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SOLUTION OF TRANSIENT OPTIMIZATION PROBLEMS BY USING AN ALGORITHM BASED ON NONLINEAR PROGRAMMING

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SOLUTION OF TRANSIENT OPTIMIZATION PROBLEMS BY USING AN ALGORITHM BASED ON NONLINEAR PROGRAMMING

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ABSTRACT

A new algorithm is presented for solution of dynamic optimization problems which are nonlinear in the state variables and linear in the control variables. It is shown that the optimal control is bang-bang. A nominal bang-bang solution is found which satisfies the system equations and constraints, and influence functions are generated which check the optimality of the solution. Nonlinear optimization (gradient search) techniques are used to find the optimal solution. The algorithm is used to find a minimum time acceleration for a turbofan engine.

I. INTRODUCTION

In recent years, linear-quadratic regulator theory has been developed for the design of multi-input, multi-output control systems. An account of the theory and application is given, for example, in reference 1. Use of the theory has been facilitated by computer programs such as those described in references 2 and 3, which rapidly and efficiently calculate the optimal feedback control gains, given the system description and performance index.

In addition to the design of regulators, the problem of minimizing the time required to transfer from one set point to another is also important; this problem cannot be solved systematically by use of linear quadratic regulator theory. Minimum-time trajectory optimization has been considered by many investigators for many years. Athans and Falb (ref. 4) give a good account of the literature dealing with time optimal systems in their chapter 7. However, nearly all of their discussion is concerned with problems having only a single control variable. Furthermore, the systems are assumed to be linear and time invariant, and the control limits are not dependent on the state or time. It is shown that the optimal control for such problems is bang-bang; i.e., the control always operates at either the upper or the lower limit.

Wolske (ref. 5) considered the problem of fuel-optimal control of a dynamic system which is nonlinear in the state and linear in the control. The controls are assumed to be bounded in magnitude, and the resulting optimal control is bang-bang. The problem is solved by linearizing about a nominal history, which is neither optimal nor feasible (i.e., it does not satisfy the terminal constraints). The optimality condition and terminal constraints are expressed as linear inequalities, and linear programming techniques are used to improve the solution until a feasible optimum is attained.

In this report, a new algorithm is developed for solution of dynamic optimization problems which are nonlinear in the state variables, and linear in the control variables. Specifically, the problem considered is to minimize a performance index subject to satisfaction of the system dynamic equations, a set of terminal constraints (the number of which may be less than or equal to the number of states) and path inequality constraints. The performance index, system equations and path constraints are all linear in the control variables.

It is shown that the optimal control is bang-bang, except for possible singular arcs, which are not considered. The algorithm requires that a nominal bang-bang solution be found that satisfies the system dynamic equations and terminal constraints. Once such a feasible solution has been found, influence functions are generated which check the optimality of the solution and determine whether or not additional control switches are needed. Nonlinear optimization (gradient search) techniques are used to find the values of the control switching times which result in a minimizing solution. The nonlinear optimization technique described in reference 6 is used to generate the numerical results presented in this report.

This algorithm is then used to find a minimum time acceleration history for the F100 engine, a two-spool turbofan engine used to power the F15 and F16 aircraft. A piecewise-linear engine model is used. The linear models used in this report were obtained by linearization of a nonlinear model at five equilibrium points, and were taken from reference 7. The linear model which applies at a given time in the trajectory is determined by calculating a normalized "distance" from the current state to the state at each of the equilibrium points; the linear model associated with the

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closest equilibrium point is then used. The model has three states and four controls. In addition, there are linear state/control constraints which correspond to speed, temperature, pressure, and mechanical control limits.

II. A NONLINEAR OPTIMIZATION PROBLEM, LINEAR IN CONTROL

Problem Statement

We consider a fairly general dynamic optimization problem, which is subject to one important restriction - the performance index, system equations, and path constraints must all be linear functions of the control. As will be shown later, this leads to a bang-bang solution. We wish to find the vector control history $u(t)$ which minimizes the scalar functional

$$J = \phi(x(t_f),t_f) + \int_{t_o}^{t_f} (a(x,t) + b^T(x,t)u)dt \tag{2.1}$$

subject to the vector system differential equations

$$\dot{x} = f(x,t) + g(x,t)u \tag{2.2}$$

and path inequality constraints

$$c_i(x,t) + d_i^T(x,t)u \leq 0 \qquad i = 1,2,\ldots,q \tag{2.3}$$

The initial state and time are assumed to be specified, i.e.,

$$x(t_o) = x_o$$

while the terminal state and time are subject to the p terminal constraints $(p \leq n + 1)$

$$\psi_i(x(t_f),t_f) = 0 \qquad i = 1,2,\ldots,p \leq (n + 1) \tag{2.4}$$

In the above, x is the $(n \times 1)$ state vector, and u is the $(r \times 1)$ control vector. The functions ϕ, a and c_i are scalar functions of x and t, while b and d_i are vector functions of dimension $(r \times 1)$. The vector function f and matrix function g have dimension $(n \times 1)$ and $(n \times r)$, respectively. The terminal time t_f may be either fixed or free.

The path constraints (2.3) serve to bound the allowable values of the control. A path constraint is said to be active if $c_i + d_i^T u = 0$; it is said to be inactive if $c_i + d_i^T u < 0$. State variable inequality constraints are not considered in this paper; the results presented herein are extended to include state variable inequality constraints in reference 8.

NECESSARY CONDITIONS FOR A LOCAL MINIMUM

Using the techniques employed in reference 9, a Hamiltonian function is defined as

$$H \triangleq a(x,t) + b^T(x,t)u + \lambda^T[f(x,t) + g(x,t)u] + \sum_{i=1}^{q} \mu_i\left[c_i(x,t) + d_i^T(x,t)u\right] \tag{2.5}$$

where λ and μ are undetermined Lagrange multiplier vector functions of time, having dimension $(n \times 1)$ and $(q \times 1)$, respectively.

