Least Squares Polynomial Fits and Their Accuracy

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SHUTTLE PROGRAM

LEAST SQUARES POLYNOMIAL FITS AND THEIR ACCURACY

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PREFACE

The material included in this internal note was developed by William M. Lear of TRW under contract with the Mathematical Physics Branch (MPB) at the Johnson Space Center (JSC). The material was presented to MPB in the form of a TRW publication, number 30759-6003-TU-00, dated June 1, 1977. The material was prepared for JSC publication by Paul H. Mitchell of the MPB, and the text is taken from the referenced TRW document in its entirety, with the following changes.

(a) The coefficient of n in equation 13 (page 4) was changed from $(C_{22} + 2C_{12})$ to $(C_{22} + 2C_{13})$ to be consistent with the expansion of equation 12 and the following definitions.

(b) First person pronouns were changed to third person pronouns on pages 11 and 18.

(c) The concluding remarks were labeled as "8. CONCLUSIONS" on page 19.

CONTENTS

LEAST SQUARES POLYNOMIAL FITS AND THEIR ACCURACY

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1. INTRODUCTION

Suppose we are given a table of data as shown at the right. And, suppose we desire to make a least-squares polynomial fit to this data. We will assume that $t_i - t_{i-1} = \Delta T$ is a constant. That is, we have evenly spaced tabular values of y^*. Note that

n	t	y*
1	t_1	y_1^*
2	t_2	y_2^*
3	t_3	y_3^*
.	.	.
.	.	.
.	.	.
N	t_N	y_N^*

$$n = \frac{t - t_1}{\Delta T} + 1 \qquad (1)$$

We will assume that n is the independent variable in what follows. To make t the independent variable we use Eq. 1 to substitute for n.

For the sake of explanation, let us assume that y^* is given by the second order polynomial

$$y_n^* = a_o + a_1 n + a_2 n^2 + q_n \qquad (2)$$

where q_n is a zero-mean, timewise uncorrelated random variable (noise) whose variance is σ_q^2, independent of n. In order to predict future measurements, we need to know a_o, a_1, and a_2. Thus our state vector is

$$\underline{x} = \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} \qquad (3)$$

Using ^ to indicate estimated value, we have for our estimated measurement

$$\hat{y}_n = \hat{a}_o + \hat{a}_1 n + \hat{a}_2 n^2 \qquad (4)$$

The measurement partial derivative matrix, $P = \partial\hat{\underline{y}}/\partial\underline{x}$, will be

$$P = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & N & N^2 \end{bmatrix} \tag{5}$$

Note that we have a linear measurements problem,

$$\hat{\underline{y}} = P\hat{\underline{x}} \tag{6}$$

That is, the estimated measurement vector is a linear combination of the state vector elements.

The least-squares solution for $\hat{\underline{x}}$ is given by

$$W = (P^TP)^{-1}P^T \tag{7}$$

$$\hat{\underline{x}} = \hat{\underline{x}} + W(\underline{y}^* - \hat{\underline{y}}) \tag{8}$$

$$C = \sigma_q^2(P^TP)^{-1} \tag{9}$$

where Eq. 7 is the equation for the residual weighting matrix, and Eq. 9 is the equation for the state error covariance matrix, $C = E[(\hat{\underline{x}}-\underline{x})(\hat{\underline{x}}-\underline{x})^T]$. Eq. 8 for $\hat{\underline{x}}$ is iterated until converged. But from Eq. 6 and Eq. 7, Eq. 8 becomes

$$\hat{\underline{x}} = \hat{\underline{x}} + (P^TP)^{-1}P^T(\underline{y}^* - P\hat{\underline{x}})$$

$$= \hat{\underline{x}} + (P^TP)^{-1}P^T\underline{y}^* - (P^TP)^{-1}P^TP\hat{\underline{x}}$$

or

$$\boxed{\hat{\underline{x}} = (P^TP)^{-1}P^T\underline{y}^*} \tag{10}$$

which requires no iteration to find $\hat{\underline{x}}$, and is the least-squares equation for solving for the state vector when $\hat{\underline{y}} = P\hat{\underline{x}}$. In the following sections we will give the analytic expressions for $(P^TP)^{-1}$ for various orders of polynomials.

