroutines is given in Appendix B.

It is interesting to note that a negative rotation in the single axis rotation matrices of equations (2), (3) and (4) will result in the formation of the transpose of the matrix. However the transpose of M in equation (7) is formed from reversing the order of multiplication and transposing the individual axis rotation equations, i.e.

$$M^T = (X\ Y\ Z)^T = (\ Y\ Z)^T X^T = Z^T\ Y^T\ X^T. \tag{12}$$

Hence the transposes of the matrices of (9) are easily formed by reversing the order of multiplication and transposing each single axis rotation equation. Using the notation in equations(10) and (11) above equation (12) could be written,

$$M^T(\theta_x, \theta_y, \theta_z) = M(-\theta_z, -\theta_y, -\theta_x). \tag{13}$$

It is recommended to avoid confusion that the forward transformation be computed and simply transposed to yield the reverse transformation matrix. All matrices of Appendix A are in the forward form, i.e. $X = M\bar{x}$ and formed from (9).

2.2 TRANSFORMATION MATRICES USING THE HAMILTON QUATERNION

The transformation matrix of equation (1) can be written as a function of the Hamilton Quaternion;

$$
\begin{aligned}
q_1 &= \cos \omega/2 \\
q_2 &= \cos \alpha \sin \omega/2 \\
q_3 &= \cos \beta \sin \omega/2 \\
q_4 &= \cos \gamma \sin \omega/2 ,
\end{aligned}
\tag{14}
$$

where ω is the rotation angle about the rotation axis with α, β, and γ direction angles with the x, y and z axes respectively. Notice also that $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The rotation angle, ω, is assumed positive according to the right-hand rule of axis rotation. The matrix M becomes

$$
M = \begin{pmatrix}
(q_1^2 + q_2^2 - q_3^2 - q_4^2) & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\
2(q_2q_3 + q_1q_4) & (q_1^2 - q_2^2 + q_3^2 - q_4^2) & 2(q_3q_4 - q_1q_2) \\
2(q_2q_4 - q_1q_3) & 2(q_3q_4 + q_1q_2) & (q_1^2 - q_2^2 - q_3^2 + q_4^2)
\end{pmatrix} . \tag{15}
$$

For a more detailed discussion of the derivation of equation (15), see Reference 1. Using functional notation, equation (15) can be written,

$$
M = M(q_1, q_2, q_3, q_4). \tag{16}
$$

Unlike the Euler angle rotational sequences to describe the transformation matrix of equation (1), only two quaternions can be found from equation (15). The two quaternions are:

$$\begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \quad \text{and} \quad \begin{matrix} -q_1 \\ -q_2 \\ -q_3 \\ -q_4 \end{matrix} \tag{17}$$

These two quaternions represent a positive rotation about the rotation axis pointing in one direction and a positive rotation about the same line of rotation pointing in the opposite direction. Both quaternions of (17) satisfy equation (15).

The utility subroutine "QMAT" generates the transformation matrix from a given quaternion. The "QMAT" algorithm generates the matrix as given in equation (15) without duplicating any arithmetic operations. The subroutine "MATQ" extracts the positive quaternion, i.e., $q_1 > 0$, from the transformation matrix and normalizes the results to guarantee an orthogonal matrix. In order to avoid any discontinuity in extracting the quaternion from the transformation matrix, the procedure as described in Reference 2 is used.

Early works by Hamilton (Reference 3) presented the quaternion as having a scalar and a vector part, i.e.,

$$q_1 = S \qquad \vec{V} = (q_2, q_3, q_4) \tag{18}$$

and equation (16) could be expressed as,

$$M = M(q_1, q_2, q_3, q_4) = M(S, \vec{V}). \tag{19}$$

For a given quaternion the following relationship is true (from (17) above),

$$M(S, \vec{V}) = M(-S, -\vec{V}). \qquad (20)$$

The transpose of the transformation matrix is given by,

$$M^T(S, \vec{V}) = M(-S, \vec{V}) = M(S, -\vec{V}). \qquad (21)$$

2.3 EULER ANGLE AND QUATERNION RELATIONSHIPS.

By examination of equations (10) and (16) the equality,

$$M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = M(\theta_1, \theta_2, \theta_3) = M(q_1, q_2, q_3, q_4) \qquad (22)$$

