January, 1977

ESL-R-720

Nasa Langley Research Grants
Grant NSG-1312
Grant NSG-1018

(NASA-CR-153281) LINEAR TRACKING SYSTEMS WITH APPLICATIONS TO AIRCRAFT CONTROL SYSTEM DESIGN (Massachusetts Inst. of Tech.) 124 p HC A06/MF A01 CSCL 01C

N77-27127

Unclas
G3/08 36749

LINEAR TRACKING SYSTEMS WITH APPLICATIONS TO AIRCRAFT CONTROL SYSTEM DESIGN

Wing Hong Lee
Michael Athans
David Castanon
Frank Bacchioloni

Electronic Systems Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Department of Electrical Engineering and Computer Science

January 1977 ESL-R-720

LINEAR TRACKING SYSTEMS WITH APPLICATIONS TO AIRCRAFT CONTROL SYSTEM DESIGN

by

Wing Hong Lee
Michael Athans
David Castanon
Frank Bacchioloni

Electronic Systems Laboratory
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Abstract

A class of optimal linear time-invariant tracking systems, both in continuous time and discrete time, of which the number of inputs (which are restricted to be step functions) is equal to the number of system outputs, are dealt with in this report. Along with derivation of equations and design procedures, two discretization schemes are presented, constraining either the control or its time derivative, to be a constant over each sampling period. Descriptions are given for the linearized model of the F-8C aircraft longitudinal dynamics, and the C* handling qualities criterion, which then serve as an illustration of the applications of these linear tracking designs. A suboptimal reduced state design is also presented. Numerical results are given for both the continuous time and discrete time designs.

Acknowledgement

This research was carried out in the M.I.T. Electronic Systems Laboratory with support extended by NASA Langley Research Center in part under grant NSG-1018 and in part under grant NSG-1312. We are indebted to Mr. Jarrell E. Elliott and Mr. Joseph Gera of NASA Langley Research Center for their constructive comments.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

This report deals with a class of optimal linear tracking systems, in continuous time and in discrete time. An application to the control of an aircraft is discussed, along with a suboptimal design.

The type of designs that this report focuses attention on is compensator design for a class of multi-input-multi-output linear time-invariant dynamical systems in which the number of controls is equal to the number of system outputs, and the system output vector is required to track a reference input vector of step functions with zero steady state error. It is found that the resulting designs are conceptually the proportional-integral-derivative (P-I-D) controller designs. Sandell [1], and Sandell and Athans [2], present a continuous time compensator design for such systems. Their designs suffer from the practical drawback that the initial jump in the control is very often too fast and may rate-saturate the control actuator. We pursue this design in continuous time, but eliminate the above stated effect by imposing continuity of the control everywhere. Holley and Bryson [3] give a regulator design with non-zero set points, and it is found to be of the same structure as the one that we are ensuing. Owing to the need for implementing control systems in digital computers, we concentrate much of our effort into getting a parallel design of a discrete time compensator.

The NASA F-8C digital-fly-by-wire (DFBW) aircraft is a high performance aircraft and is considered a very convenient testbed vehicle for many flight control systems. One of the purposes of this research is to investigate and to characterize the idea of designing a conceptual P-I-D controller for the linearized longitudinal dynamics of this aircraft, using a C* handling qualities criterion [10], [11].

The C* parameter of an aircraft is a linear combination of the pitch rate and the normal acceleration (and possibly the pitch angle) of the aircraft. We shall discuss its significance and motivation later, along with a description of the linearized models of the F-8C aircraft longitudinal dynamics. Briefly we can view the design problem as a tracking problem [15] as follows: the pilot, by shifting the command stick position, inputs a desired steady state value of the C* parameter to the compensator, which is then required to track the system C* response to this value, by minimizing an appropriate penalty function, with zero steady state error.

Athans, et al. [8] discusses briefly the desire of using the C* criterion in the control of the F-8C aircraft. Honeywell staff [12], [13] discuss a C*-design using model following techniques. Unfortunately, their design procedures are not transparent enough to us, at least from a theoretical point of view.

It is our desire that eventually our design procedure will be incorporated into the multiple model adaptive control (MMAC) scheme assumed in [8]. Henceforth, we follow a design philosophy similar to [8] and use the same set of data that it employs. We exploit this further in Chapters III and IV.

The remaining chapters of this report are organized as follows.

In Chapter II, we present most of the theoretical results required for the deterministic designs in this report, together with some discussions on these theoretical issues. First we derive the equations for the P-I-D controller in continuous time, as a modification of the results in [1] or [2]. Then we obtain similar results in the discrete time case. Owing to physical motivations, we discuss two discretization routines where we treat (1) the control, (2) the rate of the control, as a constant over each sampling period. In other words, method (1) uses a zero-order hold and method (2) a first-order hold. As we shall discuss later, due to sudden changes of the control, i.e. high rate of change, method (1) may not be physically satisfactory for aircraft implementations. However, the difficulty of getting a proper linear realization of method (2) may also impose additional problems.

After the completion of the work in this research, it has come to our attention that T.A.S.C. also came up to some P-I controller designs in [4] for VTOL aircraft control. The last part of this chapter is devoted to brief comparisons of these results.

In Chapter III, the general motivations of applying the C* criterion in the controller design of the F-8C aircraft linearized longitudinal dynamics are discussed. We present a review of the linearized models of the F-8C aircraft longitudinal dynamics (at 16 flight conditions), and the C* criterion. We then formulate the C* tracking problem with reference to the results in Chapter II.

Reference [8] presents a good approximation of the control gains and transient responses of the linearized F-8C aircraft full state longitudinal dynamics by a reduced state model. This approximation is also applicable to our present problem. The task of Chapter IV is to give an account of such a reduced state model. Theoretical comparisons between this model and the full state model are made.

In Chapter V, we present some numerical results, both in continuous time and discrete time. Deterministic designs, simulations, mismatching of flight conditions, and full state system versus reduced state system in performance are the major issues that we touch upon.

Finally, in Chapter VI, we discuss the overall results and draw some final conclusions. Also we outline some possibilities for future research.

CHAPTER II

P-I-D CONTROLLER DESIGNS

In this chapter, procedures are given for designing P-I-D controllers for multi-input multi-output linear time-invariant systems. Although results are first given for continuous time systems, the main parts of this chapter are devoted to discussions of discrete time systems, discretized from the continuous time systems via two different discretization schemes with sampling period T:

(1) For each integer k, we constrain the control to be a constant over the sampling interval [kT, (k+1)T).

(2) For each integer k, we constrain the time derivative of the control to be a constant over the sampling interval [kT, (k+1)T).

For the discrete time systems obtained by scheme (1), we derive three different sets of procedures for designing the P-I-D control gains. The first procedure is the discrete time equivalent of the continuous time design procedure; the other two are simply approximations of the control gains of the continuous time systems.

One drawback in using scheme (1) is that it introduces a series of steps in the control, and hence, a series of impulses in the control rate, which might then lead to rate saturation of the control actuator in many physical systems.

Scheme (2) is discussed in some detail. There, the derivation of the design procedures bears many similarities to that for scheme (1), although required conditions are somewhat stronger.

We divide this chapter into five sections. In Section 2.1, a design procedure for the continuous time case, which is almost identical to that given in [1], is presented. Sections 2.2 and 2.3 are discussions on the discretization scheme (1), the first of which deals with the scheme itself and a direct design procedure of the discrete time systems, while the latter is concerned with approximations of the continuous time design. Section 2.4 deals with discretization scheme (2) and Section 2.5 contains some final remarks and a comparison with the results in [4].

2.1 Continuous Time Designs

As a first result, we present a design procedure in the continuous time case, which can be proved by adding in a constraint that the control has to be continuous for all time, and then using the method employed by [1] or [2]. For this reason, the proof is omitted here. A derivation of this procedure, which is of a somewhat different flavor, can also be obtained if we attack this problem by using the results in [3].

<u>Proposition 2.1</u> Suppose that we are given a linear time-invariant dynamical system in continuous time:

$$\left.\begin{aligned} \dot{\underline{x}}(t) &= \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \qquad (2.1) \\ \underline{y}(t) &= \underline{C}\,\underline{x}(t) \qquad (2.2) \end{aligned}\right\} \text{System I}$$

where $\underline{x}(t) \in R^n$, $\underline{u}(t) \in R^m$, $\underline{y}(t) \in R^m$, and $\underline{A}$, $\underline{B}$, $\underline{C}$ are $n\times n$, $n\times m$, $m\times n$ real constant matrices respectively.

We assume that

(i) rank $\underline{A} = n$

(ii) $[\underline{A}, \underline{B}, \underline{C}]$ is a minimal realization

(iii) rank $\underline{C}\,\underline{A}^{-1}\underline{B} = m$.

We wish to minimize a cost functional,

$$J(\underline{u}) = \int_0^\infty \left[(\underline{z}_o - \underline{y}(\tau))'\,\underline{Q}(\underline{z}_o - \underline{y}(\tau)) + \dot{\underline{u}}'(\tau)\underline{R}\,\dot{\underline{u}}(\tau)\right]d\tau \qquad (2.3)$$

where $\underline{z}_o$ is an arbitrary step input, and $\underline{Q} \geq 0$, $\underline{R} > 0$ are $n\times n$ positive semi-definite and $m\times m$ positive definite matrices respectively. Then there exists a control $\underline{u}(t)$ such that the performance index J is minimized. The inputs are tracked with zero steady state error for all step inputs, and the closed loop system is stable. The control is given by

$$\underline{u}(t) = \int_0^t \underline{L}(\underline{z}_o - \underline{y}(\tau))d\tau + \underline{N}(\underline{x}(t) - \underline{x}_o) + \underline{u}(0) \qquad (2.4)$$

where

$$\underline{L} = (\underline{K}_2 - \underline{K}_1\underline{A}^{-1}\underline{B})(\underline{C}\ \underline{A}^{-1}\underline{B})^{-1} \qquad (2.5)$$

$$\underline{N} = (\underline{K}_1 + \underline{L}\ \underline{C})\underline{A}^{-1} \qquad (2.6)$$

The matrices $\underline{K}_1$, $\underline{K}_2$ are found from the positive definite solution of the algebraic matrix Riccati equation

$$\begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}'_{12} & \underline{P}_{22} \end{bmatrix} \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{A}' & \underline{0} \\ \underline{B}' & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}'_{12} & \underline{P}_{22} \end{bmatrix}$$

$$- \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}'_{12} & \underline{P}_{22} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \underline{R}^{-1} [\underline{0} \quad \underline{I}] \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}'_{12} & \underline{P}_{22} \end{bmatrix} + \begin{bmatrix} \underline{C}'\underline{Q}\ \underline{C} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \qquad (2.7)$$

$$= \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$

with

$$[\underline{K}_1 \quad \underline{K}_2] = -\underline{R}^{-1} [\underline{0} \quad \underline{I}] \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}'_{12} & \underline{P}_{22} \end{bmatrix} \qquad (2.8)$$

where $\underline{P}_{11}$, $\underline{P}_{12}$, $\underline{P}_{22}$ are matrices of dimensions $n\times n$, $n\times m$, $m\times m$ respectively.

The structure of such a control system is shown in Figure 2.1.

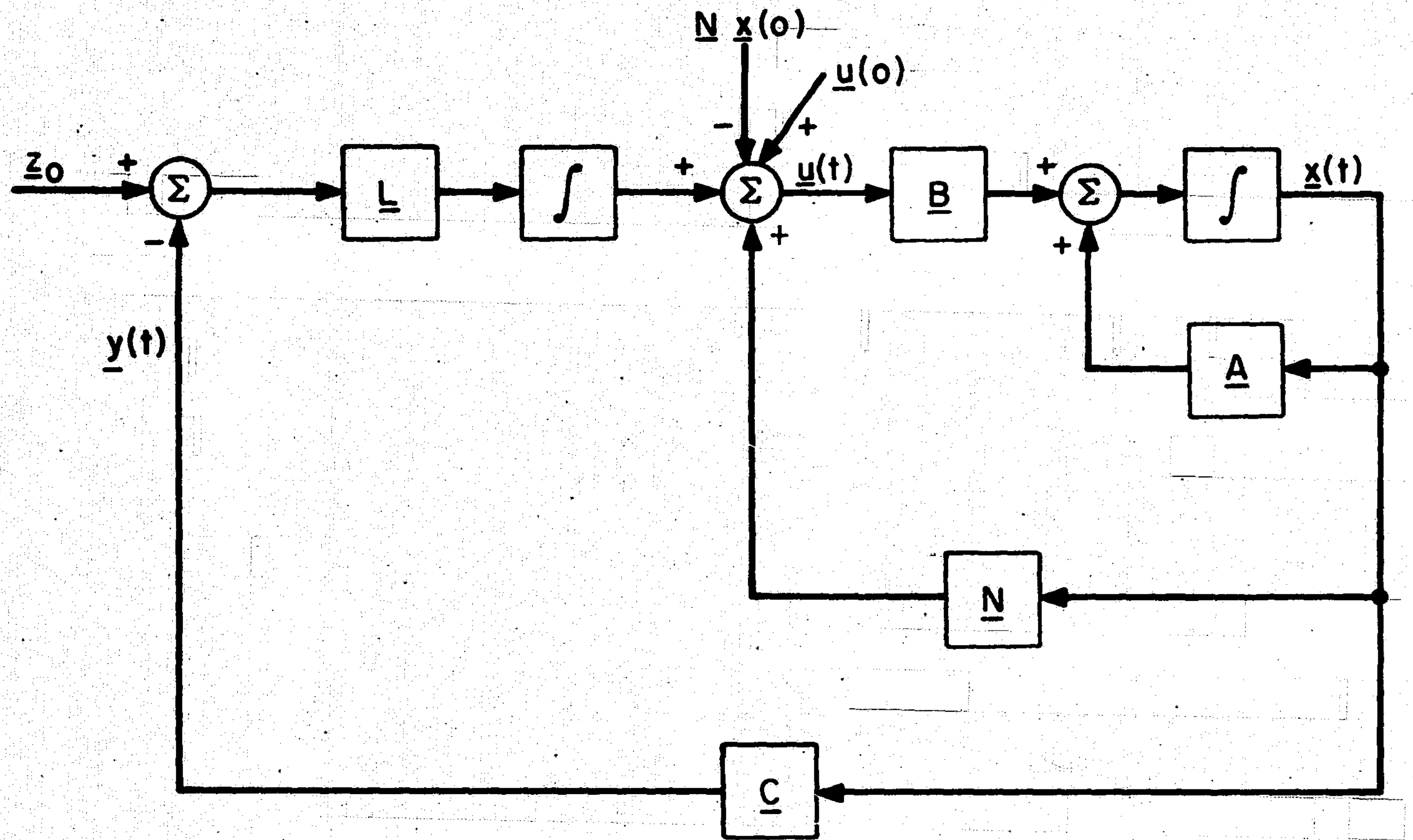


Fig. 2.1 Continuous Time Linear Tracking (P-I-D) Compensator Design

Note that in equation (2.4) $\underline{u}(0)$ can be arbitrary. In particular, there are the following choices.

The choice of $\underline{u}(0^+)$ in [1] optimizes the initial value to the integrator, in the sense that the cost functional in equation (2.3) is minimized with this particular choice. However, this procedure does not pay any regard to $u(0^-)$, or to be specific, the rate of change of the control at time $t = 0$, which is not penalized in the performance index at all. Such an initial jump in control may rate saturate the control system and lead to subsequent failure of the future control. On the other hand, [2] picks a choice such that the control, $\underline{u}(t)$, at all time t is independent of $\underline{x}(0)$. The problems of rate saturation are also present here, again due to the difference between $\underline{u}(0^+)$ and $\underline{u}(0^-)$. In order to eliminate rate saturation, we need to impose continuity of the control, in particular, if we constrain

$$\underline{u}(0^+) = \underline{u}(0^-) \tag{2.9}$$

Then it is easy to see that the control is continuous for all time t. This is indeed our choice for this report.

Now we present a similar design procedure for a discrete time system with a similar performance index.

2.2 Discrete Time Designs, Scheme (1)

In Section 2.1, a design procedure for continuous time systems is discussed. A similar procedure can be derived for a class of discrete time systems, namely, systems with a constant control over each sampling period, and a similar performance index. We discuss this in the following proposition. Then we summarize the relationship between this result and the previous proposition.

Proposition 2.2 Suppose that we are given a linear time-invariant dynamical system in discrete time (with sampling period T)

$$\underline{x}\left((k+1)T\right) = \underline{A}_d\underline{x}(kT) + \underline{B}_d\underline{u}(kT) \tag{2.10}$$

$$\underline{y}(kT) = \underline{C}_d\underline{x}(kT) \tag{2.11}$$

where $\underline{x}(kT) \in R^n$, $\underline{u}(kT) \in R^m$, $\underline{y}(kT) \in R^m$, and $\underline{A}_d$, $\underline{B}_d$, $\underline{C}_d$ are n×n, n×m, m×n real constant matrices respectively.

We assume that

(iv) rank $(\underline{A}_d - \underline{I}) = n$

(v) $[\underline{A}_d, \underline{B}_d, \underline{C}_d]$ is a minimal realization

(vi) rank $\underline{C}_d(\underline{A}_d - \underline{I})^{-1}\underline{B}_d = m$.

We want to minimize a cost functional,

$$J_d(\underline{u}) = \sum_{j=0}^{\infty} \Big[(\underline{z}_o - \underline{y}(jT))' \underline{Q}_d(\underline{z}_o - \underline{y}(jT)) + (\underline{u}((j+1)T) - \underline{u}(jT))' \underline{R}_d(\underline{u}((j+1)T) - \underline{u}(jT))\Big] \tag{2.12}$$

where $\underline{z}_o$ is an arbitrary step input, and $\underline{Q}_d \geq 0$, $\underline{R}_d > 0$ are n×n positive semi-definite and m×m positive definite matrices respectively.

Then there exists a control sequence such that the performance index J_d is minimized. The inputs are tracked with zero steady state error for all step inputs and the closed loop system is stable. The control is given by

$$\underline{u}(kT) = \sum_{j=0}^{k-1} \underline{L}_d(\underline{z}_o - \underline{y}(jT)) + \underline{N}_d(\underline{x}(kT) - \underline{x}(0)) + \underline{u}(0) \qquad (2.13)$$

where

$$\underline{L}_d = (\underline{K}_{2_d} - \underline{K}_{1_d}(\underline{A}_d - \underline{I})^{-1}\underline{B}_d)(\underline{C}_d(\underline{A}_d - \underline{I})^{-1}\underline{B}_d)^{-1} \qquad (2.14)$$

$$\underline{N}_d = (\underline{K}_{1_d} + \underline{L}_d\underline{C}_d)(\underline{A}_d - \underline{I})^{-1} \qquad (2.15)$$

The matrices $\underline{K}_{1_d}$, $\underline{K}_{2_d}$ are found from the positive definite solution of the algebraic matrix equation

$$\begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} = \begin{bmatrix} \underline{C}'_d\underline{Q}_d\underline{C}_d & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} + \begin{bmatrix} \underline{A}'_d & \underline{0} \\ \underline{B}'_d & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix}$$

$$- \begin{bmatrix} \underline{A}'_d & \underline{0} \\ \underline{B}'_d & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \left[[\underline{0} \quad \underline{I}] \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} + \underline{R}_d \right]$$

$$\times \; [\underline{0} \quad \underline{I}] \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix} \qquad (2.16)$$

with

$$\begin{bmatrix} \underline{K}_{1_d} & \underline{K}_{2_d} \end{bmatrix} = - \left[[\underline{0} \quad \underline{I}] \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}_{11_d} & \underline{P}_{12_d} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} + \underline{R}_d \right]^{-1} [\underline{0} \quad \underline{I}]$$

$$\times \begin{bmatrix} \underline{P}_{11_d} & \underline{P}_{12_d} \\ \underline{P}'_{12_d} & \underline{P}_{22_d} \end{bmatrix} \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix} \tag{2.17}$$

where $\underline{P}_{11_d}$, $\underline{P}_{12_d}$, $\underline{P}_{22_d}$ are matrices of dimensions n×n, n×m, m×m respectively.

Proof

Let

$$\tilde{\underline{x}}(kT) = \underline{x}(kT) - \underline{x}_o \tag{2.18a}$$

$$\tilde{\underline{u}}(kT) = \underline{u}(kT) - \underline{u}_o \tag{2.18b}$$

$$\tilde{\underline{v}}(kT) = \underline{u}((k+1)T) - \underline{u}(kT) \tag{2.18c}$$

where

$$\underline{x}_o = -(\underline{A}_d - \underline{I})^{-1}\underline{B}_d\underline{u}_o \tag{2.19}$$

$$\underline{u}_o = -(\underline{C}_d(\underline{A}_d - \underline{I})^{-1}\underline{B}_d)^{-1}\underline{z}_o \tag{2.20}$$

which is easily seen to be the unique solution of

$$\underline{x}_o = \underline{A}_d\, \underline{x}_o + \underline{B}_d\, \underline{u}_o \tag{2.21a}$$

$$\underline{z}_o = \underline{C}_d \underline{x}_o \tag{2.21b}$$

Then the dynamics of the system can be written as

$$\begin{bmatrix} \tilde{\underline{x}}((k+1)T) \\ \\ \tilde{\underline{u}}((k+1)T) \end{bmatrix} = \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \\ \underline{0} & \underline{I} \end{bmatrix} \begin{bmatrix} \tilde{\underline{x}}(kT) \\ \\ \tilde{\underline{u}}(kT) \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \\ \underline{I} \end{bmatrix} \underline{v}(kT) \tag{2.22}$$

and the performance index given in equation (2.12) is equivalent to

$$J = \sum_{j=0}^{\infty} \Big\{ [\tilde{\underline{x}}'(jT) \quad \tilde{\underline{u}}'(jT)] \begin{bmatrix} \underline{C}_d' \underline{Q}_d \underline{C}_d & \underline{0} \\ \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \tilde{\underline{x}}(jT) \\ \\ \tilde{\underline{u}}(jT) \end{bmatrix}$$

$$+ \underline{v}'(jT)\, \underline{R}_d\, \underline{v}(jT) \Big\} \tag{2.23}$$

Hence the problem is reduced to an optimal linear quadratic regulator problem. Since $\underline{C}_d' \underline{Q}_d \underline{C}_d \geq 0$, $\underline{R}_d > 0$, by [5], there exists a solution to this regulator problem given by

$$\underline{v}(kT) = [\underline{K}_{1_d} \quad \underline{K}_{2_d}] \begin{bmatrix} \tilde{\underline{x}}(kT) \\ \\ \tilde{\underline{u}}(kT) \end{bmatrix} \tag{2.24}$$

where $\underline{K}_{1_d}$, $\underline{K}_{2_d}$ are obtained by equations (2.16) and (2.17).

Note that

$$\underline{B}_d\, \tilde{\underline{u}}(kT) = \tilde{\underline{x}}((k+1)T) - \underline{A}_d\, \tilde{\underline{x}}(kT) \qquad (2.25)$$

and assumption (vi) implies that $\underline{B}_d$ is of rank m. Therefore there exists a m×n matrix $\underline{F}_d$ such that

$$\underline{F}_d\underline{B}_d = \underline{I} \qquad (2.26)$$

which gives us

$$\tilde{\underline{u}}(kT) = \underline{F}_d\, \tilde{\underline{x}}((k+1)T) - \underline{F}_d\underline{A}_d\, \tilde{\underline{x}}(kT) \qquad (2.27)$$

Substituting this into equation (2.24), we get

$$\underline{v}(kT) = (\underline{K}_{1_d} - \underline{K}_{2_d}\, \underline{F}_d\underline{A}_d)\, \tilde{\underline{x}}(kT) + \underline{K}_{2_d}\underline{F}_d\, \tilde{\underline{x}}((k+1)T) \qquad (2.28)$$

or equivalently,

$$\underline{v}(kT) = (\underline{K}_{1_d} - \underline{K}_{2_d}\underline{F}_d(\underline{A}_d - \underline{I}))\, \tilde{\underline{x}}(kT) + \underline{K}_{2_d}\underline{F}_d(\tilde{\underline{x}}((k+1)T) - \tilde{\underline{x}}(kT)) \qquad (2.29)$$

Now choose $\underline{L}_d$ such that

$$\underline{K}_{1_d} - \underline{K}_{2_d}\underline{F}_d(\underline{A}_d - \underline{I}) = -\underline{L}_d\underline{C}_d \qquad (2.30)$$

which gives

$$\underline{F}_d = \underline{K}_{2_d}^{-1}(\underline{K}_{1_d} + \underline{L}_d\underline{C}_d)(\underline{A}_d - \underline{I})^{-1} \tag{2.31}$$

But from equation (2.26)

$$\underline{I} = \underline{K}_{2_d}^{-1}(\underline{K}_{1_d} + \underline{L}_d\underline{C}_d)(\underline{A}_d - \underline{I})^{-1}\underline{B}_d \tag{2.32}$$

Therefore,

$$\underline{L}_d = (\underline{K}_{2_d} - \underline{K}_{1_d}(\underline{A}_d - \underline{I})^{-1}\underline{B}_d)(\underline{C}_d(\underline{A}_d - \underline{I})^{-1}\underline{B}_d)^{-1} \tag{2.33}$$

Using equation (2.30) and noting that

$$\underline{C}_d\ \underline{\tilde{x}}(kT) = \underline{y}(kT) - \underline{z}_o \tag{2.34a}$$

and

$$\underline{\tilde{x}}((k+1)T) - \underline{\tilde{x}}(kT) = \underline{x}((k+1)T) - \underline{x}(kT) \tag{2.34b}$$

we get

$$\underline{v}(kT) = \underline{L}_d(\underline{z}_o - \underline{y}(kT)) + \underline{K}_{2_d}\underline{F}_d(\underline{x}((k+1)T) - \underline{x}(kT)) \tag{2.35}$$

i.e.

$$\underline{u}((k+1)T) - \underline{u}(kT) = \underline{L}_d(\underline{z}_o - \underline{y}(kT)) + \underline{K}_{2_d}\underline{F}_d(\underline{x}((k+1)T) - \underline{x}(kT)) \tag{2.36}$$

which can also be written as

$$\underline{u}(kT) = \sum_{j=0}^{k-1} \underline{L}_d(\underline{z}_o - \underline{y}(jT)) + \underline{N}_d(\underline{x}(kT) - \underline{x}(0)) + \underline{u}(0) \qquad (2.37)$$

where

$$\underline{N}_d = \underline{K}_{2_d}\underline{F}_d \qquad (2.38)$$

and this completes the proof of this proposition.

Notice how similar this solution is to the one in proposition 2.1. The structure of the compensator thus constructed is shown in Figure 2.2.

A natural question would be what equivalence can one get between this discrete time system and system I. We answer this question in the following proposition.

<u>Proposition 2.3</u> Suppose that we discretize system I in proposition 2.1 with sampling period T such that for each integer k, the control $\underline{u}(t)$ is constrained to be a constant over the sampling interval [kT, (k+1)T), i.e. for each t in this interval,

$$\underline{u}(t) = \underline{u}(kT); \quad kT \le t \le (k+1)T \qquad (2.39)$$

Then we have a sample-data realization as follows,

$$\left.\begin{aligned} \underline{x}((k+1)T) &= \underline{A}_d\underline{x}(kT) + \underline{B}_d\,\underline{u}(kT) \qquad (2.40) \\ \underline{y}(kT) &= \underline{C}_d\,\underline{x}(kT) \qquad (2.41) \end{aligned}\right\}\ \text{System II}$$

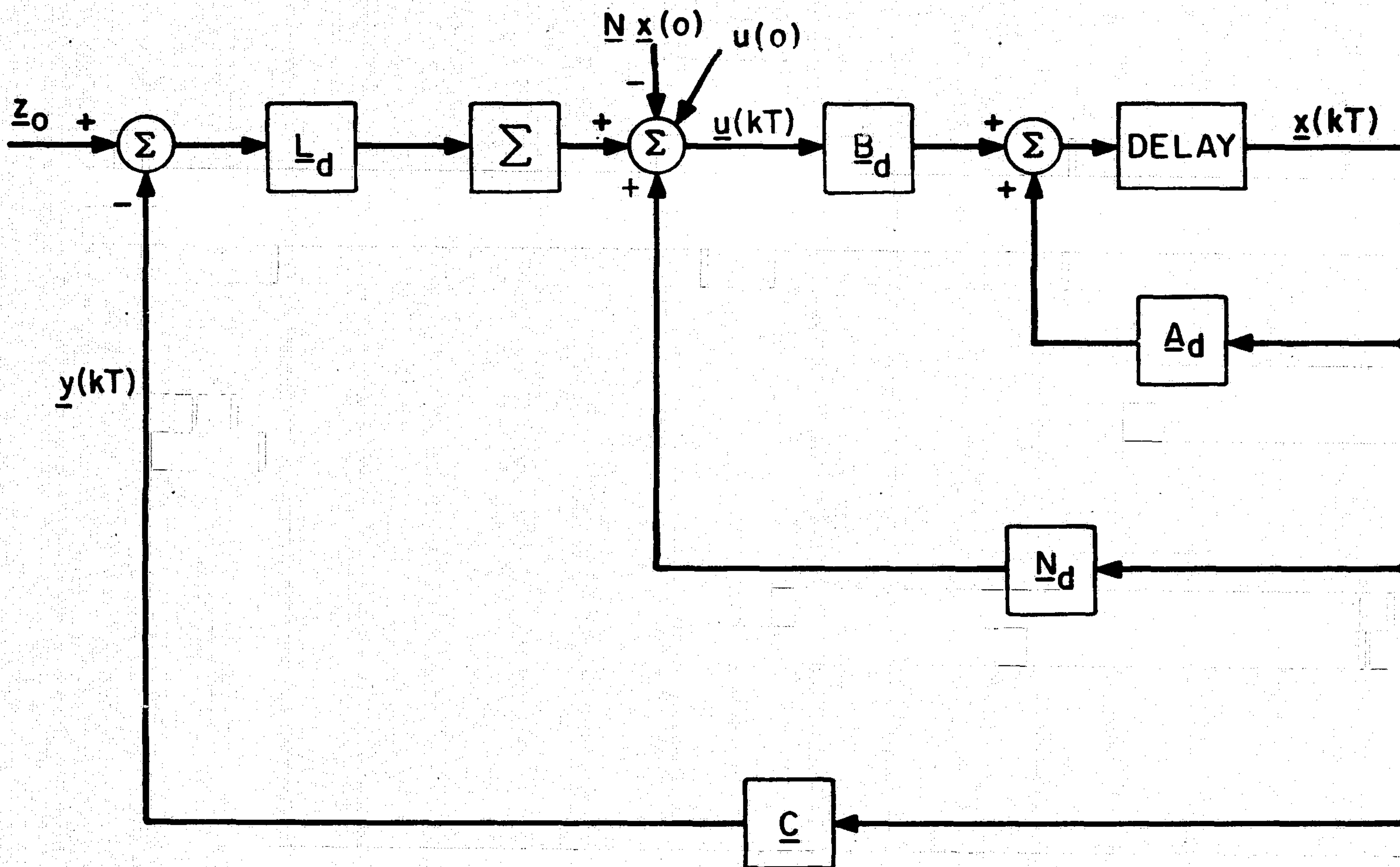


Fig. 2.2 Discrete Time Linear Tracking (P-I-D) Compensator Design

with $\underline{A}_d$, $\underline{B}_d$, $\underline{C}_d$ matrices given by the relations

$$\underline{A}_d = e^{\underline{A}T} \tag{2.42}$$

$$\underline{B}_d = \int_0^T e^{\underline{A}\tau} d\tau \; \underline{B} \tag{2.43}$$

$$= \underline{A}^{-1}(e^{\underline{A}T} - \underline{I})\underline{B}$$

$$\underline{C}_d = \underline{C} \tag{2.44}$$

where $\underline{A}$, $\underline{B}$, $\underline{C}$ are the matrices defined by system I, eqs. (2.1),(2.2).

If in addition, we have for all integer k

$$\mathrm{Im}[\lambda_i(\underline{A}) - \lambda_j(\underline{A})] \neq k\,\frac{2\pi}{T} \tag{2.45}$$

whenever

$$\mathrm{Re}[\lambda_i(\underline{A}) - \lambda_j(\underline{A})] = 0 \tag{2.46}$$

for $i \neq j$, where $\lambda_i(\underline{A})$, $\lambda_j(\underline{A})$ are (distinct) eigenvalues of $\underline{A}$, then the assumptions (i), (ii), (iii), in proposition 2.1 imply the assumptions (iv), (v), (vi) in proposition 2.2.

<u>Proof</u>

Here we only prove the validity of assumptions (iv) and (vi) due to (i) and (iii). The other parts of this proposition are all contained in [6], hence their proofs are omitted.

Since assumption (i), i.e.

$$\mathrm{Rank}\ \underline{A} = n$$

implies that for all i, $0 < i \leq n$

$$\lambda_i(\underline{A}T) \neq 0$$

therefore

$$e^{\lambda_i(\underline{A}T)} - 1 \neq 0$$

But for every $0 < j \leq n$, there exists an $0 < i \leq n$ such that

$$\lambda_j(e^{\underline{A}T} - \underline{I}) = e^{\lambda_i(\underline{A}T)} - 1$$

i.e.

$$\text{rank}(\underline{A}_d - \underline{I}) = n$$

which proves assumption (iv).

Now note that

$$\underline{C}_d(\underline{A}_d - \underline{I})^{-1}\underline{B}_d = \underline{C}(e^{\underline{A}T} - \underline{I})^{-1}\underline{A}^{-1}(e^{\underline{A}T} - \underline{I})\underline{B}$$

$$= \underline{C}\,\underline{A}^{-1}\underline{B}$$

Assumption (vi) is then obvious from (iii).

From now on we shall refer to this discrete time system as system II. Note that this system does not give us an equivalent† performance index (equation (2.12)) to that in system I (equation (2.3)).