Now y_n^* is given by

$$\begin{aligned} y_n^* &= y_n + q_n \\ y_n &= a_o + a_1 n + a_2 n^2 \end{aligned} \tag{11}$$

where y_n is the error free value of the measurement. Suppose we wish to know the accuracy of our estimate of $\hat{y}_n$.

$$\begin{aligned} E[(\hat{y}_n - y_n)^2] &= E[\{(\hat{a}_o - a_o) + (\hat{a}_1 - a_1)n + (\hat{a}_2 - a_2)n^2\}^2] \\ &= E[(\hat{a}_o - a_o)^2 + (\hat{a}_1 - a_1)^2 n^2 + (\hat{a}_2 - a_2)^2 n^4 \\ &\quad + 2(\hat{a}_o - a_o)(\hat{a}_1 - a_1)n + 2(\hat{a}_o - a_o)(\hat{a}_2 - a_2)n^2 \\ &\quad + 2(\hat{a}_1 - a_1)(\hat{a}_2 - a_2)n^3] \end{aligned} \tag{12}$$

But

$$E[(\hat{a}_o - a_o)^2] = C_{11}$$

$$E[(\hat{a}_1 - a_1)^2] = C_{22}$$

$$E[(\hat{a}_2 - a_2)^2] = C_{33}$$

$$E[(\hat{a}_0-a_0)(\hat{a}_1-a_1)] = C_{12} = C_{21}$$

$$E[(\hat{a}_0-a_0)(\hat{a}_2-a_2)] = C_{13} = C_{31}$$

$$E[(\hat{a}_1-a_1)(\hat{a}_2-a_2)] = C_{23} = C_{32}$$

Thus we have

$$E[(\hat{y}_n-y_n)^2] = C_{11} + 2C_{12}n + (C_{22}+2C_{13})n^2 + 2C_{23}n^3 + C_{33}n^4 \tag{13}$$

where, we remember that the matrix C was given by

$$C = \sigma_q^2(P^TP)^{-1}$$

The C_{ij} values will be functions of the total number of measurements, N, and σ_q^2. Eq. 13 will tell us how much we are able to reduce the measurement noise variance σ_q^2 when we make a least-squares fit to obtain $\hat{y}_n$. We will be particularly interested in the values of $E[(\hat{y}_n-y_n)^2]$ at n = 1, n = N, and the midpoint n = (N+1)/2.

We may also be interested in how accurately we can predict velocity and acceleration in our above example. Observe that from Eq. 1

$$\frac{dy}{dt} = \frac{dy}{dn}\frac{dn}{dt} = \frac{dy}{dn}\frac{1}{\Delta T}$$

and

$$\frac{d^2y}{dt^2} = \frac{d\dot{y}}{dn}\frac{dn}{dt} = \frac{d^2y}{dn^2}\frac{1}{\Delta T^2}.$$

From Eq. 12

$$\frac{dy}{dt} = \frac{1}{\Delta T}(a_1+2a_2n) \tag{14}$$

$$\frac{d^2y}{dt^2} = \frac{1}{\Delta T^2}2a_2 \tag{15}$$

Thus

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{1}{\Delta T^2} E[\{(\hat{a}_1-a_1) + 2(\hat{a}_2-a_2)n\}^2]$$

$$= \frac{1}{\Delta T^2} E[(\hat{a}_1-a_1)^2 + 4(\hat{a}_2-a_2)^2 n^2$$

$$+ 4(\hat{a}_1-a_1)(\hat{a}_2-a_2)n]$$

Or

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{1}{\Delta T^2} (C_{22} + 4C_{23}n + 4C_{33}n^2) \tag{16}$$

and likewise

$$E[(\hat{\ddot{y}}_n-\ddot{y}_n)^2] = \frac{4}{\Delta T^4} C_{33} \tag{17}$$

In the succeeding sections, the following equations have been used in the evaluation of P^TP.

$$1+2+3+\ldots+N = \frac{N}{2}(N+1)$$

$$1^2+2^2+3^2+\ldots+N^2 = \frac{N}{6}(N+1)(2N+1)$$

$$1^3+2^3+3^3+\ldots+N^3 = \frac{N^2}{4}(N+1)^2$$

$$1^4+2^4+3^4+\ldots+N^4 = \frac{N}{30}(N+1)(2N+1)(3N^2+3N-1)$$

$$1^5+2^5+3^5+\ldots+N^5 = \frac{N^2}{12}(N+1)^2(2N^2+2N-1)$$

$$1^6+2^6+3^6+\ldots+N^6 = \frac{N}{42}(N+1)(2N+1)(3N^4+6N^3-3N+1)$$

$$1^7+2^7+3^7+\ldots+N^7 = \frac{N^2}{24}(N+1)^2(3N^4+6N^3-N^2-4N+2)$$

$$1^8+2^8+3^8+\ldots+N^8 = \frac{N}{90}(N+1)(2N+1)(5N^6+15N^6+5N^4$$

$$-15N^3-N^2+9N-3)$$

2. ZERO ORDER POLYNOMIAL, $y_n = a_o$

Our first example is for a zero order polynomial.