can be written. Based on an equality for each element of the matrix the following nine equations must be true;

$$\begin{aligned}
\cos\theta_2 \cos\theta_3 &= q_1^2 + q_2^2 - q_3^2 - q_4^2 \\
-\cos\theta_2 \sin\theta_3 &= 2(q_2q_3 - q_1q_4) \\
\sin\theta_2 &= 2(q_2q_4 + q_1q_3) \\
\cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3 &= 2(q_2q_3 + q_1q_4) \\
\cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 &= q_1^2 - q_2^2 + q_3^2 - q_4^2 \\
-\sin\theta_1 \cos\theta_2 &= 2(q_3q_4 - q_1q_2) \\
\sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3 &= 2(q_3q_4 - q_1q_3) \\
\sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3 &= 2(q_3q_4 + q_1q_2) \\
\cos\theta_1 \cos\theta_2 &= q_1^2 - q_2^2 - q_3^2 + q_4^2 \;.
\end{aligned} \qquad (23)$$

It is possible to solve for the values of the quaternion using the trigonometric half angle identities. For this Euler sequence, i.e. $X(\theta_1)\ Y(\theta_2)\ Z(\theta_3)$, the following quaternion results;

$$
\begin{aligned}
q_1 &= -\sin\tfrac{1}{2}\theta_1 \, \sin\tfrac{1}{2}\theta_2 \, \sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1 \, \cos\tfrac{1}{2}\theta_2 \, \cos\tfrac{1}{2}\theta_3 \\
q_2 &= +\sin\tfrac{1}{2}\theta_1 \, \cos\tfrac{1}{2}\theta_2 \, \cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2 \, \sin\tfrac{1}{2}\theta_3 \, \cos\tfrac{1}{2}\theta_1 \\
q_3 &= -\sin\tfrac{1}{2}\theta_1 \, \sin\tfrac{1}{2}\theta_3 \, \cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2 \, \cos\tfrac{1}{2}\theta_1 \, \cos\tfrac{1}{2}\theta_3 \\
q_4 &= +\sin\tfrac{1}{2}\theta_1 \, \sin\tfrac{1}{2}\theta_2 \, \cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3 \, \cos\tfrac{1}{2}\theta_1 \, \cos\tfrac{1}{2}\theta_2
\end{aligned}
\qquad (24)
$$

Appendix A gives the quaternion as a function of the Euler angles for each of the twelve Euler rotational sequence presented in Section 2.1. The special equations for the quaternion as functions of the Euler angles, like equations (24) above, are sometimes cumbersome to use, especially when multiple Euler sequences are utilized. Less coding, but perhaps more computer operations, are required by using the more general method of first generating the matrix M from the given Euler angle set and then simply extracting the quaternion from the matrix. This method is effected by first a call to "EULMAT" and then a call to "MATQ"..

3.0 REFERENCES

1. Working Paper: MDTSCO, TM No. 1.4-MPB-304, E914-8A/B-003, "Quaternions and Quaternion Transformations," David M. Henderson, 23 June 1976.

2. Transmittal Memo: MDTSCO, 1.4-MPB-229, "Improving Computer Accuracy in Extracting Quaternions," David M. Henderson, 9 March 1976.

3. Sir William Rowan Hamilton, LLD, LL.D. MRIA, D.C.L. CANTAB., "Elements of Quaternions" 2 Volumes, Chelsea Publishing Company, New York, N. Y., 3rd Edition 1969, Library of Congress 68-54711 #8284-0219-1.