---

†equivalent to the extent that $\underline{u}(t)$ is kept to be a constant over each sampling interval while evaluating the cost functional in equation (2.3).

2.3 Two Approximate Schemes for Discretizing the Continuous Time Compensator Design

The objective of this section is to investigate two ways of how the continuous time linear tracking compensator design presented in Section 2.1 can be discretized into discrete time designs by making some approximations.

The discretized system that we want to characterize here is one which has a constant control over each sampling interval. To be specific, given a continuous time system and a control law which tracks the system output to a step input, $\underline{z}_0$,

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \tag{2.47}$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) \tag{2.48}$$

$$\underline{u}(t) = \underline{L}\int_0^t (\underline{z}_0 - \underline{y}(\tau))d\tau + \underline{N}(\underline{x}(t) - \underline{x}(0)) + \underline{u}(0) \tag{2.49}$$

where $\underline{L}$, $\underline{N}$ are matrices specified by equations (2.5)-(2.8), we propose to find a control law with control gains $\underline{L}_D$, $\underline{N}_D$, given by

$$\underline{u}(kT) = \underline{L}_D \sum_{j=0}^{k-1} (\underline{z}_0 - \underline{u}(jT)) + \underline{N}_D(\underline{x}(kT) - \underline{x}(0)) + \underline{u}(0) \tag{2.50}$$

$$\underline{u}(t) = \underline{u}(kT) \quad \text{for } kT \leq t < (k+1)T \tag{2.51}$$

which represents an approximation of the continuous time control law (equation (2.49)), and is applicable to the sampled dynamics of the system, with sampling period T,

$$\underline{x}((k+1)T) = \underbrace{e^{\underline{A}T}}_{\underline{A}_D}\underline{x}(kT) + \underbrace{\int_0^T e^{\underline{A}\tau}d\tau\ \underline{B}}_{\underline{B}_D}\ \underline{u}(kT) \tag{2.52}$$

$$\underline{y}(kT) = \underline{C}\ \underline{x}(kT) \tag{2.53}$$

We proceed to find $\underline{L}_D$, $\underline{N}_D$ by two different sets of approximations:

(i) First-order approximation

We assume that approximately,

$$\underline{L}\int_0^{kT} (\underline{z}_0 - \underline{y}(\tau))d\tau \approx T\ \underline{L}\sum_{j=0}^{k-1} (\underline{z}_0 - \underline{y}(jT)) \tag{2.54}$$

Here as a first-order approximation, we take

$$\underline{L}_D = T\ \underline{L} \tag{2.55}$$

$$\underline{N}_D = \underline{N} \tag{2.56}$$

(ii) Approximation with constant integral over sampling interval and pole-allocation

Noting that $\underline{N}$, and $\underline{N}_D$ are feedback gains on the states in the continuous time and discrete time systems, respectively, we try to make use of the pole allocation method. In closed-loop form, the system dynamics can be written as

$$\dot{\underline{x}}(t) = (\underline{A} + \underline{B}\ \underline{N})\underline{x}(t) + \underline{B}\ \underline{L}\int_0^t (\underline{z}_o - \underline{y}(\tau))d\tau + \underline{B}(\underline{u}(0)-\underline{N}\ \underline{x}(0)) \tag{2.57}$$

for the continuous time, and

$$\underline{x}((k+1)T) = (\underline{A}_D + \underline{B}_D\underline{N}_D)\underline{x}(kT) + \underline{B}_D\underline{L}_D \sum_{j=0}^{k-1} (\underline{z}_0 - \underline{y}(jT))$$

$$+ \underline{B}_D(\underline{u}(0) - \underline{N}_D\underline{x}(0)) \tag{2.58}$$

for the discrete time. We assume that

$$\int_0^t (\underline{z}_0 - \underline{y}(\tau))d\tau \approx T \sum_{j=0}^{k-1} (\underline{z}_0 - \underline{y}(jT)) \tag{2.59}$$

for $kT \leq t < (k+1)T$, and

$$\underline{B}(\underline{u}(0) - \underline{N}\,\underline{x}(0)) = \underline{B}_D(\underline{u}(0) - \underline{N}_D\underline{x}(0)) \tag{2.60}$$

which is exactly true when $\underline{u}(0)$ and $\underline{x}(0)$ are zero. Therefore, we may now write

$$\underline{A}_D + \underline{B}_D\underline{N}_D = e^{(\underline{A} + \underline{B}\,\underline{N})T} \tag{2.61}$$

$$\underline{B}_D\underline{L}_D = T\int_0^T e^{(\underline{A} + \underline{B}\,\underline{N})\tau}d\tau\underline{B}\,\underline{L} \tag{2.62}$$

which give us, since $\underline{B}$, hence $\underline{B}_D$ is of maximal rank, a (minimum norm) solution

$$\underline{N}_D = (\underline{B}_D'\underline{B}_D)^{-1}\underline{B}_D'[e^{(\underline{A} + \underline{B}\,\underline{N})T} - \underline{A}_D] \tag{2.63}$$

$$\underline{L}_D = T(\underline{B}_D'\underline{B}_D)^{-1}\underline{B}_D' \int_0^T e^{(\underline{A} + \underline{B}\,\underline{N})\tau}d\tau\underline{B}\,\underline{L} \tag{2.64}$$

2.4 A Discretization Scheme Which Reduces Rate Saturation

In the last two sections, we discussed a discretization scheme which constrains the control to be a constant over each sampling interval, and some compensator designs associated with it. These designs, however, are susceptible to control rate saturation because the derived controls are step functions. As an alternative, in this section, we present a second discretization scheme which constrains the time derivative of the control to be a constant over each sampling interval, but enjoys an equivalent† performance index to that in system I, subject to this constraint. This has the merit that the problem of control rate saturation can be reduced considerably.

<u>Proposition 2.4</u> If we discretize system I with sampling period T by constraining the time rate of the control, $\dot{\underline{u}}(t)$, to be a constant over the sampling period [kT, (k+1)T) for each integer k, i.e.

$$\dot{\underline{u}}(t) = \dot{\underline{u}}(kT) = \text{constant} \quad \text{for } kT \leq t < (k+1)T \tag{2.65}$$

then we get an equivalent realization in discrete time,

$$\left.\begin{aligned} \begin{bmatrix} \underline{x}((k+1)T) \\ \underline{u}((k+1)T) \end{bmatrix} &= \underbrace{\begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix}}_{\underline{\Phi}} \begin{bmatrix} \underline{x}(kT) \\ \underline{u}(kT) \end{bmatrix} + \underbrace{\begin{bmatrix} \underline{D}_o \\ T\,\underline{I} \end{bmatrix}}_{\underline{D}} \dot{\underline{u}}(kT) & (2.66) \\ \underline{y}(kT) &= [\underline{C} \quad \underline{0}] \begin{bmatrix} \underline{x}(kT) \\ \underline{u}(kT) \end{bmatrix} & (2.67) \end{aligned}\right\} \text{System III}$$

---

†equivalent to the extent that $\dot{\underline{u}}(t)$ is kept to be constant over each sampling interval while evaluating the cost functional in equation (2.3).

where $\underline{A}_d$, $\underline{B}_d$ are matrices from system II, given in propositions 2.2 and 2.3, the $\underline{C}$ matrix is from system I, and $\underline{D}_o$ is given by

$$\underline{D}_o = \int_o^T \int_o^s e^{\underline{A}\tau} d\tau \, ds \, \underline{B} \tag{2.68}$$

where $\underline{A}$, $\underline{B}$ are from system I.

Moreover, we obtain an equivalent performance index,

$$\begin{aligned} J = \sum_{j=0}^{\infty} \Big\{ & [(\underline{x}(jT) - \underline{x}_o)' \quad (\underline{u}(jT) - \underline{u}_o)'] \, \hat{\underline{Q}} \begin{bmatrix} \underline{x}(jT) - \underline{x}_o \\ \\ \underline{u}(jT) - \underline{u}_o \end{bmatrix} \\ & + z[(\underline{x}(jT) - \underline{x}_o)' \quad (\underline{u}(jT) - \underline{u}_o)'] \hat{\underline{M}} \, \dot{\underline{u}}(jT) \\ & + \dot{\underline{u}}(jT) \hat{\underline{R}} \, \dot{\underline{u}}(jT) \end{aligned} \tag{2.69}$$

where

$$\hat{\underline{Q}} = \begin{bmatrix} \int_o^T e^{\underline{A}'t} \underline{C}' \underline{Q} \, \underline{C} \, e^{\underline{A}t} dt & \underline{0} \\ \\ \underline{0} & \underline{0} \end{bmatrix} \tag{2.70}$$

$$\hat{\underline{M}} = \begin{bmatrix} \int_o^T e^{\underline{A}'t} \underline{C}' \underline{Q} \, \underline{C} \, [\int_o^t \int_o^s e^{\underline{A}\tau} d\tau ds] \underline{B} \, dt \\ \\ \underline{0} \end{bmatrix} \tag{2.71}$$

$$\hat{\underline{R}} = \underline{R} \, T \tag{2.72}$$

with

$$\underline{x}_o = -(\underline{A}_d - I)^{-1} \underline{B}_d \underline{u}_o \tag{2.73}$$

$$\underline{u}_o = -(\underline{C}(\underline{A}_d - \underline{I})^{-1} \underline{B}_d)^{-1} \underline{z}_o \tag{2.74}$$

which are the steady state values of $\underline{x}(t)$ and $\underline{u}(t)$ in both continuous time and discrete time.

Proof

Firstly note that $\underline{x}_o$, $\underline{u}_o$ given by equations (2.73) and (2.74) are unique solutions to any one set of the three following sets of equations, thus establishing the equivalence of steady state values in continuous time and discrete time.

$$\alpha \begin{cases} \underline{0} = \underline{A}\,\underline{x}_o + \underline{B}\,\underline{u}_o \\ \underline{z}_o = \underline{C}\,\underline{x}_o \end{cases} \tag{2.75}$$

$$\beta \begin{cases} \underline{x}_o = \underline{A}_d \underline{x}_o + \underline{B}_d \underline{u}_o \\ \underline{z}_o = \underline{C}\,\underline{x}_o \end{cases} \tag{2.76}$$

$$\mu \begin{cases} \begin{bmatrix} \underline{x}_o \\ \underline{u}_o \end{bmatrix} = \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{x}_o \\ \underline{u}_o \end{bmatrix} + \begin{bmatrix} \underline{D}_o \\ T\,\underline{I} \end{bmatrix} \dot{\underline{u}}_o \\ \dot{\underline{u}}_o = \underline{0} \end{cases} \tag{2.77}$$

It is easy to see that equations (2.73) and (2.74) indeed give the unique solution of β, and β and μ have the same solutions for $\underline{x}_o$, $\underline{u}_o$. Therefore, all we need to do is to show that they are also solutions to α. Solving α, we get

$$\underline{x}_o = -\underline{A}^{-1}\underline{B}\,\underline{u}_o \tag{2.78}$$

$$\underline{u}_o = -(\underline{C}\,\underline{A}^{-1}\underline{B})^{-1}\underline{z}_o \tag{2.79}$$

But from equations (2.42) and (2.43), we know that

$$(\underline{A}_d - \underline{I})^{-1}\underline{B}_d = \underline{A}^{-1}\underline{B} \tag{2.80}$$

which implies that equations (2.73) and (2.74) are also solutions to α.

System I can be realized as

$$\frac{d}{dt}\begin{bmatrix} \underline{x}(t) - \underline{x}_o \\ \underline{u}(t) - \underline{u}_o \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix}\begin{bmatrix} \underline{x}(t) - \underline{x}_o \\ \underline{u}(t) - \underline{u}_o \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix}\dot{\underline{u}}(t) \tag{2.81}$$

$$\underline{y}(t) - \underline{z}_o = [\underline{C} \quad \underline{0}]\begin{bmatrix} \underline{x}(t) - \underline{x}_o \\ \underline{u}(t) - \underline{u}_o \end{bmatrix} \tag{2.82}$$

$$J(u) = \int_o^\infty \left\{ [(\underline{x}(t) - \underline{x}_o)' \ (\underline{u}(t) - \underline{u}_o)'] \begin{bmatrix} \underline{C}'\underline{Q}\,\underline{C} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}\begin{bmatrix} \underline{x}(t) - \underline{x}_o \\ \underline{u}(t) - \underline{u}_o \end{bmatrix} + \dot{\underline{u}}'(t)\,\underline{R}\,\dot{\underline{u}}(t) \right\} dt \tag{2.83}$$

Constraining $\dot{\underline{u}}(t) = \dot{\underline{u}}(kT)$ for $kT \leq t < (k+1)T$, we get by [7] an equivalent discrete time realization,

$$\begin{bmatrix} \underline{x}((k+1)T) - \underline{x}_o \\ \underline{u}((k+1)T) - \underline{u}_o \end{bmatrix} = \exp\left\{ \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} T \right\} \begin{bmatrix} \underline{x}(kT) - \underline{x}_o \\ \underline{u}(kT) - \underline{u}_o \end{bmatrix}$$

$$+ \int_o^T \exp\left\{ \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} \tau \right\} d\tau \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \dot{\underline{u}}(kT) \qquad (2.84)$$

$$\underline{y}(kT) - \underline{z}_o = [\underline{C} \quad \underline{0}] \begin{bmatrix} \underline{x}(kT) - \underline{x}_o \\ \underline{u}(kT) - \underline{u}_o \end{bmatrix} \qquad (2.85)$$

and an equivalent performance index

$$J = \sum_{j=0}^{\infty} \left\{ \left[(\underline{x}(jT) - \underline{x}_o)' \ (\underline{u}(jT) - \underline{u}_o)' \right] \tilde{\underline{Q}} \begin{bmatrix} \underline{x}(jT) - \underline{x}_o \\ \underline{u}(jT) - \underline{u}_o \end{bmatrix} \right.$$

$$+ z[(\underline{x}(jT) - \underline{x}_o)' \ (\underline{u}(jT) - \underline{u}_o)'] \tilde{\underline{M}} \dot{\underline{u}}(jT) \qquad (2.86)$$

$$\left. + \dot{\underline{u}}(jT) \ \tilde{\underline{R}} \ \dot{\underline{u}}(jT) \right\}$$

where

$$\tilde{\underline{Q}} = \int_o^T \exp\left\{ \begin{bmatrix} \underline{A}' & \underline{0} \\ \underline{B}' & \underline{0} \end{bmatrix} t \right\} \begin{bmatrix} \underline{C}'\underline{Q}\,\underline{C} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \exp\left\{ \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} t \right\} dt \qquad (2.87)$$

$$\tilde{\underline{M}} = \int_o^T \exp\left\{ \begin{bmatrix} \underline{A}' & \underline{0} \\ \underline{B}' & \underline{0} \end{bmatrix} t \right\} \begin{bmatrix} \underline{C}'\underline{Q}\,\underline{C} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \left[\int_o^t \exp\left\{ \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} s \right\} ds \right] \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} dt \qquad (2.88)$$

$$\tilde{\underline{R}} = \underline{R}T + [\underline{0} \quad \underline{I}] \int_0^T \int_0^t \exp\left\{\begin{bmatrix} \underline{A}' & \underline{0} \\ \underline{B}' & \underline{0} \end{bmatrix} s\right\} ds \begin{bmatrix} \underline{C}'\underline{Q}\,\underline{C} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$$

$$\times \left[\int_0^t \exp\left\{\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} s\right\} ds\right] dt \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} \tag{2.89}$$

After some algebra, we see that

$$\exp\left\{\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} T\right\} = \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \underline{0} & \underline{I} \end{bmatrix} \tag{2.90}$$

$$\int_0^T \exp\left\{\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{0} & \underline{0} \end{bmatrix} \tau\right\} d\tau \begin{bmatrix} \underline{0} \\ \underline{I} \end{bmatrix} = \begin{bmatrix} \underline{D}_o \\ T\,\underline{I} \end{bmatrix} \tag{2.91}$$

$$\tilde{\underline{Q}} = \hat{\underline{Q}} \tag{2.92}$$

$$\tilde{\underline{M}} = \hat{\underline{M}} \tag{2.93}$$

$$\tilde{\underline{R}} = \hat{\underline{R}} \tag{2.94}$$

where $\underline{D}_o$, $\hat{\underline{Q}}$, $\hat{\underline{M}}$, $\hat{\underline{R}}$ are given by equations (2.68), (2.70)-(2.72), thus completing the proof of this proposition.

From now on, we shall call this system, given in proposition 2.4, system III. We are now in a position to obtain a compensator design such that the performance index J in equation (2.69) is minimized. It turns out that we get a very similar expression for $\underline{u}(kT)$ as in proposition 2.2 for system II. The results are given in proposition 2.5. Since the technique in proving them is similar to the one we used in proposition 2.2, we only give the major steps in the proof.

Proposition 2.5

a) Suppose that we are given system III (in proposition 2.4), and we want to minimize the performance index given by equation (2.69), then the optimal control rate $\dot{\underline{u}}(kT)$ is given by the expression

$$\dot{\underline{u}}(kT) = [\tilde{\underline{K}}_1 \quad \tilde{\underline{K}}_2] \begin{bmatrix} \underline{x}(kT) - \underline{x}_o \\ \underline{u}(kT) - \underline{u}_o \end{bmatrix} \tag{2.95}$$

where $\tilde{\underline{K}}_1$, $\tilde{\underline{K}}_2$ are matrices of dimensions $m \times n$, $m \times m$ respectively, and are found by the relation

$$[\tilde{\underline{K}}_1 \quad \tilde{\underline{K}}_2] = -\left\{ \hat{\underline{R}}^{-1}\hat{\underline{M}} + [\hat{\underline{R}} + \underline{D}'\underline{G}\,\underline{D}]^{-1}[\underline{\Phi} - \underline{D}\,\hat{\underline{R}}^{-1}\hat{\underline{M}}'] \right\} \tag{2.96}$$

in which $\underline{G}$ is the solution to a steady state matrix difference equation

$$\underline{G} = [\underline{\Phi} - \underline{D}\,\hat{\underline{R}}^{-1}\hat{\underline{M}}']' \, [\underline{G} - \underline{G}\,\underline{D}(\hat{\underline{R}} + \underline{D}'\underline{G}\,\underline{D})^{-1}\underline{D}'\underline{G}]\,[\underline{\Phi} - \underline{D}\,\hat{\underline{R}}^{-1}\hat{\underline{M}}']$$
$$+ [\hat{\underline{Q}} - \hat{\underline{M}}\,\hat{\underline{R}}^{-1}\hat{\underline{M}}'] \tag{2.97}$$

where the matrices $\underline{\Phi}$, $\underline{D}$, $\hat{\underline{R}}$, $\hat{\underline{M}}$, $\hat{\underline{Q}}$ are all given in proposition 2.4, assuming that

(vii) $[\underline{\Phi}, \underline{D}]$ is controllable.

b) If in addition, we assume that

(viii) rank $[\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1 - \underline{I}] = n$

(ix) rank $[\underline{C}(\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1 - \underline{I})^{-1}(\underline{B}_d + \underline{D}_o\tilde{\underline{K}}_2)] = m$

then the control $\underline{u}(kT)$ is given by

$$\underline{u}(kT) = \sum_{j=0}^{k-1} \tilde{\underline{L}}(\underline{z}_o - \underline{y}(jT)) + \tilde{\underline{N}}(\underline{x}(kT) - \underline{x}(0)) + \underline{u}(0) \tag{2.98}$$

where

$$\begin{aligned}\tilde{\underline{L}} &= T[\tilde{\underline{K}}_2 - \tilde{\underline{K}}_1(\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1 - \underline{I})^{-1}(\underline{B}_d + \underline{D}_o\tilde{\underline{K}}_2)] \\ &\quad \times[\underline{C}(\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1 - \underline{I})^{-1}(\underline{B}_d + \underline{D}_o\tilde{\underline{K}}_2)]^{-1}\end{aligned} \tag{2.99}$$

$$\tilde{\underline{N}} = (T\,\tilde{\underline{K}}_1 + \tilde{\underline{L}}\,\underline{C})\;(\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1 - \underline{I})^{-1} \tag{2.100}$$

<u>Proof</u>

For (a), the result is immediate by using [7]. For (b), from equation (2.66) and (2.95), we obtain

$$\begin{aligned}(\underline{x}((k+1)T) - \underline{x}_o) &= [\underline{A}_d + \underline{D}_o\tilde{\underline{K}}_1](\underline{x}(kT) - \underline{x}_o) \\ &\quad + [\underline{B}_d + \underline{D}_o\tilde{\underline{K}}_2](\underline{u}(kT) - \underline{u}_o)\end{aligned} \tag{2.101}$$

Assumption (ix) implies that there exists a m×n matrix $\tilde{\underline{F}}$ such that

$$\tilde{\underline{F}}(\underline{B}_d + \underline{D}_o\tilde{\underline{K}}_2) = \underline{I} \tag{2.102}$$

Hence using equations (2.101), (2.102) and (2.66) in equation (2.95), we obtain

$$\frac{1}{T}\left(\underline{u}((k+1)T) - \underline{u}(kT)\right) = [\underline{\tilde{K}}_1 - \underline{\tilde{K}}_2\underline{\tilde{F}}(\underline{A}_d + \underline{D}_o\underline{\tilde{K}}_1)](\underline{x}(kT) - \underline{x}_o)$$

$$+ \underline{\tilde{K}}_2\underline{\tilde{F}}(\underline{x}((k+1)T) - \underline{x}_o) \qquad (2.103)$$

This implies

$$\underline{u}(kT) = \sum_{j=0}^{k-1} T[\underline{\tilde{K}}_1 - \underline{\tilde{K}}_2\underline{\tilde{F}}(\underline{A}_d + \underline{D}_o\underline{\tilde{K}}_1 - \underline{I})](\underline{x}(jT) - \underline{x}_o)$$

$$+ T\,\underline{\tilde{K}}_2\underline{\tilde{F}}(\underline{x}(kT) - \underline{x}(0)) + \underline{u}(0) \qquad (2.104)$$

After some manipulations, this reduces to equation (2.98) and thus completes the proof.

Note that in synthesis, this compensator design has the same structure as the one for system II. Also note that the validity of assumptions (viii) and (ix), unlike assumptions (i) - (vii) depends not only on the system matrices, but also on the weighting matrices in the performance index. Furthermore, equation (2.66) gives

$$\underline{x}((k+1)T) = \underline{A}_d\underline{x}(kT) + (\underline{B}_d - \frac{1}{T}\underline{D}_o)\underline{u}(kT) + \frac{1}{T}\underline{D}_o\underline{u}((k+1)T) \qquad (2.105)$$

The presence of the last term on the right side of equation (2.105) <u>appears</u> to be a serious drawback in using the control law which we have just formulated in proposition 2.5, because computation of $\underline{u}((k+1)T)$ in this scheme requires knowledge of $\underline{x}((k+1)T)$. It appears that the system has become <u>noncausal</u>, and some approximations must be used to make it causal.

Two possible ones, which are both ad hoc, are

a) approximate $\dot{\underline{u}}(kT)$ by $\frac{1}{T}(\underline{u}(kT) - \underline{u}(k-1)T))$, giving

$$\underline{x}((k+1)T) = \underline{A}_d\underline{x}(kT) + (\underline{B}_d + \frac{1}{T}\underline{D}_0)\ \underline{u}(kT) - \frac{1}{T}\underline{D}_0\underline{u}((k-1)T) \tag{2.106}$$

If T is large, the deviation introduced by such an approximation might be considerable, especially for the first few time steps, where the system is by no means near steady state.

b) simply use

$$\underline{x}((k+1)T) = \underline{A}_d\underline{x}(kT) + \underline{B}_d\underline{u}(kT) \tag{2.107}$$

but again, this may lead to rate-saturation.

Depending on the actual system, any such approximation can induce instability, a highly undesirable outcome in controller design. Moreover, one might argue that whichever of the above approximations we take, we are in effect generating step like controls. Hence all our considerations which led us to proposition 2.5 are essentially violated: we are still faced with a problem of rate saturation and may even have worsened the situation by introducing approximations.

On closer examination, however, all these problems are nonexistent, if we do exactly what proposition 2.5 says - its actual goal is a continuous time compensator which generates a sequence of ramp functions as the control signals. Specifically,

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \tag{2.108}$$

$$\underline{u}(t) = \underline{u}(kT) + (t-kT)\dot{\underline{u}}(kT) \tag{2.109}$$

for $kT \leq t < (k+1)T$. Note that $\underline{u}(kT)$, $\dot{\underline{u}}(kT)$ only depend on the values of $\underline{z}_0$, $\underline{u}(0)$, $\underline{x}(0)$, $\underline{x}(T)$,..., $\underline{x}(kT)$, and $\underline{y}(0)$, $\underline{y}(T)$, ...,$\underline{y}((k-1)T)$ (equations (2.98), (2.95)). These computations can all be carried out by a digital computer in a sampled data fashion, provided one has a device to generate $\underline{u}(t)$, given in equation (2.109) (e.g. an integrator). Using equations (2.95), (2.73), (2.74) and (2.80), this can be written as

$$\underline{u}(t) = \underline{u}(kT) + (t - kT)[\underline{\tilde{K}}_1\underline{x}(kT) + \underline{\tilde{K}}_2\underline{u}(kT) + (\underline{\tilde{K}}_2 - \underline{\tilde{K}}_1\underline{A}^{-1}\underline{B})(\underline{C}\,\underline{A}^{-1}\underline{B})^{-1}\underline{z}_0] \tag{2.110}$$

There does not exist any non-causality problem at all.

2.5 Final Remarks

In this chapter we have presented several deterministic designs of compensators that can be used to control a linear time-invariant system to track a step input in continuous time, and discrete time. In the next two chapters, we shall concern ourselves with the formulation of the linearized longitudinal dynamics of the F-8C aircraft and the C* handling qualities criterion, and then formulate a control problem using some of the results obtained in this chapter.

Comparisons among these various designs are made in Chapter 5, via simulations. It is shown there that the approximate schemes presented in Section 2.3 are in general less effective than the scheme in Section 2.2.

Before we close this chapter, let us compare briefly the servo controller designs by T.A.S.C. [4] with these results.

Firstly let it be said that in places where there are similarities, T.A.S.C.'s approach looks more similar to [2], while ours is closer to [1]. In particular, instead of penalizing $\underline{z}_0 - \underline{y}(t)$ directly, they penalize $\underline{x}_0 - \underline{x}(t)$, which is clearly equivalent as shown in [1] or [2] or even in the proof of proposition 2.2. Also [2] shows that the techniques can be extended to the case when one puts an additional penalty term on $\underline{u}(t) - \underline{u}_0$, which is often done in [4]. Furthermore, if one cares to do the algebra, one can easily see that the invertibility assumption of the matrix $\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix}$ by T.A.S.C. (and [2]) is necessary for the assumptions (i) and (iii) in proposition 2.1, and the same holds for the corresponding results, too. Having said this, we shall consider the above to be equivalent conditions.

The basic similarities between the two approaches are the issues of P-I controller designs. Indeed, the derivations and results of T.A.S.C.'s continuous time and discrete time designs are the same as our results for system I and II respectively, except for the choice of $\underline{u}(0)$. They consider also issues of P-I-I controller, which we do not touch at all.

The discretization scheme of the continuous time system and the cost penalty presented in [4] is completely the same as that derived in [7], which we referenced. Hence T.A.S.C. considered the case that $\underline{u}(t)$ is constrained to be step functions, and no detailed consideration has been given to problems of rate saturation. We constrain $\underline{u}(t)$ to be ramp functions, and derive the corresponding optimal control law, and discuss also the apparent non-causality problem.

CHAPTER III

F-8C AIRCRAFT LONGITUDINAL DYNAMICS AND THE C* CRITERION

In this chapter, we describe briefly a linearized deterministic model and a C* handling qualities criterion of the F-8C aircraft longitudinal dynamics. Then we formulate a control problem (of the tracking type) on this linear model, using the C* criterion, in such a way that we can apply the results developed in Chapter II.

3.1 Linear Equations of the F-8C Aircraft Longitudinal Dynamics

The F-8C aircraft is a nonlinear system whose dynamics depends on a set of parameters, including its Mach number, altitude and dynamic pressure. Each such set of parameter defines an equilibrium flight condition for the aircraft. J. Gera [14] linearized the nonlinear longitudinal dynamics about various selected flight conditions to arrive at linear equations for the longitudinal motion of the F-8C aircraft.

In this report, we elect to use sixteen flight conditions, namely, flight conditions 5-20, from [14], which represent a clean wing-down configuration of the vehicle. Note that this is the same set of data used by [8]. Indeed one of the objectives of this research is to investigate the feasibility of incorporating the resulting controller design of this report to the MMAC scheme [8].

In the linear model, there are four state variables inherent from the longitudinal dynamics of the aircraft:

State Variables	Units
$q(t)$ pitch rate	rad/sec
$V(t)$ velocity error	ft/sec
$\alpha(t)$ angle of attack (from trimmed value)	rad.
$\theta(t)$ pitch attitude (from trimmed value)	rad.

and $\delta_e(t)$ is the perturbed elevator angle (in rad.), from its trimmed value, driving these states. The linearized dynamics, for each flight condition, are given by

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ V(t) \\ \alpha(t) \\ \theta(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & \cdots & a_{14} \\ \vdots & & & \vdots \\ a_{31} & \cdots & \cdots & a_{34} \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\underline{A}_{LNG}} \begin{bmatrix} q(t) \\ V(t) \\ \alpha(t) \\ \theta(t) \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\underline{B}_{LNG}} \delta_e(t) \qquad (3.1)$$

and in the servo-system, the actuator dynamics are represented by

$$\dot{\delta}_e(t) = -12\ \delta_e(t) + 12\ \delta_{ec}(t) \qquad (3.2)$$

where $\delta_{ec}(t)$ is the commanded elevator angle (in rad.). Furthermore, we include in the states a wind disturbance state $w(t)$ (in rad.). Choosing $\delta_{ec}(t)$ as a control, the complete dynamics can now be written as,

$$\dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ u(t) \qquad (3.3)$$

where

$$\underline{x}(t) = \begin{bmatrix} q(t) \\ V(t) \\ \alpha(t) \\ \theta(t) \\ \delta_e(t) \\ w(t) \end{bmatrix} \tag{3.4}$$

$$u(t) = \delta_{ec}(t) \tag{3.5}$$

$\underline{A}$, $\underline{B}$ are 6x6, 6x1 matrices respectively, given by

$$\underline{A} = \begin{bmatrix} \underline{A}_{LNG} & \underline{B}_{LNG} & a_{13} \\ & & a_{23} \\ (4x4) & (4x1) & a_{33} \\ & & 0 \\ \underline{0} & -12 & 0 \\ (2x4) & 0 & \omega \end{bmatrix} \tag{3.6}$$

$$\underline{B} = \begin{bmatrix} \underline{0} \\ (4x1) \\ 12 \\ 0 \end{bmatrix} \tag{3.7}$$

where ω is the windpole depending on the flight condition, given by

$$\omega = -2 \frac{V_o}{P} \tag{3.8}$$

where V_o is the velocity of the aircraft, i.e.

$$V_o = \text{Mach no. x velocity of sound (alt.)} \tag{3.9}$$

and

$$P = \begin{cases} 2500 \text{ ft.} & \text{if alt.} > 2500 \text{ ft.} \\ 200 \text{ ft.} & \text{if alt.} = 0 \end{cases} \tag{3.10}$$

We assume that our sensor takes two measurements, one on pitch rate, $q(t)$, and the other on normal acceleration, $a_z(t)$, which is given, about equilibrium, (positive in the upward direction) by

$$a_z(t) = \frac{V_o}{g} \left(q(t) - \dot{\alpha}(t)\right) \tag{3.11}$$

where g is the acceleration due to gravity (32.2 ft. $\sec^{-2}$).

Equations (3.11) and (3.1), and the fact that $a_{31} = 1$, $a_{34} = 0$ for all the flight conditions, give us a linear combination of the state variables, i.e.,

$$a_z(t) = \frac{V_o}{g} \left(-a_{32} V(t) - a_{33} \alpha(t) - b_3 \delta_e(t)\right) \tag{3.12}$$

Therefore, in the form of an observation matrix equation, we write

$$\underline{y}(t) = \underline{C}\, \underline{x}(t) \tag{3.13}$$

where $\underline{C}$ is a 2 x 6 matrix given by

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{V_o a_{32}}{g} & -\frac{V_o a_{33}}{g} & 0 & -\frac{V_o b_3}{g} & 0 \end{bmatrix} \tag{3.14}$$

and

$$\underline{y}(t) = \begin{bmatrix} q(t) \\ a_z(t) \end{bmatrix} \tag{3.15}$$

This completes a brief deterministic description of the linearized F-8C aircraft longitudinal dynamics.

3.2 F-8C Aircraft C* Criterion

In this section we summarize and discuss briefly the significance of a C* handling qualities criterion in the longitudinal dynamics of a F-8C aircraft.

In today's technology, the implementation of flight control systems in an aircraft has been widespread. In order for an engineer to make sound judgement on the merits or short-comings of a control design, he needs a well defined criterion on the handling qualities of the aircraft. A proper choice of such criterion is clearly essential to demonstrate the effectiveness of a good flight control system.

It used to be a common practice in classical designs to use a "short period damping ratio" type of criterion. This is becoming out-of-date for the following reasons:

(1) it assumes that only one variable is predominantly sensed by the pilot, namely, the normal acceleration;

(2) it assumes that the short period response of the system may be represented by a linear second order system;

(3) it does not give an exact picture of what the aircraft is really doing in the transient period.

For a high performance aircraft such as the F-8C in which maneuvers are frequent due to its agility, this is certainly not satisfactory. It has been observed that normal acceleration, as well as pitch attitude and pitch rate, are all important cues to a pilot, and their relative importance depends on the velocity of the aircraft [10]. Specifically, normal acceleration is more important when the velocity is high, and the pitch variables predominate when it is low. Moreover, an aircraft is more complex than any simple representation by a second order linear system. Also it seems reasonable to rely on a criterion in the time domain, rather than one in the frequency domain, partly because of its ease to perceive, and partly because of the past success of modern control theory.