The necessary conditions for a local minimum are

$$\dot{\lambda}^T = -\frac{\partial a}{\partial x} - \frac{\partial b^T}{\partial x}u - \lambda^T\frac{\partial f}{\partial x} - \lambda^T\frac{\partial g}{\partial x}u - \sum_{i=1}^{q}\left[\mu_i\frac{\partial c_i}{\partial x} + \mu_i\frac{\partial d^T}{\partial x}u\right] \tag{2.6a}$$

$$b^T + \lambda^T g + \sum_{i=1}^{q}\mu_i d_i^T = 0 \tag{2.6b}$$

$$\mu_i \geq 0; \; \mu_i = 0 \qquad \text{if} \qquad c_i + d_i^T u < 0 \tag{2.6c}$$

In addition, the Lagrange multipliers must satisfy the following terminal conditions.

$$\lambda^T(t_f) = \nu^T\frac{\partial\psi}{\partial x_f}(x(t_f),t_f) + \frac{\partial\phi}{\partial x_f}(x(t_f),t_f) \tag{2.7}$$

where ν is a $(p \times 1)$ undetermined parameter vector. If the terminal time is not specified, another necessary condition which applies is

$$H(t_f) = -\frac{\partial\phi(x(t_f),t_f)}{\partial t_f} - \nu^T\frac{\partial\psi(x(t_f),t_f)}{\partial t_f} \tag{2.8}$$

The optimal control is determined by satisfying (2.3) and (2.6b and c). For given x and t, the functions b, c, d, g, and λ are all constant. Therefore, the determination of the optimal u (for each x and t) is simply a linear programming problem, which can be readily solved by using the Simplex method (ref. 10). Except for singular arcs, which are not considered, the optimal control is always determined by r active constraints from (2.3). The optimization problem is simply to choose the r constraints which result in satisfaction of (2.6b and c).

The r constraints which satisfy (2.6b and c) change from time to time along the trajectory. When a change of constraints occurs, the control variables jump discontinuously from one boundary to another. Such control is referred to as bang-bang control, and the points at which the jumps occur are called junction points.

DETERMINATION OF OPTIMAL TRAJECTORY

We now assume that the linear programming problem has been solved to yield the active constraints and the optimal control. If we assume that the active constraints are constraints 1 through r and form these into a vector-matrix representation, i.e.,

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_r \end{pmatrix}, \quad D^T = \begin{pmatrix} d_1^T \\ d_2^T \\ \cdot \\ \cdot \\ \cdot \\ d_r^T \end{pmatrix}$$

then we can solve for the optimal control from

$$u = - D^{-T}C \tag{2.9}$$

where the superscript (-T) denotes the inverse of the transpose. Also, equation (2.6b) can be rewritten using vector-matrix representation for the active constraints as

$$b^T + \lambda^T g + \mu^T D^T = 0$$

and this equation can be solved for μ to yield

$$\mu = - D^{-1}(b + g^T\lambda) \tag{2.10}$$

Substitution of (2.9) and (2.10) into (2.2) and (2.6a) results in a simpler version of (2.2) and (2.6a)

$$\dot{x} = f - gD^{-T}C \tag{2.11}$$

$$\dot{\lambda}^T = - \frac{\partial}{\partial x} (a - b^T D^{-T}C) - \lambda^T \frac{\partial}{\partial x} (f - gD^{-T}C) \tag{2.12}$$

III. A NEW ALGORITHM

In this section, the nature of the optimal control strategy derived in Section II is used as the basis for a new algorithm for the solution of such optimization problems.

First, a feasible solution is obtained which satisfies all path and terminal constraints, and for which the controls always lie along r constraint boundaries. The Euler Lagrange equations are not utilized in the determination of this feasible solution; it may or may not be a local minimum. Then, it is shown that the Lagrange multiplier time history can be easily and uniquely calculated from this feasible solution. The optimality of the feasible solution is established by consideration of functions of the Lagrange multipliers. An improvement scheme which makes use of nonlinear optimization (gradient search) techniques to converge to a local minimum is proposed for use in those cases in which the initial feasible trajectory is not a local minimum.

Feasible Trajectory

In order for a trajectory to be a local minimum solution to (2.1), equations (2.11) and (2.12) of Section II must be satisfied, and the control must satisfy constraints (2.3) and optimality conditions (2.6b and c). In addition, terminal constraints (2.4) and (2.7) must be satisfied, and equation (2.8) must be satisfied if the terminal time t_f is free. In developing the numerical algorithm for the solution of this problem, it will be assumed initially that t_f is free. Later, we will show how to modify the algorithm for the case in which t_f is specified.

We define a feasible trajectory as one which satisfies the system differential equations (2.2) and the terminal constraints (2.4), and where the control $u(t)$ is consistent with the necessary conditions for optimality - that is, the control is determined by r of the path constraints (2.3) at all points along the trajectory. It is not necessary that the control be determined by the same r constraints at all points along the trajectory - in fact, we will usually require the control to be determined by several different constraint sets, as will be seen shortly. Such a trajectory may or may not be a local minimum solution for the performance index (2.1).

In order to obtain a feasible trajectory, there are p terminal constraints (eq. (2.4)) which must be satisfied. In general, there must be p degrees of freedom available in order to satisfy the p terminal constraints. In order to provide these degrees of freedom, it will be assumed that there are p segments in the trajectory. For each segment, the control is determined by choosing the r constraints which are active. The set of active constraints may be chosen arbitrarily for each segment except that the constraint sets may not be identical for any two adjacent segments. The durations of the p segments are variable, and provide the p degrees of freedom necessary to satisfy the p terminal constraints.

The choice of the constraints to be active for the various trajectory segments should be made carefully. There is no guarantee that a solution exists for arbitrary choice of the active constraints, or for any choice of active constraints, for that matter.

The combination of p degrees of freedom and p terminal constraints is known as a multipoint boundary value problem, and must generally be solved iteratively. Two widely used classes of methods for solution of such problems are Newton-Raphson methods and gradient methods.

