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**GRAVITATIONAL POTENTIAL ENERGY
OF THE EARTH:
A SPHERICAL HARMONIC APPROACH**

DAVID PARRY RUBINCAM

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A SPHERICAL HARMONIC APPROACH

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ABSTRACT

A spherical harmonic equation for the gravitational potential energy of the earth is derived for an arbitrary density distribution by conceptually bringing in mass-elements from infinity and building up the earth shell upon spherical shell. The zeroth degree term in the spherical harmonic equation agrees with the usual expression for the energy of a radial density distribution. The second degree terms give a maximum nonhydrostatic energy in the mantle and crust of -2.77×10^{29} ergs, an order of magnitude below McKenzie's (1966) estimate. This figure is almost identical with Kaula's (1963) estimate of the minimum shear strain energy in the mantle, a not unexpected result on the basis of the virial theorem. If the earth is assumed to be a homogeneous viscous oblate spheroid relaxing to an equilibrium shape, then a lower limit to the mantle viscosity of 1.3×10^{20} poises is found by assuming the total geothermal flux is due to viscous dissipation. This number is almost six orders of magnitude below MacDonald's (1966) estimate of the viscosity and removes his objection to convection. If the nonequilibrium figure is dynamically maintained by the earth acting as a heat engine at one per cent efficiency, then the viscosity is 10^{22} poises, a number preferred by some (e.g. Cathles (1975)) as the viscosity of the mantle.

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GRAVITATIONAL POTENTIAL ENERGY OF THE EARTH:

A SPHERICAL HARMONIC APPROACH

INTRODUCTION

Many quantities of geophysical interest, such as the earth's density distribution, gravitational potential, and moments of inertia, may be expressed in terms of spherical harmonics or integrals of spherical harmonic coefficients (Kaula, 1968). Since the earth's gravitational potential energy has received attention (and spherical harmonic treatments) in the past, particularly from MacDonald (1966), McKenzie (1966), and Kaula (1967), it should prove worthwhile to derive a general equation for the energy from the viewpoint of spherical harmonics.

We will derive here an equation for the gravitational potential energy for an arbitrary density distribution and make a few simple applications with some remarks. Specifically, we will investigate the energy released when a homogeneous earth differentiates into a mantle and core; compute the nonhydrostatic part of the energy contained in the gravity anomalies in the mantle and crust; show its relation to the elastic energy; and estimate a lower limit on the viscosity of the mantle. We also hope to clear up at least some of the questions surrounding the subject of the earth's gravitational potential energy.

DERIVATION

The gravitational potential energy is given by

$$U = -\frac{1}{2} G \int_V \int_V \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dv dv'.$$

(See Mac Donald (1966, p. 230), or an elementary textbook on electricity and magnetism for the analogous electrostatic energy. The gravitational potential energy is seldom discussed in geophysics textbooks.)

The singularity which occurs when $\vec{r} = \vec{r}'$ appears to make the evaluation of the integral difficult. We can get around this problem, however, in the following manner. We can conceptually assemble the earth shell upon spherical shell by bringing in matter from infinity and depositing it on the earth's surface, computing the work necessary to bring in each shell. The sum of the work done bringing in all the shells then gives us $-U$.

Let us assume for simplicity that the earth is spherical and nonrotating. Neither of these assumptions is restrictive and both will be discussed later. Let us further assume that the earth's density distribution may be expressed in terms of normalized spherical harmonics:

$$\rho(\vec{r}) = \rho(r, \phi, \lambda) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{i=1}^2 \bar{\rho}_{\ell m i}(r) \bar{Y}_{\ell m i}(\phi, \lambda) \quad (1)$$

with

$$\bar{Y}_{\ell m 1}(\phi, \lambda) = \bar{P}_{\ell m}(\cos \phi) \cos m\lambda, \quad \bar{Y}_{\ell m 2}(\phi, \lambda) = \bar{P}_{\ell m}(\cos \phi) \sin m\lambda,$$

and $\bar{\rho}_{001}(r) = \rho_0(r)$, $\bar{\rho}_{002}(r) = 0$. The $\bar{P}_{\ell_m}(\cos \phi)$ are the associated Legendre polynomials, normalized so that

$$\int_{-\pi/2}^{\pi/2} [\bar{P}_{\ell_m}(\cos \phi)]^2 \sin \phi \, d\phi = 4 - 2 \delta_{0m}$$

and

$$\int_{\text{unit sphere}} [\bar{Y}_{\ell_m}(\phi, \lambda)]^2 \, dA = 4\pi$$

where dA represents an element of area.