Thus it is intuitive that in order to develop a more satisfactory criterion of aircraft handling, we may want to first establish a time history envelope for one or more chosen parameters of the aircraft dynamics. Such an idea is well illustrated in [10], [11], by a so-called C* handling quality criterion, and indeed this is the step that we take.

We give a brief formulation of this criterion in terms of the F-8C aircraft longitudinal dynamics below.

A variable C*(t) is defined to be a linear combination of the normal acceleration, $a_z(t)$, and the pitch rate, q(t), which is given by the following relation (in g's)

$$C^*(t) = k_1 \, a_z(t) + k_2 \, q(t) \tag{3.16}$$

where k_1, k_2 are some arbitrary pair of constants, and can be viewed as the relative weightings of the importance of these two state variables in the pilot's sensing. Arbitrarily, k_1 is chosen to be unity, and correspondingly, k_2 is given the value

$$k_2 = \frac{\text{cross-over velocity (ft/sec)}}{g \qquad (\text{ft/sec}^2)} \tag{3.17}$$

where the cross-over velocity is the aircraft speed at which the normal acceleration and the pitch rate give equal cues to the pilot. Approximately, k_2 is thus found to be equal to 10 with respect to the F-8C aircraft dynamics, or equivalently,

$$C^*(t) = a_z(t) + 10\, q(t) \qquad \text{(in g's)} \tag{3.18}$$

Figure 3.1 shows a time history envelope for the C* parameters thus defined, representing an acceptable range of the C* transient response to a step input from the pilot.

3.3 Continuous Time Formulation of a C* Tracking Design

Thus far we have given almost all the preliminaries required for a deterministic compensator design, such that the C* transient response of the F-8C aircraft to a step input is made to fall within the C* time history envelope, e.g. along the middle dotted line of the envelope in Fig. 3.1. In what follows, our task is to give a physical motivation for the overall design, and then discuss the various theoretical issues. In the next section we shall extrapolate these ideas into the discrete time designs for the F-8C aircraft.

When a pilot attempts to control an aircraft, most probably he applies an impluse like action to the command stick, inducing a step like shift in the stick position. This in turn is transformed into a pilot's desired input value to the dynamics by some servo-mechanisms. With regard to the C* criterion, we imagine that such a step input thus generated would be a

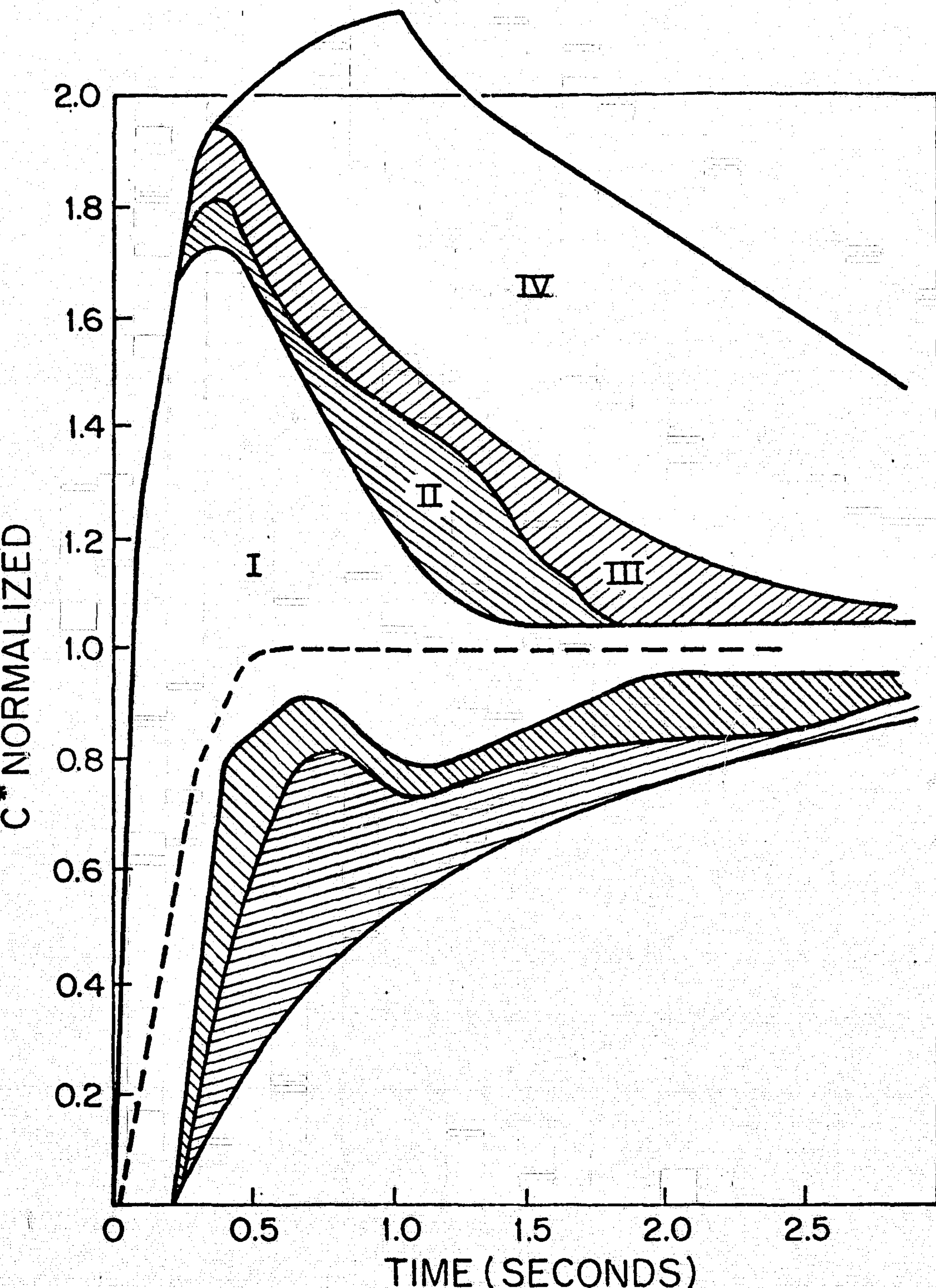


Fig. 3.1 C* Time History Envelope (Regions I and IV are acceptable for the C* response)

desired steady state C* response that the pilot wants to have the short period response of the aircraft to follow. We summarize these in Fig. 3.2.

Note that in this model we have assumed that a C* desired value is generated by the pilot at time t_o, represented by $C^*_{d_p}$, and there is no time delay in the pilot's reflexes and motion, nor in the servomechanisms of the aircraft. From now on, we shall assume implicitly this part of the model, and only consider C^*_d, the desired C* response, as a step input to the aircraft at time t_o. Without loss of generality, we take $t_o = 0$.

The formulation of the control problem that we are aiming at is now clear. Specifically, we are given the linear longitudinal dynamics of the F-8C aircraft, described by the equations (3.3) - (3.11) in section 3.1, and we want to find an optimal control function

$$u(t) = \delta_{ec}(t) \qquad\qquad t \in [0, \infty) \tag{3.19}$$

such that the following performance index is minimized:

$$J(u) = \int_0^\infty \left[(C^*_d - C^*(t))^2 Q + \delta^2_{ec}(t) R \right] dt \tag{3.20}$$

where Q, R are some chosen non-negative numbers.

We divide the specification of this problem into two parts:

(i) Statement

The deterministic descriptions of the dynamical system

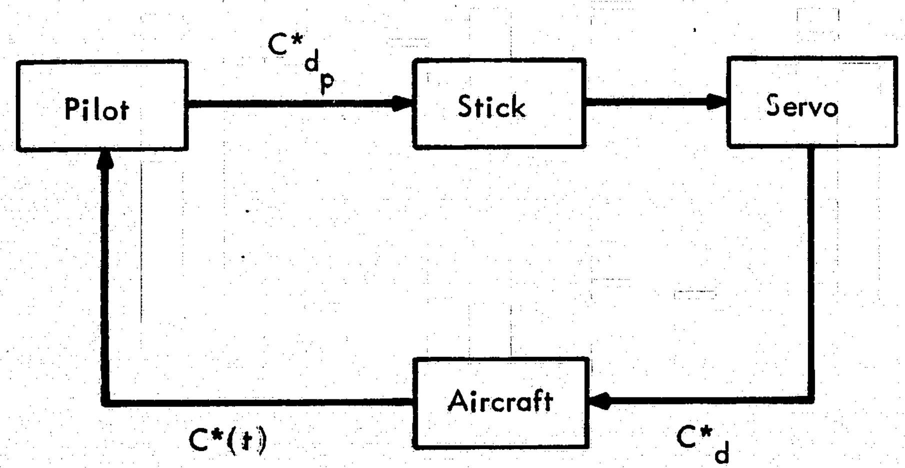


Fig. 3.2 C* Pilot-Aircraft Interaction Model

are

$$\underline{\dot{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ u(t) \tag{3.21}$$

$$C^*(t) = \underline{C^*}\ \underline{x}(t) \tag{3.22}$$

where $\underline{A}, \underline{B}, \underline{x}(t), u(t)$ are described by equations (3.4), (3.5) and (3.7), (3.8). In particular, $\underline{C^*}$ is a 1 x 6 matrix, given by

$$\underline{C^*} = \begin{bmatrix} 10 & -\frac{V_o}{g} a_{32} & -\frac{V_o}{g} a_{33} & 0 & -\frac{V_o}{g} b_3 & 0 \end{bmatrix} \tag{3.23}$$

In addition, we are given a desired steady state value of $C^*(t)$, namely C^*_d, which serves as an input (step at t=0) to the system. We want to find an optimal, everywhere continuous control u(t) such that this performance index is minimized,

$$J(u) = \int_0^\infty \left[(C^*_d - C^*(t))^2\ Q + \dot{u}^2(t) R \right] dt \tag{3.24}$$

where Q, R are positive real numbers.

(ii) Solution

It is easy to see that this belongs to the class of problems examined in proposition 2.1, with scalar output and control, if we assume that

i) $\underline{A}$ is non-singular;

ii) $[\underline{A}, \underline{B}, \underline{C^*}]$ is minimal;

iii) $\underline{C^*}\underline{A}^{-1}\underline{B}$ is invertible.

For the purpose of this chapter, let us simply assume them to hold and we shall see in Chapter V that all the flight conditions satisfy ii) and iii), and except for a few of them, i) is also satisfied. The exceptions are due to numerical approximations in the data.

By proposition 2.1, we are then able to find a scalar constant L and a 1 x 6 matrix $\underline{N}$ such that

$$u(t) = L\int_0^t (C_d^* - C^*(\tau))\, d\tau + \underline{N}(\underline{x}(t) - \underline{x}(0)) + u(0) \qquad (3.25)$$

is the optimal control solution. Fig. 3.3 shows the structure of the controlled aircraft, where N^q, N^v, N^α, N^Θ, N^{δ_e}, N^w are entries in the $\underline{N}$ matrix.

3.4 Discrete Time Formulation of the C* Tracking Design

Recall that in Chapter II, we discuss two schemes of discretizing the continuous time dynamics, i.e. system II and III. Here, for each of them, we outline a formulation of a control problem, on the F-8C aircraft longitudinal dynamics, which is equivalent to the one just given for continuous time. Because of the possibility of too many redundant discussions with Chapter II, and the many similarities to the continuous time problem, much of the detail here is omitted. For simplicity of discussions, we assume conditions such as controllability, observability and invertibility of matrices implicitly whenever they are needed.

1) Discretization using Scheme (1)

The continuous time linearized F-8C aircraft longitudinal dynamics, described in section 3.1, can be discretized by constraining u(t) (i.e., $\delta_{ec}(t)$) to be a constant over each sampling period T, i.e. for each k and $t \in [kT, (k+1)T)$,

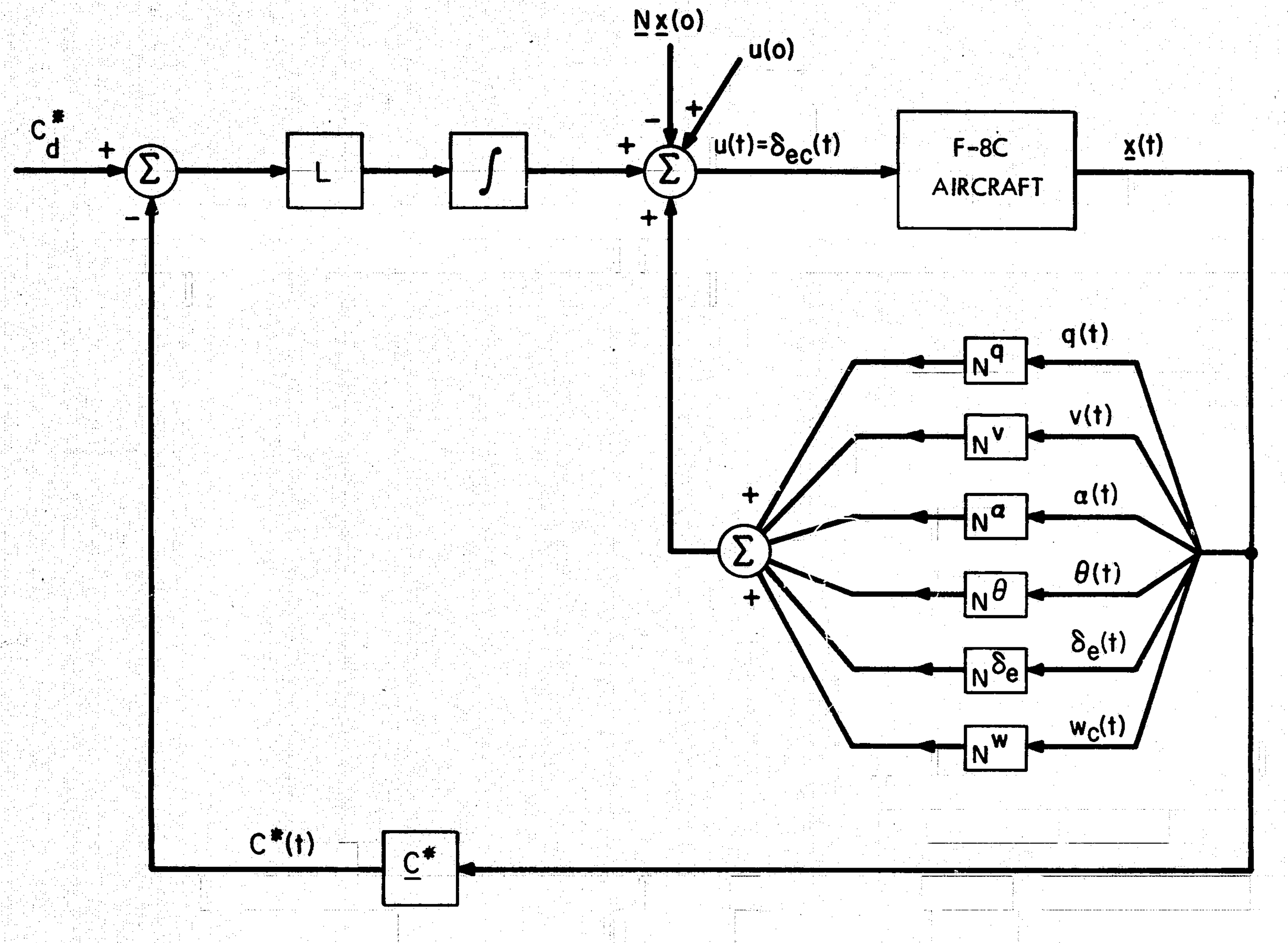


Fig. 3.3 Continuous Time C* Controller Structure for the F-8C Aircraft

$$u(t) = u(kT) = u_k \tag{3.26}$$

with the application of proposition 2.3. Specifically, we get a deterministic discrete time realization

$$\underline{x}_{k+1} = \underline{A}_d \underline{x}_k + \underline{B}_d u_k \tag{3.27}$$

$$C^*_k = \underline{C}^* \underline{x}_k \tag{3.28}$$

where

$$\underline{A}_d = e^{\underline{A}T} \tag{3.29}$$

$$\underline{B}_d = \int_o^T e^{\underline{A}\tau}\, d\tau\, \underline{B} \tag{3.30}$$

Suppose in addition, we are given a steady state desired value of $C^*(t)$, say C^*_d, and we are required to minimize the performance index

$$J(u) = \sum_{j=0}^{\infty} [(C^*_d - C^*_j)^2 Q_d + (u_{j+1} - u_j)^2 R_d] \tag{3.31}$$

where Q_d, R_d are positive real numbers. Then, by applying proposition 2.2 we can find a scalar L_d, and a 1 x 6 matrix $\underline{N}_d$ such that the optimal discrete time control law is given by

$$u_k = L_d \sum_{j=0}^{k-1} (C^*_d - C^*_j) + \underline{N}_d(\underline{x}_k - \underline{x}_o) + u_o \tag{3.32}$$

Instead of considering a performance index such as equation (3.32), one might want to use one of the two approximations of the continuous time design, discussed in Section 2.3. Then the control law is given by

$$u_k = L_D \sum_{j=0}^{k-1} (C_d^* - C_j^*) + \underline{N}_D(\underline{x}_k - x_o) + u_o \tag{3.33}$$

where L_D is a scalar, N_D is a 1x6 matrix computed by using either equations (2.56) and (2.57) or equations (2.63) and (2.64)..

2) Discretization using Scheme (2)

Alternatively, we may discretize the continuous time problem by keeping $\dot{\delta}_{ec}(t)$ to be a constant over each sampling period T, i.e. for each k and t ε [kT, (k+1)T),

$$\dot{u}(t) = \dot{u}(kT) = \dot{u}_k \tag{3.34}$$

Then by proposition 2.4, we obtain the following discrete time realization for the deterministic C* dynamics,

$$\begin{bmatrix} \underline{x}_{k+1} \\ \\ u_{k+1} \end{bmatrix} = \underline{\Phi} \begin{bmatrix} \underline{x}_k \\ \\ u_k \end{bmatrix} + \underline{D}\,\dot{u}_k \tag{3.35}$$

$$C_k^* = [\underline{C}^* \quad 0] \begin{bmatrix} \underline{x}_k \\ \\ u_k \end{bmatrix} \tag{3.36}$$

where

$$\Phi = \begin{bmatrix} \underline{A}_d & \underline{B}_d \\ \\ \underline{0} & \underline{I} \end{bmatrix} \tag{3.37}$$

$$\underline{D} = \begin{bmatrix} \int_o^T \int_o^S e^{\underline{A}\tau} \, d\tau \, ds \, \underline{B} \\ \\ T \end{bmatrix} \tag{3.38}$$

Matrices $\underline{A}_d$, $\underline{B}_d$ are given by equations (3.38), (3.39). Moreover, we arrive at a performance index equivalent to that for continuous time (equation (3.25)),

$$J = \sum_{j=0}^{\infty} [(\underline{x}_j' - \underline{x}_d') \, \hat{\underline{Q}}(\underline{x}_j - \underline{x}_d) + (\underline{x}_j' - \underline{x}_d')\hat{\underline{M}} \, \dot{u}_j + \dot{u}_j^2 \, \hat{R}] \tag{3.39}$$

where

$$\hat{\underline{Q}} = Q \int_o^T e^{\underline{A}'t} \, \underline{C}^{*\prime}\underline{C}^* \, e^{\underline{A}t} \, dt \tag{3.40}$$

$$\hat{M} = Q \int_o^T e^{\underline{A}'t} \, \underline{C}^{*\prime}\underline{C}^* \, [\int_o^t \int_o^s e^{\underline{A}\tau} \, d\tau \, ds]\underline{B} \, dt \tag{3.41}$$

$$\hat{R} = RT \tag{3.42}$$

and

$$\underline{x}_d = \underline{A}^{-1}\underline{B}(\underline{C}^*\underline{A}^{-1}\underline{B})^{-1}C_d^* \tag{3.43}$$

Applying proposition 2.5, the optimal sequence u_k such that this performance is minimized is given by

$$u_k = \tilde{L} \sum_{j=0}^{k-1} (C_d^* - C_j^*) + \underline{\tilde{N}}(\underline{x}_k - \underline{x}_o) + u_o \tag{3.44}$$

where $\tilde{L}$ is a scalar and $\underline{\tilde{N}}$ is a 1 x 6 matrix found by the procedures given in that proposition. Along the line, we also obtain a 1 x 6 matrix $\underline{\tilde{K}}_1$ and a scalar $\tilde{K}_2$ such that the optimal control rate for $t \in [kT, (k+1)T)$ is given by

$$\dot{u}(t) = \dot{u}(kT) = \dot{u}_k = \underline{\tilde{K}}_1(\underline{x}_k - \underline{x}_d) + \underline{\tilde{K}}_2(u_k - u_d) \qquad (3.45)$$

where

$$u_d = -(\underline{C}^*\underline{A}^{-1}\underline{B})^{-1}C_d^* \qquad (3.46)$$

and the control generated should be a sequence of continuous time ramp functions, to be applied to the continuous time system, given by

$$u(t) = u_k + (t - kT)\dot{u}_k \qquad (3.47)$$

Thus we have completed the discussions on full state control tracking designs for the F-8C aircraft longitudinal dynamics, using the C* handling qualities criterion. In Chapter IV, we present a suboptimal design, by reducing the number of states, which we compare with in Chapter V on the basis of numerical results. However, we only consider system I (continuous time) and system II (discrete time), and leave system III for future research.

CHAPTER IV

A SUBOPTIMAL REDUCED STATE DESIGN OF THE F-8C AIRCRAFT LONGITUDINAL DYNAMICS

4.1 Preliminary Remarks

In [8], it is sketchily pointed out that a reduced state design can be an extremely good approximation to the transient behavior of the full state F-8C aircraft longitudinal dynamics, particularly in the suboptimal regulator design. The purpose of this chapter is to exploit this idea in the C* design formulated in Chapter III. Treatment here includes a brief account of the desirability of such a reduced state design, and some detail of the design equations. We compare and contrast this design with the optimal one along the theoretical issues. Numerical comparisons are given in the next chapter.

Here we derive how one can go from a full state model to a reduced state model in continuous time by making approximations and linear transformations. To obtain a reduced state model in discrete time, we remark that this can be accomplished by discretizing the continuous time reduced state model using either one of the methods given in Chapter II, as we have done for the full state model in Chapter III. Bearing this in mind, we restrict ourselves to discussions in the continuous time case only for the rest of this chapter, since the discussions in Chapter III on discrete time systems serve the purpose equally well here, provided that one remembers to deal with a reduced state system instead of an optimal one.

Note that notations used in Chapter III are also used here with the same meanings; no attempt is made to redefine them.

4.2 Reasons for Studying the Reduced State Model

From a study of the components of the system matrices, $\underline{A}$, of the different flight conditions, it is noted that two of the states, namely, the velocity error, V, and the pitch attitude, θ, seem to have very small effect on the short term transient responses of the F-8C aircraft. This is further confirmed by an analysis of the eigenvalues and eigenvectors of the system matrices. Apparently, we can eliminate these two variables from the model design, without degrading much of the short period performance of the aircraft.

Such an intuitive idea is pursued in [8], in the design of a suboptimal feedback regulator. It is noted that the feedback gains on V and θ are very small in the full state regulator design. Indeed, feedback gains on the other state variables, i.e. q, α, δ_e, δ_{ec}, w, are approximately the same as those from a reduced state model design, which uses only these five variables as states. Different plots of closed-loop poles, and gains, plus a very detailed analysis of simulations, show that the transient responses of the aircraft in both cases appear to be very similar.

It is intuitive that if a reduced state design is acceptable, we may expect that the overall engineering design (implementations, etc.) can be simplified. Eliminating the state variables V and θ from our full state model in Chapter III, we arrive at the following (deterministic) dynamical representation:

$$\frac{d}{dt}\begin{bmatrix} q(t) \\ \alpha(t) \\ \delta_e(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & b_1 & a_{13} \\ a_{31} & a_{33} & b_3 & a_{33} \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}\begin{bmatrix} q(t) \\ \alpha(t) \\ \delta_e(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 12 \\ 0 \end{bmatrix} \delta_{ec}(t) \qquad (4.1)$$

We take a step further from this by replacing the state variable $\alpha(t)$ by a new (but familiar) state variable, $a_z(t)$, the normal acceleration. This can be easily accomplished by techniques such as matrix transformation, giving us

$$\begin{bmatrix} \dot{q}(t) \\ \dot{a}_z(t) \\ \dot{\delta}_e(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & -\frac{g}{V_o}\frac{a_{13}}{a_{33}} & b_1-\frac{a_{13}}{a_{33}}b_3 & a_{13} \\ -a_{33}\frac{V_o}{g} & a_{33} & 12\, b_3\frac{V_o}{g} & -\frac{V_o}{g}a_{33}^2 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}\begin{bmatrix} q(t) \\ a_z(t) \\ \delta_e(t) \\ w(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -12\, b_3\frac{V_o}{g} \\ 12 \\ 0 \end{bmatrix} \delta_{ec}(t) \qquad (4.2)$$

To simplify notations, we write this as

$$\dot{\underline{x}}_r(t) = \underline{A}_r\, \underline{x}_r(t) + \underline{B}_r\, u(t) \qquad (4.3)$$

Hence the C* parameter is given by

$$C^*(t) = \underbrace{[10 \quad 1 \quad 0 \quad 0]}_{\underline{C}^*_r}\underline{x}_r(t) \tag{4.4}$$

This transformation is justified for the following reasons:

(i) Computations, whether on-line or off-line, are reduced;

(ii) Since we make measurements on the pitch rate and normal acceleration, and the C* parameter depends only on these two states, therefore such a transformation provides us the convenience of having a constant observation matrix $\underline{C}$, and a constant matrix $\underline{C}^*$ for all flight conditions;

(iii) We can form $C^*(t)$ directly from the measurements.

We remark further that except for the use of equations (4.3) and (4.4), a C* tracking design on this model can use the same performance index used in the full state system design, i.e.

$$J = \int_0^\infty [(C^*_d - C^*(t))^2 Q + \dot{u}^2(t)R]\, dt \tag{4.5}$$

Using this, and applying proposition 2.1, we thus obtain a scalar L_r, and and a 1 x 4 constant matrix

$$\underline{N}_r = [n_q \quad n_{a_z} \quad n_{\delta_e} \quad n_w] \tag{4.6}$$

such that a suboptimal deterministic design is achieved, with a control law given by

$$u(t) = L_r \int_0^t (C_d^* - C^*(\tau))d\tau + \underline{N}_r(\underline{x}_r(t) - \underline{x}_r(0)) + u(0) \qquad (4.7)$$

Fig. 4.1 shows the structure of this reduced state controller, and is to be compared with Fig. 3.3 for the full state design, where N^q_r, $N^{a_z}_r$, $N^{\delta_e}_r$, N^w_r are entries in the N_r matrix. Equivalently, we can write this as

$$u(t) = L_r \int_0^t (C_d^* - C^*(\tau))d\tau + \underline{N}_f(\underline{x}(t) - \underline{x}(0)) + u(0) \qquad (4.8)$$

where

$$\underline{N}_f = \underline{N}_r \begin{bmatrix} \multicolumn{3}{c}{\underset{(2\times 6)}{C}} \\ \hdots & \hdots & \hdots \\ \underset{(2\times 4)}{0} & \vdots & \underset{(2\times 2)}{I} \end{bmatrix} \qquad (4.9)$$

and $\underline{x}(\cdot)$ is the full state vector.

We shall analyze the effect of such a suboptimal design on the full state aircraft model, by studying some numerical examples in Chapter V.

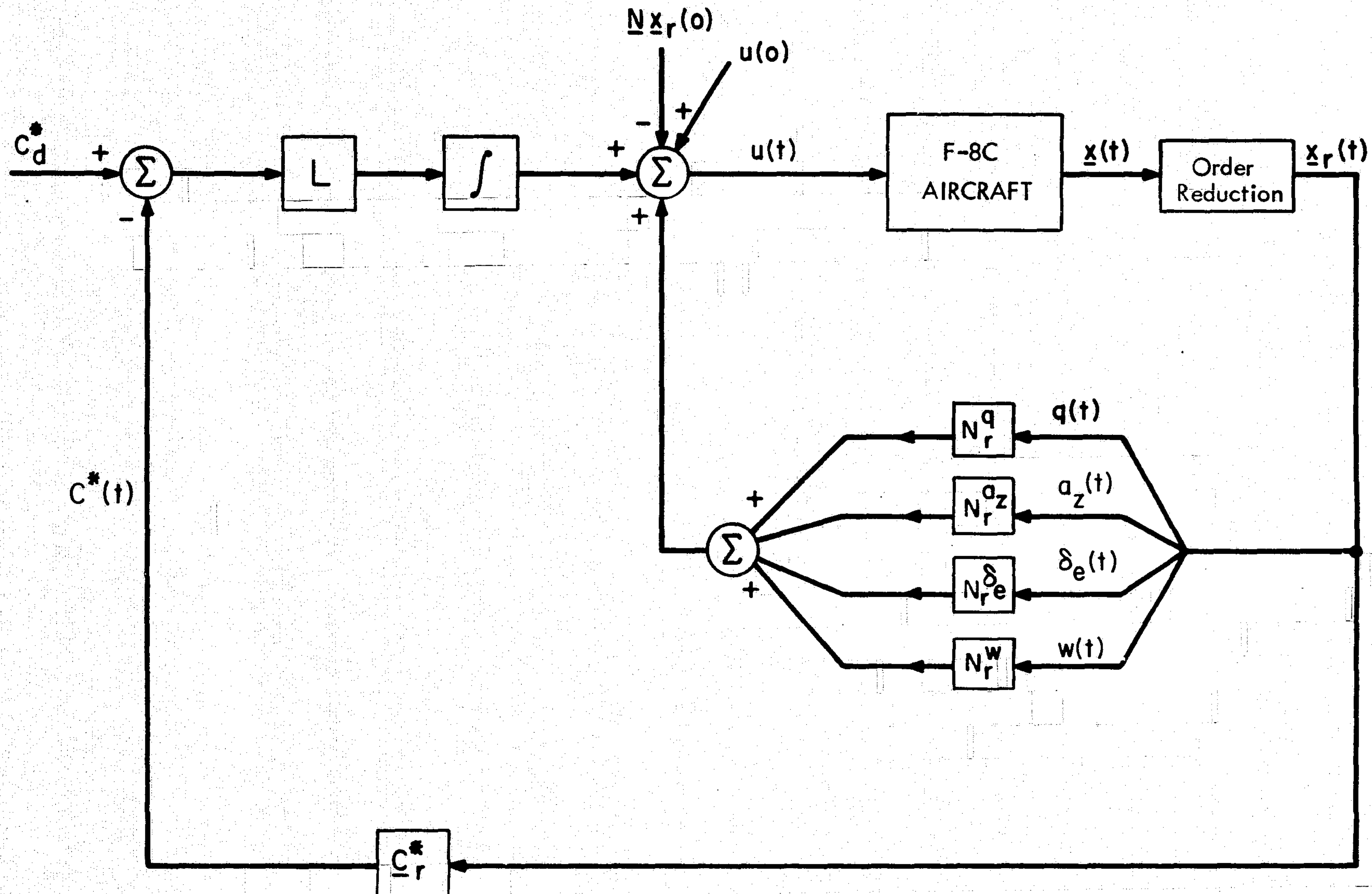


Fig. 4.1 Continuous Time Reduced State C* Controller Structure for the F-8C Aircraft

CHAPTER V

NUMERICAL EXAMPLES

5.1 Overview

In this chapter, we present some numerical results on the F-8C aircraft linearized longitudinal dynamics that illustrate our work on the deterministic design, both in continuous time and in discrete time. We selected to use sixteen of the flight conditions (5-20) from Gera [14], which represent linearized dynamics of the F-8C aircraft in clean wing down configurations. Table 5-1 shows each of the sixteen flight conditions (altitude, Mach no., dynamic pressure). Throughout this chapter, we emphasize the fact that the full state model of the F-8C aircraft (see Chapter III) is very well approximated by the reduced state model discussed in Chapter IV. Thus a major part of this chapter contains results for the reduced state designs, although brief notes of comparison with the full state model are given along the discussions.

We remark that in linear simulations on a digital computer, one in general does not encounter the problem of rate saturation of the control which is discussed in some detail in the later part of Chapter II. Therefore, for the purposes of this report, we only use System II (See propositions 2.2, 2.3) and leave aside System III (See proposition 2.4). Also in Section 2.3, we presented two discrete time designs which make use of and represent (approximate) discretizations of the continuous time design. We shall not give any numerical results for them, but instead we point out here that simulated results have shown that both of them are less effective

Flight Condition	Altitude	Mach. No.	Dynamic Pressure (PSF)
5	sea level	0.3	133.2
6	" "	0.53	416
7	" "	0.7	726
8	" "	0.86	1098
9	" "	1.0	1480
10	20,000 ft.	0.4	109
11	" "	0.6	245
12	" "	0.8	434
13	" "	0.9	550
14	" "	1.2	978
15	40,000 ft.	0.7	135
16	" "	0.8	176
17	" "	1.9	223
18	" "	1.2	397
19	" "	1.4	537
20	" "	1.6	703

Table 5.1 Flight Condition Parameters

than the discrete time design derived directly from the dynamical system (See propositions 2.2, 2.3), though the first order approximation method gives rather satisfactory results. Indeed the one obtained by pole-allocation is far from satisfactory and should not be used in practice.

There are five different designs to be investigated in this chapter, including two continuous time designs, and three discrete time designs. However, only the ones that deal with the reduced state model will be discussed in detail. We summarize them in Table 5.2a.