Fixed Final Time

It was assumed in the above discussion that the terminal time t_f is free. If t_f is fixed, then the duration of the final segment does not provide a degree of freedom. Therefore, there must be (p + 1) segments for p terminal constraints, and the durations of the first p segments provide the necessary degrees of freedom.

CALCULATION OF LAGRANGE MULTIPLIERS

We consider first the case in which the terminal time is free. Suppose a feasible trajectory has been found. We will show that the Lagrange multipliers λ and μ can be uniquely calculated, as a function of time. Once the multipliers have been calculated, the local minimality of the feasible trajectory can be checked. If the feasible trajectory is not a local minimum, an iterative improvement scheme is developed which converges to a local minimum.

Let the i^{th} trajectory segment have initial time t_{i-1} and final time t_i. The control for the i^{th} segment was determined by a set of r active constraints, and the control just prior to t_i is denoted by $u(t_i^-)$. Just after $t = t_i$, the control is determined by different active constraints, and is given by $u(t_i^+) \neq u(t_i^-)$. It may be shown (ref. 9) that the Hamiltonian must be continuous at $t = t_i$. Therefore, we must have

$$\left[b^T(t_i) + \lambda^T g(t_i)\right]\left[u(t_i^+) - u(t_i^-)\right] = 0$$

$$i = 1,2,\ldots,(p-1) \qquad (3.1)$$

Equation (3.1) must be satisfied at each of the (p - 1) junction points between trajectory segments. Also, as shown in reference 9, the terminal λ and H must be given by (2.7) and (2.8). If we use (2.5) for H, and substitute for u and λ from (2.9) and (2.7) respectively, equation (2.8) is converted to

$$\nu^T \frac{d\psi}{dt}(t_f) + \frac{d\phi(t_f)}{dt_f} + a(t_f) - b^T(t_f)D^{-T}(t_f)C(t_f) = 0 \qquad (3.2)$$

Equations (3.1) and (3.2) give p equations which must be satisfied, and there are p multipliers which may be varied. Thus, we have a multipoint boundary value problem, similar to that which must be solved iteratively to determine a feasible trajectory. However, in the present case, iterative solution is not required. Instead we can solve for the parameters ν as follows:

First, we find (p + 1) backward solutions of the $\dot{\lambda}$ equation (2.12) for $\nu^T = (0,\ldots,0)$, $\nu^T = (1,0,\ldots,0)$, $\nu^T = (0,1,0,\ldots,0),\ldots,$ $\nu^T = (0,0,\ldots,0,1)$ where

$$\lambda^T(t_f) = \frac{\partial\phi}{\partial x_f} + \nu^T \frac{\partial\psi}{\partial x_f}$$

These backward solutions are called

$$\lambda^{(o)}(t),\ \lambda^{(1)}(t),\ \ldots,\ \lambda^{(p)}(t) \qquad (3.3)$$

Then $\lambda(t)$ is given by

$$\lambda(t) = \lambda^{(o)}(t) + \Lambda(t)\nu \qquad (3.4)$$

where $\Lambda(t)$ is defined as

$$\Lambda(t) \triangleq \left[\lambda^{(1)}(t) \,\vdots\, \ldots \,\vdots\, \lambda^{p}(t)\right] \qquad (3.5)$$

There are (p - 1) equations for the continuity of H:

$$\left[u(t_i^+) - u(t_i^-)\right]^T\Big[b(t_i) + g^T(t_i)\lambda^{(o)}(t_i)$$

$$+ g^T(t_i)\Lambda(t_i)\nu\Big] = 0 \qquad i = 1,2,\ldots,(p-1) \qquad (3.6)$$

and at t_f, we must have

$$a(t_f) - b^T(t_f)D^{-T}(t_f)C(t_f) + \frac{d\phi}{dt}(t_f) + \left(\frac{d\psi}{dt}(t_f)\right)^T \nu = 0 \qquad (3.7)$$

these equations may be put in matrix form as follows:

Define vectors r_i and q_i and matrices Q and R by

$$q_i^T \triangleq \left[u(t_i^+) - u(t_i^-)\right]^T g^T(t_i)\Lambda(t_i)$$

$$i = 1,\ldots,(p-1)$$

$$r_i^T \triangleq \left[u(t_i^+) - u(t_i^-)\right]^T\left[b(t_i) + g^T(t_i)\lambda^{(o)}(t_i)\right]$$

$$q_p^T \triangleq \left(\frac{d\psi(t_f)}{dt}\right)^T$$

$$r_p^T \triangleq a(t_f) - b^T(t_f)D^{-T}(t_f)C(t_f) + \frac{d\phi}{dt}(t_f) \qquad (3.8)$$

$$Q \triangleq \begin{pmatrix} q_1^T \\ \hline \vdots \\ \hline q_p^T \end{pmatrix} \quad R \triangleq \begin{pmatrix} r_1^T \\ \hline \vdots \\ \hline r_p^T \end{pmatrix} \qquad (3.9)$$

Then ν may be calculated from

$$\nu = - Q^{-1}R \qquad (3.10)$$

Once the ν are known, (2.7) can be used to calculate $\lambda^T(t_f)$, and $\lambda(t)$ can be obtained by integrating (2.12) backward in time, or by using (3.4).

Fixed Final Time

In the above development, it was assumed that the terminal time t_f is free. In the event that t_f is fixed, the equations for calculation of ν are easily modified. In this case, (3.2) is not applicable. Instead, there are p of equations (3.6), instead of (p - 1), since there are (p + 1) segments for this case. The p equations (3.6) are sufficient to calculate the p parameters, ν.

Improvement of Feasible Trajectory

The Lagrange multipliers can be used to determine if the initial feasible trajectory is a local minimum. The optimal control is obtained as a function of time by using equations (2.6b and c), with $x(t)$ and $\lambda(t)$ as determined from the feasible trajectory. If $u_{opt}(t)$ is identical to the control time history $u_{fe}(t)$ utilized in the feasible trajectory, then the initial feasible trajectory is a local minimum, and no further calculation need be made. On the other hand, if $u_{opt}(t)$ differs from $u_{fe}(t)$ for even a portion of the trajectory, then the feasible trajectory is not a local minimum.