If we bring in a mass $dm = \rho(r, \phi, \lambda) dA dr$ from infinity and place it on the surface of a partially-assembled earth of mass M and radius r (see Figure 1), then the work involved in doing this is $V(r, \phi, \lambda) dm$, where V is the gravitational potential. Bringing in more masses until we have built a spherical shell of thickness dr requires work

$$-dU = dr \int_{\text{surface}} V(r, \phi, \lambda) \rho(r, \phi, \lambda) \, dA \quad (2)$$

where $dA = r^2 \sin \phi \, d\phi d\lambda$. (Strictly speaking, the layer does work on itself; but this is of order $(dr)^2$ in the infinitesimals and may be neglected.)

The potential V may be expressed in terms of spherical harmonics (Kaula, 1968, p. 64):

$$V(\Delta, \phi, \lambda) = \frac{GM}{\Delta} \left[\sum_{\ell_{mi}} \bar{c}'_{\ell_{mi}} \left(\frac{r}{\Delta} \right)^{\ell} \bar{Y}_{\ell_{mi}}(\phi, \lambda) \right]$$

where Δ is the radial distance to some external point and where

$$\bar{C}'_{\ell_{mi}} = \frac{4\pi \int_0^r x^{\ell+2} \bar{\rho}_{\ell_{mi}}(x) dx}{(2\ell+1) M r^{\ell}},$$

with $\bar{C}'_{\ell_{m1}} = \bar{C}'_{\ell_m}$ and $\bar{C}'_{\ell_{m2}} = \bar{S}'_{\ell_m}$. The $\bar{C}'_{\ell_{mi}}$ refer of course to the partially-assembled earth and should not be confused with the present-day potential coefficients.

The potential at $\Delta = r$ may then be written

$$V(r, \phi, \lambda) = 4\pi G \sum_{\ell_{mi}} \left[\frac{\int_0^r x^{\ell+2} \bar{\rho}_{\ell_{mi}}(x) dx}{(2\ell+1) r^{\ell+1}} \right] \bar{Y}_{\ell_{mi}}(\phi, \lambda).$$

Substituting this expression along with (1) into (2) and working out the integral yields

$$-dU = 16\pi^2 G dr \sum_{\ell_{mi}} \frac{\bar{\rho}_{\ell_{mi}}(r) r^{-\ell+1}}{(2\ell+1)} \int_0^r x^{\ell+2} \bar{\rho}_{\ell_{mi}}(x) dx$$

by the orthogonality of spherical harmonics.

Adding on more shells until we reach the final radius R_E of the earth gives us our expression for the gravitational potential energy:

$$U = \sum_{\ell_{mi}} U_{\ell_{mi}}$$

where

$$U_{\ell_{mi}} = -\frac{16\pi^2 G}{(2\ell+1)} \int_0^{R_E} \bar{\rho}_{\ell_{mi}}(r) r^{-\ell+1} \int_0^r x^{\ell+2} \bar{\rho}_{\ell_{mi}}(x) dx dr. \quad (3)$$

We note in passing that our expression differs from McKenzie's (1966) equation (10); his expression deals with a layer of mass added to the surface of a deformed earth with uniform density. (See Appendix 1).

Our equation differs also from that of Kaula's (1967) eq. (3) for the "energy", or degree variance, which may be written

$$\sigma_l^2 = \sum_m \{ \bar{C}_{l_m} + \bar{S}_{l_m}^2 \} = \sum_{mi} \bar{C}_{l_{mi}}^2 .$$

The two are related through our eq. (6) (see below).

RADIAL DENSITY DISTRIBUTION

As an example of eq. (3), consider the case of a spherical earth with a radially symmetric density distribution. Then $\bar{\rho}_{001}(r) = \rho_0(r)$, $\bar{\rho}_{l_{mi}}(r) = 0$ for $l_{mi} \neq 001$, and the energy is simply

$$U = - 16 \pi^2 G \int_0^{a_E} \rho_0(r) r \int_0^r \rho_0(x) x^2 dx dr , \quad (4)$$

where a_E is the mean radius of the earth. (We keep a_E and the final radius R_E distinct for reasons made clear below.) This expression agrees with that of Urey (1952, p. 174). For an earth with constant density $\bar{\rho}_E$ the above equation works out to be

$$U = - \frac{3}{5} \frac{GM_E^2}{a_E} = - 2.24 \times 10^{39} \text{ ergs}$$

where M_E is the mass of the earth and we have used the numerical values given in Stacey (1969, Appendix E).