We selected three of the sixteen flight conditions for simulations in order to demonstrate the effects of mismatching in the C*-controlled flight trajectory. Namely, they are flight conditions 7, 11 and 12, representing variations in dynamic pressures, with flight condition 7 being the highest, and 11, the lowest. We assume that the aircraft is in flight condition 12, while control gains from one of these three flight conditions are used. We remark that all of these flight conditions are subsonic, where the C*-theory plays the most significant role. We shall assume that initially all the states are zero and the desired C* steady state value is a step of 1 g. Also $u(0) = \delta ec\ (0)$ is chosen to be zero. We summarize them in Table 5.2b.

From now on, for simplicity purposes, with reference to Tables 5.2 a and b, we shall refer to the simulation i, $1 \leq i \leq 3$, for the design Ω, $\Omega = A,B,C,D,E$, as the simulation Ω-i. We remark that simulations 1 and 2 represent mismatches in gain scheduling. Where appropriate, we shall attach plots of simulations to the discussions. All the simulations are for 6.25

Design	Time Set	Simulated Model	Gains	Simulations
A	continuous	reduced state	from reduced state model	A
B	"	full state	" " " "	B
C	discrete	reduced state	" " " "	C
D	"	full state	" " " "	D
E	"	full state	from full state model	E

Table 5.2a Summary of the Designs Simulated

Simulation	Flight Condition	Control Gains (F/C)	Initial Conditions of States	C*d(g's)
1	12	7	all zero	1
2	12	11	(i) " "	1
3	12	12	" "	1

Table 5.2b Summary of the Simulations for Each Design in Table 5.2a

seconds, with a time step of 0.125 second, which is the sampling period for discretization.

For each of the two reduced state designs (designs A and C), we show a mismatch stability table of the sixteen flight conditions, which indicates whether the flight would be stable when control gains computed for flight condition i ($5 \leq i \leq 20$), are scheduled to be used in controlling the aircraft in flight condition j ($5 \leq j \leq 20$). These tables, together with the simulations, form the basis for the mismatch stability analysis in this report.

We arrange the rest of the chapter as follows:
Section 5.2 deals with the continuous time designs (designs A and B). There brief notes are given for the choice of the weightings in the cost function, and mismatching of the flight conditions. A very detailed analysis for the reduced state design is presented. Control gains are computed for the reduced state and the full state models. The same is done in Section 5.3 for the discrete time designs (designs C and D). In addition, a full state model with full state control gains (design E) is also discussed. In Section 5.4, we draw some conclusions and add some remarks.

For simplicity and reference purposes, all notations used in this chapter are the same as those in Chapters III and IV, unless otherwise stated.

It should be noted that use has been made of Sandell and Athens [16] and Kleinman [17], in the computing work done for this report.

5.2 Continuous Time Designs

The first step that one should take is to choose the appropriate weightings, Q and R, for the cost function given by equation (3.25).

In doing so, we pick R from a consideration of the rate saturation of the actuator dynamics;

$$R = \frac{1}{\dot{\delta}^2_{ec_{max}}} \tag{5.1}$$

where

$$\dot{\delta}_{ec_{max}} = 0.435 \text{ rad/sec (25 deg/sec)} \tag{5.2}$$

Then we chose Q in such a way that the short period C* responses of the closed loop dynamics fall more or less in the middle of the C* time history envelope shown in Fig. 3.1, i.e., along the dotted line. We found out, by trial and error, that a good choice for Q is

$$Q = 1 \quad (g^{-2}) \tag{5.3}$$

Figure 5.1 shows representative C* responses.

(i) Design A

Using this pair of values of Q, eq. (5.3), and R, eqs. (5.1) and (5.2), one can then proceed with the (continuous time) design discussed in Chapter IV for the reduced state model. That is given equations (4.3), (4.4), and (4.5), one is to find the control $\delta_{ec}(t)$ given by equation (4.7).

It is observed that, for all flight conditions, the matrices $\underline{A}_r$, $\underline{B}_r$, $\underline{C}^*_r$ satisfy all the three assumptions required by proposition 2.1. We can therefore apply the design procedures given by this proposition and obtain the gains L_r, $\underline{N}_r$ directly.

Figure 5.2 shows a plot of the resultant dominant complex poles of the closed loop system for all the flight conditions. The magnitude of

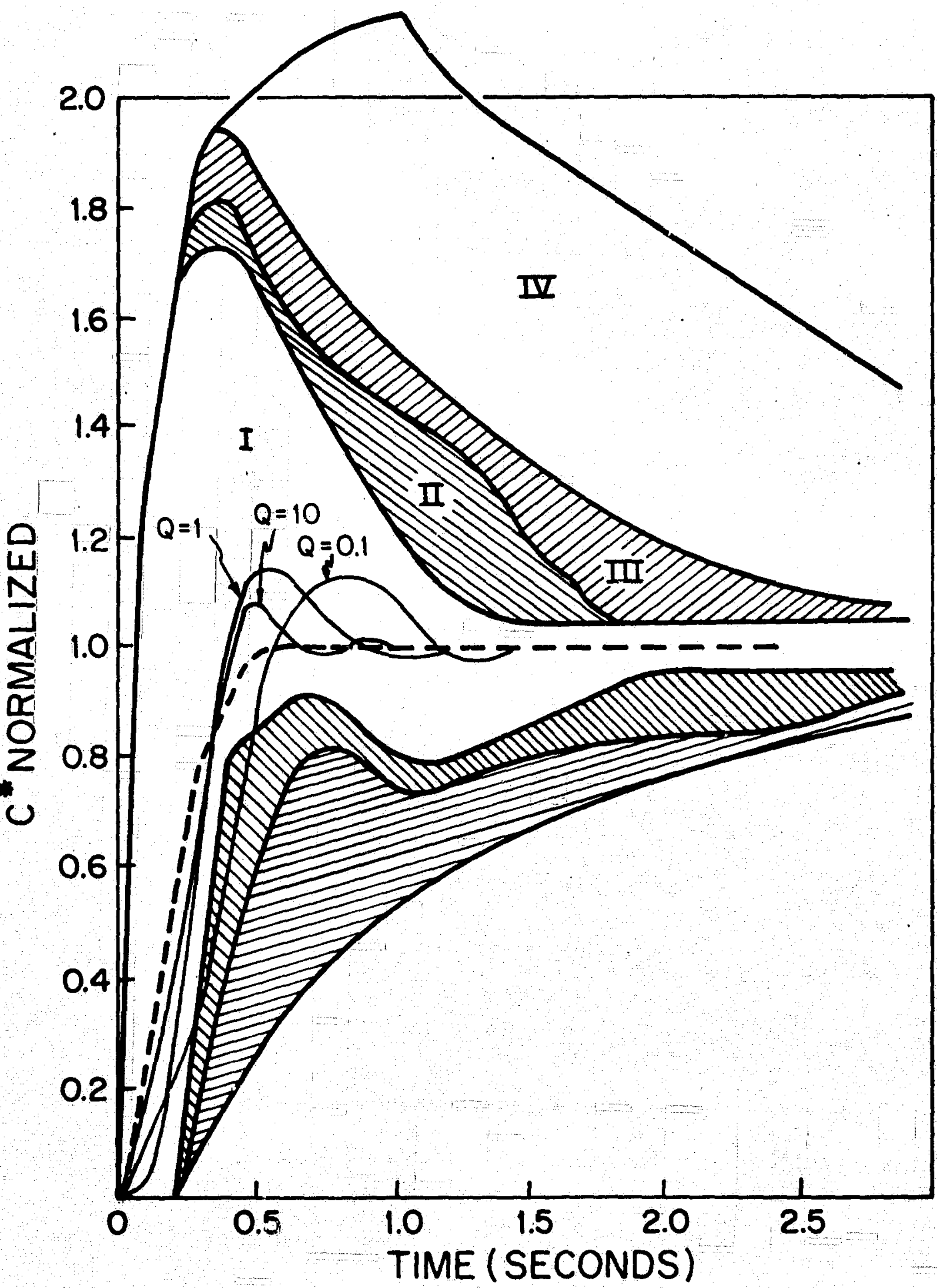


Fig. 5.1 Illustrations of how Choice Of Q is Made. Shown Above are Simulations for 3 Values of Q on Flight Condition 13. In General, Larger Values of Q Induce Faster Modes of C* Responses. Q = 1 is Chosen for this Report

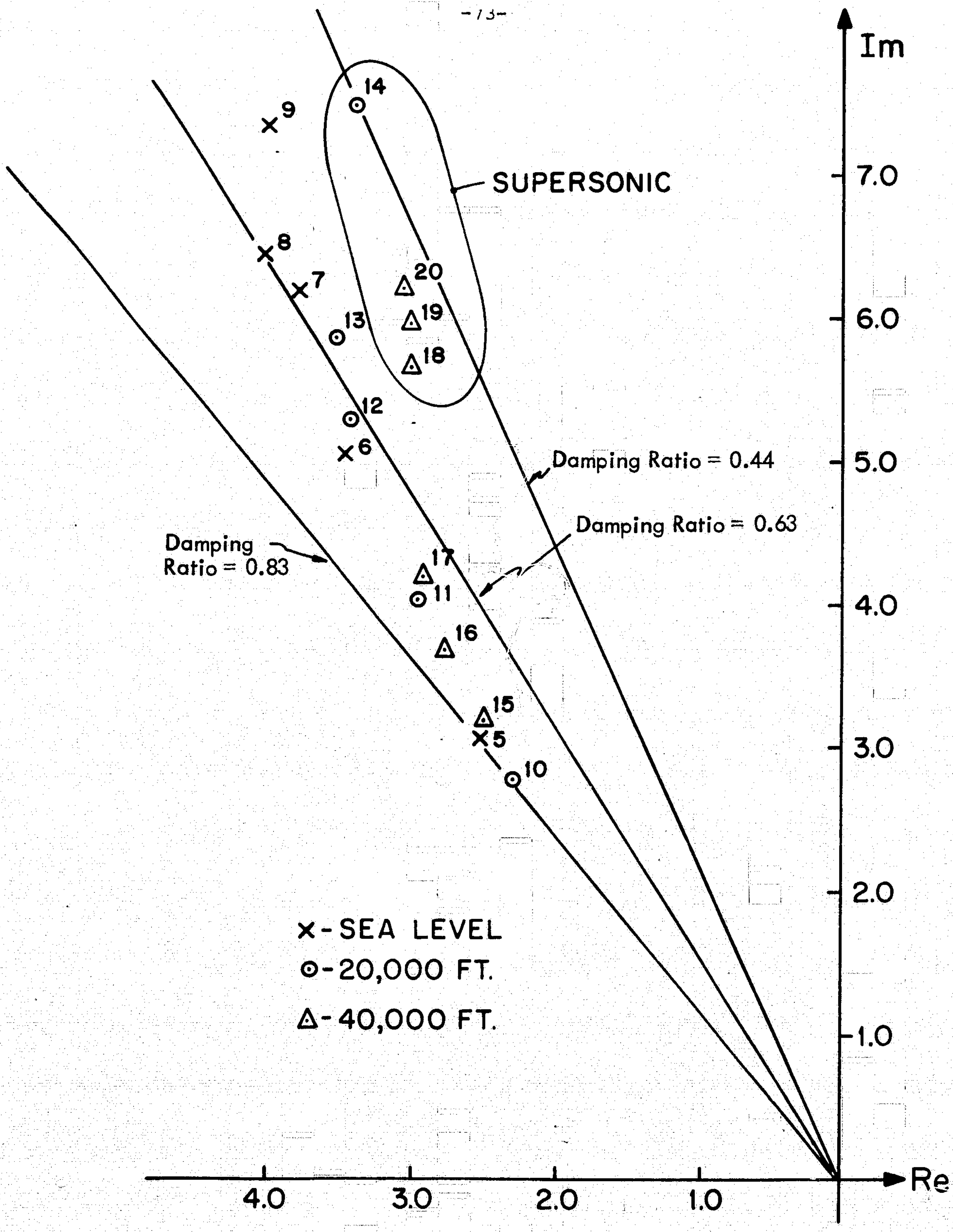


Fig. 5.2 A Plot of Continuous Time Closed Loop Poles of the Reduced State System using C* Reduced State Gains; the Poles Correspond to the Optimal Short Period Dynamic for each Flight Condition

the complex pole increases as the dynamic pressure increases, at each altitude. These, of course, correspond to the short period dynamics. For the purposes of comparing this C* design with some other traditional handling criteria of aircrafts, we draw in three straight lines to show the range of damping ratios through these flight conditions. One such criterion is the Cornell Aeronautical Laboratory "thumbprint" [10]. This "thumbprint" represents combinations of short period natural frequency (ω_n) and damping ratio (ζ) that were preferred by pilots of several variable stability aircrafts, and is valid on the assumption that the predominant variable sensed by the pilot is normal acceleration, a_z. Figure 5.3 shows such a thumbprint plot together with three vertical lines corresponding to the damping ratios in Fig. 5.2. Comparing these two figures, we see that all the flight conditions lie more or less within the thumbprint, except flight conditions 9 and 14 which represent transition behavior between subsonic and supersonic flights. The fact that flight conditions 18-20 lie close to the boundary is not surprising since they are supersonic. Most important of all, we observe that flight conditions 5, 10 and 15 lie very close to the boundary. This is explained by the fact that they have low Mach numbers, and indeed, the pitch variables are more important cues to the pilot.

Table 5.3 is a mismatch stability table, in the sense that the aircraft is in flight condition i, $5 \leq i \leq 20$, but one is using control gains calculated from flight condition j, $5 \leq j \leq 20$. Specifically, the closed loop dynamics are

$$\dot{\underline{x}}(t) = \underline{A}_{r_i} \underline{x}(t) + \underline{B}_{r_i} u(t) \tag{5.4}$$

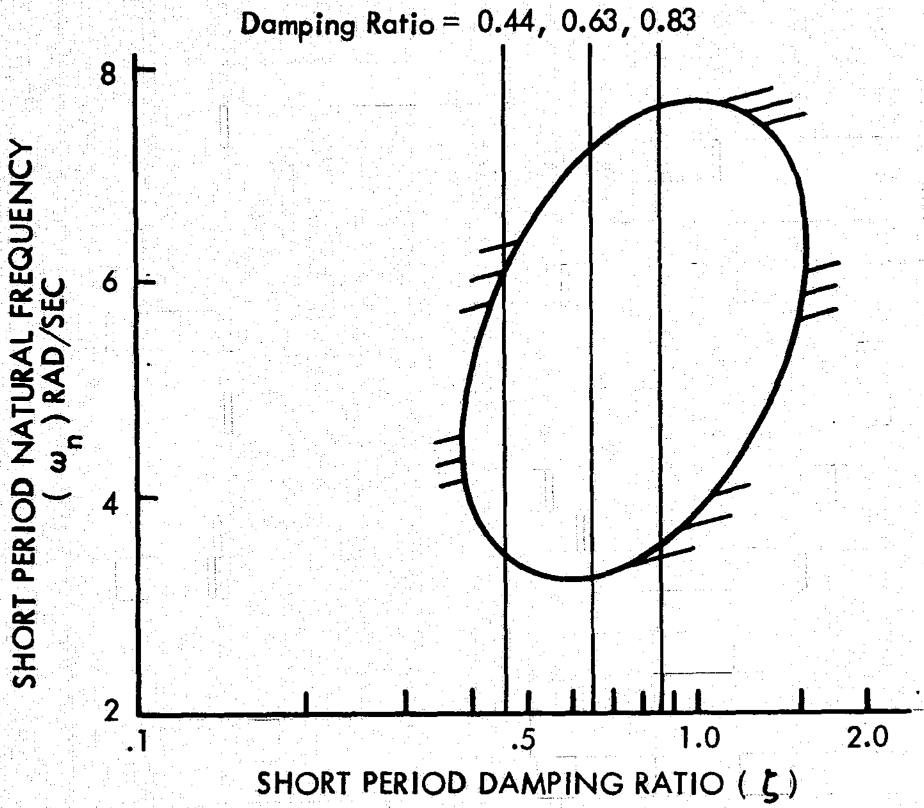


Fig. 5.3 Cornell Aeronautical Laboratory "Thumbprint"

STABILITY SUMMARY TABLE

TRUE FC	CONTROLLER 5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
6	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
7	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
8	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
9	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
10	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
13	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
14	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
15	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
16	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
17	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
18	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
19	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
20	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*

U=UNSTABLE
*=STABLE

Table 5.3 Mismatch Stability Table for Design A
(All mismatches are stable.)

$$u(t) = L_{r_j} \int_0^t (C_d^* - C^*(\tau))d\tau + \underline{N}_{r_j} (\underline{x}(t) - \underline{x}(0)) + u(0) \qquad (5.5)$$

where $\underline{A}_{r_i}$, $\underline{B}_{r_i}$ are $\underline{A}_r$, $\underline{B}_r$ matrices of flight condition i, and L_{r_j} and $\underline{N}_{r_j}$ are control gains from flight condition j.

It is seen from Table 5.3 that all mismatches give stable flight for the system. Also we can see from the simulations A-1 to A-3 (Figures 5.4 - 5.9) that the C* and δ_{e_c} responses are in general very satisfactory, with acceptable control rates. When there is not any mismatch (i.e., i,j=12), the system gives the best response, as is expected. Observe how nicely the C* response fits into the C* envelope in simulation A-2 (Figures 5-6, 5-7). Mismatches give a little slower response, and more overshoot.

We attach also plots of the L and N gains against Mach numbers and dynamic pressures (psf) (Figures 5-10 - 5-19). Notice that in each plot the subsonic flight conditions are readily divided into three groups according to altitude, and the supersonic flight conditions form a single group by themselves.

(ii) Design B

In Chapter IV it was discussed how one can apply the reduced state control gains computed in design A to a full state model, in equations (4.8) and (4.9). We study such a design strategy by doing simulations.

It is seen that the responses in all cases are almost identical to the corresponding ones for design A. Also we remark that this design gives stable flights for all mismatches. This suggests very strongly that the reduced state design is a good design for the system, at least as far as

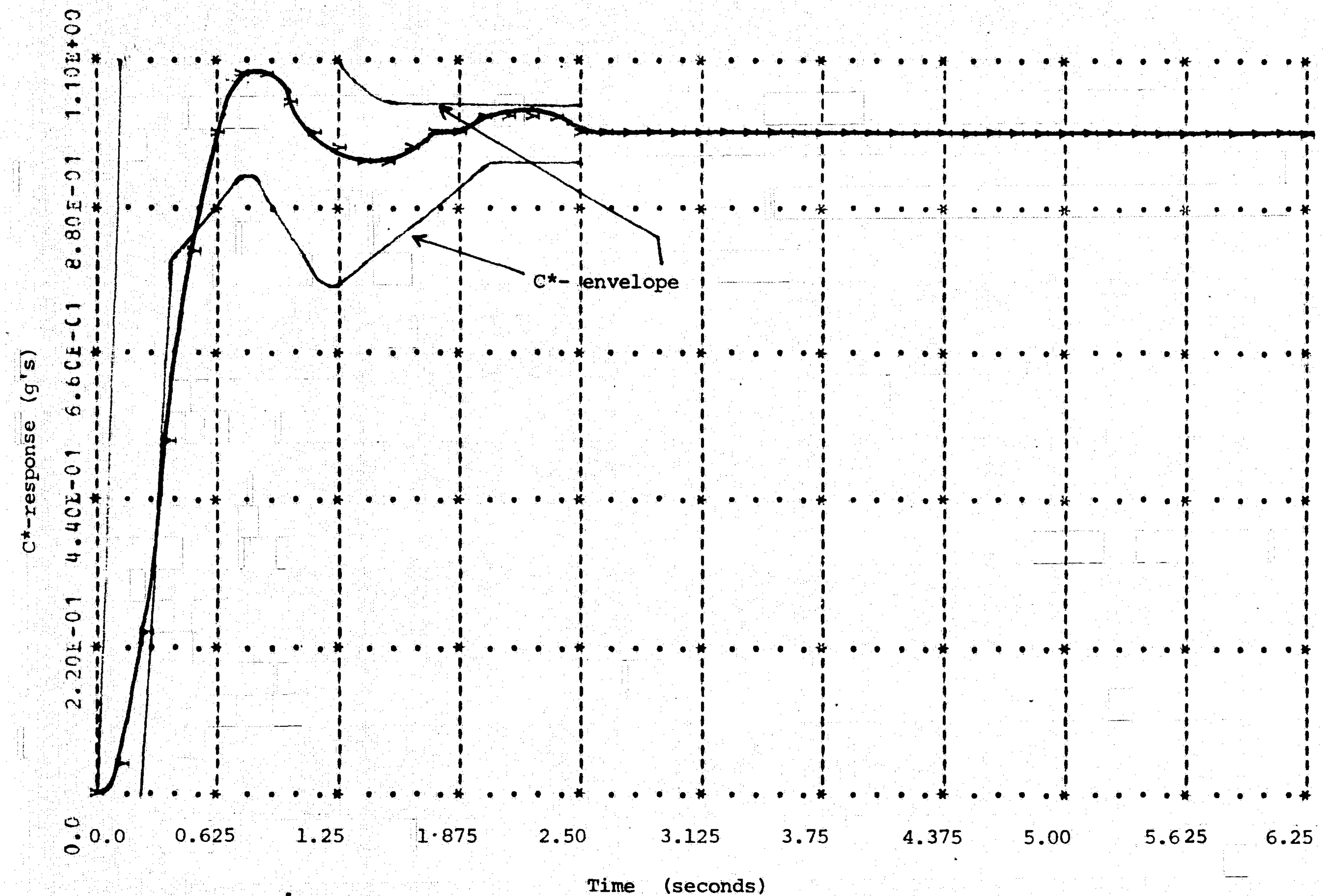


Fig. 5.4 Simulation A-1; C*-response to a unit step C_d^* input, Mismatch (12-7)

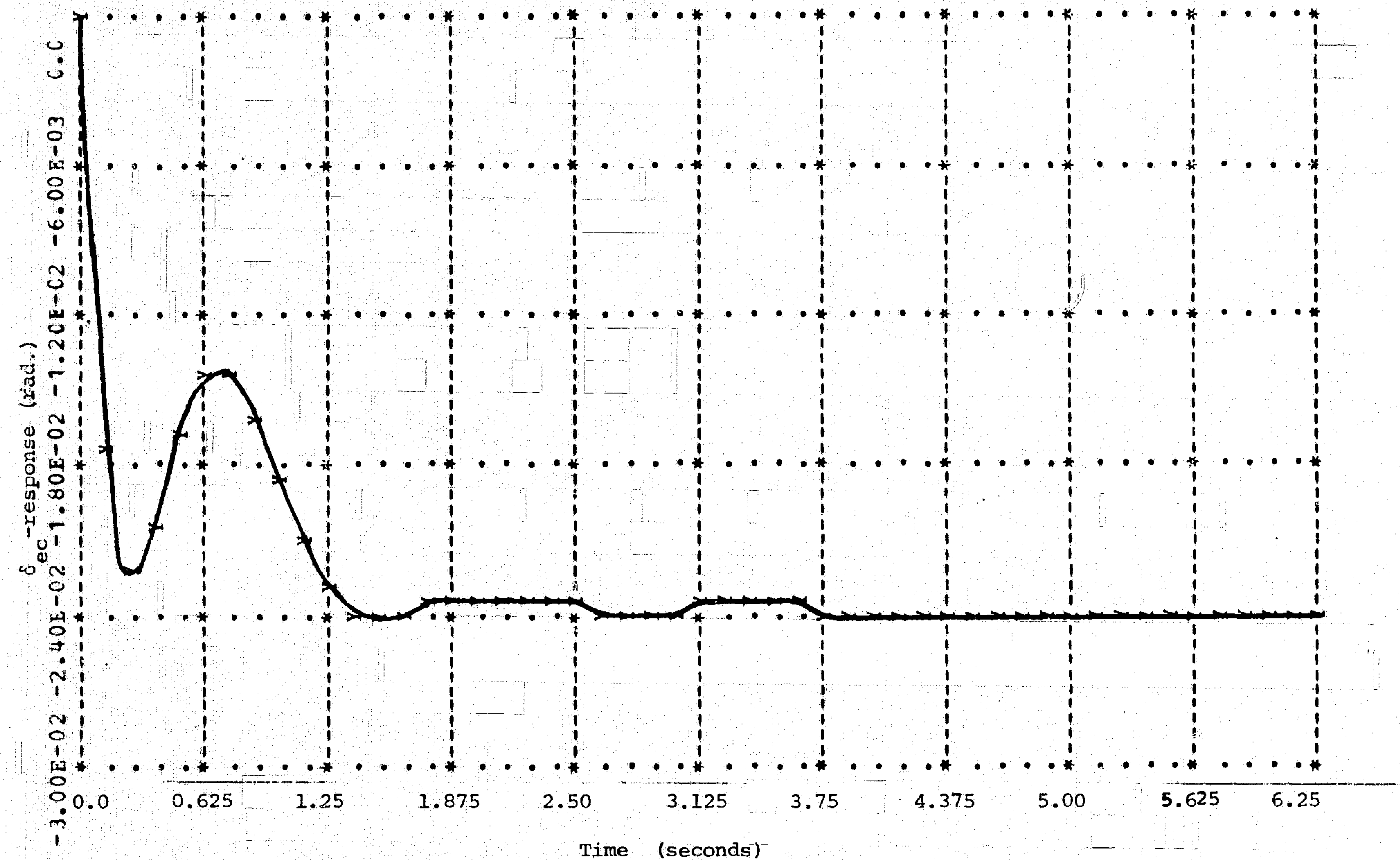


Fig. 5-5 Simulation A-1; δ_{ec}-reponse to a unit step C_d^* input; Mismatch (12-7)

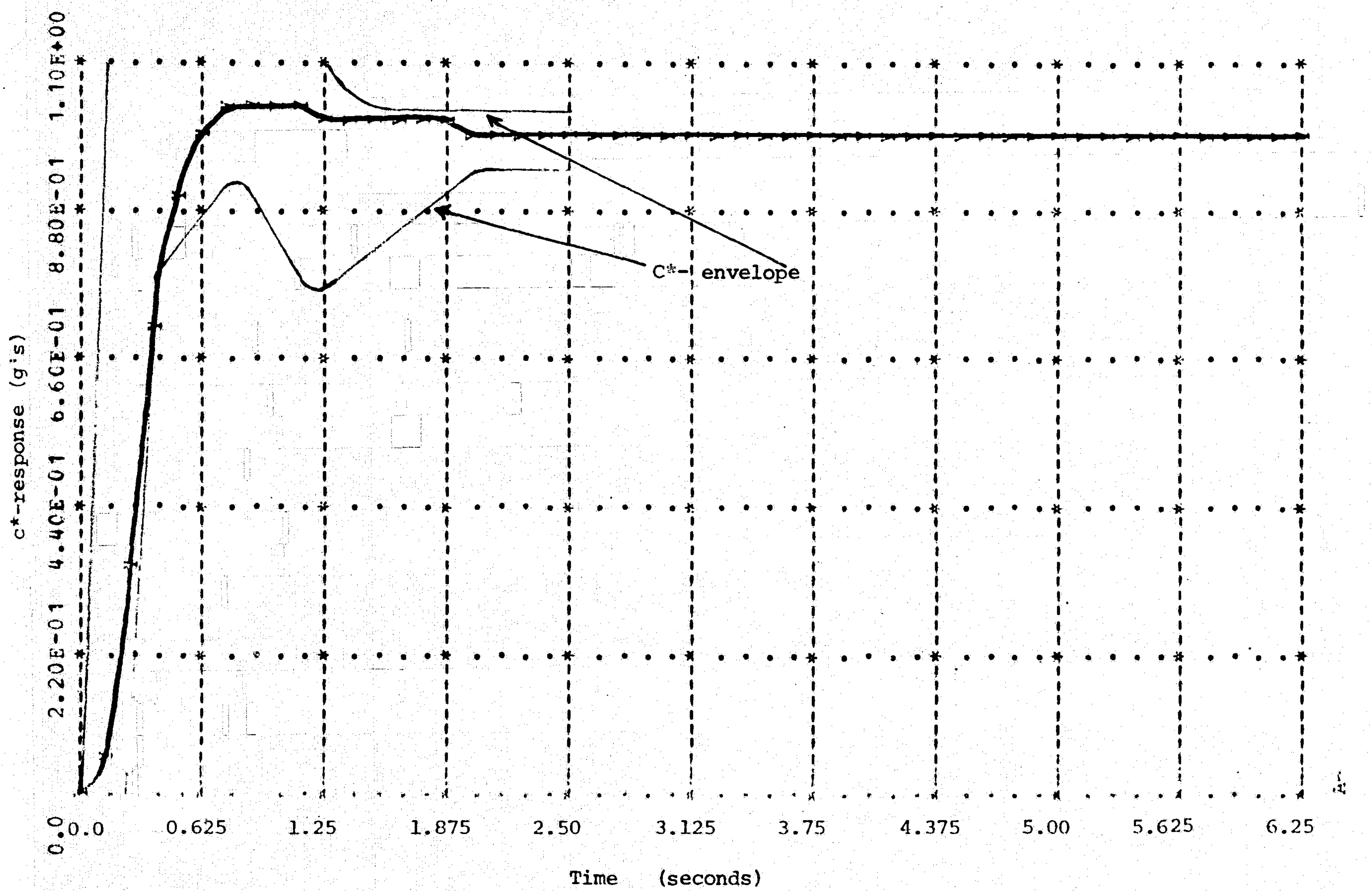


Fig. 5.6 Simulation A-2 C*-response to a unit step C_d^* input Mismatch (12-11)

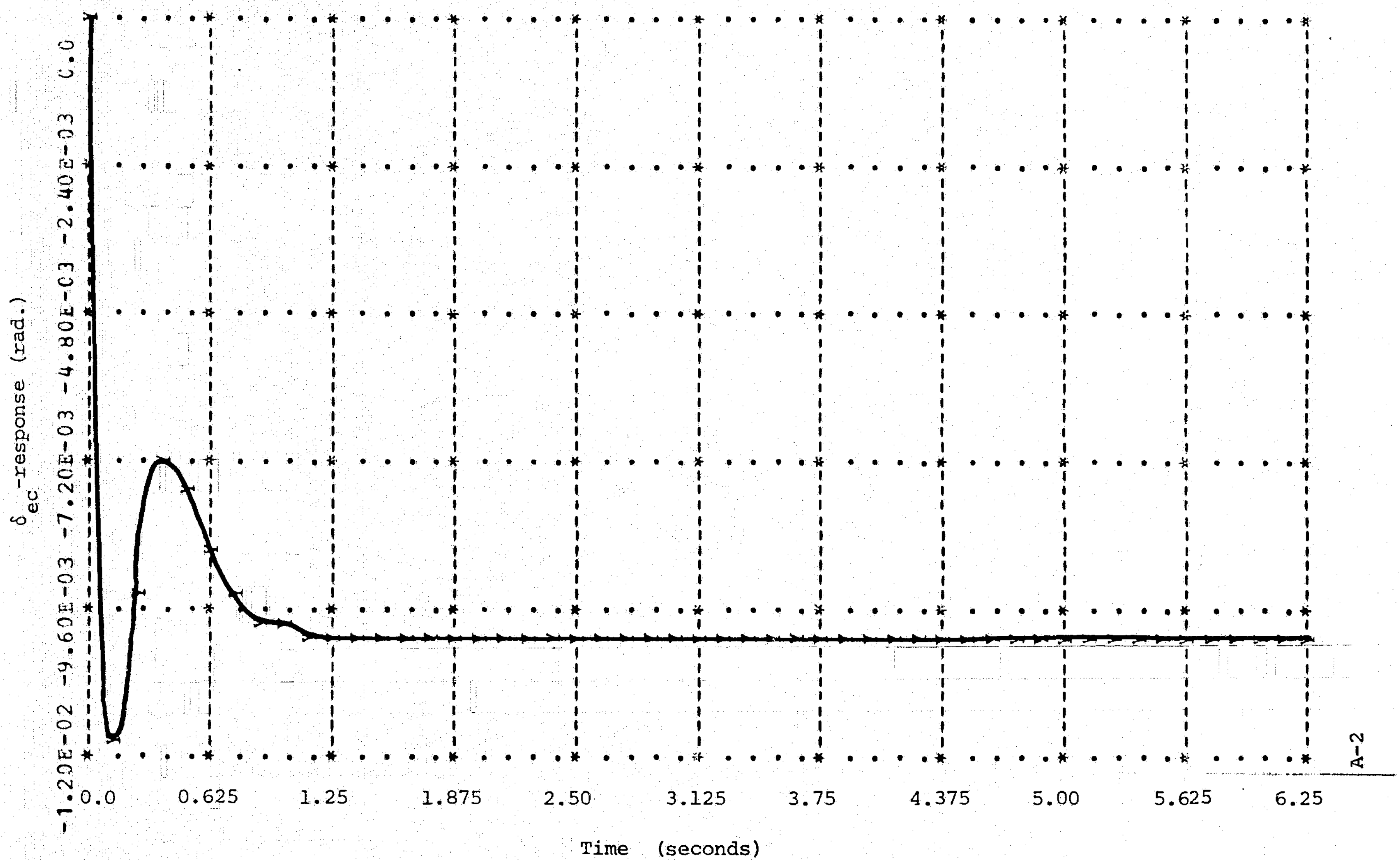


Fig. 5.7 Simulation A-2; δ_{ec}-response to a unit step C_d^* input; Mismatch (12-11)