Suppose, for example, the control $u_{opt}(t)$ differs from $u_{fe}(t)$ during trajectory segment k. Then the performance index (2.1) can be improved by splitting segment k into two parts, and using control as follows.

$$u(t) = u_{opt}(t), \; t_{k-1} \leq t \leq t_{sw}$$
$$u(t) = u_{fe}(t), \; t_{sw} \leq t \leq t_k \qquad (3.11)$$

where t_{sw} should be chosen to be only slightly greater than t_{k-1}, so that the modified trajectory differs only slightly from the initial feasible trajectory. Because of the modified control history, the new trajectory will not satisfy the p terminal constraints (2.4). Therefore, the original p junction times should be adjusted, while holding t_{sw} fixed, so that the terminal constraints are satisfied. Since the trajectory was modified only slightly from the initial feasible trajectory, the process of reconverging the trajectory should be accomplished easily.

Once a modified trajectory has been obtained and the terminal constraints satisfied, the Lagrange multiplier time history may be calculated for the modified trajectory in the same manner as it was calculated for the initial feasible trajectory. From the Lagrange multipliers, the gradient of the performance index with respect to the switching time, i.e., $\partial J/\partial t_{sw}$ can be obtained. It is shown in reference 8 that

$$\frac{\partial J}{\partial t_{sw}} = H(t_{sw}^-) - H(t_{sw}^+) = \lambda^T(t_{sw})g(t_{sw}) \left[u(t_{sw}^-) - u(t_{sw}^+)\right] + b^T(t_{sw})\left[u(t_{sw}^-) - u(t_{sw}^+)\right] \qquad (3.12)$$

The set of all such switching points t_{sw} and corresponding gradients, can be used in conjunction with a nonlinear search technique to search for the values of t_{sw} such that the gradient vector $\partial J/\partial t_{sw}$ is equal to, or nearly equal to, zero. At this point, a local minimum solution has been achieved.

IV. PIECEWISE-LINEAR MODELS

An important special case of the problem considered in Sections II and III is when the performance index, system equations and path constraints are all linear in both the state and the control. This case occurs frequently in practice because linear approximations to complex dynamic systems are readily available. Furthermore, solutions to linear problems are easier and less costly to obtain because the system and Euler-Lagrange equations can be integrated in closed form.

If the actual system is only slightly nonlinear, a single linear approximation to the nonlinear system may suffice over the full operating range. However, if greater accuracy is desired and/or the actual system is very nonlinear, a series of linear models may be used, each of which is obtained by linearizing about a different equilibrium point. Linear equations are still used to describe the system at each state point, but coefficients in the linear model vary from point to point. Such a model is called a piecewise linear model.

Suppose, for example, the nonlinear system is linearized about a number of equilibrium points. In the neighborhood of each equilibrium point, we have

$$\dot{x} = F_j(x - x_{ej}) + G_j(u - u_{ej}) \qquad (4.1)$$

where x_{ej} and u_{ej} are the equilibrium values of state and control at the equilibrium point j. The system matrices F_j and G_j also differ, in general, for each equilibrium point. The path constraints may also be linearized about each equilibrium point to yield

$$c_{ij}^T(x - x_{ej}) + d_{ij}^T(u - u_{ej}) + e_{ij} \leq 0$$
$$i = 1,2,\ldots,q \qquad (4.2)$$

The path constraint vectots c_{ij} and d_{ij} also differ for each equilibrium point.

With a piecewise linear model, the system is described by line r equations at each state point, but the linear system coefficients vary from one point to another. A question that arises is which equilibrium model applies best for a given state, x? It is natural to choose the equilibrium point which is closest to x in some sense. Since the various states do not necessarily have the same physical dimension, a normalized distance function is used to determine which equilibrium point is closest. For a given state x, the distance functions

$$I_j = (x - x_{ej})^T W(x - x_{ej}) \tag{4.3}$$

are calculated for each equilibrium point j, and the equilibrium point is chosen for which I_j is a minimum.

Since the I_j are continuous functions of x, model switches (say, from j to k) occur when $I_j = I_k$ or

$$(x - x_{ej})^T W(x - x_{ej}) = (x - x_{ek})^T W(x - x_{ek})$$

Necessary Conditions for Optimality

The problem to be solved is to minimize

$$J = \phi(x(t_f), t_f) + \int_{t_o}^{t_f} (a^T x + b^T u) dt$$

subject to the system differential equations

$$\dot{x} = F_j(x - x_{ej}) + G_j(u - u_{ej})$$

and path inequality constraints

$$c_{ij}^T(x - x_{ej}) + d_{ij}^T(u - u_{ej}) + e_{ij} \leq 0$$

$$i - 1, 2, \ldots, q$$

whe e the linear model index j is determined fron

$$\min_j (x - x_{ej})^T W(x - x_{ej})$$

The Hamiltonian is defined as

$$H \triangleq a^T x + b^T u + \lambda^T \left\{ F_j(x - x_{ej}) + G_j(u - u_{ej}) + \sum_{i=1}^{q} \mu_i \left[c_{ij}(x - x_{ej}) + d_{ij}^T(u - u_{ej}) + e_{ij} \right] \right\}$$

and the Euler-Lagrange equations are

$$\dot{\lambda}^T = - a^T - \lambda^T F_j - \sum_{i=1}^{q} \mu_i c_{ij}^T$$

where the model index j is the same function of time as determined by integration of the system equations. In addition, it is shown in reference 9 that the Lagrange multipliers are discontinuous at model switching points, the jump in λ being given by

$$\lambda(t_s^+) = \lambda(t_s^-) + \epsilon\alpha_{jk}$$

where

$$\alpha_{jk} \triangleq 2\left(x_k^T - x_j^T\right) W \tag{4.4}$$

and t_s is the time at which the model subscript changes from j to k. Furthermore, since the switching time is not specified a priori, the Hamiltonian must be continuous at $t = t_s$. Therefore, we must have

$$(\lambda(t_s^-) + \epsilon\alpha_{jk})^T \dot{x}(t_s^+) = \lambda^T(t_s^-)\dot{x}(t_s^-) \tag{4.5}$$

which gives

$$\epsilon = \frac{\lambda^T(t_s^-)\left[\dot{x}(t_s^-) - \dot{x}(t_s^+)\right]}{\alpha_{jk}^T \dot{x}(t_s^+)} \tag{4.6}$$

V. DEMONSTRATIVE EXAMPLE

The ideas developed in Sections II through IV are illustrated in this section with the aid of a numberical example. A problem will be solved in which the system equations are nonlinear in the state and linear in the control.