APPENDIX A

The twelve Euler matrices for each rotation sequence are given based on the single axis rotation equations (2), (3) and (4). The Euler matrices transform vectors from the system that has been rotated into vectors in the stationary system. Also presented here are the equations for the quaternion as a function of the Euler angles and the Euler angles as a function of the matrix elements for each rotation sequence.

(1) $M = M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = XYZ$

Axis Rotation Sequence: 1, 2, 3

$$M = \begin{bmatrix} \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 & \sin\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2 \\ -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = \sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_3 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_4 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{23}}{m_{33}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{13}}{\sqrt{1-m_{13}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{12}}{m_{11}}\right)$$

(2) $M = M(X(\theta_1),\ Z(\theta_2),\ Y(\theta_3)) = XZY$

Axis Rotation Sequence: 1, 3, 2

$$M = \begin{bmatrix} \cos\theta_2\cos\theta_3 & -\sin\theta_2 & \cos\theta_2\sin\theta_3 \\ \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_3 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{32}}{m_{22}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{12}}{\sqrt{1-m_{12}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{13}}{m_{11}}\right)$$

(3) $M = M(X(\theta_1), Y(\theta_2), X(\theta_3)) = XYX$

Axis Rotation Sequence: 1, 2, 1

$$M = \begin{bmatrix} \cos\theta_2 & \sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_3 - \sin\theta_1\cos\theta_2\sin\theta_3 & -\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3 \\ -\cos\theta_1\sin\theta_2 & +\sin\theta_1\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 & -\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = \cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = \cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_3 = \sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = \sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{21}}{-m_{31}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{12}}{m_{13}}\right)$$

(4) $M = M(X(\theta_1), Z(\theta_2), X(\theta_3)) = XZX$

Axis Rotation Sequence: 1, 3, 1

$$M = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 \\ \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = \cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = \cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_3 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = \sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{31}}{m_{21}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{13}}{-m_{12}}\right)$$

(5) $M = M(Y(\theta_1), X(\theta_2), Z(\theta_3)) = YXZ$

Axis Rotation Sequence: 2, 1, 3

$$M = \begin{bmatrix} \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\cos\theta_2 \\ \cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 & -\sin\theta_2 \\ \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix}$$

$$q_1 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_3 = \sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_4 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{31}}{m_{33}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{23}}{\sqrt{1-m_{23}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{21}}{m_{22}}\right)$$

(6) $M = M(Y(\theta_1), Z(\theta_2), X(\theta_3)) = YZX$

Axis Rotation Sequence: 2, 3, 1

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_2 & \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$q_4 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{31}}{m_{11}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{21}}{\sqrt{1-m_{21}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{23}}{m_{22}}\right)$$

(7) $M = M(Y(\theta_1), X(\theta_2), Y(\theta_3)) = YXY$

Axis Rotation Sequence: 2, 1, 2

$$M = \begin{bmatrix} -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 & \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 \\ \sin\theta_2\sin\theta_3 & \cos\theta_2 & -\sin\theta_2\cos\theta_3 \\ -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_3 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_4 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{12}}{m_{32}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{21}}{-m_{23}}\right)$$

(8) $M = M(Y(\theta_1), Z(\theta_2), Y(\theta_3)) = YZY$

Axis Rotation Sequence: 2, 3, 2

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_2\cos\theta_3 & \cos\theta_2 & \sin\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = +\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_3 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_4 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{32}}{-m_{12}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{23}}{m_{21}}\right)$$

(9) $M = M(Z(\theta_1), X(\theta_2), Y(\theta_3)) = ZXY$

Axis Rotation Sequence: 3, 1, 2

$$M = \begin{bmatrix} -\sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 \\ \cos\theta_1\sin\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ -\cos\theta_2\sin\theta_3 & \sin\theta_2 & \cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$\theta_1 = \tan^{-1}\left(\frac{-m_{12}}{m_{22}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{m_{32}}{\sqrt{1-m_{32}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{-m_{31}}{m_{33}}\right)$$