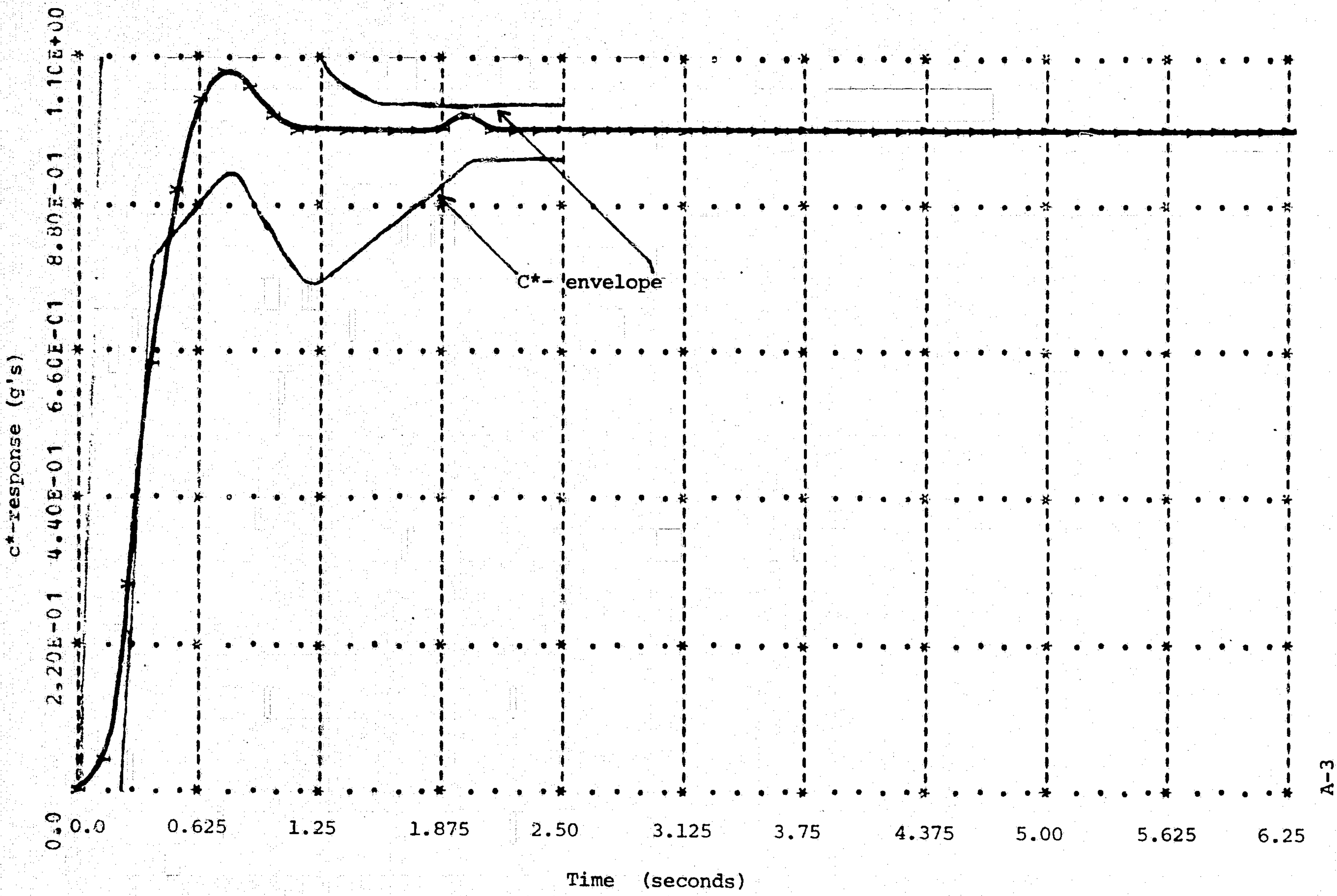


Fig. 5.8 Simulation A-3; C*-response to a unit step C^*_d input; match (12-12)

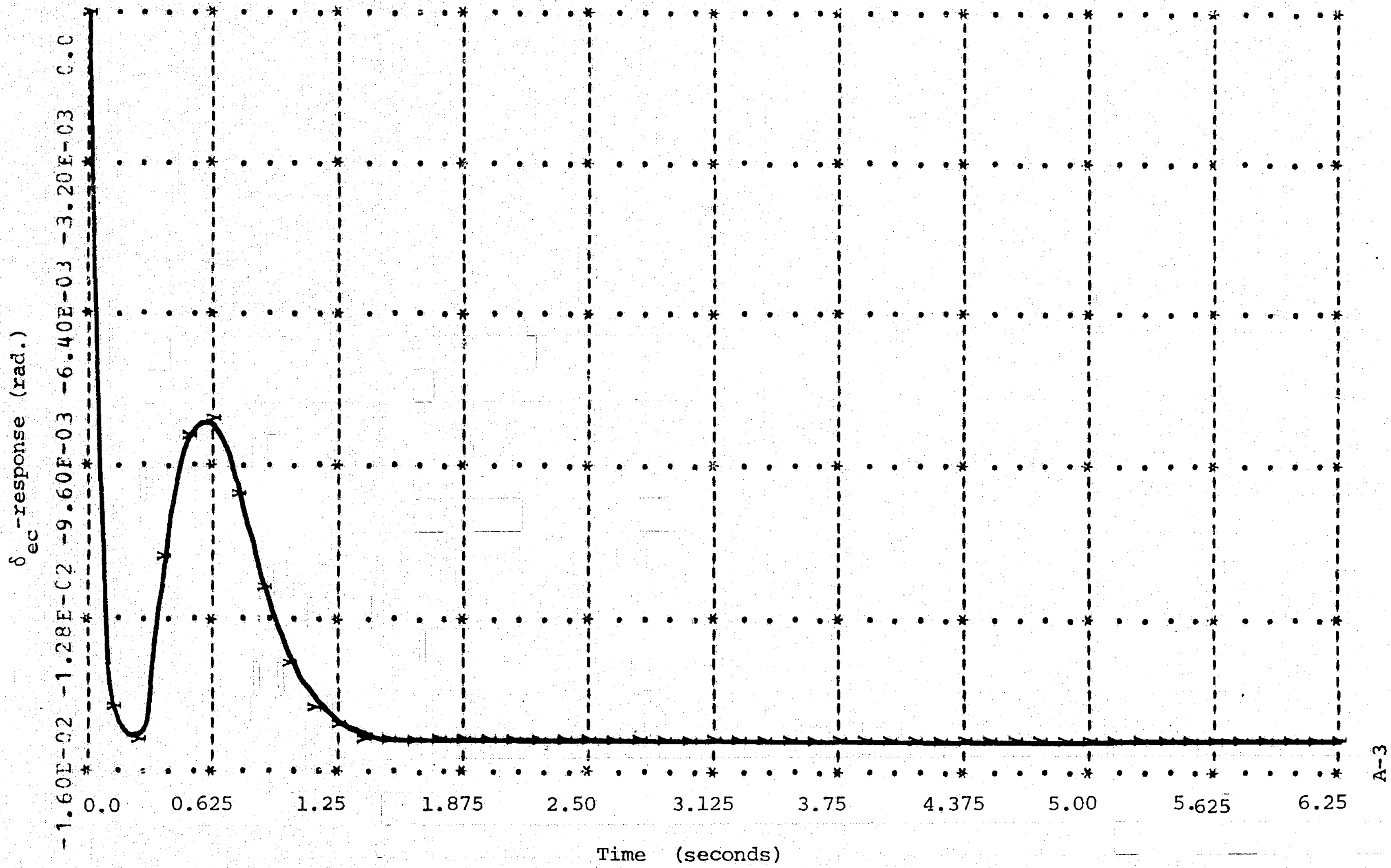


Fig. 5.9 Simulation A-3; δec-response to a C^*_d input; match (12-12)