Problem Statement

Consider the problem of finding $u(t)$ which transfers the system

$$\dot{x} = - x^2 + u$$
$$\dot{y} = - 4y + u \tag{5.1}$$

subject to the control limits $|u| \leq 2$ from initial conditions $(x_o, y_o) = (1,1)$ to terminal conditions $(x_f, y_f) = (0,0)$ in minimum time.

We have in (2.1), (2.3) and (2.4)

$$\begin{aligned} &\phi = t_f, \ a = b = 0 \\ &C_1 = -2, \ d_1 = -1 \\ &C_2 = -2, \ d_2 = 1 \\ &\psi_1 = x_f, \ \psi_2 = y_f \end{aligned} \tag{5.2}$$

The Hamiltonian (2.5) is

$$H = \lambda_x(-x^2 + u) + \lambda_y(-4y + u) + \mu_1(-2 - u) + \mu_2(-2 + u) \tag{5.3}$$

and the Euler Lagrange equations are

$$\begin{aligned} \dot{\lambda_x} &= 2\lambda_x x \\ \dot{\lambda_y} &= 4\lambda_y \end{aligned} \tag{5.4}$$

The control is determined from

$$\min_u (\lambda_x + \lambda_y)u \tag{5.5}$$

which results in

$$u = -2 \ \mathrm{sgn}(\lambda_x + \lambda_y) \tag{5.6}$$

The terminal conditions on the Lagrange multipliers are

$$\begin{aligned} \lambda(t_f) &= \nu \\ H(t_f) &= -1 \end{aligned} \tag{5.7}$$

Initial Feasible Solution

Since there are two terminal constraints and the terminal time is free, the initial feasible trajectory must have two segments. The only possible control for these segments is $u = \pm 2$. Therefore, we assume tentatively that the optimal control history is given by

$$\begin{aligned} u &= -2, \ 0 \leq t \leq t_1 \\ u &= 2, \ t_1 \leq t \leq t_f \end{aligned} \tag{5.8}$$

with t and t_f to be determined such that $x_f = y_f = 0$. The state equations (5.1) can be integrated in closed form when (5.8) is used as the control. The result is

$$x(t_1) = \frac{1 - \sqrt{2} \tan \sqrt{2}\, t_1}{1 + \frac{1}{\sqrt{2}} \tan \sqrt{2}\, t_1}$$

$$y(t_1) = -\frac{1}{2}(1 - e^{-4t_1}) + e^{-4t_1} \tag{5.9}$$

Iterative solution of (5.9) for t_1 and t_f such that $x(t_f) = y(t_f) = 0$ results in

$$t_1 = 0.56, \ t_f = 0.69$$

CALCULATION OF LAGRANGE MULTIPLIERS

For this problem, we have $\partial\phi/\partial x = 0$ and $\partial\psi/\partial x = I$. Therefore, $\lambda(t_f) = \nu$. We must integrate (5.4) backward with three sets of initial conditions - $\lambda^{(0)}(t_f) = \begin{pmatrix}0\\0\end{pmatrix}$, $\lambda^{(1)}(t_f) = \begin{pmatrix}1\\0\end{pmatrix}$, $\lambda^{(2)}(t_f) = \begin{pmatrix}0\\1\end{pmatrix}$ in order to find $\Lambda(t_1)$ and $\lambda^{(0)}(t_1)$. Equations (5.4) may be integrated in closed form as follows:

$$\left.\begin{aligned} \lambda_y(t) &= \lambda_y(t_f)e^{4(t-t_f)} \\ \lambda_x(t) &= \frac{-2\lambda_x(t_f) - 2\lambda_y(t_f) - \lambda_y(t)(-4y(t) + 2)}{-x^2(t) + 2} \end{aligned}\right\} t \geq t_1$$

Therefore, we obtain

$$\lambda^{(0)}(t_1) = \begin{pmatrix}0\\0\end{pmatrix}, \ \lambda_1(t_1) = \begin{pmatrix}1.035\\0\end{pmatrix}, \ \lambda_2(t_1) = \begin{pmatrix}0\\0.595\end{pmatrix}$$

Equations (3.6) and (3.7) give

$$4(1,1)\begin{pmatrix}1.035 & 0\\0 & 0.595\end{pmatrix}\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix} = 0$$

and

$$(2,2)\begin{pmatrix}\nu_1\\\nu_2\end{pmatrix} = -1$$

Which can be expanded to give

$$\begin{aligned} 1.035\nu_1 + 0.595\nu_2 &= 0 \\ \nu_1 + \nu_2 &= -0.5 \end{aligned}$$