(10) $M = M(Z(\theta_1), Y(\theta_2), X(\theta_3)) = ZYX$

Axis Rotation Sequence: 3, 2, 1

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 \\ \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 \\ -\sin\theta_2 & \cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 \end{bmatrix}$$

$$q_1 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3 + \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3$$

$$q_2 = -\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 + \sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2$$

$$q_3 = +\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_2 + \sin\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_3$$

$$q_4 = +\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\tfrac{1}{2}\theta_3 - \sin\tfrac{1}{2}\theta_2\sin\tfrac{1}{2}\theta_3\cos\tfrac{1}{2}\theta_1$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{21}}{m_{11}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{-m_{31}}{\sqrt{1-m_{31}^2}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{32}}{m_{33}}\right)$$

(11) $M = M(Z(\theta_1), X(\theta_2), Z(\theta_3)) = ZXZ$

Axis Rotation Sequence: 3, 1, 3

$$M = \begin{bmatrix} -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & -\sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 \\ \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\sin\theta_2 \\ \sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & \cos\theta_2 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_3 = +\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{13}}{-m_{23}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{33}^2}}{m_{33}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{31}}{m_{32}}\right)$$

(12) $M = M(Z(\theta_1), Y(\theta_2), Z(\theta_3)) = ZYZ$

Axis Rotation Sequence: 3, 2, 3

$$M = \begin{bmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 \\ -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 & \cos\theta_2 \end{bmatrix}$$

$$q_1 = +\cos\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$q_2 = -\sin\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_3 = +\sin\tfrac{1}{2}\theta_2\cos(\tfrac{1}{2}(\theta_1 - \theta_3))$$

$$q_4 = +\cos\tfrac{1}{2}\theta_2\sin(\tfrac{1}{2}(\theta_1 + \theta_3))$$

$$\theta_1 = \tan^{-1}\left(\frac{m_{23}}{m_{13}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{33}^2}}{m_{33}}\right)$$

$$\theta_3 = \tan^{-1}\left(\frac{m_{32}}{-m_{31}}\right)$$

APPENDIX B

The following subroutines with a brief description of their use are presented in this appendix.

(1) "EULMAT" - Generates the transformation matrix from a given set of Euler angles and an axis rotation sequence.

(2) "MATEUL" - Extracts the Euler angles from the given transformation matrix and an axis rotation sequence.

(3) "QMAT" - Generates the transformation matrix from a given quaternion.

(4) "MATQ" - Extracts the quaternion from a given transformation matrix.

(5) "YPRQ" - Generates the quaternion directly from the yaw-pitch-roll Euler angles.

(6) "POSNOR" - Computes the positive-normalized quaternion from the given quaternion.

NAME:	EULMAT
PURPOSE:	Generates a 3 x 3 transformation matrix from a given sequence and Euler angle set.
INPUT:	ISEQ - Rotation Sequence (Integer Array (3); i.e., 1, 2, 3)
	EUL - Euler Angles in radians, in "ISEQ" Order; ARRAY (3)
OUTPUT:	A - The 3 x 3 transformation matrix
ALGORITHM REFERENCE:	Appendix A; Euler Sequences (1) thru (12).