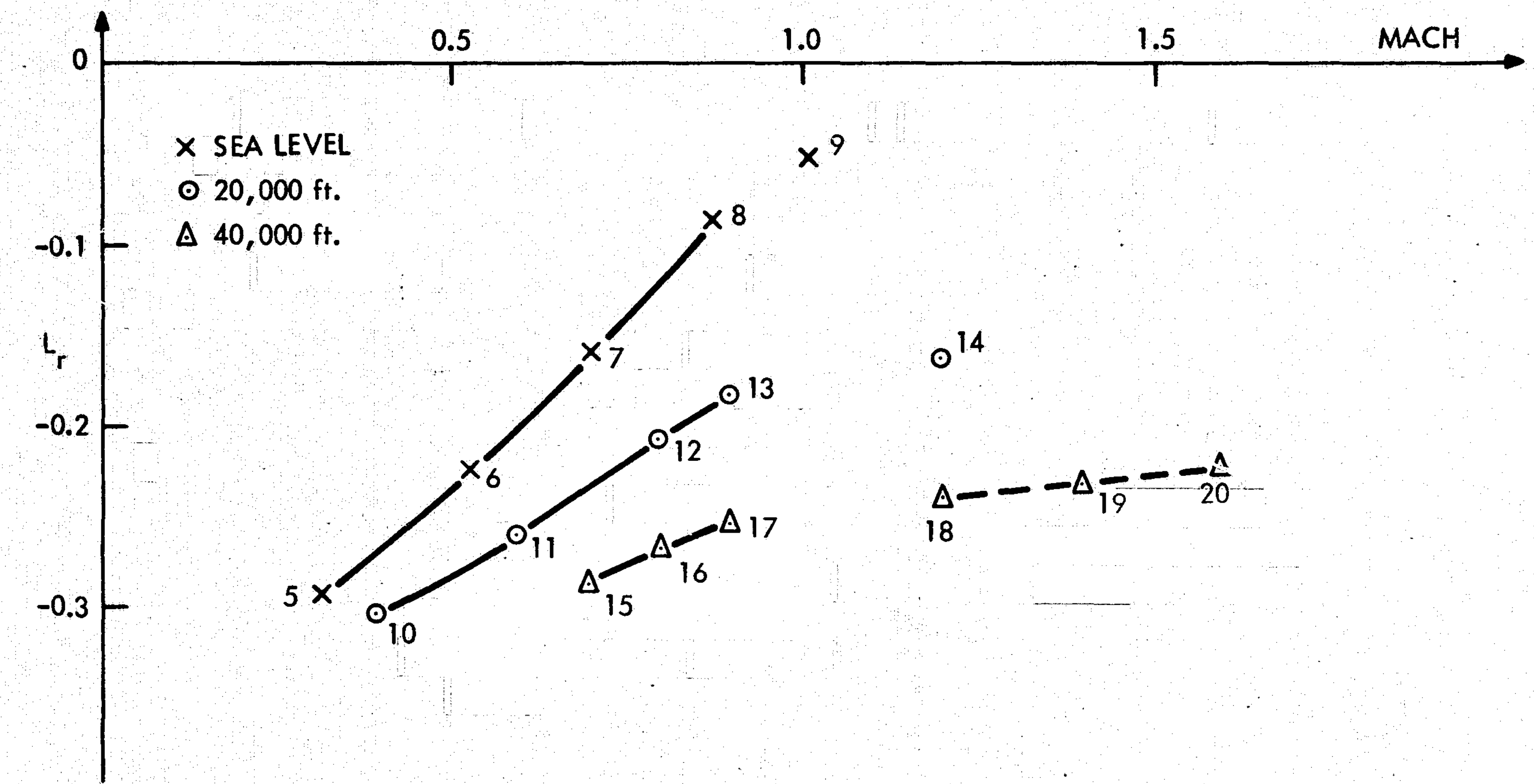


Fig. 5.10 (Continuous Time) L_r Gains vs. Mach Numbers

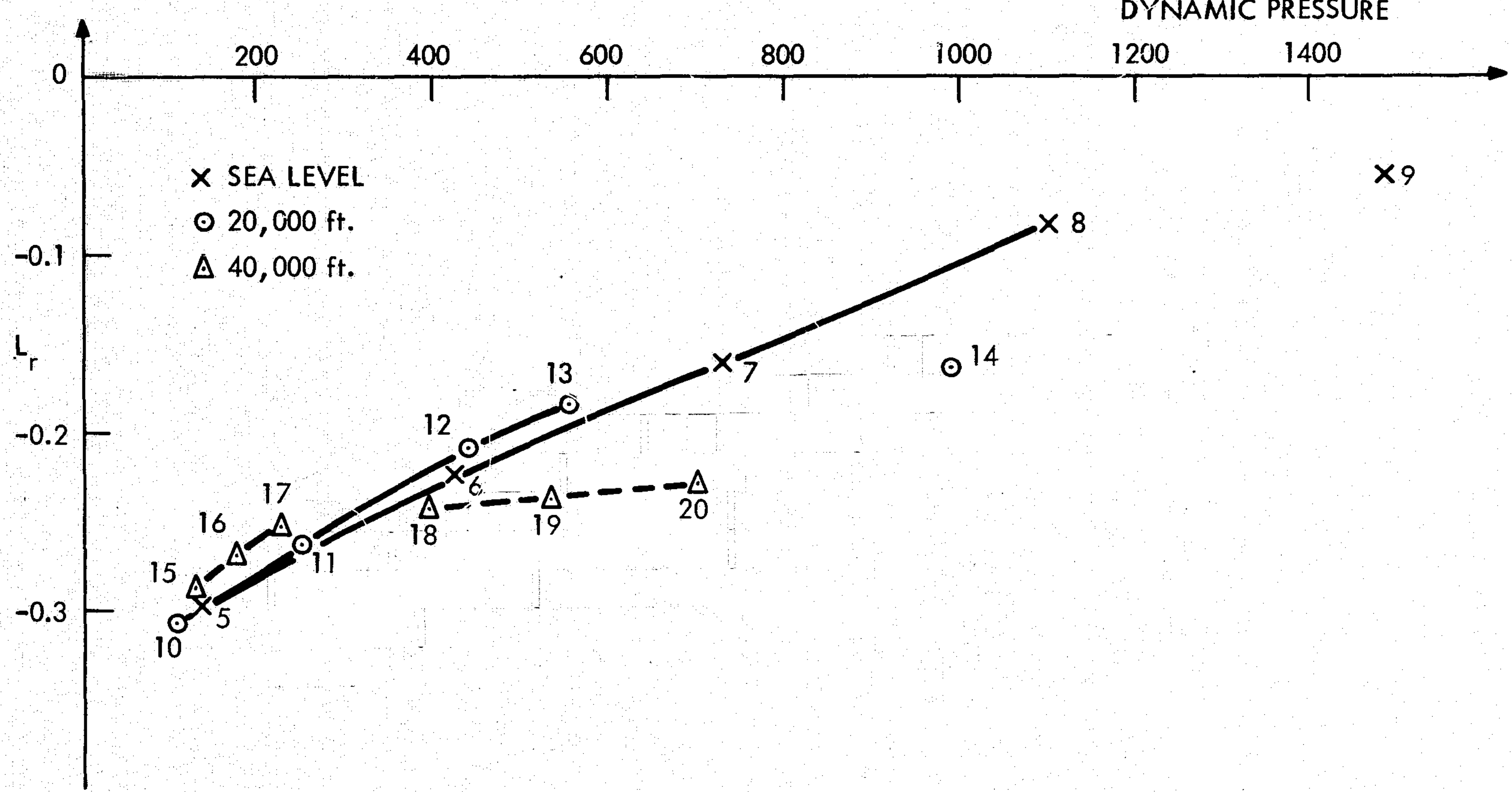


Fig. 5.11 (Continuous Time) L_r Gains vs. Dynamic Pressures

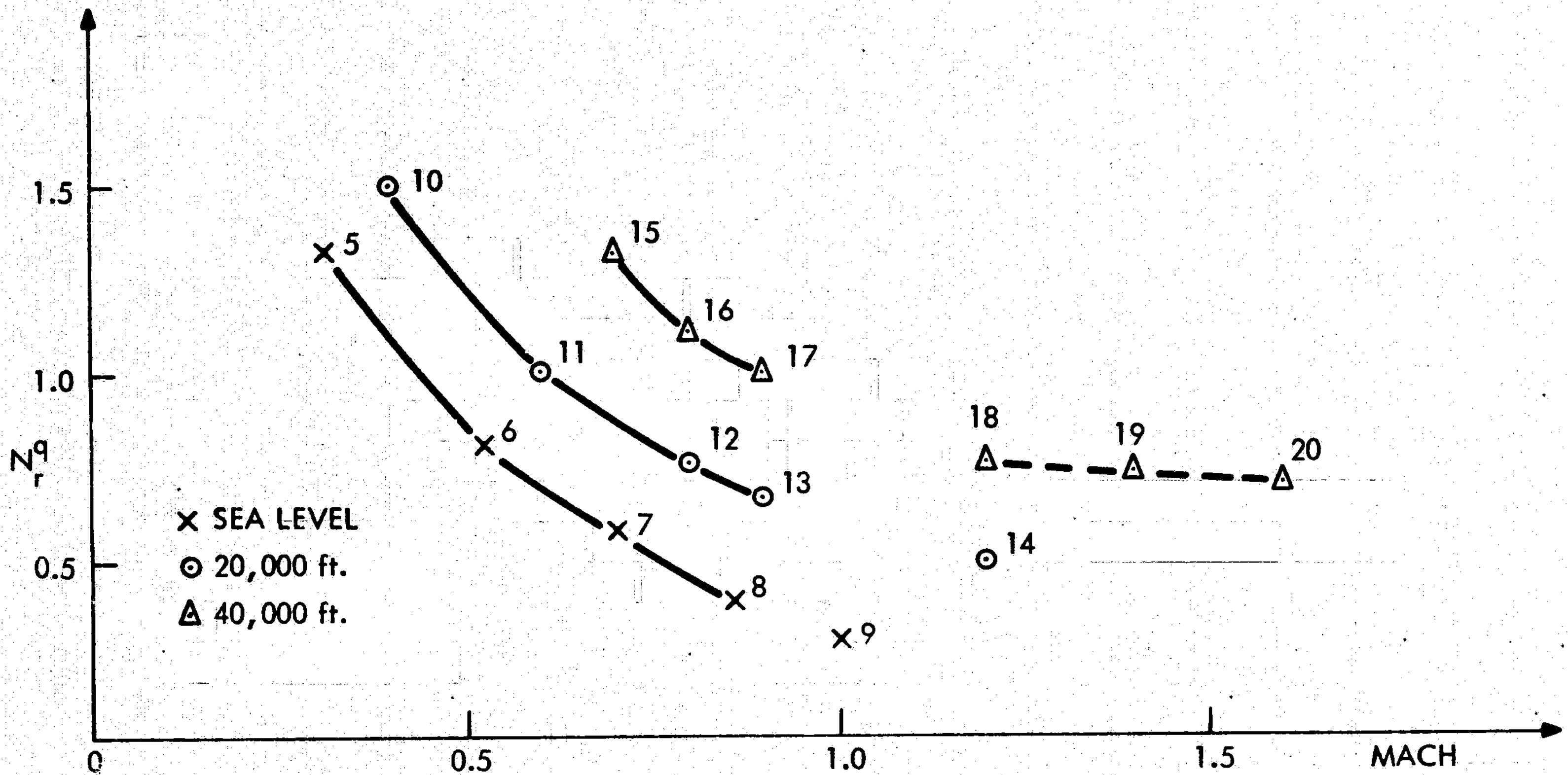


Fig. 5.12 (Continuous Time) N_r^q Gains vs. Mach Numbers

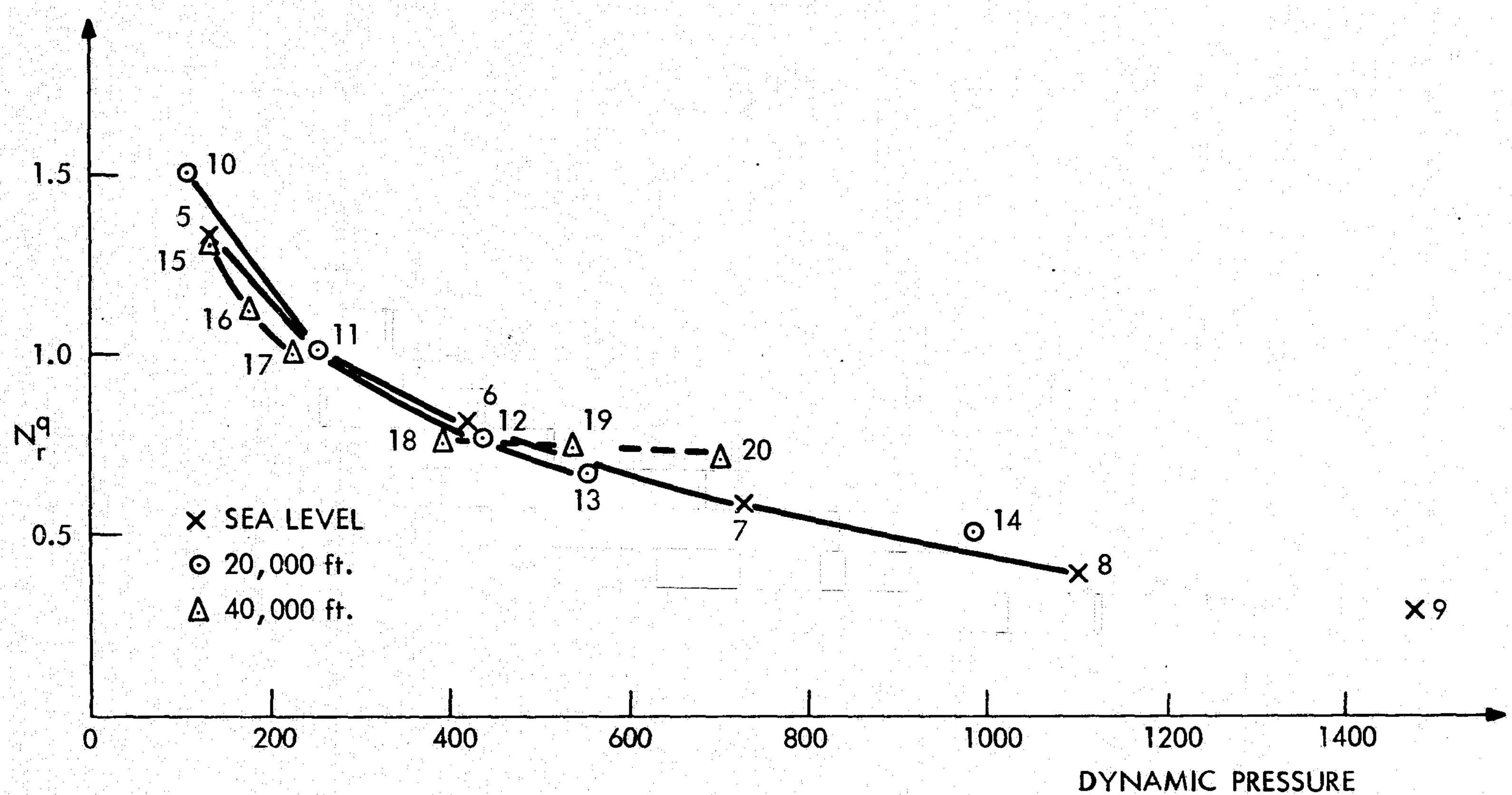


Fig. 5.13 (Continuous Time) N_r^q Gains vs. Dynamic Pressures

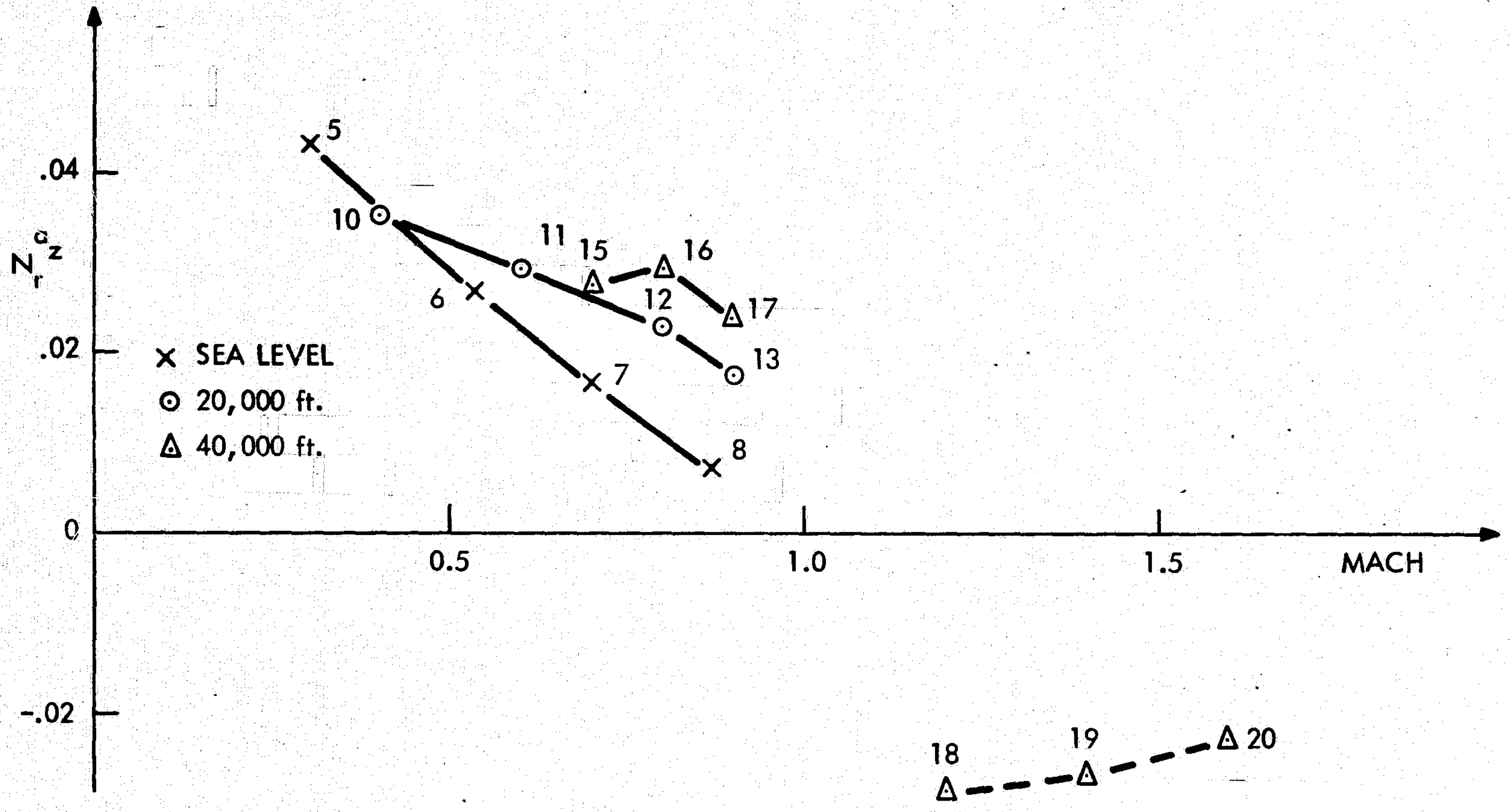


Fig. 5.14 (Continuous Time) $N_r^{a_z}$ Gains vs. Mach Numbers

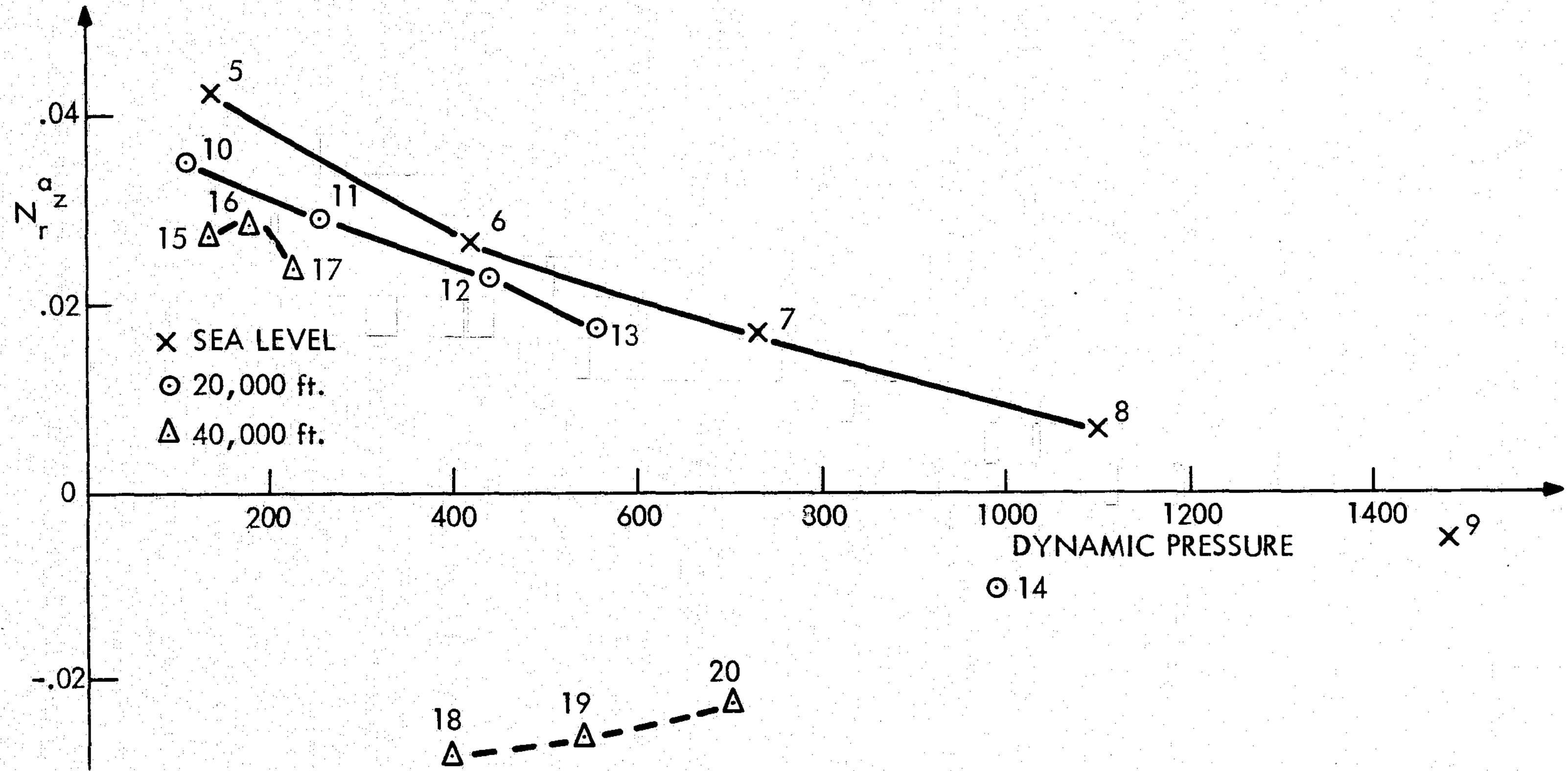


Fig. 5.15 (Continuous Time) $N_r^{a_z}$ Gains vs. Dynamic Pressures

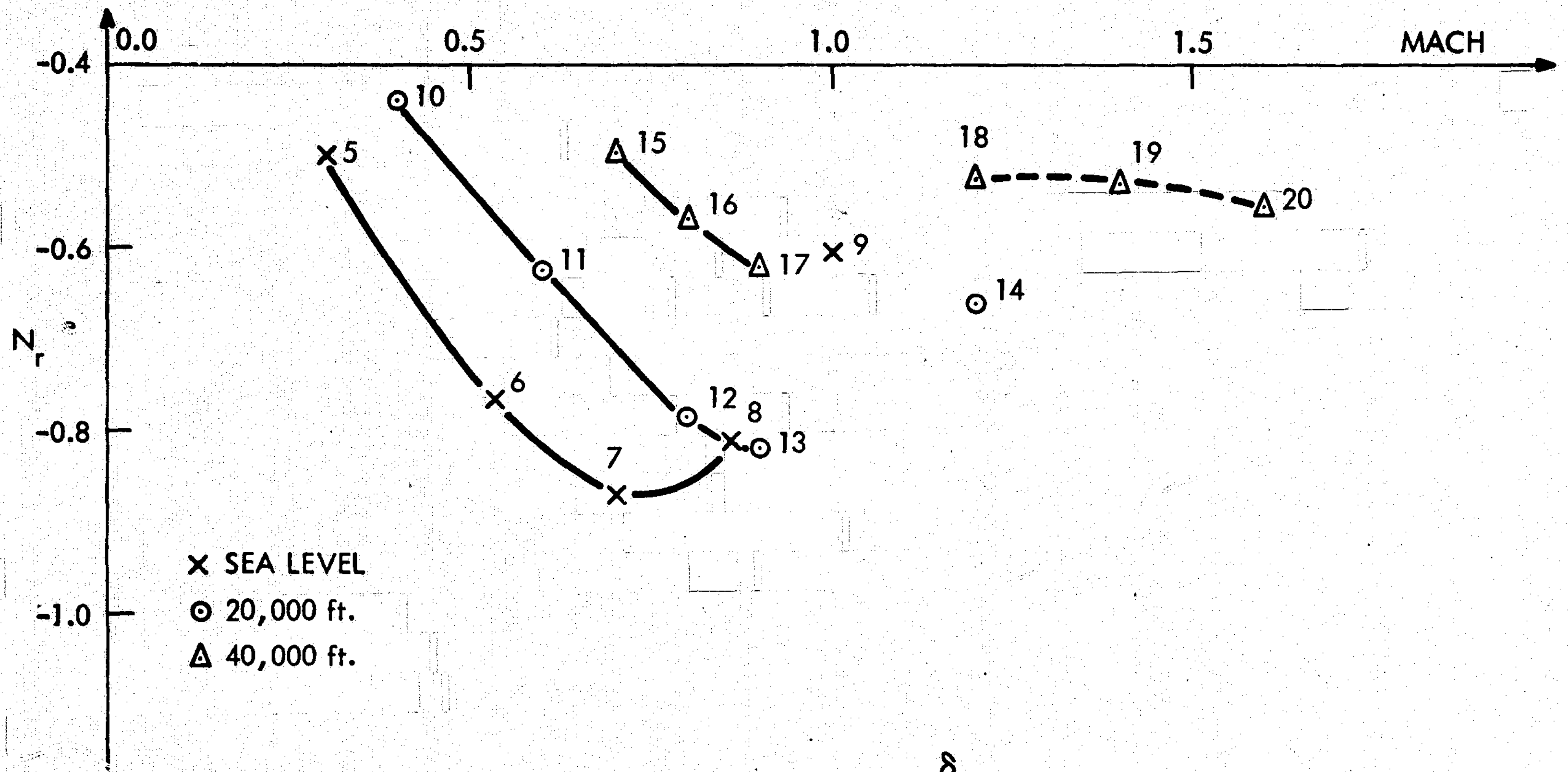


Fig. 5.16 (Continuous Time) $N_r^{\delta_e}$ Gains vs. Mach Numbers

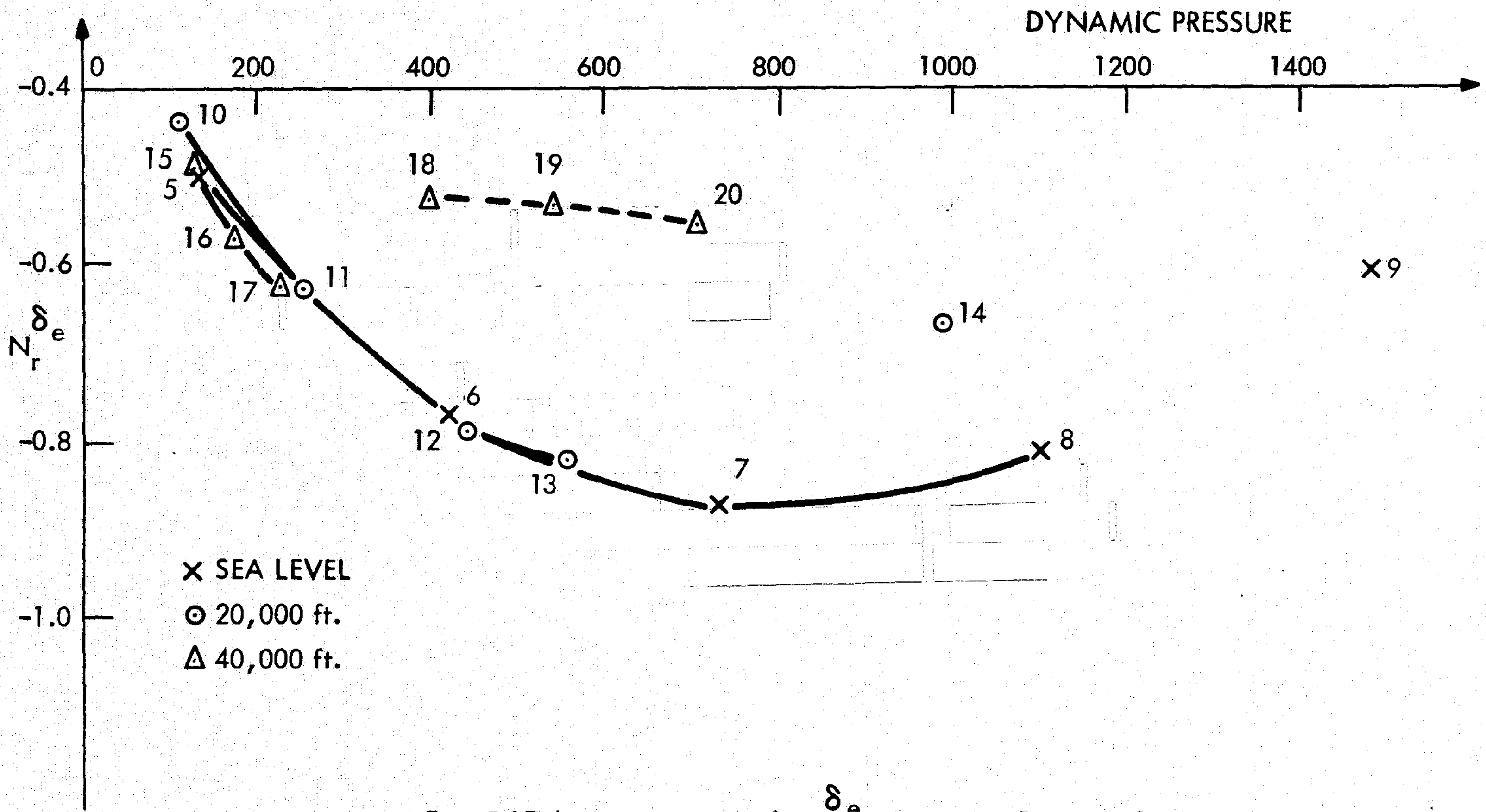


Fig. 5.17 (Continuous Time) $N_r^{\delta_e}$ Gains vs. Dynamic Pressures

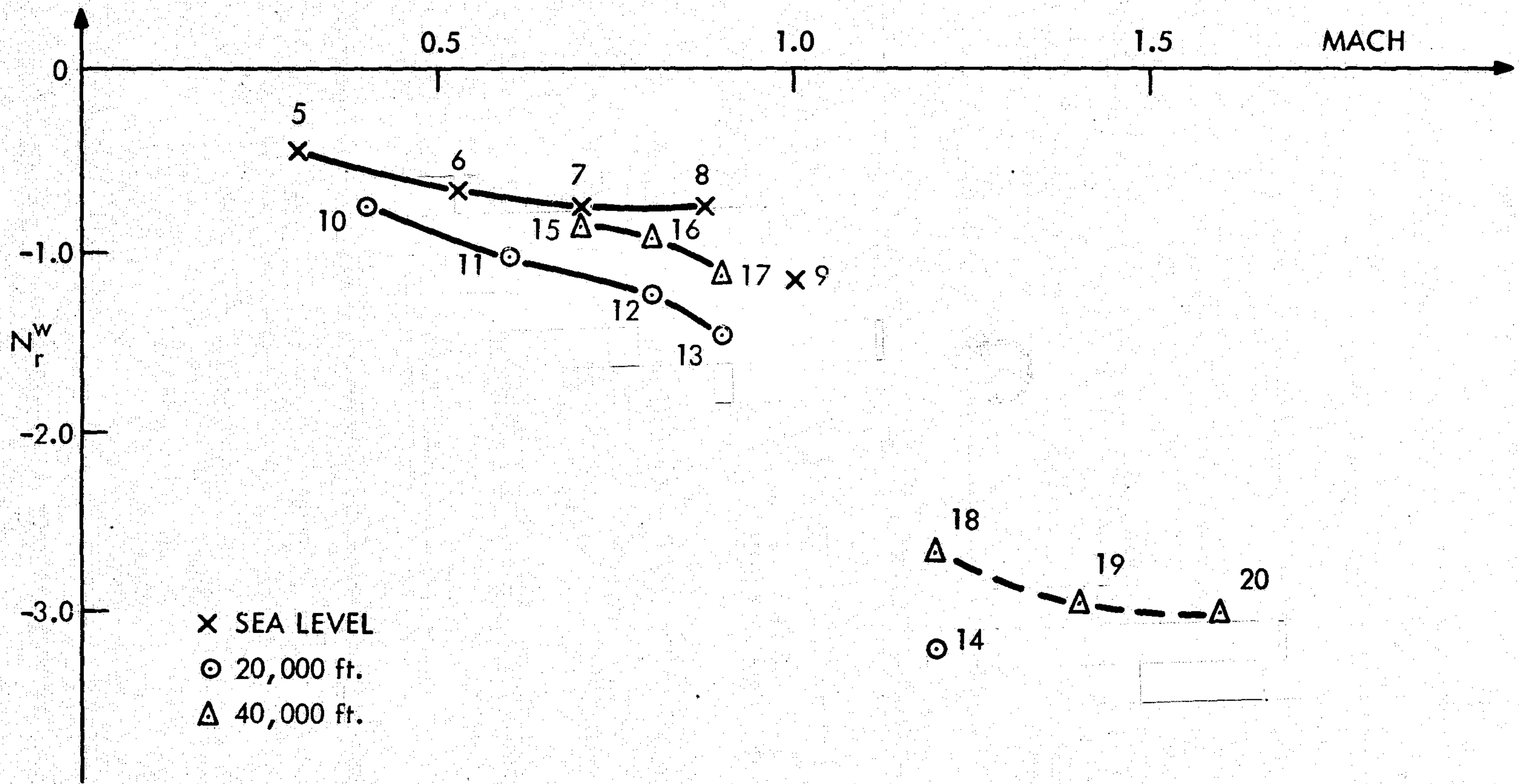


Fig. 5.18 (Continuous Time) N_r^w Gains vs. Mach Numbers

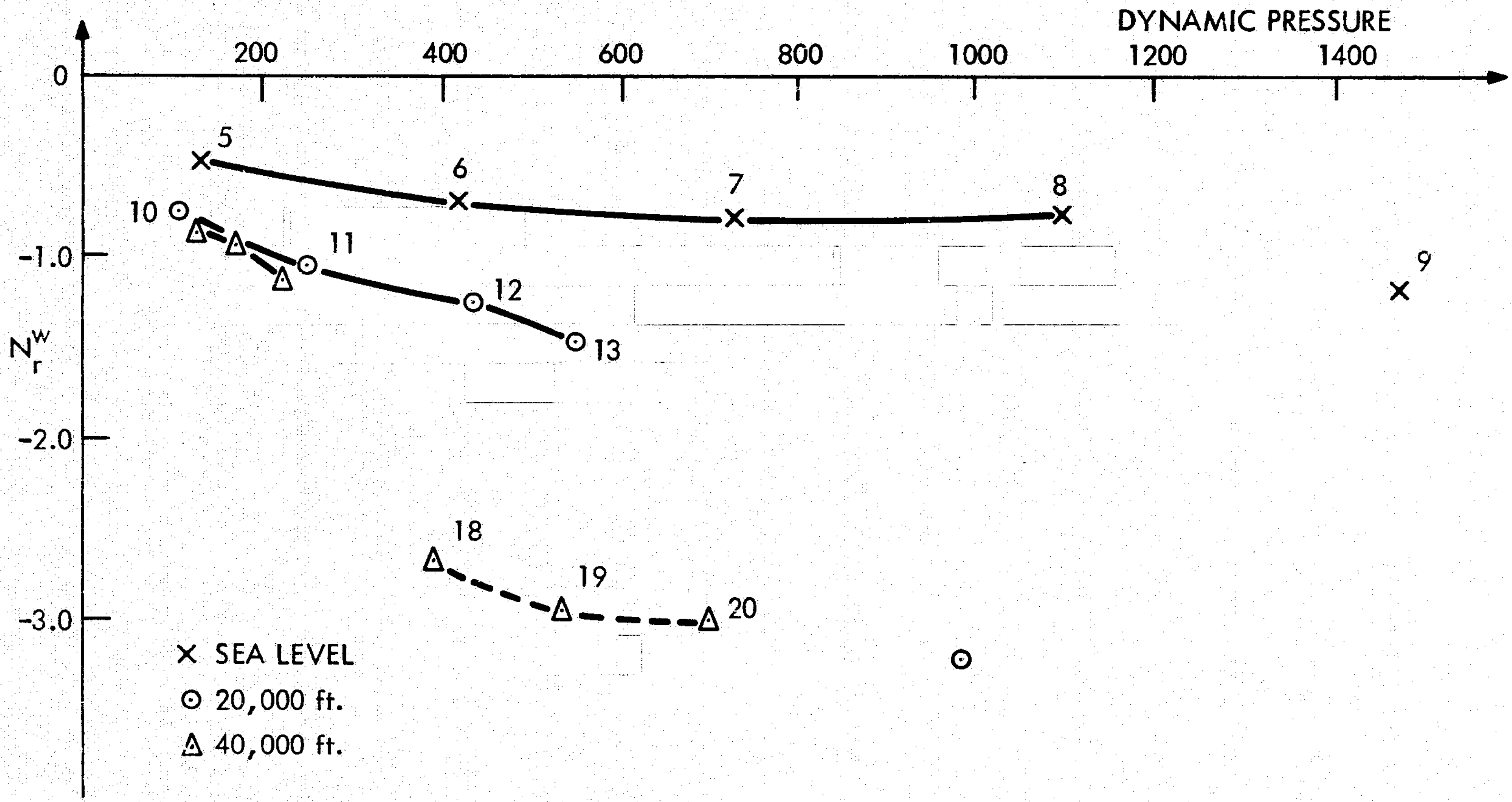


Fig. 5.19 (Continuous Time) N_r^w Gains vs. Dynamic Pressures

continuous time is concerned. As an example, we attach Figures 5.20, 5.21 which show simulation B-3 of the matched flight condition 12.

5.3 Discrete Time Designs

To pick appropriate values for the weightings Q_d, R_d, in the cost function given by equation (3.39), one might want to do it in such a way that this cost function approximates that for continuous time, i.e., equation (3.21). We elect to use the following values:

$$Q_d = TQ \tag{5.6}$$

$$R_d = R/T \tag{5.7}$$

where T is the sampling period, which is 1/8 sec. in our design, and the choice of Q and R is given by equations (5.1)-(5.3). This was done because we can use the approximations

$$\dot{u}(kT) \simeq \frac{u((k+1)T)-u(kT)}{T} \tag{5.8}$$

$$\int_0^{kT} f(\tau)d\tau \simeq T \sum_{j=0}^{k-1} f(jT) \tag{5.9}$$

for any real continuous time function $f(\cdot)$ and hence $(C^*_d - C^*(\cdot))^2 Q$. Thus, we use the following weighting parameters in Eq. (3.39)

$$Q_d = \frac{1}{8} \tag{5.10}$$

$$R_d = 8 \times 0.435 \tag{5.11}$$

(i) Design C

Using the above values of Q_d and R_d, and the discretized reduced state model obtained by applying the discretization scheme in proposition 2.3 to the continuous time reduced state system given by equations (4.3), (4.4), we can follow the design procedures in proposition 2.3

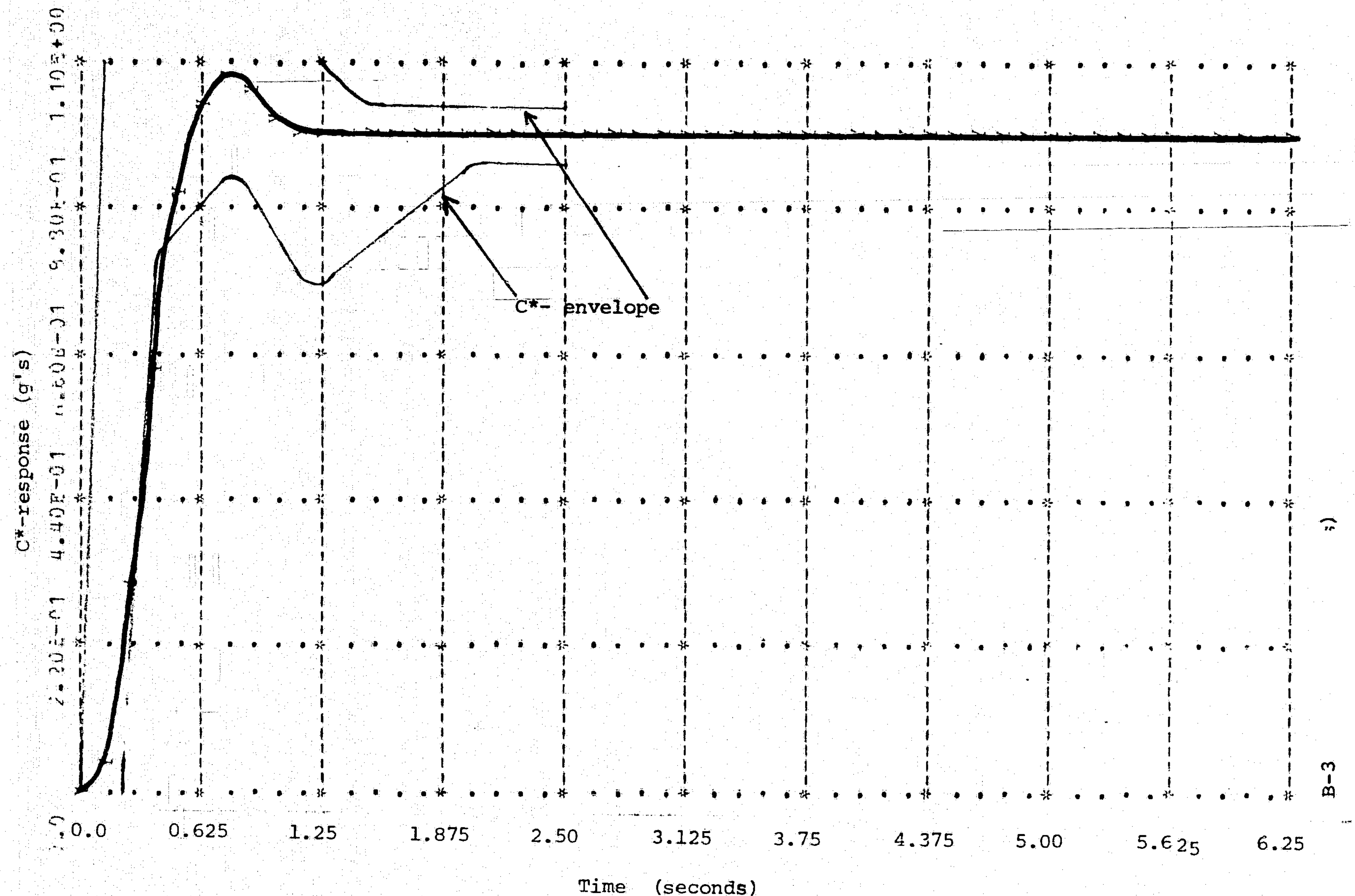


Fig. 5.20 Simulation B-3; C*-response to a unit step C^*_d input; Match (12-12)

B-3

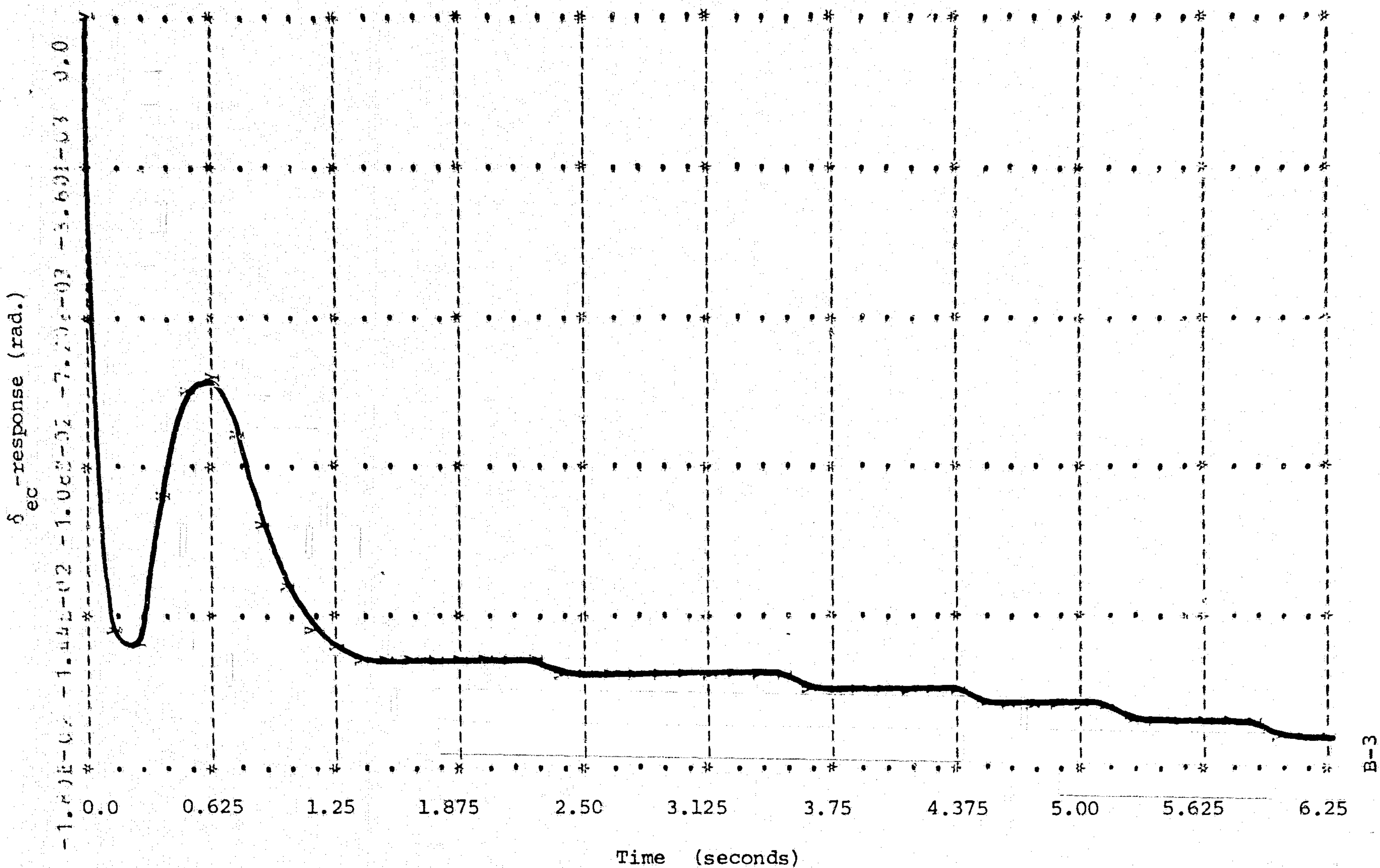


Fig. 5.21 Simulation B-3; δec-response to a unit step C^*_d input; match (12-12)

to find control gains L_{d_r}, and $\underline{N}_{d_r}$. That is, we may write,

$$\underline{x}_r((k+1)T) = \underline{A}_{d_r}\underline{x}_r(kT) + \underline{B}_{d_r}u(kT) \tag{5.12}$$

$$u(kT) = L_{d_r}\sum_{j=0}^{k-1}(C^*_d - C^*(jT)) + \underline{N}_{d_r}(\underline{x}_r(kT) - \underline{x}_r(0)) + u(0) \tag{5.13}$$

Indeed the assumptions required by proposition 2.2 are found to be satisfied by all the sixteen flight conditions.

Considering mismatches, we form the dynamics

$$\underline{x}_r((k+1)T) = \underline{A}_{d_{r_i}}\underline{x}_r(kT) + \underline{B}_{d_{r_i}}u(kT) \tag{5.14}$$

$$u(kT) = L_{d_{r_j}}\sum_{\ell=0}^{k-1}(C^*_d - C^*(\ell T)) + \underline{N}_{d_{r_j}}(\underline{x}_r(kT) - \underline{x}_r(0)) + u(0) \tag{5.15}$$

where $5\leq i,j\leq 20$ and $\underline{A}_{d_{r_i}}$, $\underline{B}_{d_{r_i}}$ are $\underline{A}_{d_r}$, $\underline{B}_{d_r}$ matrices of flight condition i, and $L_{d_{r_j}}$, $\underline{N}_{d_{r_j}}$ are control gains L_{d_r}, $\underline{N}_{d_r}$ from flight condition j. A mismatch stability table for this design is shown in Table 5.4. Here instead of universal mismatch stability, as in the continuous time case (Table 5.3), there are about 10% of the mismatches that are unstable. In particular, most of them are mismatches of controllers from flight conditions that have low dynamic pressures to flight conditions of high Mach numbers (and high dynamic pressures). Of course the system is always stable whenever there is no mismatching.

Looking at the simulations (Figures 5.22-5.27), we observe that they all show very desirable responses. Indeed they are almost the same as those for design A, although now we have a little more overshoot. The

STABILITY SUMMARY TABLE

TRUE FC	CONTROLLER 5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
5	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
6	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
7	U	*	*	*	*	U	*	*	*	*	U	U	*	*	*	*
8	U	U	*	*	*	U	U	*	*	*	U	U	U	*	*	*
9	U	U	*	*	*	U	U	U	U	*	U	U	U	*	*	*
10	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
11	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
13	U	*	*	*	*	U	*	*	*	*	U	*	*	*	*	*
14	U	*	*	*	*	U	U	*	*	*	U	U	U	*	*	*
15	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
16	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
17	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
18	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
19	*	*	*	*	*	U	*	*	*	*	*	*	*	*	*	*
20	U	*	*	*	*	U	*	*	*	*	*	*	*	*	*	*

U=UNSTABLE
*=STABLE

Table 5.4 Mismatch Stability Table for Design C
(e.g. if the aircraft is in FC 9 (row) and one use the controller of FC 7 (column), the closed loop aircraft will be stable; if one uses controller of FC 12 (column), the closed loop aircraft will be unstable.)

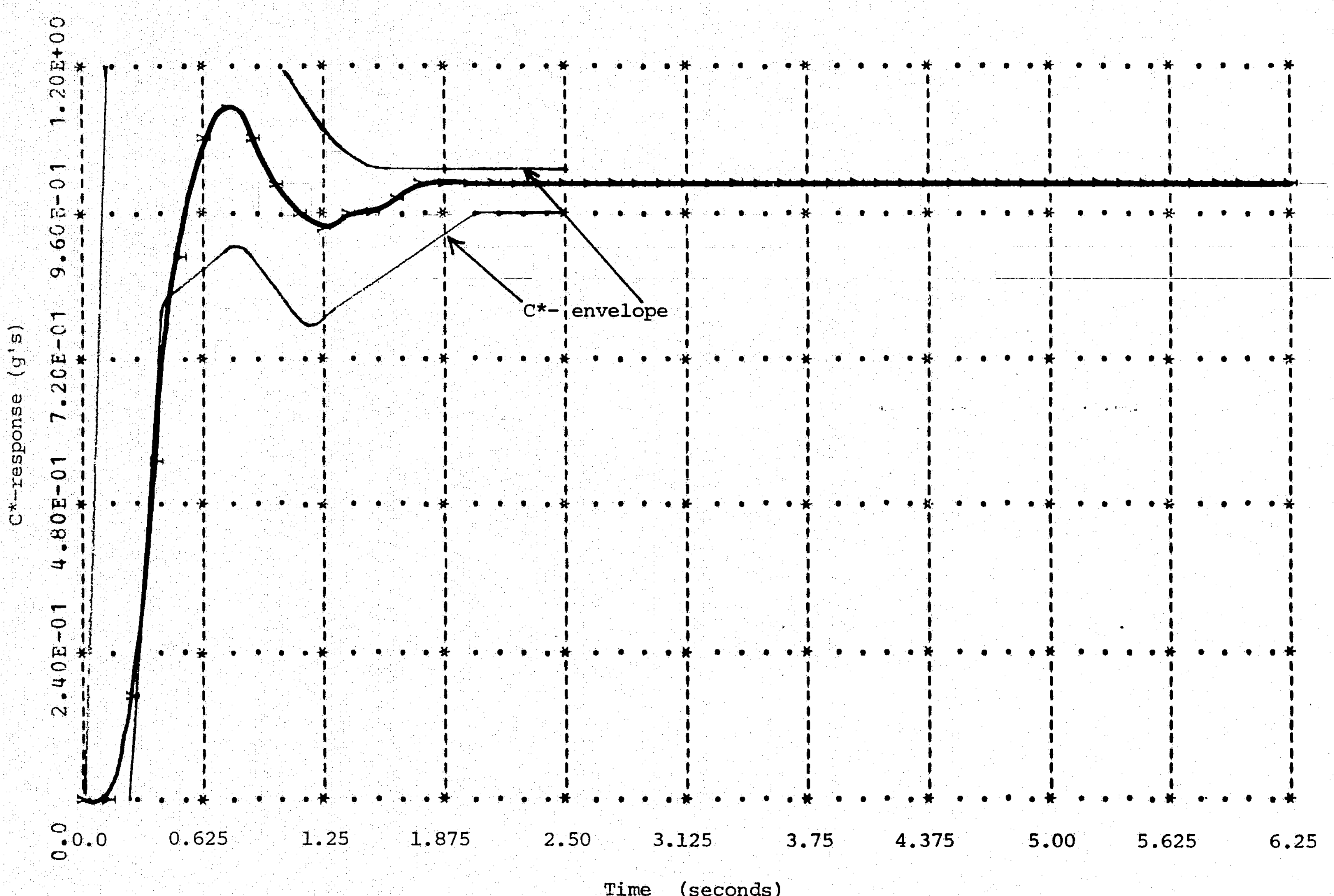


Fig. 5.22 Simulation C-1; C*-response to a C^*_d input; Mismatch (12-7)

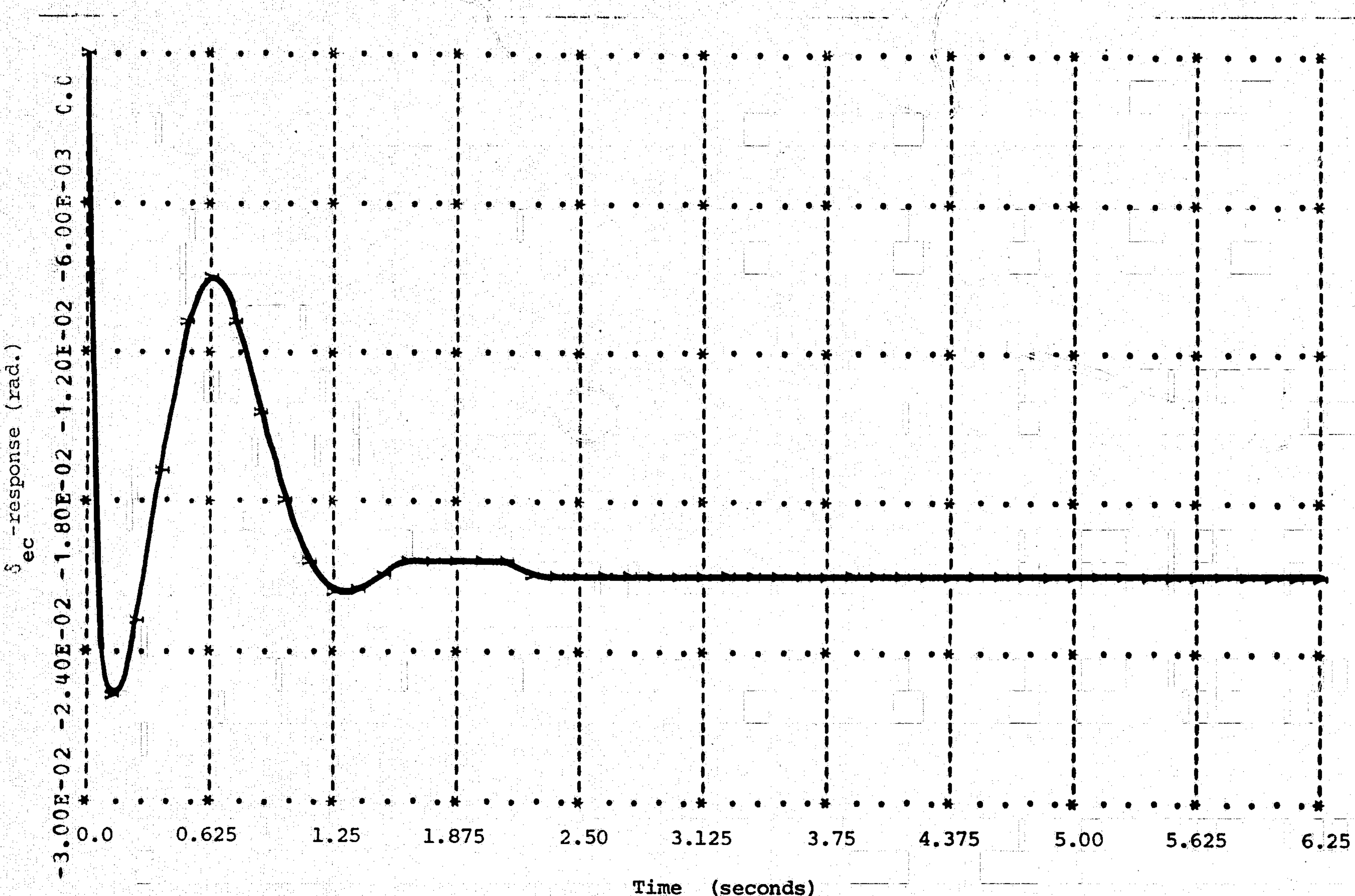


Fig. 5.23 Simulation C-1; δec-response to a C^*_d input; Mismatch (12-7)