An earth differentiated into a mantle and core with constant respective densities ρ_M and ρ_C and no change in mass or radius is a little more complex. In this case the interior integral in eq. (4) is

$$\rho_C \frac{r^3}{3} \text{ for } 0 \leq r \leq a_C$$

and

$$\rho_C \frac{a_C^3}{3} + \rho_M \frac{(r^3 - a_C^3)}{3} \text{ for } a_C < r \leq a_E$$

where a_C is the radius of the core. Using this information the energy is

$$U = -\frac{3}{10} G \left\{ \frac{[m_1 - m_2] [2m_1 - 3m_2]}{a_C} + \frac{m_3 [2m_3 + 5(m_1 - m_2)]}{a_E} \right\}$$

$$= -2.45 \times 10^{39} \text{ ergs}$$

where

$$m_1 = \frac{4\pi}{3} \rho_C a_C^3$$

$$m_2 = \frac{4\pi}{3} \rho_M a_C^3$$

$$m_3 = \frac{4\pi}{3} \rho_M a_E^3$$

and our numerical values have come once again from Stacey (1969). Hence if a homogeneous earth differentiates into the present-day mantle and core,

approximately 2.1×10^{30} ergs is released as heat, in rough agreement with Urey (1952, pp. 174-176), who used a somewhat different model of density and took compressibility into account.

ENERGY IN THE MANTLE AND CRUST

We now turn our attention to the gravitational potential energy contained in the gravity anomalies.

The density variations $\bar{\rho}_{\ell_{mi}}(r)$ give rise to the observed $\bar{C}_{\ell_{mi}}$ of the earth's gravitational field. By choosing various models for the $\bar{\rho}_{\ell_{mi}}(r)$ and using the known $\bar{C}_{\ell_{mi}}$ as constraints, we may use eq. (3) to estimate the gravitational potential energy in the mantle and crust, where most if not all the gravity anomalies are believed to reside. In particular, if we choose $\bar{\rho}_{\ell_{mi}}(r) \propto (r/R_E)^n$, where n is an adjustable parameter, then

$$\bar{\rho}_{\ell_{mi}}(r) = (2\ell + 1)(n + \ell + 3) \bar{C}_{\ell_{mi}} r^n \frac{M_E}{4\pi R_E^{n+3}}$$

as may be found by substituting $\bar{\rho}_{\ell_{mi}}(r)$ in the equation for $\bar{C}_{\ell_{mi}}$; giving

$$U_{\ell_{mi}} = - \frac{(2\ell + 1)(n + \ell + 3)}{(2n + 5)} \bar{C}_{\ell_{mi}} \frac{GM_E^2}{R_E}. \quad (5)$$

If we require that $n \rightarrow \infty$, then we are dealing with a surface density distribution (Dirac delta function), and eq. (5) becomes

$$U_{\ell_{mi}}^{\max} = - \frac{(2\ell + 1)}{2} \bar{C}_{\ell_{mi}}^2 \frac{GM_E^2}{R_E}, \quad (6)$$

which is the maximum energy contained in each harmonic, since the material is as far from the center of the earth as it can be. The above equation may be shown to agree with McKenzie's (1966) eq. (10) after some corrections have been made to his expression (see below).

Most of the nonradial gravitational potential energy resides in the \bar{C}_{201} term, due to the rotational equilibrium flattening of the earth. What we would really like to know at this point is the energy due to nonhydrostatic equilibrium. To find it let us do the following: we stop the rotation of the earth and assume that it relaxes to a spherical shape with radius a_E , but retains the same non-hydrostatic $\bar{C}_{\ell m i}$ as it did before. (This will not be strictly true, of course, but makes for a simple case to analyze.) Then the gravitational potential energy contained in the important second degree ($\ell = 2$) harmonics is from eq. (6) (going back to the C and S form for the potential coefficients)

$$U_2^{max} = - (\Delta \bar{C}_{20}^2 + \bar{C}_{22}^2 + \bar{S}_{22}^2) \frac{5}{2} \frac{GM_E^2}{a_E} ,$$

where we have been careful to use $\Delta \bar{C}_{20}$, the nonhydrostatic part of \bar{C}_{20} . Both \bar{C}_{21} and \bar{S}_{21} have been set equal to zero, in accordance with the small amplitude of the Chandler wobble. Using the entries in Table 1 of Kaula (1967) as values accurate enough for our purposes, namely

$$\Delta \bar{C}_{20} = 4.70 \times 10^{-6}$$

$$\bar{S}_{22} = 1.34 \times 10^{-6}$$

$$\bar{C}_{22} = 2.40 \times 10^{-6}$$

we obtain $U_2^{\max} = -2.77 \times 10^{29}$ ergs as the maximum amount of energy contained in the second degree harmonics. This is smaller by an order of magnitude than McKenzie's (1966) estimate. The discrepancy will be commented upon below.