EULER ANGLES TO THE TRANSFORMATION MATRIX

```
@FOR,IS     EULMAT,EULMAT
FOR S3E3-02/19/77-06:24:23 (,0)

   SUBROUTINE EULMAT     ENTRY POINT 000237

   STORAGE USED: CODE(1) 000250; DATA(0) 000104; BLANK COMMON(2)

   EXTERNAL REFERENCES (BLOCK, NAME)

    0003    SIN
    0004    COS
    0005    NERR3S

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001     000135 100L         0001     000013 120G          0001     00001
    0001     000142 162G         0001     000144 155G          0001     00014
    0001     000077 30L          0000 R 000033 B              0000 R 00005
    0000     000057 INJP$        0000 I 000046 J              0000 I 00004
    0000 R 000047 SINA           0000 R 000053 TEMP           0000 R 00000


00101                SUBROUTINE EULMAT(ISEQ,EUL,A)
00103                DIMENSION ISEQ(3),EUL(3),A(3,3)
00104                DIMENSION X(3,3,3),B(3,3)
00106                DO 100 K=1,3
00110                DO 10 I=1,3
00113                DO 5 J=1,3
00116                X(I,J,K)=0.0
00117                IF(I.EQ.J) X(I,J,K)=1.0
00121              5 CONTINUE
00123             10 CONTINUE
00125                IF(ISEQ(K).LE.0) GO TO 100
00127                SINA=SIN(EUL(K))
00130                COSA=COS(EUL(K))
00131                IF(ISEQ(K).EQ.2) GO TO 20
00133                IF(ISEQ(K).EQ.3) GO TO 30
00135                X(2,2,K)=COSA
00136                X(2,3,K)=-SINA
00137                X(3,2,K)= SINA
00140                X(3,3,K)=COSA
00141                GO TO 100
00142             20 X(1,1,K)=COSA
00143                X(1,3,K)= SINA
00144                X(3,1,K)=-SINA
00145                X(3,3,K)=COSA
00146                GO TO 100
00147             30 X(1,1,K)=COSA
00150                X(1,2,K)=-SINA
00151                X(2,1,K)= SINA
00152,               X(2,2,K)=COSA
```

EULER ANGLES TO THE TRANSFORMATION MATRIX

(CONTINUED)

```
00153      100 CONTINUE
00155          DO 400 L=1,2
00157          M=3-L
00161          DO 300 I=1,3
00164          DO 300 J=1,3
00167          TEMP=0.0
00172          DO 250 K=1,3
00173          IF(L.EQ.1) HOLD=X(K,J,3)
00175          IF(L.EQ.2) HOLD=B(K,J)
00177          IF(ABS(HOLD).LT.1.0E-10) GO TO 250
00201          IF(ABS(X(I,K,M)).LT.1.0E-10) GO TO 250
00203          TEMP=TEMP+X(I,K,M)*HOLD
00204      250 CONTINUE
00206          IF(L.EQ.1) B(I,J)=TEMP
00210          IF(L.EQ.2) A(I,J)=TEMP
00212      300 CONTINUE
00215      400 CONTINUE
00217          RETURN
00220          END

END OF COMPILATION:        NO. DIAGNOSTICS.
```

NAME: MATEUL

PURPOSE: Extracts the Euler angles from the given transformation matrix and the required Euler rotational sequence.

INPUT: ISEQ - Rotation sequence, (Integer Array (3), i.e., 1,2,3.)

A - The 3 x 3 transformation

OUTPUT: EUL - The Euler angles, in "ISEQ" order; ARRAY(3).

ALGORITHM REFERENCE: Appendix A; Euler angles as a function of the matrix elements, sequences (1) thru (12).

TRANSFORMATION MATRIX TO THE EULER ANGLES

I

```
@FOR,IS     MATEUL,MATEUL
FOR S3E3-02/19/77-06:24:25 (,0)

     SUBROUTINE MATEUL     ENTRY POINT 000335

     STORAGE USED: CODE(1) 000353; DATA(0) 000052; BLANK COMMON(2)

     EXTERNAL REFERENCES (BLOCK, NAME)

      0003    SQRT
      0004    ATAN2
      0005    NERR3$

     STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

      0001     000054 10L         0001     000066 15L         0001     00117
      0001     000115 30L         0001     000220 40L         0001     00222
      0001     000251 55L         0001     000257 60L         0001     00227
      0000 R 000004 BSIGN         0000 R 000005 CSIGN         0000 R 00001
      0000 R 000013 ENUM          0000 I 000000 I             0000 I 00002
      0000 I 000001 J             0000 I 000012 JJ            0000 I 00003


00101        1*            SUBROUTINE MATEUL(ISEQ,A,EUL)
00103        2*            DIMENSION A(3,3),EUL(3)
00104        3*            DIMENSION ISEQ(3)
00105        4*            I=ISEQ(1)
00106        5*            J=ISEQ(2)
00107        6*            K=ISEQ(3)
00110        7*            IEOK=0
00111        8*            IF(I.EQ.K) IEOK=4095
00113        9*            BSIGN=1.0
00114       10*            CSIGN=1.0
00115       11*            IF(I.EQ.1) GO TO 10
00117       12*            IF(I.EQ.2) GO TO 20
00121       13*            IF(J.NE.1) GO TO 5
00123       14*            BSIGN=-1.0
00124       15*            IF(IECK.NE.0) L=2
00126       16*            GO TO 30
00127       17*          5 CSIGN=-1.0
00130       18*            IF(IECK.NE.0) L=1
00132       19*            GO TO 30
00133       20*         10 IF(J.NE.2) GO TO 15
00135       21*            BSIGN=-1.0
00136       22*            IF(IECK.NE.0) L=3
00140       23*            GO TO 30
00141       24*         15 CSIGN=-1.0
00142       25*            IF(IECK.NE.0) L=2
00144       26*            GO TO 30
00145       27*         20 IF(J.NE.3) GO TO 25
00147       28*            BSIGN=-1.0
```