$$P = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{18}$$

$$(P^T P)^{-1} = \frac{1}{N} \tag{19}$$

$$E[(\hat{y}_n - y_n)^2] = \sigma_q^2/N \tag{20}$$

3. FIRST ORDER POLYNOMIAL, $y_n = a_o + a_1 n$

In this case

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ . & . \\ . & . \\ . & . \\ 1 & N \end{bmatrix} \tag{21}$$

$$(P^T P)^{-1} = \frac{2}{N(N^2-1)} \begin{bmatrix} (N+1)(2N+1) & -3(N+1) \\ -3(N+1) & 6 \end{bmatrix} \tag{22}$$

$$E[(\hat{y}_n - y_n)^2] = \frac{2\sigma_q^2}{N(N^2-1)} [(N+1)(2N+1)-6(N+1)n+6n^2] \tag{23}$$

$$= \frac{2\sigma_q^2}{N(N+1)} (2N-1) \quad \text{for } n = 1,\ n = N \tag{24}$$

$$= \sigma_q^2/N \quad \text{for } n = (N+1)/2 \tag{25}$$

Note that $n = (N+1)/2$ yields the minimum value for $E[(\hat{y}_n - y_n)^2]$. That is, our best value of $\hat{y}_n$ occurs at the midpoint of the fit interval.

$$E[(\hat{\dot{y}}_n - \dot{y}_n)^2] = \frac{12\sigma_q^2}{\Delta T^2 N(N^2-1)} \tag{26}$$

Notice that the total time, T, over which the fit is made is $(N-1)\Delta T$.

4. SECOND ORDER POLYNOMIAL, $y_n = a_o+a_1n+a_2n^2$

$$P = \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & N & N^2 \end{bmatrix} \tag{27}$$

$$(P^TP)^{-1} = \frac{3}{N(N^2-1)(N^2-4)} \begin{bmatrix} (N+1)(N+2)(3N^2+3N+2) & -6(N+1)(N+2)(2N+1) & 10(N+1)(N+2) \\ -6(N+1)(N+2)(2N+1) & 4(2N+1)(8N+11) & -60(N+1) \\ 10(N+1)(N+2) & -60(N+1) & 60 \end{bmatrix} \tag{28}$$

$$E[(\hat{y}_n-y_n)^2] = \frac{3\sigma_q^2}{N(N^2-1)(N^2-4)} [(N+1)(N+2)(3N^2+3N+2) \tag{29}$$

$$-12(N+1)(N+2)(2N+1)n + 12(7N^2+15N+7)n^2 - 120(N+1)n^3 + 60n^4]$$

$$= \frac{3\sigma_q^2}{N(N+1)(N+2)} (3N^2-3N+2) \quad \text{for } n = 1,\ n = N \tag{30}$$

$$= \frac{3\sigma_q^2}{4N(N^2-4)} (3N^2-7) \quad \text{for } n = (N+1)/2 \tag{31}$$

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{12\sigma_q^2}{\Delta T^2 N(N^2-1)(N^2-4)} [(2N+1)(8N+11)-60(N+1)n+60n^2] \tag{32}$$

$$= \frac{12\sigma_q^2}{\Delta T^2 N(N^2-1)(N^2-4)} (16N^2-30N+11) \quad \text{for } n = 1,\ n = N \tag{33}$$

$$= \frac{12\sigma_q^2}{\Delta T^2 N(N^2-1)} \quad \text{for } n = (N+1)/2 \tag{34}$$

Note that Eq. 34 is identical to its counterpart for the linear fit, Eq. 26.

$$E[(\hat{\ddot{y}}_n-\ddot{y}_n)^2] = \frac{720\sigma_q^2}{\Delta T^4 N(N^2-1)(N^2-4)} \tag{35}$$