Solving for ν_1 and ν_2 yields

$$\nu_1 = 0.676, \ \nu_2 = -1.176$$

Therefore, from (5.7) we have

$$\lambda_x(t_f) = 0.676, \ \lambda_y(t_f) = -1.176 \tag{5.10}$$

For this problem, the state equations and Euler-Lagrange equations can be integrated in closed form. The result is

$$\left.\begin{aligned}
\lambda_y(t) &= -\,1.176e^{4(t-t_f)} \\
\lambda_x(t) &= \frac{-1 - \lambda_y(t)(-4y(t) + 2)}{-x^2(t) + 2} \\
x(t) &= \frac{-0.252 + 1.414 \tanh \sqrt{2}\,(t - t_1)}{1 - 0.178 \tanh \sqrt{2}\,(t - t_1)} \\
y(t) &= 0.5\;1 - e^{-4(t-t_1)} - 0.340\, e^{-4(t-t_1)}
\end{aligned}\right\} t_1 \le t \le t_f \tag{5.11}$$

$$\left.\begin{aligned}
\lambda_y(t) &= -\,0.699\, e^{4(t-t_1)} \\
\lambda_x(t) &= \frac{1 - \lambda_y(t)(4y(t) + 2)}{x^2(t) + 2} \\
x(t) &= \frac{1 - 1.414 \tan \sqrt{2}\, t}{1 + 0.707 \tan \sqrt{2}\, t} \\
y(t) &= -\,0.5(1 - e^{-4t}) + e^{-4t}
\end{aligned}\right\} 0 \le t \le t_1 \tag{5.12}$$

Optimality of Initial Feasible Solution

The initial feasible trajectory was obtained under the tentative assumption that the optimal u(t) is given by (5.8). According to (5.6), this control strategy is optimal only if

$$\begin{aligned}
\lambda_x + \lambda_y &> 0, \quad 0 \le t \le t_1 \\
\lambda_x + \lambda_y &< 0, \quad t_1 \le t \le t_f
\end{aligned} \tag{5.13}$$

To determine if this is the case, $(\lambda_x + \lambda_y)$ is calculated as a function of time from (5.11) and (5.12). The result is plotted in figure 1, and shows that for this problem, the initial feasible trajectory is a local minimum. Therefore, no further improvement needs to be made. The optimal trajectory, x(t) and y(t), is shown in figure 2.

VI. APPLICATION TO TURBOFAN CONTROL

In this chapter, the algorithm described in Sections III and IV is used to find optimal trajectories for an aircraft turbofan engine. Specifically, values of the control variables are found as a function of time, which allow the engine to accelerate from one steady state power setting to another as rapidly as possible, while adhering to the engine constraints.

ENGINE MODEL

Because of the virtual impossibility of using nonlinear feedback control theory for realistic systems, control system softwave is usually developed using linear models. For turbofan engine control system design, nonlinear dynamic simulations such as reference 11 are linearized about various equilibrium conditions, and linear models obtained. This process produces equations of the form

$$\dot{x} = F(x - x_e) + G(u - u_e) \tag{6.1}$$

where x_e and u_e are equilibrium values of state and control, respectively. Other engine variables which are not modeled as states are also of interest. Such variables will be called outputs, and denoted by y. The linearized output equations are given by

$$y = y_e + C(x - x_e) + D(u - u_e) \tag{6.2}$$

If the outputs have upper (or lower) bounds which must not be exceeded, then combined state/control path inequality constraints of the form

$$\begin{aligned}
y_e + C(x - x_e) + D(u - u_e) - y_{max} &\le 0 \\
y_{min} - y_e - C(x - x_e) - D(u - u_e) &\le 0
\end{aligned} \tag{6.3}$$

are produced. Mechanical limits on the control variables also have the general form of (6.3) with C = 0.

TRANSIENT PERFORMANCE

We consider the problem of minimizing the time required to accelerate the F100 engine from one equilibrium power setting to another as rapidly as possible. In solving this problem, a piecewise-linear model or the F100 engine will be used, having three state variables and four control variables. There are four equilibrium linear models at power level angle (PLA) settings of 36, 52, 67 and 83 degrees. The coefficient values in (6.1) to (6.3) are different for each of the models. The linear model which applies at a given time is selected by minimizing the quadratic function

$$I_i = (x - x_{ei})^T W (x - x_{ei}) \tag{6.4}$$

with respect to i; the model whose equilibrium state is "closest" to the actual state at that time is chosen to represent the engine.

The trajectory must also satisfy path inequality constraints given by

$$c_i^T x + d_i^T u + e_i \le 0, \quad i = 1, \ldots, q \tag{6.5}$$

The coefficients c_i, d_i, and e_i are different for each equilibrium model. Some of the path inequality constraints correspond to engine physical limits, others to control mechanical limits. Specifically, the constraints which apply to this problem are:

(1) Turbine inlet temperature cannot exceed the equilibrium value at the PLA = 83° equilibrium point by more than 50 degrees.

(2) Fan and compressor speeds cannot exceed the equilibrium values at the PLA = 83° equilibrium point by more than 50 RPM.

(3) Fan and compressor surge margins must not be less than five percent.

(4) Inlet guide vanes, compressor vanes, exhaust nozzle area and fuel flow rate must not exceed their mechanical limits.

The problem to be solved is stated as follows. Find controls $u(t)$, $0 \leq t \leq t_f$ which minimize the terminal error function.

$$\phi(x_f) = \sum_{i=1}^{3} \left(\frac{x_i - x_{id}}{x_{id}}\right)^2 \qquad (6.6)$$

while satisfying the system equations (6.1) and path constraints (6.3). A sequence of solutions to such problems for different acceleration times t_f may be used to find the minimum time solution for a given value of terminal error. Necessary conditions for an optimal solution are given in Section II. The problem is solved by using the new algorithm described in Sections III and IV.

RESULTS

Consider the problem of minimizing the terminal error for an acceleration from a part-power condition (PLA = 36°) to intermediate thrust. The final time is specified to be $t_f = 0.6$ seconds. The problem variables (states, outputs and controls) for the optimal trajectory are shown in figure 3. The state variables, fan speed, compressor speed and augmentor pressure, are shown as functions of time in figure 3(a), (b), and (c). It can be seen that the states approach the desired final values smoothly and with no overshoot.

The outputs are shown as a function of time in figures 3(d) to (g). Because of the way in which the outputs are defined in equation (6.3), these variables are in general discontinuous at model switching points and points of discontinuous control. However, if the piecewise linear model is a good representation of the engine, the discontinuities in the outputs at model switching points should be small. In figure 3, the discontinuities in the outputs are, for the most part, so small that they are not visible. These discontinuities are substantial only for the fan and compressor surge margins.