TRANSFORMATION MATRIX TO THE EULER ANGLES (CONTINUED)

```
00150         IF(IEOK.NE.0) L=1
00152         GO TO 30
00153      25 CSIGN=-1.0
00154         IF(IEOK.NE.0) L=3
00156      30 DO 100 N=1,3
00161         FNSGN=1.0
00162         FDSGN=1.0
00163         IF(N.EQ.2) GO TO 70
00165         IF(N.EQ.1) GO TO 55
00167         IF(IEOK.NE.0) GO TO 40
00171         FNSGN=BSIGN
00172         JJ=I
00173         GO TO 45
00174      40 JJ=L
00175         IF(BSIGN.GT.0.0) FDSGN=-1.0
00177      45 FNUM=FNSGN*A(I,J)
00200         FDEN=FDSGN*A(I,JJ)
00201         GO TO 90
00202      50 IF(IEOK.NE.0) GO TO 55
00204         FNSGN=BSIGN
00205         II=K
00206         JJ=K
00207         GO TO 60
00210      55 FDSGN=BSIGN
00211         II=L
00212         JJ=I
00213      60 FNUM=FNSGN*A(J,K)
00214         FDEN=FDSGN*A(II,JJ)
00215         GO TO 90
00216      70 IF(IEOK.NE.0) GO TO 80
00220         FNUM=CSIGN*A(I,K)
00221         FDEN=SQRT(1.0-A(I,K)**2)
00222         GO TO 90
00223      80 FNUM=SQRT(1.0-A(I,I)**2)
00224         FDEN=A(I,I)
00225      90 EUL(N)=ATAN2(FNUM,FDEN)
00226     100 CONTINUE
00230         RETURN
00231         END
```

END OF COMPILATION: NO DIAGNOSTICS.

NAME: QMAT

PURPOSE: Generates the transformation matrix from the given quaternion.

INPUT: Q - The quaternion; ARRAY(4).

OUTPUT: A - The 3 x 3 transformation matrix

ALGORITHM REFERENCE: Equation (15) from Section 2.2.