It is of interest to note that the maximum accuracy of $\hat{y}_n$ is not at the midpoint of the fit interval. This can best be seen for N >> 1. Eq. 29 becomes

$$\begin{aligned} E[(\hat{y}_n-y_n)^2] &= \frac{9\sigma_q^2}{N^5}[N^4-8N^3n+28N^2n^2-40Nn^3+20n^4] \\ &= \frac{9}{N}\sigma_q^2[20(\frac{n}{N}-\frac{1}{2})^4 - 2(\frac{n}{N}-\frac{1}{2})^2+\frac{1}{4}] \end{aligned} \tag{36}$$

For $\frac{n}{N}-\frac{1}{2}=0$ $\quad E[(\hat{y}_n-y_n)^2] = \frac{9}{4N}\sigma_q^2$ (a maximum) (37)

For $\frac{n}{N}-\frac{1}{2}=\pm\frac{1}{\sqrt{20}}$ $\quad E[(\hat{y}_n-y_n)^2] = \frac{9}{5N}\sigma_q^2$ (a minimum) (38)

5. THIRD ORDER POLYNOMIAL, $y_n = a_o+a_1n+a_2n^2+a_3n^3$

$$P = \begin{bmatrix} 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & N & N^2 & N^3 \end{bmatrix} \qquad (39)$$

$$(P^TP)^{-1} = \frac{1}{N(N^2-1)(N^2-4)(N^2-9)} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \qquad (40)$$

where

$$A_{11} = 8(N+1)(N+2)(N+3)(2N+1)(N^2+N+3)$$

$$A_{12} = A_{21} = -20(N+1)(N+2)(N+3)(6N^2+6N+5)$$

$$A_{13} = A_{31} = 120(N+1)(N+2)(N+3)(2N+1)$$

$$A_{14} + A_{41} = -140(N+1)(N+2)(N+3)$$

$$A_{22} = 200(6N^4+27N^3+42N^2+30N+11)$$

$$A_{23} = A_{32} = -300(N+1)(3N+2)(3N+5)$$

$$A_{24} = A_{42} = 280(6N^2+15N+11)$$

$$A_{33} = 360(2N+1)(9N+13)$$

$$A_{34} = A_{43} = -4200(N+1)$$

$$A_{44} = 2800$$

The accuracy of $\hat{y}_n$ is given by

$$E[(\hat{y}_n-y_n)^2] = \frac{8\sigma_q^2}{N(N^2-1)(N^2-4)(N^2-9)}[(N+1)(N+2)(N+3)(2N+1)(N^2+N+3)$$
$$-5(N+1)(N+2)(N+3)(6N^2+6N+5)n+5(42N^4+213N^3+378N^2+288N+91)n^2$$
$$-10(N+1)(71N^2+175N+96)n^3+5(246N^2+525N+271)n^4 \tag{41}$$
$$-1050(N+1)n^5+350n^6]$$

$$= \frac{8\sigma_q^2}{N(N+1)(N+2)(N+3)}(2N^3-3N^2+7N-3) \quad \text{for } n = 1,\ r = N \tag{42}$$

$$= \frac{3\sigma_q^2}{4N(N^2-4)}(3N^2-7) \quad \text{for } n = (N+1)/2 \tag{43}$$

Note that Eq. 43 is identical to its counterpart in the preceding section, Eq. 31. In this case however, it is suggested that we have a minimum value this time instead of a maximum value as before.

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{40\sigma_q^2/\Delta T^2}{N(N^2-1)(N^2-4)(N^2-9)}[5(6N^4+27N^3+42N^2+30N+11)$$

$$-30(N+1)(3N+2)(3N+5)n+6(150N^2+315N+155)n^2 \tag{44}$$

$$-1260(N+1)n^3+630n^4]$$

For $n = 1$ and $n = N$

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{200\sigma_q^2/\Delta T^2}{N(N^2-1)(N^2-4)(N^2-9)}(6N^4-27N^3+42N^2-30N+11) \tag{45}$$

For $n = (N+1)/2$

$$E[(\hat{\dot{y}}_n-\dot{y}_n)^2] = \frac{25\sigma_q^2/\Delta T^2}{N(N^2-1)(N^2-4)(N^2-9)}(3N^4-18N^2+31) \tag{46}$$