Probably a reasonable lower limit on the second degree energy can be obtained from eq. (5) by setting $n = 0$, so that the anomalous density distribution is spread throughout the earth; then

$$-5.44 \times 10^{29} \text{ ergs} \leq U_2 \leq -2.77 \times 10^{29} \text{ ergs},$$

and a guess of $U_2 \cong -4 \times 10^{29}$ ergs is almost certainly right to within a factor of 2.

An estimate of the total gravitational energy in the earth for $\ell \geq 2$ can be found from Kaula's rule-of-thumb, as given in Kaula (1968, p. 77):

$$\frac{1}{(2\ell + 1)} \sum_{mi} \bar{c}_{\ell mi}^2 \cong \frac{10^{-10}}{\ell^4}.$$

From (6)

$$U_{\text{tot}}^{\max} \cong -\frac{GM_E^2}{2a_E} (10^{-10}) \sum_{\ell=2}^{\infty} \left(\frac{4}{\ell^2} + \frac{4}{\ell^3} + \frac{1}{\ell^4} \right).$$

The series appearing on the right side may be evaluated with the help of Jolley (1961, pp. 64-65, 240), and we have $U_{\text{tot}}^{\max} = -6.5 \times 10^{29}$ ergs. A guess of $U_{\text{tot}} = -1 \times 10^{30}$ ergs is probably good to within a factor of 2.

Our value of U_2^{\max} is almost identical with Kaula's (1963) estimate of 2.94×10^{29} ergs for the minimum second degree elastic shear strain energy of the mantle. This is perhaps not a coincidence: rough equality is expected on the basis of the virial theorem, if gravity and elasticity are the two sources of potential energy and the velocity of the particles making up the earth are small (see Appendix 2).

ENERGY OF AN OBLATE SPHEROID

We have assumed in the derivation for the gravitational potential energy that the earth is spherical and nonrotating. Neither of these conditions hold for the real earth, of course. However, the assumptions are not restrictive. As far as rotation is concerned, the gravitational potential energy depends only on the relative positions of the particles composing the earth and not their velocities, hence the rotation of the earth plays no part in the computation of the potential energy. Rotation is important, of course, in computing the total mechanical energy E of the earth, which is the sum of the kinetic and potential energies. For simple rotation about the polar axis this is merely

$$E = \frac{1}{2} C \omega^2 + U$$

where C is the polar moment of inertia, ω is the angular speed, and other forms of energy are ignored. As for sphericity, we may take the earth to be spherical by letting R_g be the distance from the center of the earth to the highest point on the planet (see Figure 2). In practice this will be the equatorial radius of the

earth; thus our distinction between R_E and a_E , the two symbols commonly used to denote the equatorial and mean radius, respectively. The density of the imagined spherical earth now happens to be zero in the space between the actual surface of the earth and the sphere. This poses no particular problems and is no obstacle to a spherical harmonic expansion of the density distribution.

Let us compute the gravitational potential energy to second degree for a homogeneous oblate spheroid as an illustration of this point. The result will be used later to estimate a lower limit on the viscosity of the mantle.

Let the equatorial radius of the spheroid be R_E and the eccentricity be e .

It may be shown from

$$\bar{\rho}_{\ell mi}(r) = \frac{1}{4\pi} \int \rho(r, \phi, \lambda) \bar{Y}_{\ell mi}(\phi, \lambda) dA$$

that

$$\left. \begin{array}{l} \bar{\rho}_{001}(r) = \bar{\rho}_E \\ \bar{\rho}_{201}(r) = 0 \end{array} \right\} \text{ for } 0 \leq r \leq R_E (1 - e^2)^{1/2}$$

and

$$\left. \begin{array}{l} \bar{\rho}_{001}(r) = \bar{\rho}_E \frac{(1 - e^2)^{1/2}}{e} \frac{(R_E^2 - r^2)^{1/2}}{r} \\ \bar{\rho}_{201}(r) = \frac{\sqrt{5}}{2} \bar{\rho}_E \left[\frac{(1 - e^2)^{3/2}}{e^3} \frac{(R_E^2 - r^2)^{3/2}}{r^3} - \frac{(1 - e^2)^{1/2}}{e} \frac{(R_E^2 - r^2)^{1/2}}{r} \right] \end{array} \right\} \text{ for } R_E (1 - e^2) < r \leq R_E$$

where $R_E(1 - e^2)^{1/2}$ is the polar radius. The other $\bar{\rho}_{\ell_{mi}}(r)$ through degree 2 are zero by the symmetry of the spheroid.