QUATERNION TO THE TRANSFORMATION MATRIX

I

```
OFOR,IS        QMAT,QMAT
FOR S?E3-?2/19/77-?5:?4:19 (,?)

     SUBROUTINE QMAT        ENTRY POINT 000077

     STORAGE USED: CODE(1) 000133; DATA(0) 000015; BLANK COMMON(2) 0

     EXTERNAL REFERENCES (BLOCK, NAME)

      0003    NERR3$

     STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

      0000    000007 INJP$       0000 R 000000 P2        0000 R 000001
      0000 R 000004 P6           0000 R 000005 TEMP


00101                SUBROUTINE QMAT(Q,A)
00103                DIMENSION Q(4),A(3,3)
00104                P2=Q(2)+Q(2)
00105                P3=Q(3)+Q(3)
00106                P4=Q(4)+Q(4)
00107                P5=P2*Q(2)
00110                P6=P4*Q(4)
00111                TEMP=1.0-P3*Q(3)
00112                A(1,1)=TEMP-P6
00113                A(2,2)=1.0-P5-P6
00114                A(3,3)=TEMP-P5
00115                P5=P2*Q(3)
00116                P6=P4*Q(1)
00117                A(1,2)=P5-P6
00120                A(2,1)=P5+P6
00121                P5=P2*Q(4)
00122                P6=P3*Q(1)
00123                A(1,3)=P5+P6
00124                A(3,1)=P5-P6
00125                P5=P3*Q(4)
00126                P6=P2*Q(1)
00127                A(2,3)=P5-P6
00130                A(3,2)=P5+P6
00131                RETURN
00132                END

          END OF COMPILATION:        NO  DIAGNOSTICS.
```

NAME: MATQ

PURPOSE: Extracts the positive quaternion from the given transformation matrix.

INPUT: A - The 3 x 3 transformation matrix

OUTPUT: Q - The positive quaternion; ARRAY(4).

ALGORITHM REFERENCE: See Reference 2.

I

TRANSFORMATION MATRIX TO THE QUATERNION

```
@FOR,SI       MATQ,MATQ
FOR S1E3-02/19/77-06:24:21 (,0)

     SUBROUTINE MATQ        ENTRY POINT 000203

     STORAGE USED: CODE(1) 000203; DATA(0) 000050; BLANK COMMON(2)

     EXTERNAL REFERENCES (BLOCK, NAME)

      0003     SQRT
      0004     NERR3$

     STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

      0001     000073 10L          0001   000052 137G          0001   000157
      0001     000111 35L          0001   000121 40L           0001   000163
      0000     000015 INJP$        0000 I 000006 J             0000 R 000000

00101                  SUBROUTINE MATQ(A,Q)
00103                  DIMENSION A(3,3),Q(4),T(4)
00104                  I=0
00105                  BIG=0.0
00106                  DO 40 J=1,4
00111                  Q(J)=0.0
00112                  IF(J.EQ.2) GO TO 10
00114                  IF(J.EQ.3) GO TO 20
00116                  IF(J.EQ.4) GO TO 30
00120                  Q(J)=1.0
00121                  TEMP=A(1,1)+A(2,2)+A(3,3)+1.0
00122                  T(J)=0.0
00123                  GO TO 35
00124               10 TEMP=A(1,1)-A(2,2)-A(3,3)+1.0
00125                  T(J)=A(3,2)-A(2,3)
00126                  GO TO 35
00127               20 TEMP=-A(1,1)+A(2,2)-A(3,3)+1.0
00130                  T(J)=A(1,3)-A(3,1)
00131                  GO TO 35
00132               30 TEMP=-A(1,1)-A(2,2)+A(3,3)+1.0
00133                  T(J)=A(2,1)-A(1,2)
00134               35 IF(TEMP.LT.BIG) GO TO 40
00136                  BIG=TEMP
00137                  I=J
00140               40 CONTINUE
00142                  IF(I.EQ.0) GO TO 60
00144                  Q(I)=0.50*SQRT(BIG)
00145                  IF(I.NE.1) Q(1)=ABS(0.25*T(I)/Q(I))
00147                  TEMP=0.25/Q(1)
00150                  DO 50 J=2,4
00153                  Q(J)=TEMP*T(J)
00154               50 CONTINUE
00155               60 RETURN
00157                  END

     END OF COMPILATION:          NO DIAGNOSTICS.
```

NAME: YPRQ

PURPOSE: Generates the quaternion directly from the yaw-pitch-roll Euler angles, i.e., a 3, 2, 1 Euler sequence.

INPUT: YPR - The yaw-pitch-roll Euler angles; ARRAY (3).

OUTPUT: QO - The positive quaternion, ARRAY (4).