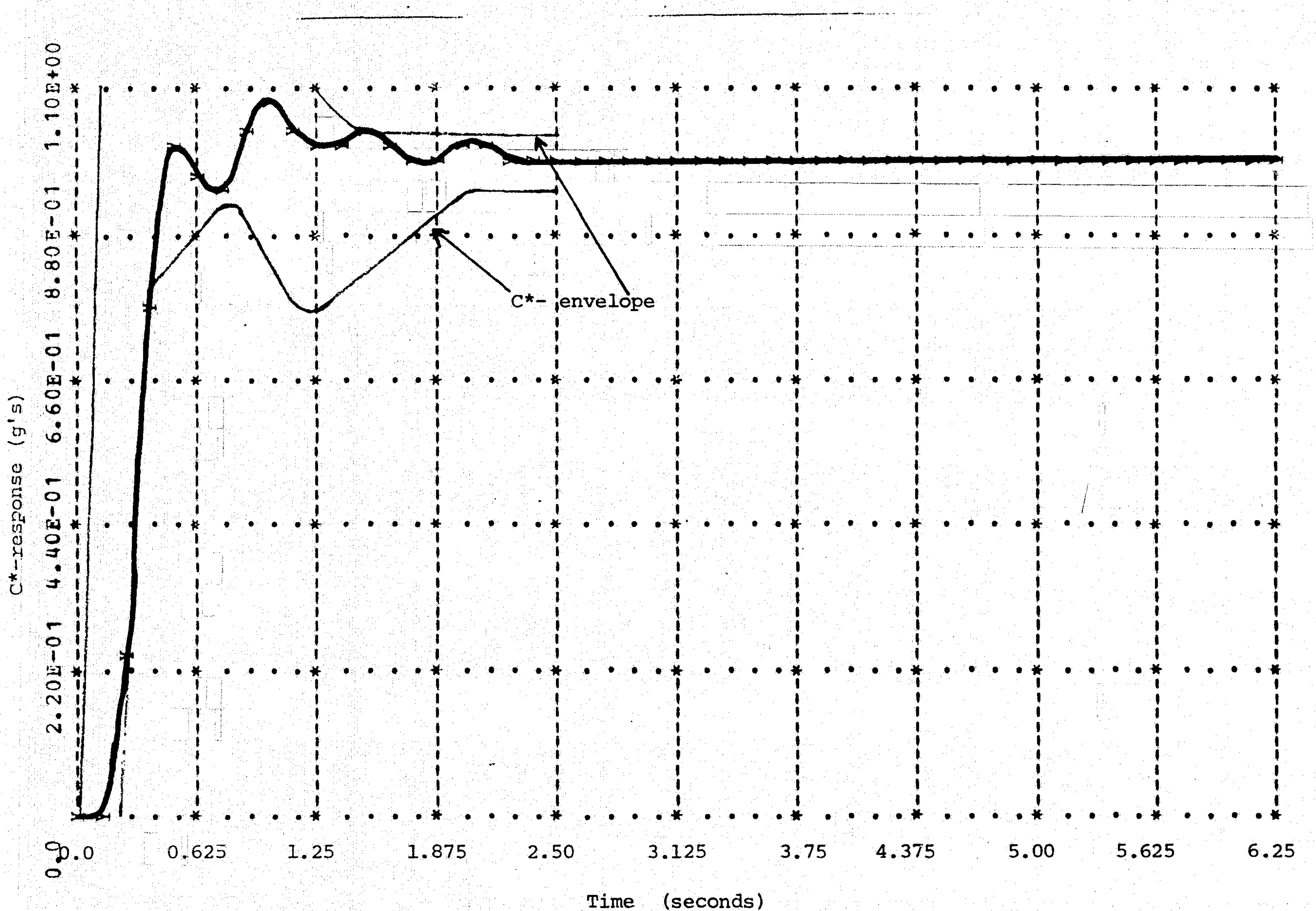


Fig. 5.24 Simulation C-2; C*-response to a C^*_d input; Mismatch (12-11)

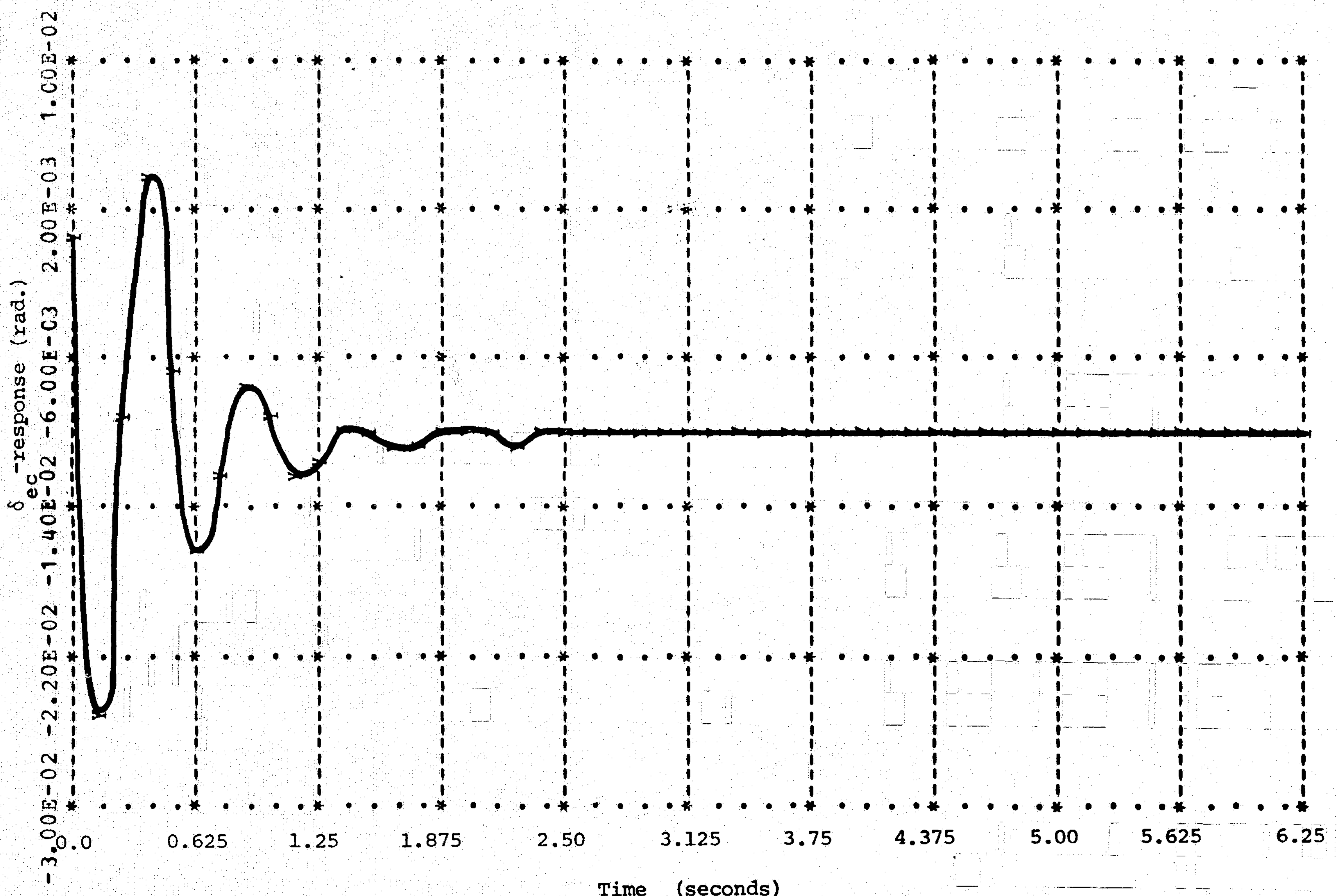


Fig. 5.25 Simulation C-2; δec-response to a unit step C^*_d input; Mismatch (12-11)

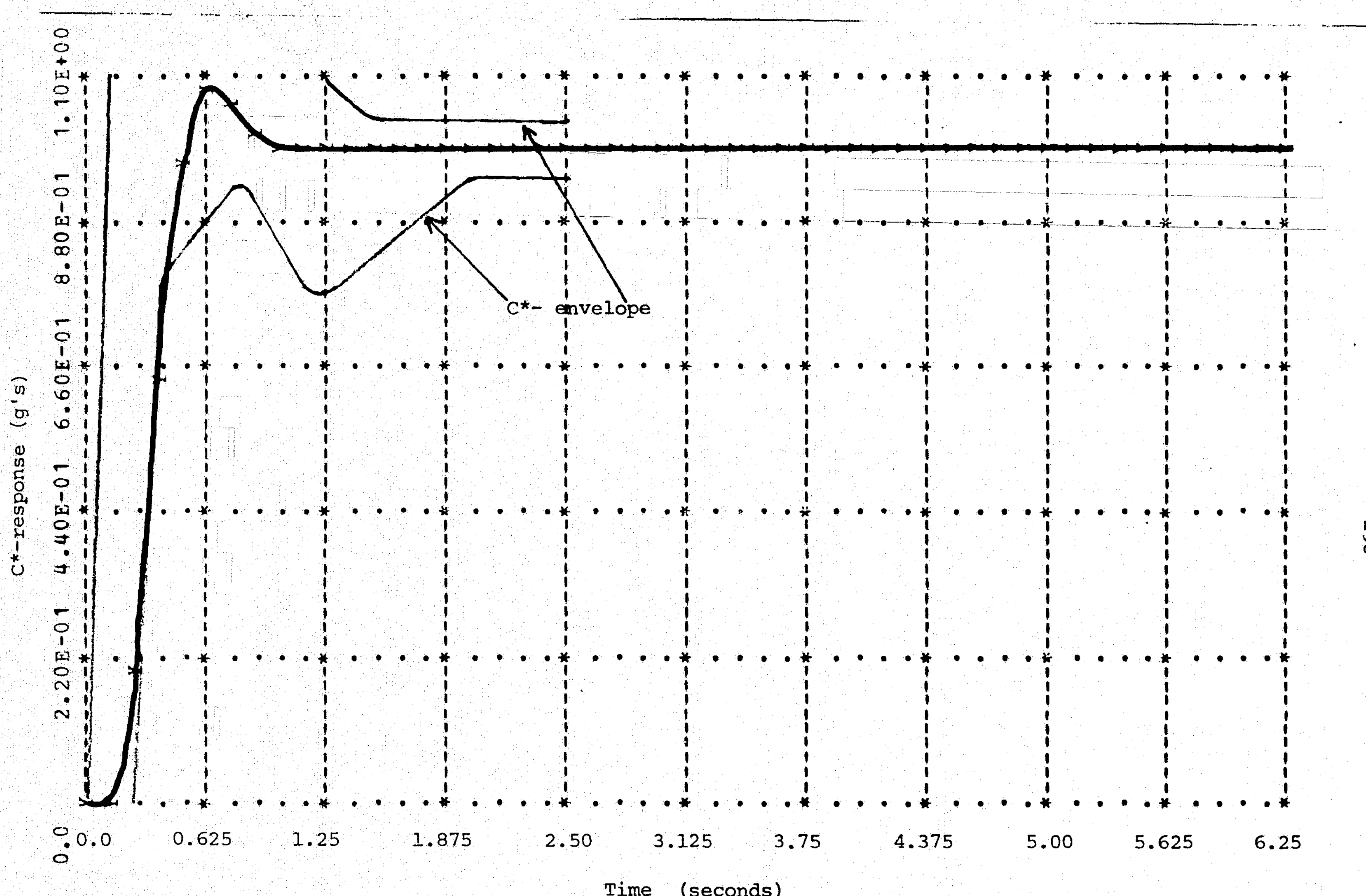


Fig. 5.26 Simulation C-3; C*-response to a unit step C^*_d input; Match (12-12)

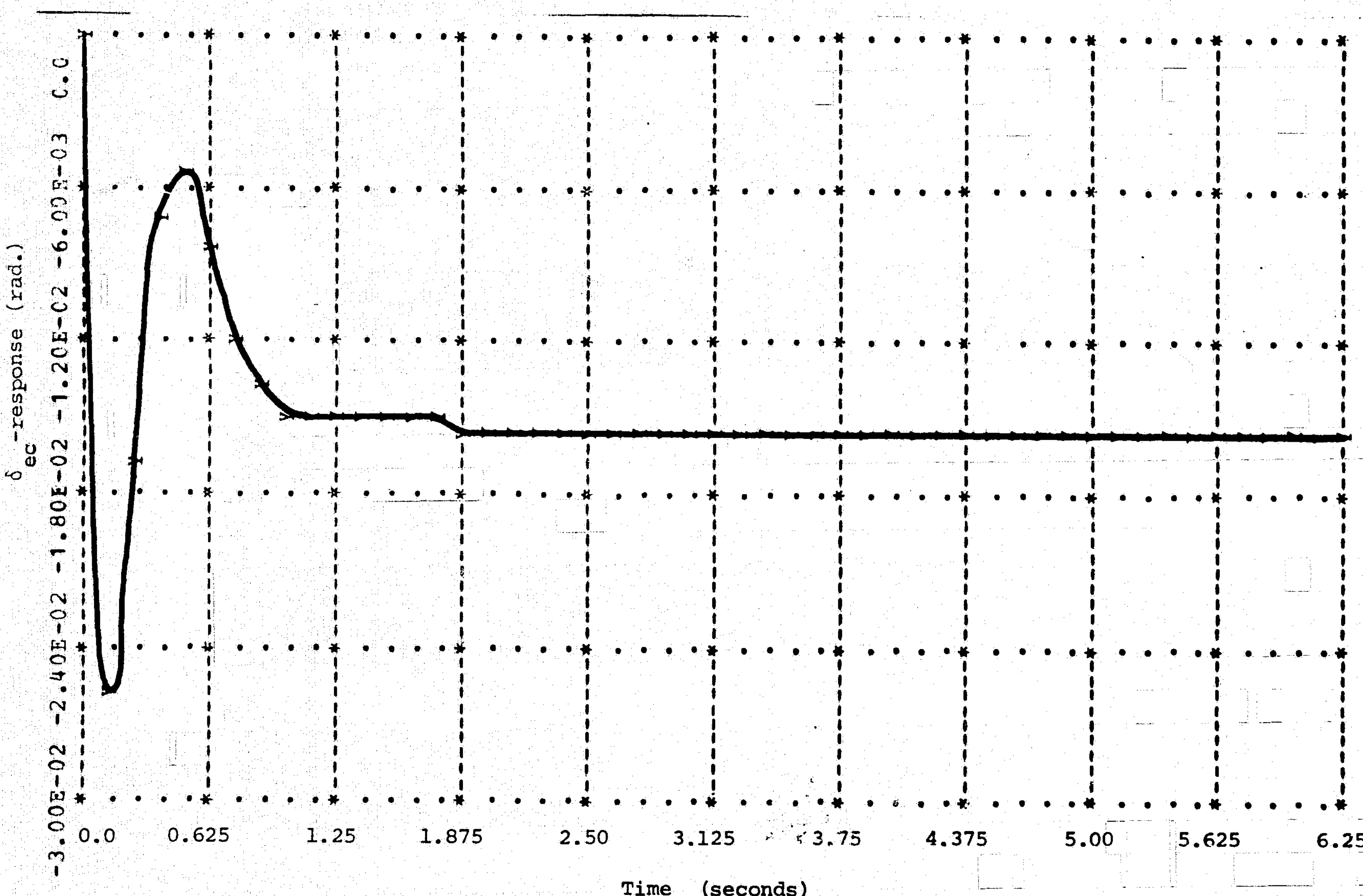


Fig. 5.27 Simulation C-3; δec-response to a unit step C^*_d input; Match (12-12)