The above expressions may be substituted in eq. (3) to give U_{001} and U_{201} . The calculations are tedious in the extreme; only the final results will be given here. They are

$$U_{001} \cong - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_E^5 \left\{ 1 - \frac{5}{6} e^2 - \frac{1}{8} e^4 + \dots \right\}$$

$$U_{201} \cong - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_E^5 \left\{ \frac{e^4}{30} + \dots \right\}$$

so that

$$U \cong U_{001} + U_{201} \cong - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_E^5 \left\{ 1 - \frac{5}{6} e^2 - \frac{11}{120} e^4 + \dots \right\} \quad (7)$$

to the fourth power of e .

This result is in complete agreement with the exact expression for a homogeneous spheroid given by Lyttleton (1953, p. 36), which is obtained by well-known integral techniques dealing with rotating liquids:

$$\begin{aligned} U &= - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_E^5 \frac{(1 - e^2)}{e} \text{Arc sin } e \\ &= - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_E^5 \left\{ 1 - \frac{5}{6} e^2 - \frac{11}{120} e^4 + \dots \right\} . \end{aligned}$$

VISCOSITY OF THE MANTLE

The shape of the earth is approximately that of an oblate spheroid. The flattening factor f , related to the eccentricity through the equation $1 - f = (1 - e^2)^{1/2}$, which best fits the earth in the least squares sense, is $f = 1/298.255$ (e.g. Kahn and O'Keefe 1974). If the earth were in hydrostatic equilibrium the flattening would be $f_h = 1/299.75$ (Kahn and O'Keefe, 1974). Hence the earth is flatter than predicted from hydrostatic theory, i.e. the equatorial bulge is too big.

Various mechanisms have been suggested for producing the excess bulge. Munk and MacDonald (1960) and MacDonald (1966) thought the excess flattening might be a fossil bulge left over from the remote past when the earth was rotating faster. This implies the earth has a "long memory" (roughly 10^7 years), or a high viscosity (about 10^{26} poises), if a linearly viscous fluid is assumed to be the appropriate rheology. Goldreich and Toomre (1969) strongly indicated that there is no fossil bulge at all; subtraction of the hydrostatic bulge shows the earth to be a distinctly triaxial object. They felt that the irregularities in the earth's gravity field might be a by-product of mantle convection and that the viscosity was several orders of magnitude smaller than 10^{26} poises. The irregularities would then steer the rotation axis to a position which maximizes the polar moment of inertia, thus producing the excess bulge. Wang (1966) thought the excess flattening might be due to heavy glaciation at the poles, which would squeeze out a bulge due to the weight of the ice. This view was

criticized by McKenzie (1966), Kaula (1967), and O'Connell (1971); they indicated the ice caps would have to be unacceptably large to produce the extra flattening. Kahn and O'Keefe (1974), however, showed that glaciation in Antarctica probably produced the gravity field's large third harmonic, giving the earth its "pear shape." Jeffreys (1970, pp. 429-432) felt the earth has finite strength.

We will not speculate upon the cause of the excess flattening here, but merely note that it exists and use it to estimate a lower limit for the viscosity of the mantle. Our argument depends upon the excess flattening, the mechanical energy, viscosity, and heat flux from the earth.

Take the earth to be a homogeneous, oblate spheroid with constant density and viscosity; thus we will make no distinction between upper and lower mantle viscosities. The viscosity may, in fact, be relatively constant throughout the mantle (Cathles, 1975, p. 3).

Let us first assume that the earth has been squashed past its equilibrium flattening and is now relaxing back to its equilibrium shape. Heat will be generated as the excess bulge subsides through viscous dissipation, subtracting energy from E , the total mechanical energy. Hence if we find \dot{E} , the rate of change of mechanical energy, we will have the heat flux due to viscous dissipation, by conservation of energy. Let us proceed to do this.

Let e_h , C_h , ω_h , and R_h denote the equilibrium values of the eccentricity, polar moment of inertia, rotational speed, and equatorial radius of the earth,

respectively. Let e , C , ω , and R_E refer to the same quantities at some time t .

Assuming conservation of mass (and volume)

$$M_E = \frac{4\pi}{3} \bar{\rho}_E R_E^3 (1 - e^2)^{1/2} = \frac{4\pi}{3} \bar{\rho}_E R_h^3 (1 - e_h^2)^{1/2},$$

conservation of angular momentum

$$C \omega = C_h \omega_h,$$

and the relationship between the equilibrium eccentricity and angular speed

(Lyttleton, 1953, p. 38)

$$\frac{\omega_h^2}{2\pi G \bar{\rho}_E} = \left(\frac{3 - 2e_h^2}{e_h^3} \right) (1 - e_h^2)^{1/2} \arcsin e - 3 \frac{(1 - e_h^2)}{e_h^2},$$