ALGORITHM REFERENCE: Appendix A, the quaternion equations for Euler sequence (10), a 3, 2, 1 sequence.

NOTE: This subroutine calls "POSNOR" to take the normal of the positive quaternion.

YAW-PITCH-ROLL EULER ANGLES TO THE QUATERNION

I

```
@FOR,SI     YPRQ,YPRQ
FOR S0E3-02/17/77-06:24:03 (,0)

    SUBROUTINE YPRQ        ENTRY POINT 000114

    STORAGE USED: CODE(1) 000121; DATA(0) 000025; BLANK COMMON(2) 0

    EXTERNAL REFERENCES (BLOCK, NAME)

     0003    POSNOR
     0004    COS
     0005    SIN
     0006    NERR3$

    STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

     0000 R 000010 CP          0000 R 000011 CR          0000 R 000007 C
     0000 R 000004 HY          0000   000016 INJP$       0000 R 000000 
     0000 R 000012 SY


 00101      1*            SUBROUTINE YPRQ(YPR,Q0)
 00103      2*            DIMENSION YPR(3),Q(4),Q0(4)
 00104      3*            HY=0.50*YPR(1)
 00105      4*            HP=0.50*YPR(2)
 00106      5*            HR=0.50*YPR(3)
 00107      6*            CY=COS(HY)
 00110      7*            CP=COS(HP)
 00111      8*            CR=COS(HR)
 00112      9*            SY=SIN(HY)
 00113     10*            SP=SIN(HP)
 00114     11*            SR=SIN(HR)
 00115     12*            Q(1)=CY*CP*CR+SY*SP*SR
 00116     13*            Q(2)=CY*CP*SR-SY*SP*CR
 00117     14*            Q(3)=CY*SP*CR+SY*CP*SR
 00120     15*            Q(4)=-CY*SP*SR+SY*CP*CR
 00121     16*            CALL POSNOR(Q,Q0)
 00122     17*            RETURN
 00123     18*            END

        END OF COMPILATION:        NO  DIAGNOSTICS.
```

NAME: POSNOR

PURPOSE: To output the positive and normalized quaternion from the given quaternion.

INPUT: Q - The quaternion; ARRAY (4).

OUTPUT: QO - The positive-normalized quaternion; ARRAY (4).

ALGORITHM REFERENCE:

1. If the sign of Q(1) is negative:

   Set QO(I) = -Q(I) for I = 1, 2, 3, 4.

2. Set QO(I) = QO(I)/TEMP

   where TEMP = $\sqrt{QO_1^2 + QO_2^2 + QO_3^2 + QO_4^2}$

SELECTS THE POSITIVE QUATERNION AND NORMALIZES

```
@FOR,IS      POSNOR,POSNOR
FOR S1E3-02/19/77-06:24:14 (,0)

    SUBROUTINE POSNOR     ENTRY POINT 000055

    STORAGE USED: CODE(1) 000067; DATA(0) 000017; BLANK COMMON(2) 0

    EXTERNAL REFERENCES (BLOCK, NAME)

     0003    SQRT
     0004    NERR3$

    STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

     0001    000016 111G        0001   000036 120G        0000 I 000002
     0000 R 000000 TEMP


00101      1*              SUBROUTINE POSNOR(Q,QO)
00103      2*              DIMENSION Q(4),QO(4)
00104      3*              TEMP=1.0
00105      4*              IF(Q(1).LT.0.0) TEMP=-1.0
00107      5*              SUM=0.0
00110      6*              DO 50 I=1,4
00113      7*              QO(I)=TEMP*Q(I)
00114      8*              SUM=SUM+QO(I)*QO(I)
00115      9*           50 CONTINUE
00117     10*              TEMP=1.0/SQRT(SUM)
00120     11*              DO 100 I=1,4
00123     12*              QO(I)=TEMP*QO(I)
00124     13*          100 CONTINUE
00126     14*              RETURN
00127     15*              END

       END OF COMPILATION:        NO  DIAGNOSTICS.
```