The optimal control strategy calls for turbine inlet temperature to have its maximum value for the entire trajectory; this is shown in figure 3(d). Fan and compressor surge margins remain well above acceptable minimums (figs. 3(f) and (g)). Thrust (fig. 3(e)) increases smoothly and monotonically with time.

The controls are shown as a function of time in figures 3(h) to (k). Fuel flow jumps at $t = 0$ from its idle value, then increases slowly to maintain constant turbine inlet temperature. The optimal value of nozzle area is constant. The inlet guide vanes and compressor variable vanes are constrained to be within ±7 degrees of the Bill-of-Material control scheduled values.

VII. CONCLUDING REMARKS

A new algorithm has been presented for solution of dynamic optimization problems which are nonlinear in the state variables, and linear in the control variables. It is shown that the optimal control for such problems is bang-bang, except for possible singular arcs, which are not considered. The algorithm requires that a nominal bang-bang solution be found that satisfies the system equations and terminal constraints. The Euler-Lagrange equations are not utilized in the determination of this feasible solution; it is generally not a local minimum. Equations are derived for the determination of the Lagrange multipliers (sensitivity functions) which correspond to the initial feasible solution. These sensitivity functions are then utilized, along with nonlinear optimization (gradient search) techniques, to converge to a local minimum.

The new algorithm has several advantages over methods currently in use for solution of such problems. First, the system dynamic equations are uncoupled from the Euler-Lagrange equations: the Euler-Lagrange equations are not utilized in the determination of the initial feasible solution. Second, use of the new algorithm minimizes the number of variables involved in the gradient search. With the new algorithm, the search variables are the constraint switching times - the times at which switches take place from one set of constraint functions to another. Other methods currently in use generally discretize the trajectory into a large number of intervals, and search for the optimal values of the controls for each interval.

LIST OF SYMBOLS

a scalar function in performance index (2.1)

b (r x 1) vector function in performance index (2.1)

c_i scalar path inequality constraint function in (2.3)

C (r x 1) vector function of active constraints

d_i (r x 1) vector path inequality constraint function in (2.3)

D	(r x r) matrix function of active constraints
e_{ij}	scalar path constraint limits
f	(n x 1) vector of system functions
F	(n x n) constant system matrix
g	(n x r) matrix of control distribution functions
G	(n x r) constant control distribution matrix
H	variational Hamiltonian
I_j	distance functions
J	performance index
n	number of states
p	number of terminal state constraints
PLA	power level angle, deg.
q	number of path inequality constraints
q_i	vectors defined by (3.10)
Q	matrices defined by (3.11)
r	number of controls
r_i	vectors defined by (3.10)
R	matrices defined by (3.11)
t	time
u	(r x 1) control vector
W	(n x n) weighting matrix
x	(n x 1) state vector
y	output variable
α_{jk}	model switching vectors
ϵ	jump parameter
ν	(p x undetermined parameter vector
λ	(n x 1) undetermined Lagrange multiplier vector
Λ	(n x n) block-diagonal system matrix
μ_i	scalar undetermined Lagrange multiplier
ϕ	scalar function of terminal conditions in (2.1)
ψ_i	scalar terminal constraint function

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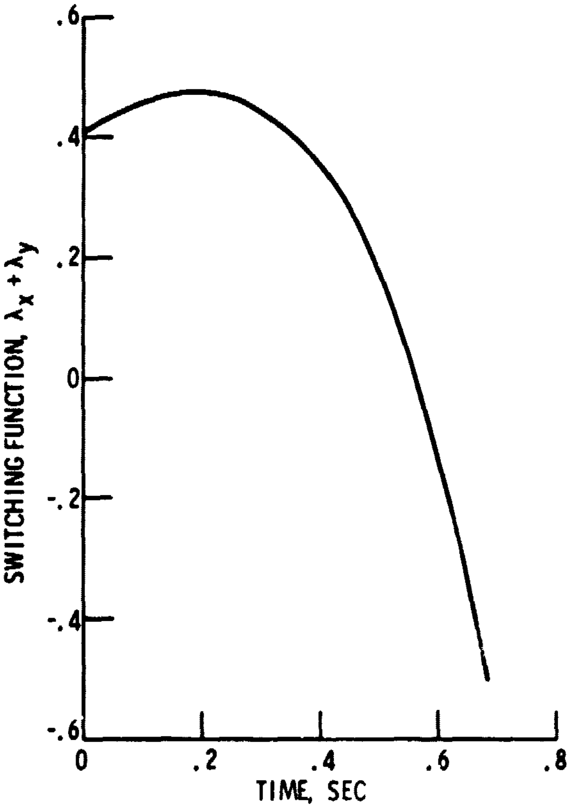

Figure 1. - Switching function as a function of time for initial feasible trajectory.