and using eq. (7), we have for the kinetic energy T and gravitational potential energy U at time t

$$T = \frac{1}{2} C \omega^2 \cong \frac{16\pi^2}{15} G \bar{\rho}_E^2 R_h^5 \left[\frac{2}{15} e_h^2 - \frac{1}{63} e_h^4 - \frac{2}{45} e_h^2 e^2 \right]$$

$$U \cong \frac{-16\pi^2}{15} G \bar{\rho}_E^2 R_h^5 \left[1 - \frac{5}{6} e_h^2 - \frac{5}{6} e_h^4 - \frac{1}{45} e^4 \right]$$

to order e^4 , where e is the only quantity in the above equations which varies with time. Adding the two equations together gives

$$\begin{aligned}
E = T + U &\cong + \frac{16\pi^2}{15} G \bar{\rho}_E^2 R_h^5 \left[-1 + \frac{29}{30} e_h^2 \right. \\
&\quad \left. + \frac{167}{210} e_h^4 + \frac{1}{45} (e^2 - e_h^2)^2 \right] \\
&\cong \frac{16\pi^2}{15} G \bar{\rho}_E^2 R_h^5 \left[-1 + \frac{29}{15} f_h + \frac{334}{105} f_h^2 + \frac{4}{45} (f - f_h)^2 \right] \quad (8)
\end{aligned}$$

where we have used $2f \cong e^2$ and $2f_h \cong e_h^2$.

Darwin (1879) worked out the relation between the viscosity and the flattening. It is for second degree terms

$$f = f_h + \Delta f$$

where

$$\Delta f = \Delta f_0 e^{\frac{-2g \bar{\rho}_E a_E t}{19 \eta}},$$

Δf_0 being the excess flattening at time $t = 0$, g the gravitational acceleration at the earth's surface, and η the viscosity. Substitution in eq. (8) yields

$$E \cong \frac{16\pi^2}{15} G \bar{\rho}_E^2 R_h^5 \left[-1 + \frac{29}{15} f_h + \frac{334}{105} f_h^2 + \frac{4}{45} (\Delta f)^2 \right]$$

Differentiation of this expression with respect to time yields

$$\dot{E} \cong - \frac{16 \pi^2}{15} G \bar{\rho}_E^2 R_h^5 \cdot \frac{16}{45} \frac{g \bar{\rho}_E a_E}{19 \eta} (\Delta f)^2 \quad (9)$$

Taking t to be the present time and $\Delta f = f - f_h = 1/298.255 - 1/299.75$ gives $\dot{E} \cong 4.05 \times 10^{40}/\eta$ ergs/sec as the present-day heat flow due to viscous dissipation. The values for f and f_h refer, of course, to the real earth and not homogeneous spheroids; but they should be good enough for our purposes.

Now \dot{E} must certainly be less than the observed total geothermal flux from the earth of 3.15×10^{20} ergs/sec (Stacey, 1969, p. 280), which is believed to be primarily due to radioactive heating. Thus

$$\dot{E} \cong 4.05 \times 10^{40}/\eta \text{ ergs/sec} \leq 3.15 \times 10^{20} \text{ ergs/sec}$$

or $\eta \geq 1.3 \times 10^{20}$ poises. This is certainly in agreement with the observed Fennoscandian uplift, which gives $\eta \cong 10^{22}$ poises, the number preferred by some as the viscosity of the mantle (e.g. Cathles, 1975, pp. 1-4).

Instead of relaxing, the excess bulge might be dynamically maintained through the earth's action as a heat engine. Goldreich and Toomre (1969) suggested that this is indeed the case: the gravity anomalies are a by-product of convection. Stacey (1967; 1969, pp. 209-210) estimated the efficiency of the earth's heat engine at less than 10 per cent. If one per cent of the earth's heat flux is used to maintain the excess bulge, then the viscosity is 10^{22} poises, in agreement with Cathles (1975). Smaller efficiencies of course yield higher viscosities.

It is worth mentioning that Paddack's (1967) observation of the nodal acceleration of an earth satellite also permits a lower limit to be put on the viscosity. The nodal rate $\dot{\Omega}_s$ of a satellite is proportional to $J_2 = -\bar{C}_{20}$ term in the gravity field expansion. Since J_2 is also proportional to f for a homogeneous oblate spheroid, we have $\dot{\Omega}_s = -Kf$, where K is a constant of no concern to us here. Differentiation with respect to time and division by $\dot{\Omega}_s$ yields

$$\frac{\ddot{\Omega}_s}{\dot{\Omega}_s} = \frac{\Delta f}{f} \frac{2 g \bar{\rho}_E a_E}{19 \eta} .$$

Paddack found the left side to be zero to one part in a million. This gives $\eta \geq 2 \times 10^{15}$ poises, which is smaller than our limit by five orders of magnitude. Thus our heat flow argument puts a more stringent lower limit on the viscosity than does the observation of the satellite.