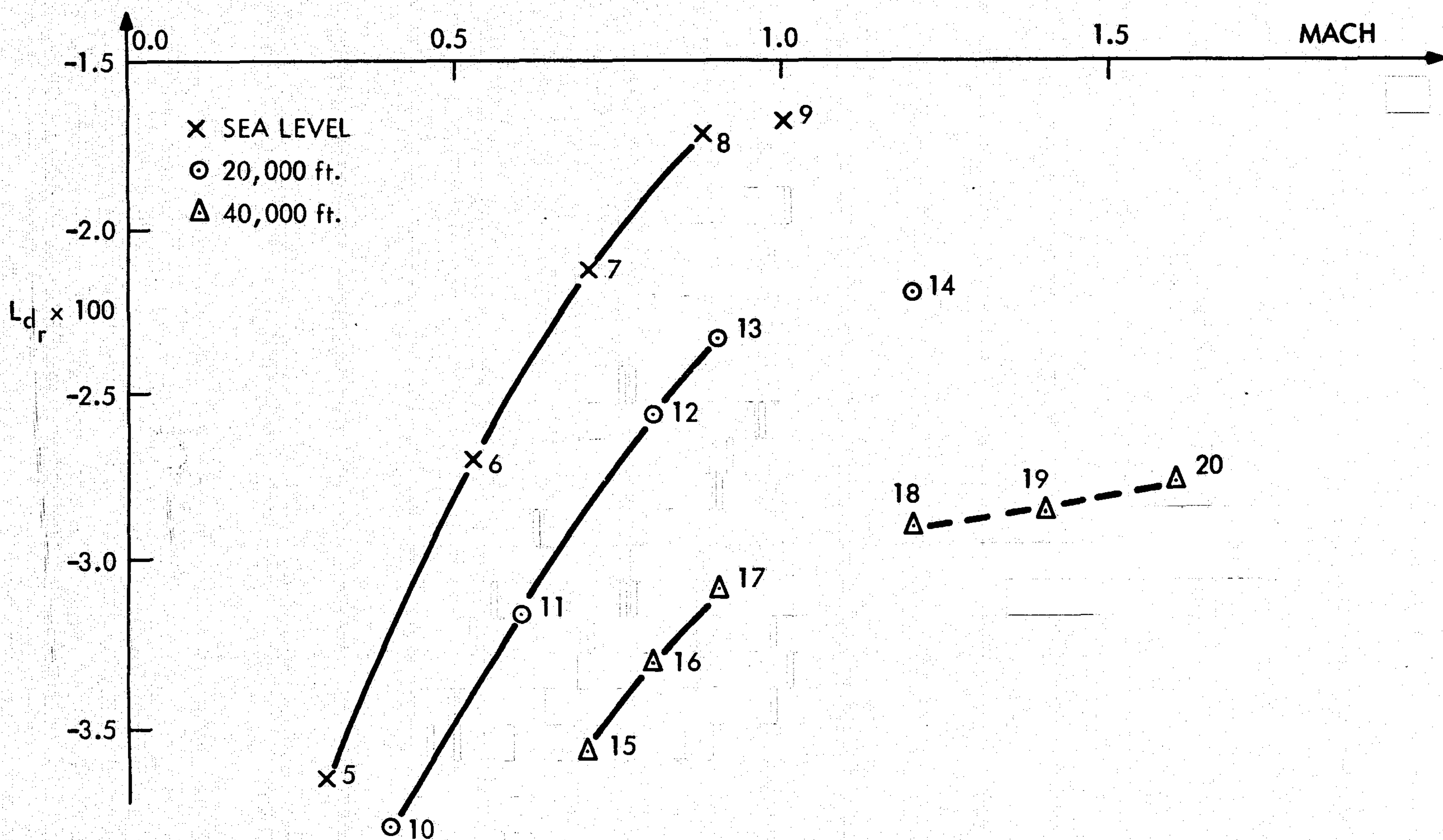


Fig. 5.28 (Discrete Time) L_{d_r} Gains vs. Mach Numbers

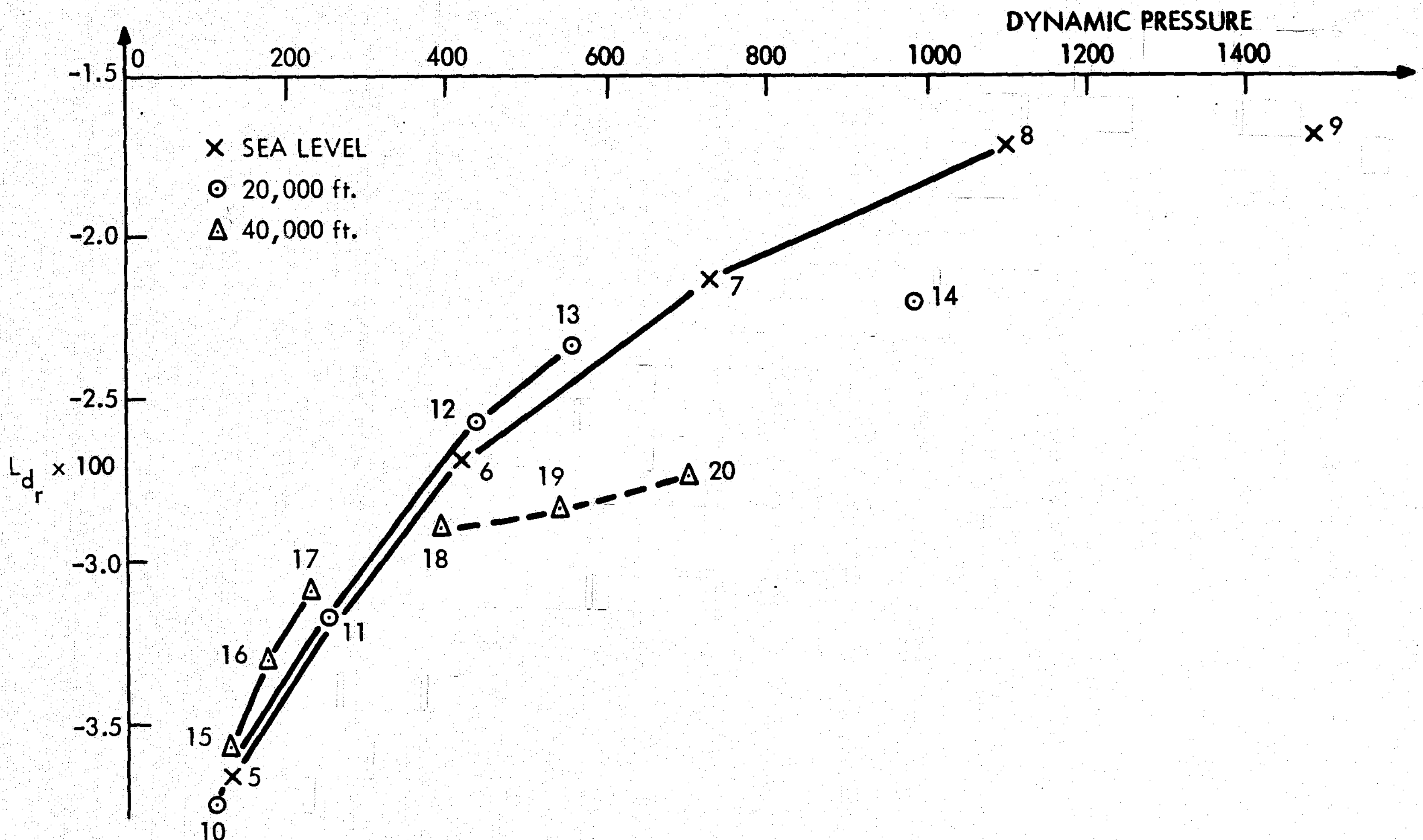


Fig. 5.29 (Discrete Time) L_{d_r} Gains vs. Dynamic Pressures

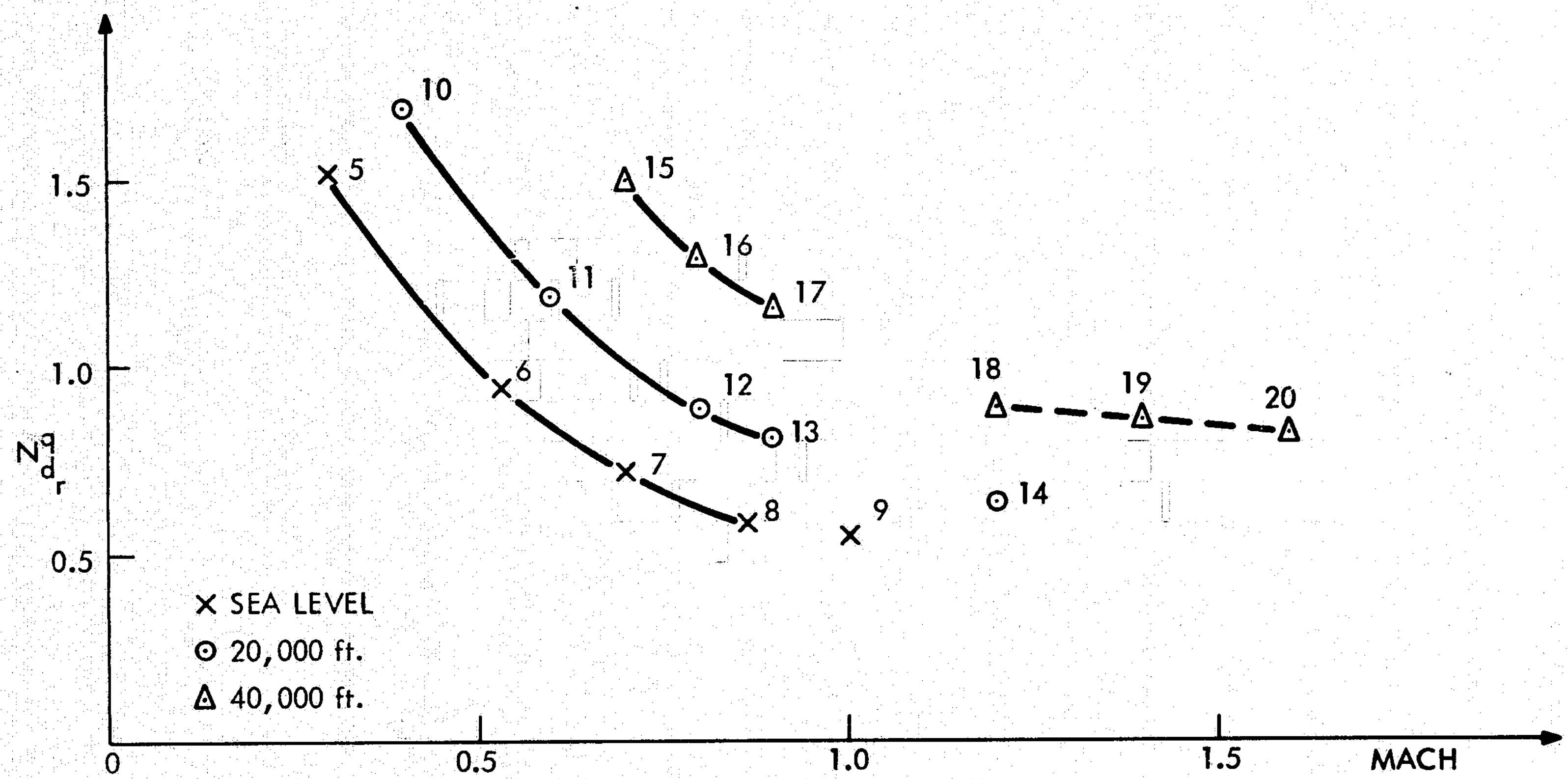


Fig. 5.30 (Discrete Time) $N^q_{d_r}$ Gains vs. Mach Numbers

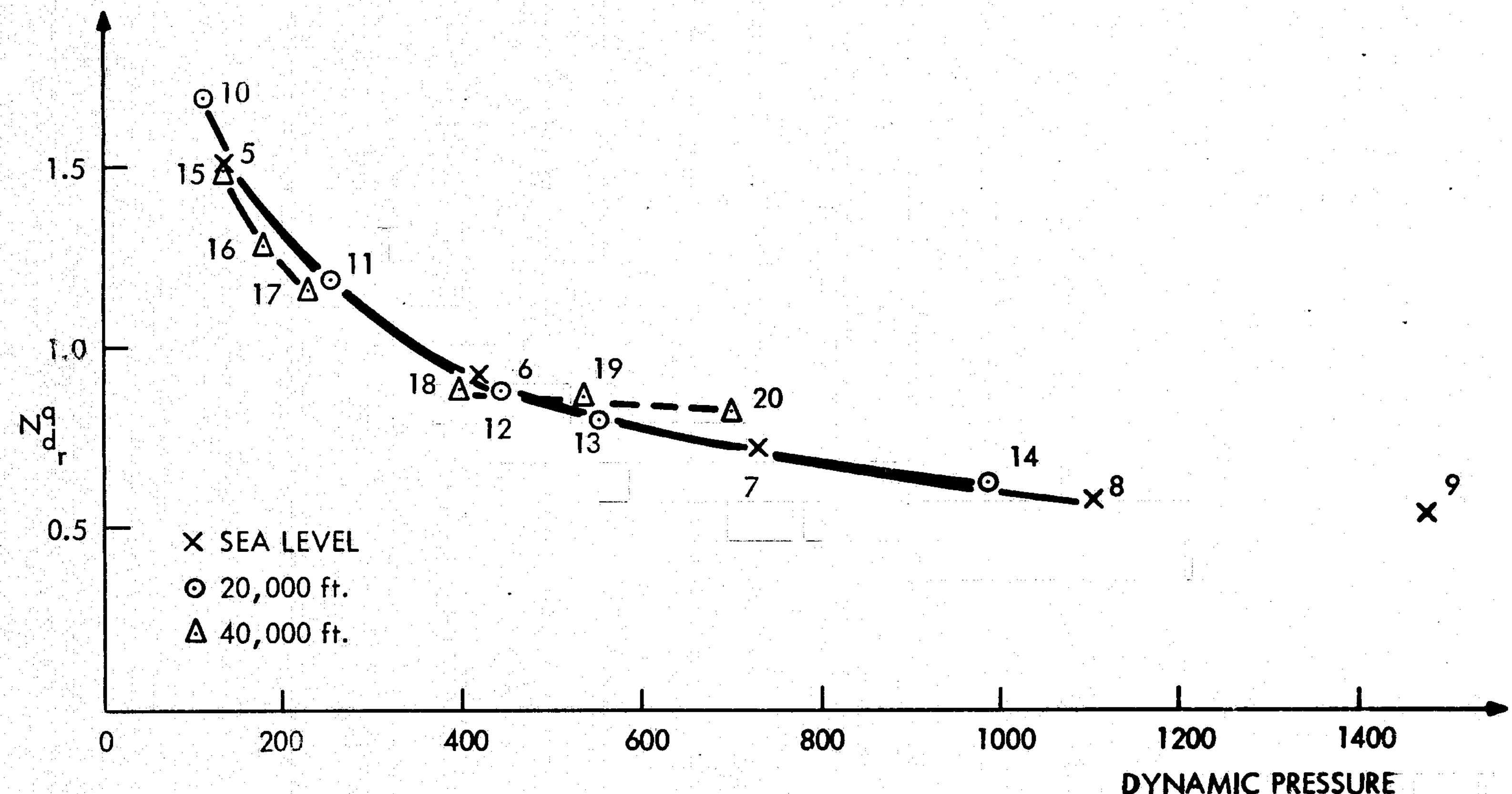


Fig. 5.31 (Discrete Time) $N^{q}_{d_r}$ Gains vs. Dynamic Pressures

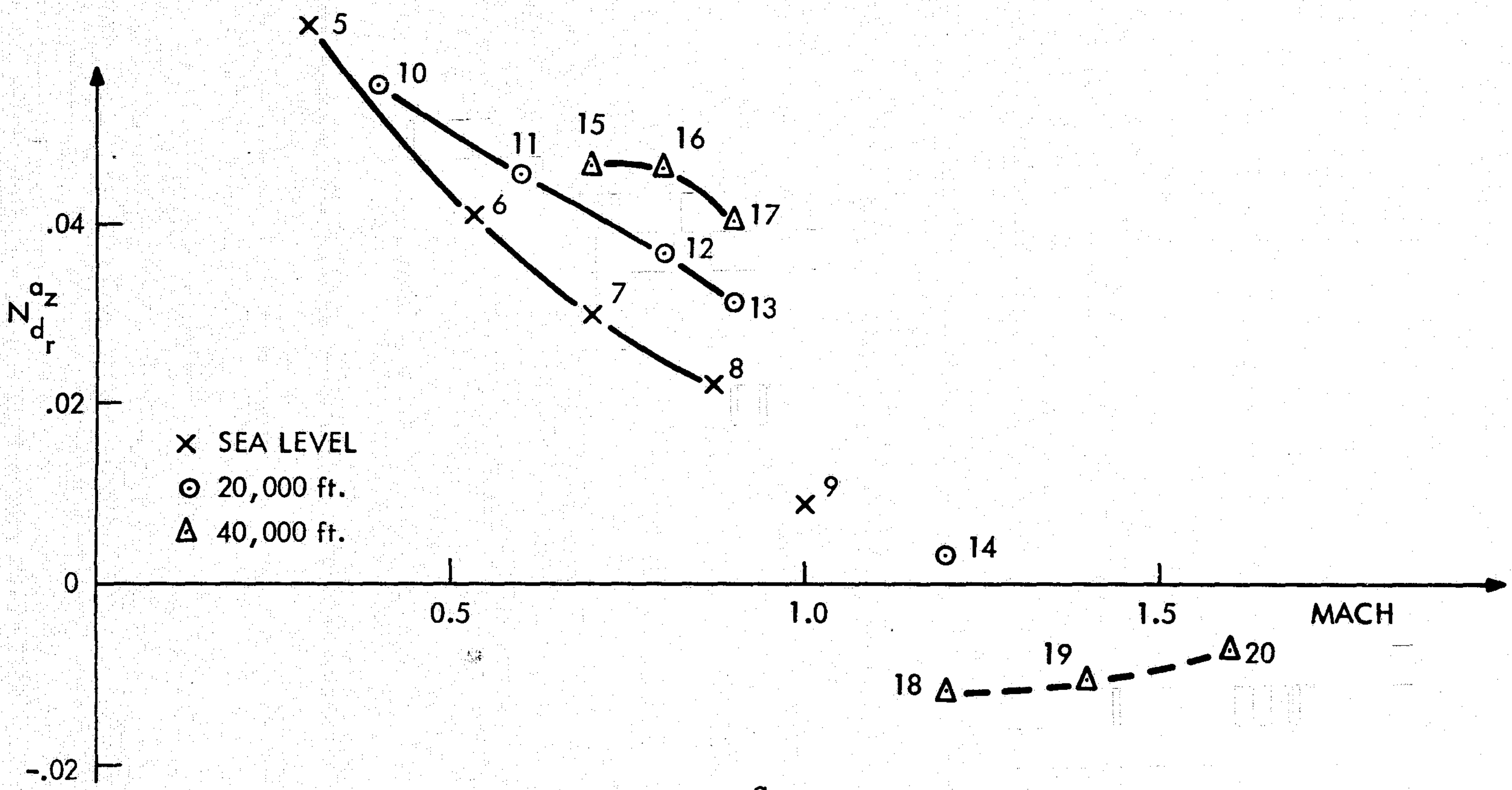


Fig. 5.32 (Discrete Time) $N_{d_r}^{a_z}$ Gains vs. Mach Numbers

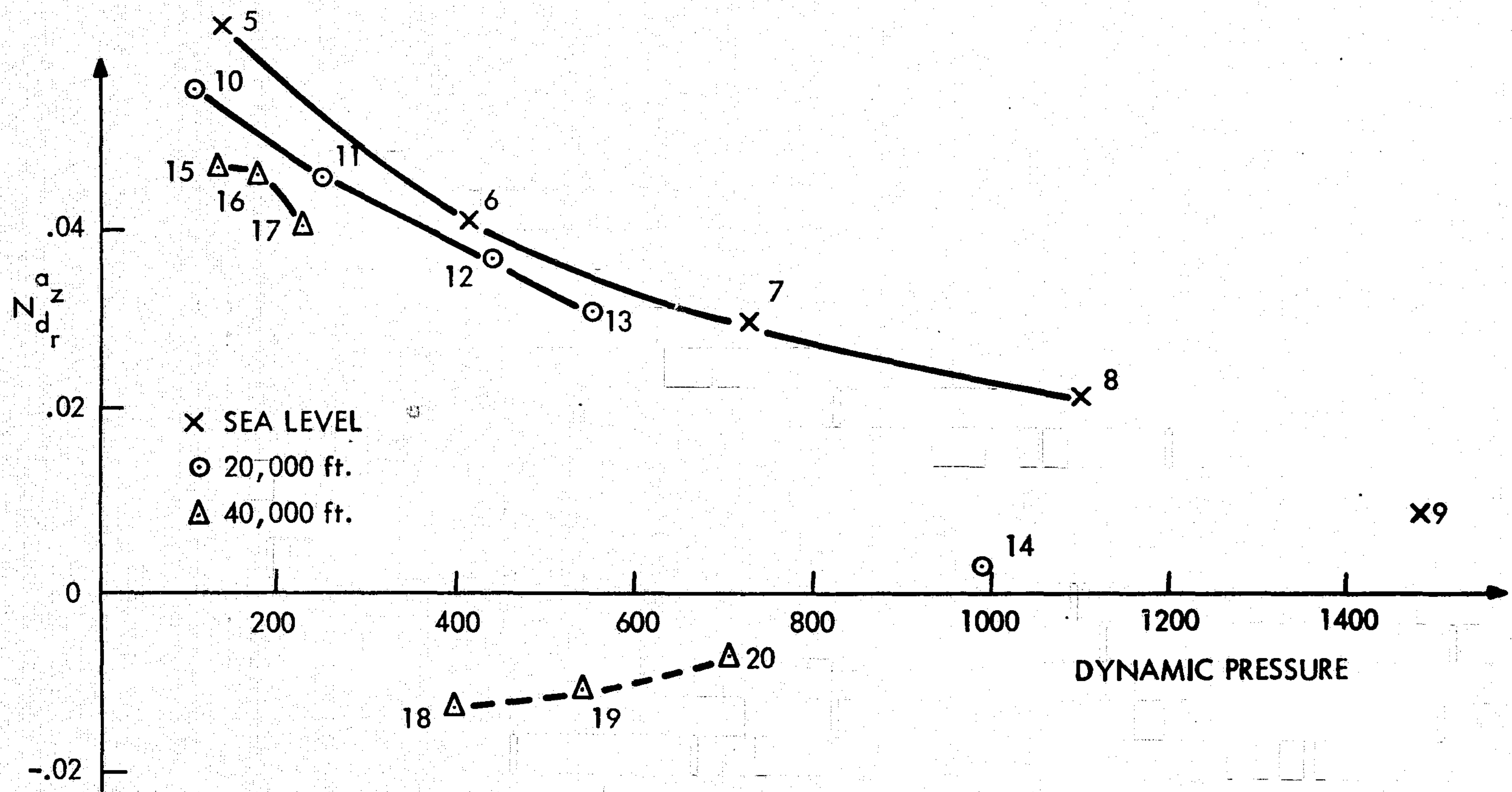


Fig. 5.33 (Discrete Time) $N_{d_r}^{a_z}$ Gains vs. Dynamic Pressures

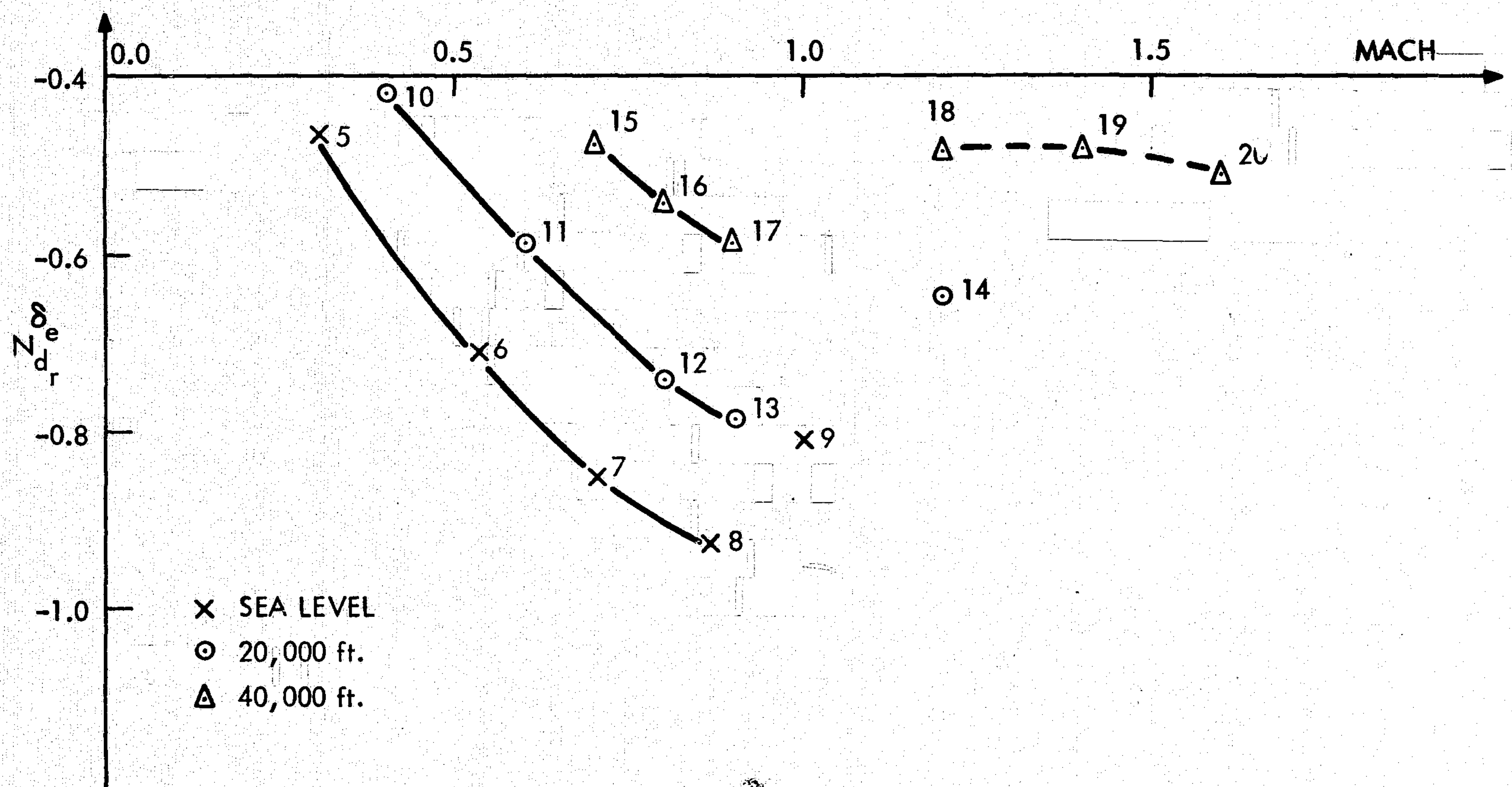


Fig. 5.34 (Dicrete Time) $N_{d_r}^{\delta_e}$ Gains vs. Mach Numbers

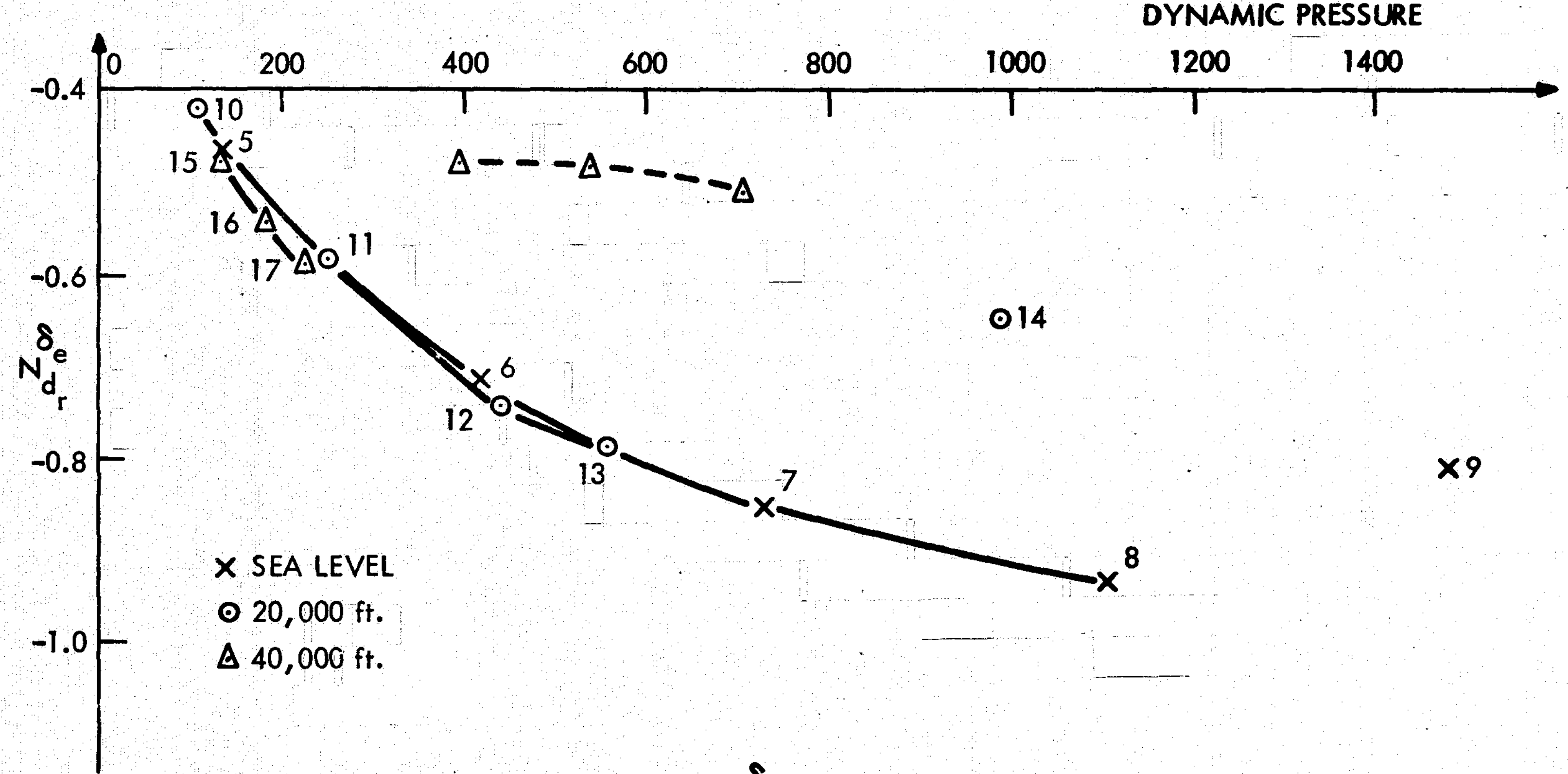


Fig. 5.35 (Discrete Time) $N_{d_r}^{\delta_e}$ Gains vs. Dynamic Pressures

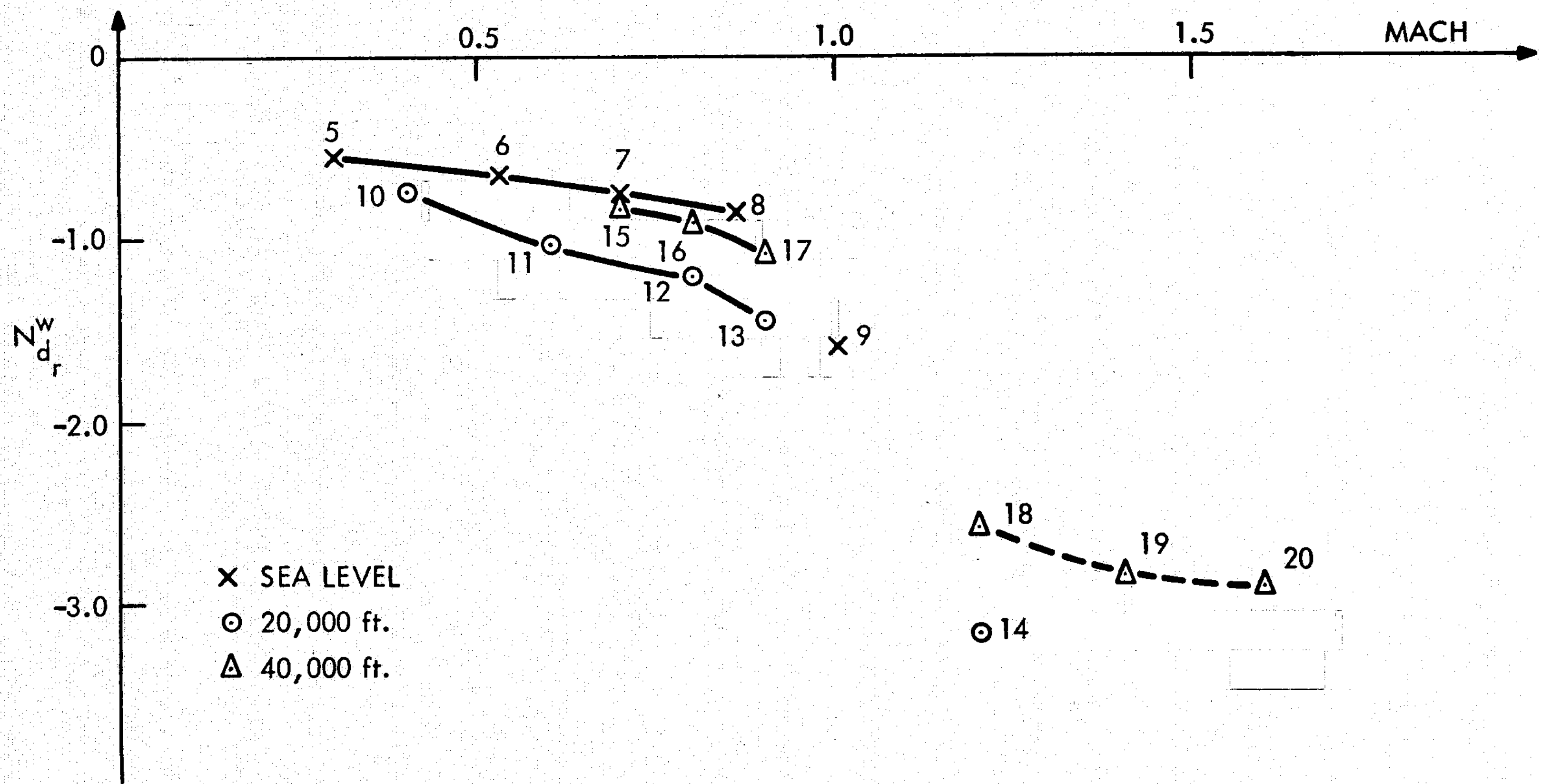


Fig. 5.36 (Discrete Time) $N^{w}_{d_r}$ Gains vs. Mach Numbers

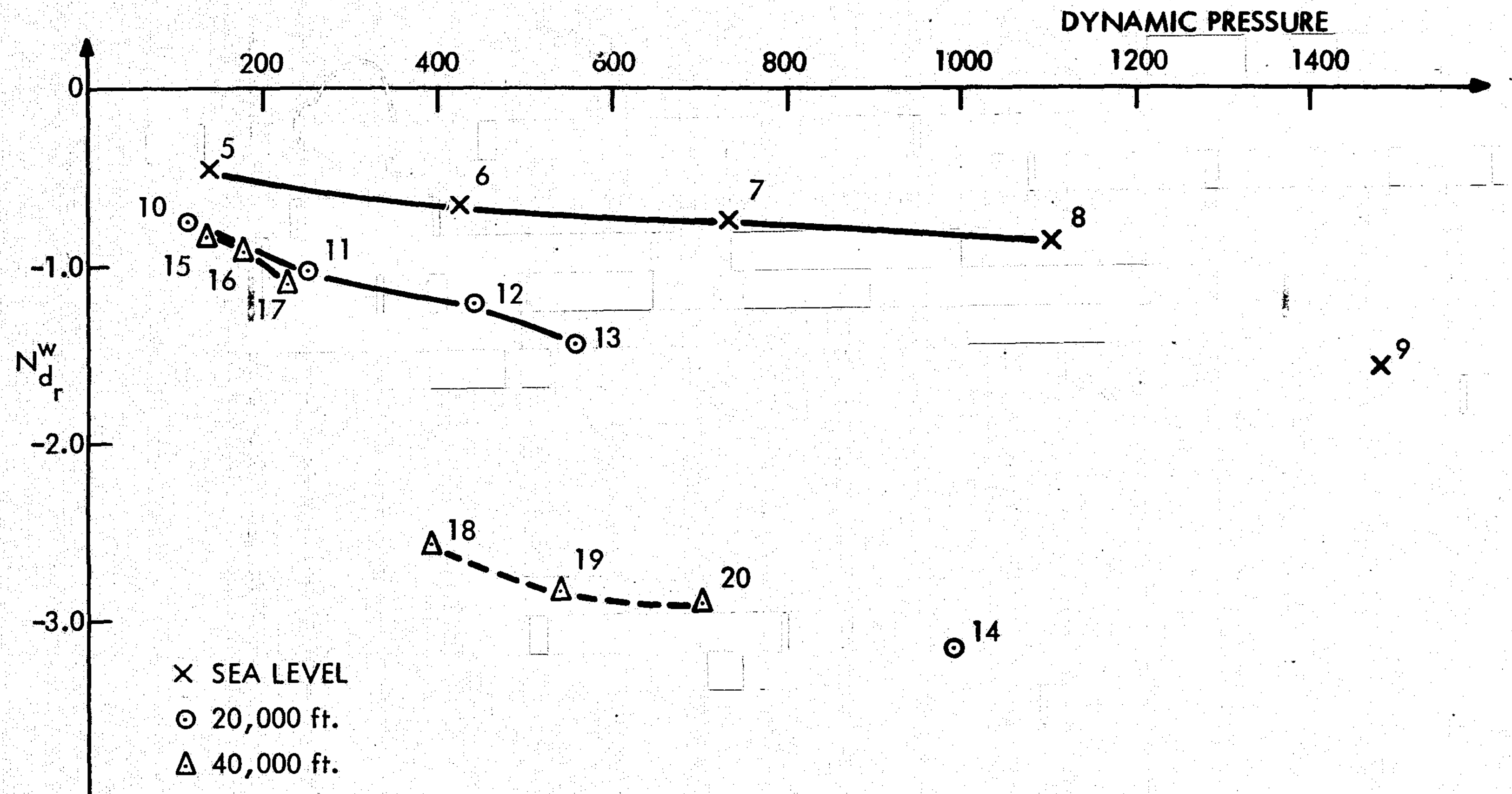


Fig. 5.37 (Discrete Time) $N^w_{d_r}$ Gains vs. Dynamic Pressures

controls differ somewhat, with higher initial rates of change here, but the slopes are more or less the same. Again these mismatches seem to impose no problem at all in the overall control of the aircraft.

We also attach plots of the L and N gains against Mach numbers and dynamic pressures (psf) (Figures 5.28-5.37). Notice that, similar to the continuous time case (design A), in each plot, the subsonic flights are readily divided into three groups according to altitude, and the supersonic flight conditions form a single group by themselves.

(ii) Design D

Similar to the continuous time case, we can use the control gains from the reduced state design C, in the full state model. That is, we generate the control by the relation

$$\underline{u}(kT) = L_{r_d} \sum_{j=0}^{k-1} (C^*_d - C^*(jT)) + \underline{N}_{f_d} (\underline{x}(kT) - \underline{x}(0)) + u(0) \qquad (5.16)$$

where

$$\underline{N}_{f_d} = \underline{N}_{r_d} \begin{bmatrix} \underline{\quad C \quad} \\ 2 \times 6 \\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1 \end{bmatrix} \qquad (5.17)$$

The simulated results are almost identical to those in design C, and in fact, are very similar to the continuous time simulations (designs A and B).

(iii) Design E

In doing a reduced state design, one is using less information than what is available to him. One may expect, therefore, that a full state design is more accurate and performs better than the reduced state design. The full state design is described by equations (3.35) to (3.40).

Comparing the simulations of such a design, with design D, it is observed that both the responses and controls almost coincide exactly in the two cases. As an example, Figures 5.28-5.29 show simulation E-3, which is a matched simulation for flight condition 12.

If one examines the gains, one can see that they are very close together for this design and the design D.

5.4 Final Remarks

In general, the reduced state design does seem to provide a very good approximation to the full state design. As far as responses are concerned, all designs (A-E) are very similar. This shows a strong correspondence between the continuous time designs and the discrete time designs. Higher initial rate of change of the control occurs for the discrete time designs, however. Nevertheless, they all seem to be acceptable.

If one compares the plots of control gains for the continuous time reduced state design (design A) with those for the discrete time reduced state design (design C), one can see that the N gains are in general rather close together, with the continuous time gains being somewhat smaller. Also the continuous time L gains are approximately one eight of those in discrete time. This accounts for much of the partial

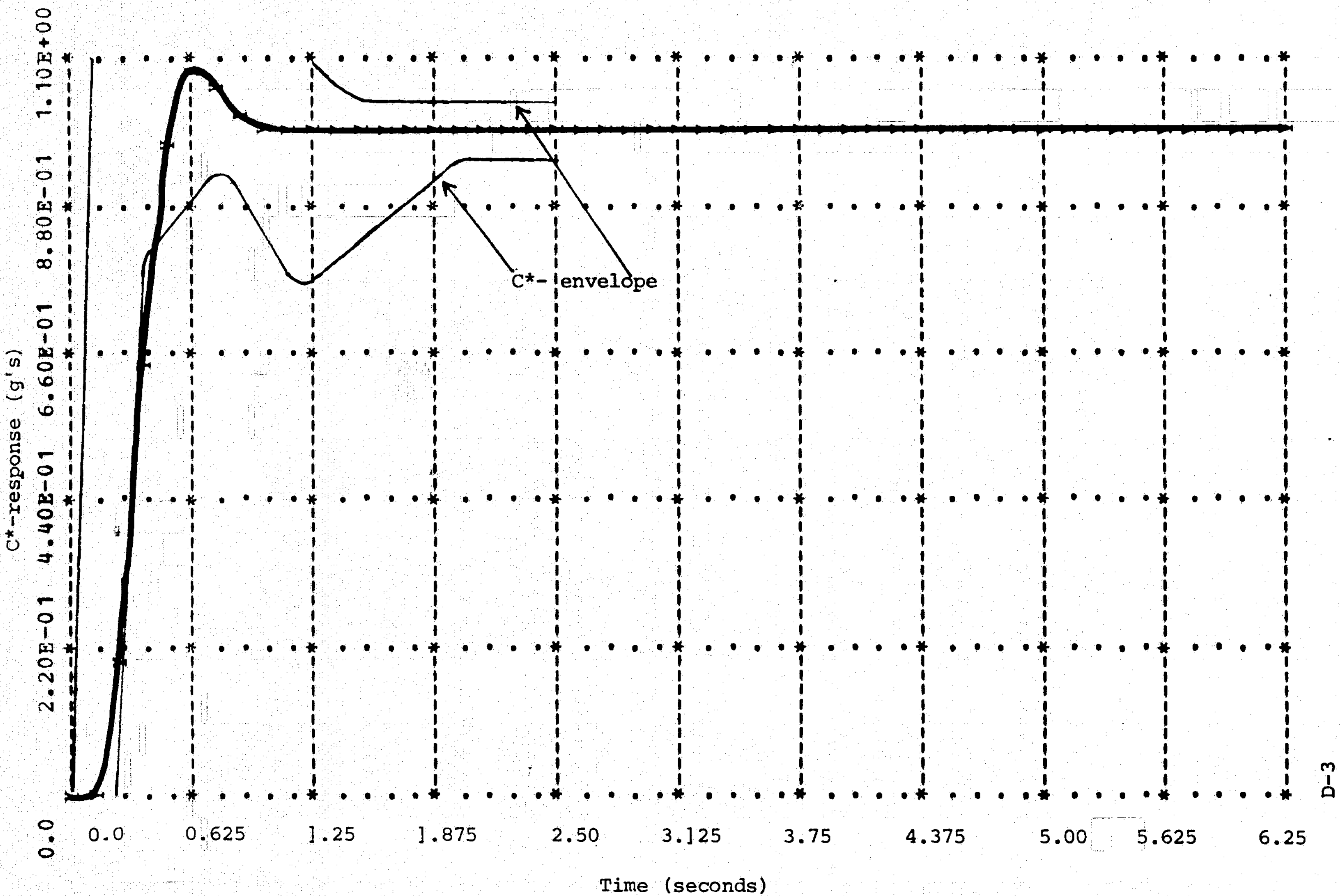
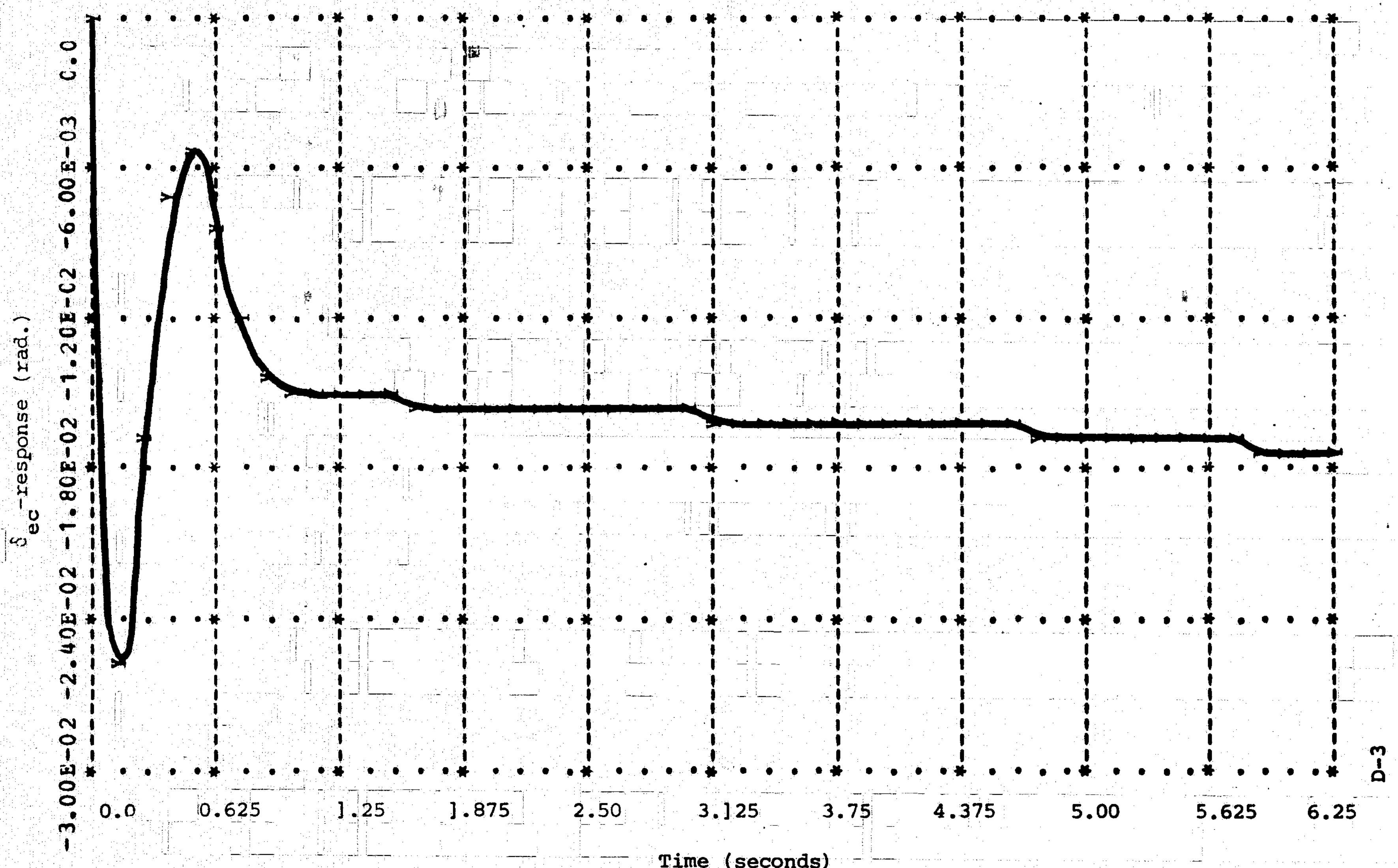


Fig. 5.38 Simulation D-3; C*-response to a unit step C^*_d input; Match (12-12)



-118-

Fig. 5.39 Simulation D-3; δec-response to a unit step C^*_d input; Match (12-12) success of the method of first order approximation of the continuous time design (although results in design C are better).

The above discussion holds, at least as far as the linearized system dynamics go. In particular, it confirms our theoretical results.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS

The approach taken in this report seems to provide satisfactory results on linear time-invariant dynamical systems, which are required to track step inputs, both from a theoretical and a practical point of view. Theoretically, it is illustrated by the propositions given in Chapter II, where we give digressions of both the continuous time and discrete time designs. Their applications to the F-8C aircraft linearized longitudinal dynamics, using a C* handling qualities criterion, provide us a lot of physical insights, as we have observed from the control gains and various simulations.

Apparently the reduced state designs possess very high practical values. Chapters III and IV illustrate in some detail how they can be implemented in the aircraft. It is believed that further study into this can be very fruitful.

The two discretization schemes discussed in Chapter II, of which one keeps the control to be a constant over each sampling period, and the other, its rate, bear very important significance in physical interpretation, because of the possibility of rate saturation in the actuator dynamics. We think the different controls computed by these two different schemes should generate different performances of the system. Since scheme II constrains the control to be continuous, it is likely that it approximates the continuous time system better. Apparently from Chapter V, the continuous time design provides the best result. However, the digital requirements in implementation limit its usefulness. On the other hand, the discrete time designs by scheme I seem to be acceptable in the linear simulations. So we expect that scheme II should be able to give more satisfactory compensator designs. More research effort should be spent in investigating them.

In this report, we do not discuss any stochastic aspects of the control designs, but we believe that they are very important also, and a lot more should be done in this direction, especially for the discrete time designs which are seldom touched upon in literature.

In developing the theoretical results for this report, we were just aiming at a P-I-D controller for the F-8C aircraft, and hence the consideration was not made very general. We believe that one can gain much further and develop more adequate theory if one takes disturbances into consideration, be they known or unknown, measurable or unmeasurable. Also for simplicity and generality, many of the conditions that we imposed can be relaxed to simply require that the matrix $\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix}$ be or a rank equal to the sum of the numbers of controls and outputs. Also, more general system or input types can be considered, e.g., type-ℓ systems and polynomial inputs. Also a study of the robustness of the systems and its sensitivity to parameter variation can be very useful, as is indicated in Chapter V.

REFERENCES

[1] N. R. Sandell, Jr., "Optimal Linear Tracking Systems," S.M. Thesis, M.I.T., 1971.

[2] N. R. Sandell, Jr. and M. Athans, "Design of a Generalized P-I-D Controller by Optimal Linear Control Theory," M.I.T., 1974.

[3] W. E. Holley and A. E. Bryson, "Multi-Input, Multi-Output Regulator Design for Constant Disturbances and Non-zero Set Points with Application to Automatic Landing in a Crosswind," SUDAAR Rpt. No. 465, August 1973.

[4] R. F. Stengel, J. R. Broussard, P. W. Berry, "The Design of Digital Adaptive Controllers for VTOL Aircraft," NASA CR-144912, T.A.S.C., March 1976.

[5] M. Athans, "The Discrete Time Linear-Quadratic-Gaussian Stochastic Control Problem," Annals of Economic and Social Measurement, 4, 1972.

[6] "Further Topics of Interest in Controllability and Observability," Course notes for 6.231, Dynamical System and Control I, handout #10, M.I.T., Fall 1975.

[7] A. H. Levis, M. Athans, "Sampled-Data Control of High Speed Trains," Technical Rpt. ESL-R-339, Electronic Systems Laboratory, M.I.T., 1968.

[8] M. Athans, et al, "The Stochastic Control of the F-8C Aircraft Using the Multiple Model Adaptive Control (MMAC) Method," Proceedings of the IEEE Conference on Decision and Control, 1975

[9] A. E. Bryson, Y. C. Ho, Applied Optimal Control, Ginn Blaisdell, 1969.

[10] H. N. Tobie, E. M. Elliot, L. G. Malcolm, "A New Longitudinal Handling Qualities Criterion," presented at the 18th Annual National Aerospace Electronics Conference, May 16-18, 1966.

[11] R. V. Brulle, D. C. Anderson, "Design Methods for Sampling Handling Qualities for Control Configured Vehicles," Volume I Technical Discussion, Technical Report AFFDL-TR-73-142, Volume I, Air Force Flight Dynamics Laboratory, Ohio, November 1973.

[12] G. L. Hartmann, J. A. Hauge, R. C. Hendrick, "F-8C Digital CCV Flight Control Laws," NASA Contractor report No. CR-2629, February 1976.

[13] G. Stein, G. L. Hartmann, R. C. Hendrick, "Adaptive Control Laws for F-8 Flight Test," Proceedings of the IEEE Conference on Decision and Control, 1975.

[14] J. Gera, "Linear Equations of Motion for F-8 DFBW Airplane at Selected Flight Conditions," NASA Langley Research Center, Hampton, Virginia, F-8 DFBW International Document, Report No. 010-74.

[15] M. Athans, P. Falb, Optimal Control, New York, McGraw-Hill, 1966.

[16] N.R. Sandell, M. Athans, "Manual of FORTRAN Computer Subroutines for Linear, Quadratic, Gaussian Designs", Modern Control Theory, Center for Advanced Engineering Study, M.I.T., 1974.

[17] D.L. Kleinman, "A Description of Computer Programs Useful in Linear System Studies", Report No. TR-75-4, U. of Connecticut, October 1975.