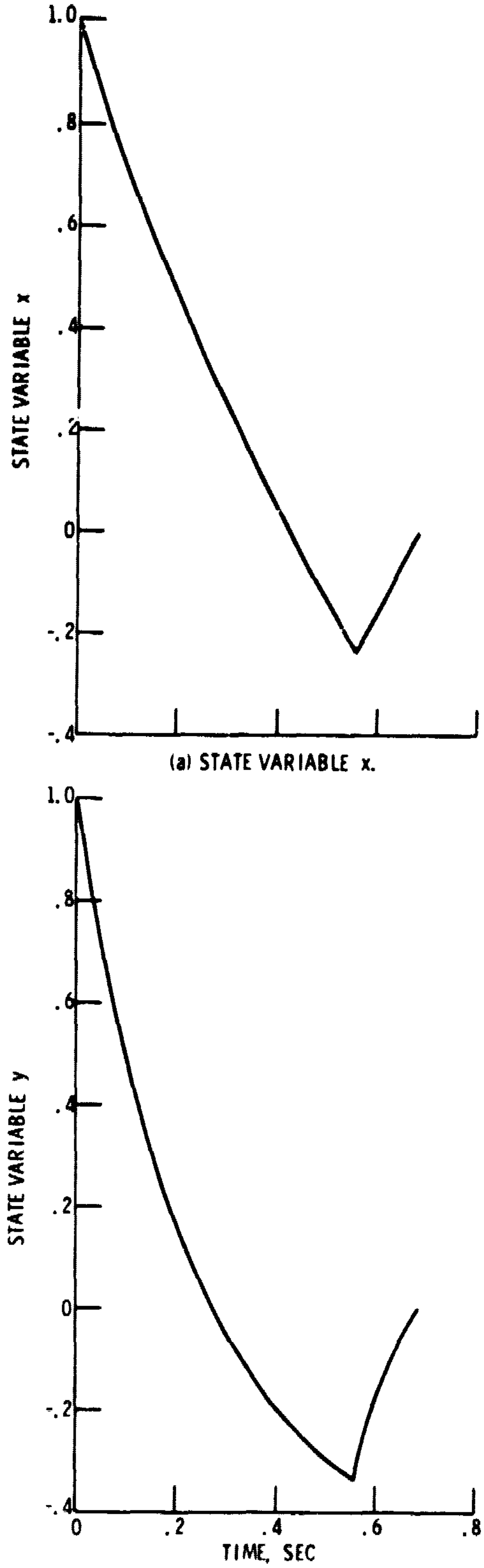

Figure 2. - Initial feasible trajectory for example problem.

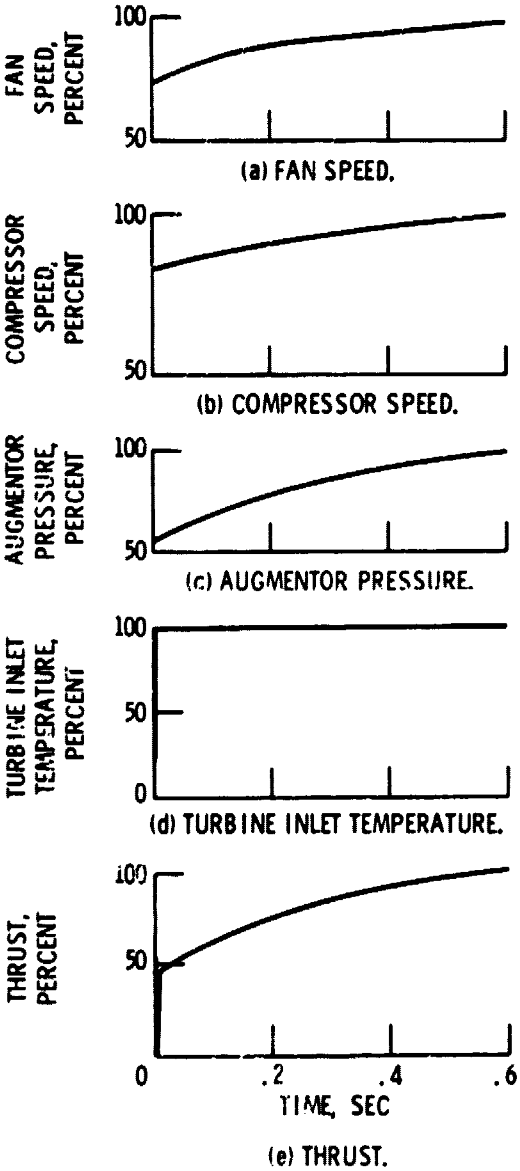

Figure 3. - Acceleration from part-power to intermediate.

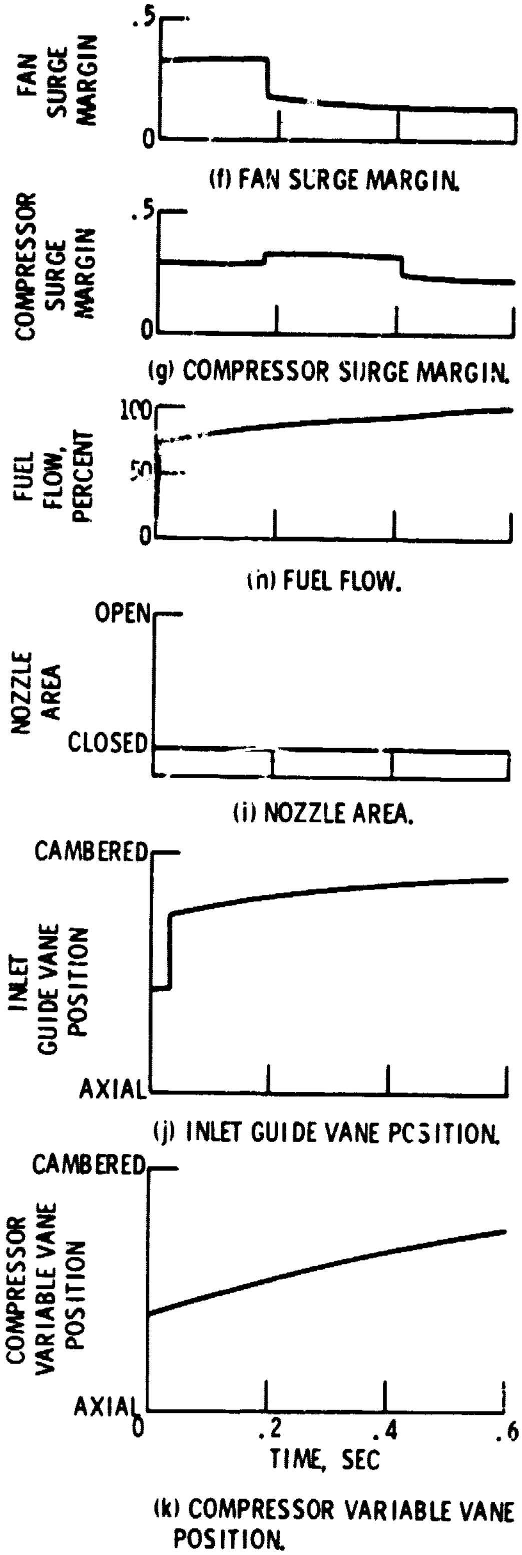

Figure 3. - Concluded.