DISCUSSION

McKenzie (1966) estimated the second degree gravitational potential energy at -2.02×10^{30} ergs, a factor of ten larger than our estimate of -2.77×10^{29} ergs. We can clear up this discrepancy by noting three things about McKenzie's analysis. First, there is a spurious factor of 4π in his equation for dm (the third equation above his eq. (10)). Second, his gravitational potential coefficients $C_{\ell mi}$ (using our notation) differ from our normalized coefficients $\bar{C}_{\ell mi}$ by the relation $C_{\ell mi} = (2 - \delta_{0m})^{1/2} \bar{C}_{\ell mi}$. He erred in using the numerical values of $\bar{C}_{\ell mi}$ for his values of $C_{\ell mi}$ in computing the energy. Last, his procedure for

deriving the gravitational energy is slightly inconsistent: in some places in the derivation a spherical surface is assumed and in others a deformed surface. If a spherical surface is consistently assumed (i.e. using his eq. (6) for the potential coefficient), and the above factors of 4π and $(2 - \delta_{0m})^{1/2}$ are noted, then his eq. (10) becomes identical with our eq. (6).

MacDonald (1966) after a lengthy analysis estimated that 2×10^{34} ergs of gravitational potential energy was pent up in the earth, a factor of 10^5 larger than our estimate of total energy release of 2×10^{29} ergs, which may be derived from our eq. (9). Use of MacDonald's figure in our viscosity argument results in $\eta = 10^{26}$ poises, in agreement with the number given on his page 227. This assumes, however, that the entire geothermal flux comes from viscous dissipation. Attributing part of the heat flow to radioactive heating results in even higher values. Such viscosities rule out convection on a meaningful time-scale; the mantle would not overturn even once in the entire history of the earth. Adoption of our much lower number does not necessarily rule out convection. But we can make no statement as to the actual value of the viscosity: in principle it could be any value above 1.3×10^{20} poises. However, a thermodynamic efficiency of one per cent yielding 10^{22} poises is suggestive.

We can answer two questions raised by Kaula (1967, pp. 790 and 792). First, he wondered whether the energy associated with the gravity anomalies would become available as heat should the anomalies disappear. The answer depends delicately on the shape of the earth and its internal density distribution.

The energy associated with the relaxation of an oblate homogeneous spheroid produces heat, as we have found in the last section; i.e. the mechanical energy decreases. But the gravitational potential energy increases when the anomalies disappear in a spherical earth, as evidenced by our eq. (5). Therefore computations involving the gravitational energy must be done with great care. Second, Kaula wondered whether the harmonics in a shell of matter interacted only with themselves to produce self-energy. Our derivation answers this question in the negative: the gravitational energy arises through the interaction of each harmonic of a shell with the same harmonic in the other material composing the earth.

We hope to have answered at least some of the questions surrounding the topic of the earth's gravitational potential energy. Future investigations should prove fruitful in clarifying still further our understanding of the physics of the earth.

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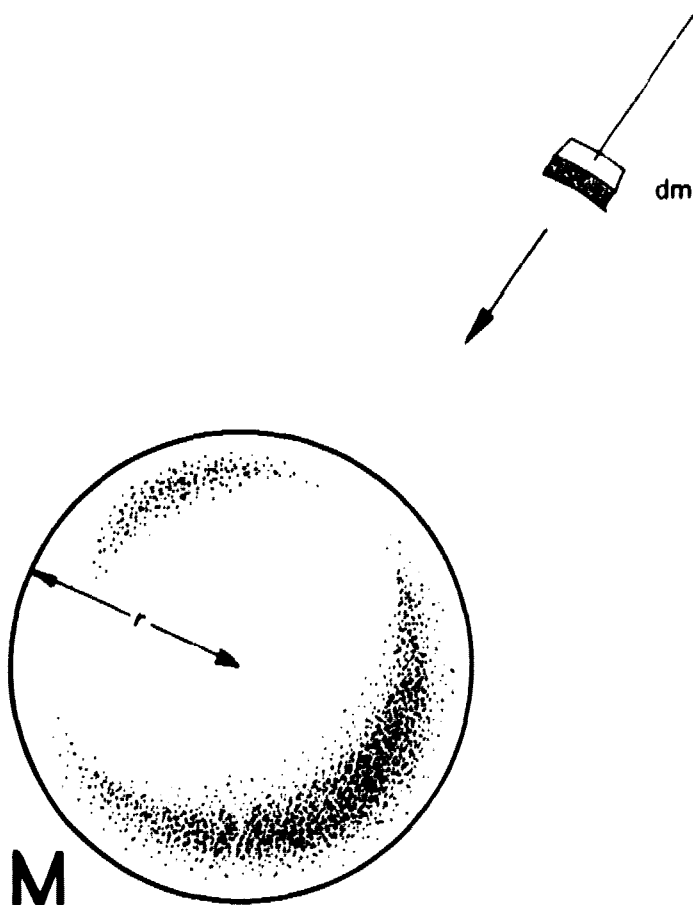


Figure 1. Bringing in a mass dm from infinity to the surface of the partially-assembled earth of mass M and radius r .

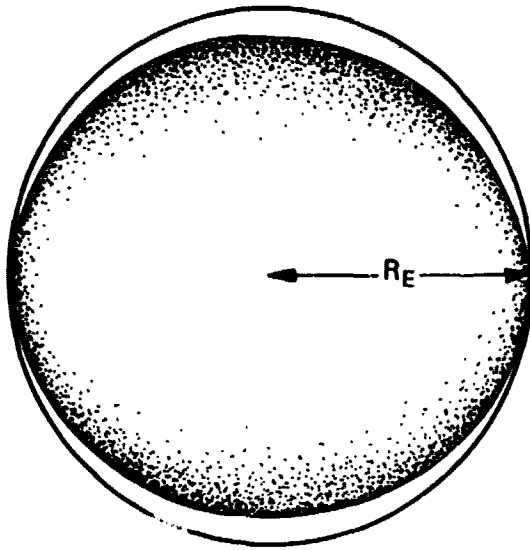


Figure 2. Imaginary sphere containing the earth. R_E is the distance from the center to the highest point on the earth.

FIGURE CAPTIONS

Figure 1. Bringing in a mass \underline{dm} from infinity to the surface of the partially-assembled earth of mass \underline{M} and radius \underline{r} .

Figure 2. Imaginary sphere containing the earth, $\underline{R_g}$ is the distance from the center to the highest point on the earth.

APPENDIX 1

McKenzie's (1966) equation (10) is, in his own notation,

$$E = \frac{-4\pi}{3} g a M.$$

$$\left[\sum_{\ell} (\ell-1) (2\ell+1) C_{\ell 0}^2 + \sum_{\ell, m \neq 0} (\ell-1) (2\ell+1) \left(\frac{C_{\ell m}^2 + S_{\ell m}^2}{2} \right) \right]$$

where g is the gravitational acceleration at the surface of the earth, and a and M are the mean radius and mass of the earth, respectively. The $C_{\ell m}$ and $S_{\ell m}$ are his potential coefficients.

The equation can be made to agree with our eq. (6), after corrections, as stated in the text. Our eq. (6) can also be derived from the equation for the potential energy for a surface distribution:

$$U = - \frac{1}{2} \int \sigma V \, dA.$$

Here σ is the surface mass density and V is the potential.

APPENDIX 2

The virial theorem states (Goldstein, 1950, p. 70):

$$-2 \langle T \rangle = \left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle$$

where T is the kinetic energy of the earth, \vec{F}_i and \vec{r}_i are the total force on and the position of the i th particle of the earth, respectively, and the angular brackets denote time - averages. We will henceforth drop the brackets.

For small perturbations in the positions of the particles the above equation becomes

$$\begin{aligned} -2 \delta T &= \sum_i \vec{F}_i \cdot \delta \vec{r}_i \\ &= \sum_i \vec{F}_i^G \cdot \delta \vec{r}_i + \sum_i \vec{F}_i^E \cdot \delta \vec{r}_i \end{aligned}$$

where we have decomposed the total force on each particle into the gravitational and elastic forces. The right side of the equation is nothing more than the work done, which is equal to minus the change in energies, so that

$$2 \delta T = \delta U + \delta W,$$

where U is the gravitational energy and δW is the elastic energy.

If

$$|\delta T| \ll |\delta U|,$$

then

$$\delta U \cong - \delta W,$$

which is the statement desired.

To test the inequality, we note that

$$\left| \delta U \right| \cong \left| U_2^{\max} \right| \cong 3 \times 10^{29} \text{ ergs.}$$

For δT to approach $\left| U_2^{\max} \right|$ in value, we would have to have particle speeds V such that

$$\delta T \cong \frac{1}{2} M_E V^2 \cong 3 \times 10^{29} \text{ ergs}$$

or $V \cong 10 \text{ cm/sec}$. Since V is more like 2 cm/year , we can assert the inequality with confidence (assuming heat plays no role in the energy balance).