Acceleration error is given by

$$E[(\hat{\ddot{y}}_n-\ddot{y}_n)^2] = \frac{1440\sigma_q^2/\Delta T^4}{N(N^2-1)(N^2-4)(N^2-9)}[(2N+1)(9N+13) \tag{47}$$

$$-70(N+1)n+70n^2]$$

For $n = 1$ and $n = N$

$$E[(\hat{\ddot{y}}_n-\ddot{y}_n)^2] = \frac{1440\sigma_q^2/\Delta T^4}{N(N^2-1)(N^2-4)(N^2-9)}(2N-1)(9N-13) \tag{48}$$

For $n = (N+1)/2$

$$E[(\hat{\ddot{y}}_n - \ddot{y}_n)^2] = \frac{720\sigma_q^2/\Delta T^4}{N(N^2-1)(N^2-4)} \tag{49}$$

For jerk we have

$$E[(\hat{\dddot{y}}_n - \dddot{y}_n)^2] = \frac{100\ 800\sigma_q^2/\Delta T^6}{N(N^2-1)(N^2-4)(N^2-9)} \tag{50}$$

6. SUMMARY OF RESULTS FOR N LARGE

Let $n = N >> 1$. Then Table 1 summarizes the previously determined results. Note that $T = (N-1)\Delta T \approx N\Delta T$.

TABLE 1. SUMMARY OF RESULTS FOR n=N or n=1

ORDER OF POLYNOMIAL	0	1	2	3
$E[(\hat{y}_n - y_n)^2]$	$\frac{\sigma_q^2}{N}$	$\frac{4\sigma_q^2}{N}$	$\frac{9\sigma_q^2}{N}$	$\frac{16\sigma_q^2}{N}$
$E[(\hat{\dot{y}}_n - \dot{y}_n)^2]$		$\frac{12\sigma_q^2/T^2}{N}$	$\frac{192\sigma_q^2/T^2}{N}$	$\frac{1200\sigma_q^2/T^2}{N}$
$E[(\hat{\ddot{y}}_n - \ddot{y}_n)^2]$			$\frac{720\sigma_q^2/T^4}{N}$	$\frac{25920\sigma_q^2/T^4}{N}$
$E[(\hat{\dddot{y}}_n - \dddot{y}_n)^2]$				$\frac{100\ 800\sigma_q^2/T^6}{N}$

Let M = order of the polynomial fit. Then it appears that we may extrapolate our results to obtain the interesting equation for $n = 1$ and $n = N$.

$$\boxed{E[(\hat{y}_n - y_n)^2] = (M+1)^2 \sigma_q^2 / N} \tag{51}$$

Also we note from Table 1, that there is a large degradation in velocity accuracy when we go to the next higher order polynomial. Thus, when we desire a velocity estimate at the end point of the fit interval, we should use the minimum order polynomial fit.

Now let $n = (N+1)/2 >> 1$. Table 2 summarizes the previously determined results.

TABLE 2: SUMMARY OF RESULTS FOR $n = (N+1)/2$

ORDER OF POLYNOMIAL	0	1	2	3
$E[(\hat{y}_n-y_n)^2]$	$\frac{\sigma_q^2}{N}$	$\frac{\sigma_q^2}{N}$	$\frac{9\sigma_q^2}{N}$	$\frac{9\sigma_q^2}{N}$
$E[(\hat{\dot{y}}_n-\dot{y}_n)^2]$		$\frac{12\sigma_q^2/T^2}{N}$	$\frac{12\sigma_q^2/T^2}{N}$	$\frac{75\sigma_q^2/T^2}{N}$
$E[(\hat{\ddot{y}}_n-\ddot{y}_n)^2]$			$\frac{720\sigma_q^2/T^4}{N}$	$\frac{720\sigma_q^2/T^4}{N}$
$E[(\hat{\dddot{y}}_n-\dddot{y}_n)]$				$\frac{100\ 800\sigma_q^2/T^6}{N}$

7. AN EXAMPLE

Let us make up a hypothetical example in which we let

$$y_n = 1 + .1n^2$$

$$\sigma_q = 1$$

Using a random number generator, we can produce a table of noisy measurements, $y_n^* = y_n + q_n$, as shown in Table 3.

TABLE 3. LEAST-SQUARES FIT OF SECOND ORDER POLYNOMIAL

n	y	q	y*	$\hat{y}$	y*-$\hat{y}$	$\hat{y}$-y
1	1.1	1.403	2.503	1.465	1.038	.365
2	1.4	-.354	1.046	1.790	-.744	.390
3	1.9	-.634	1.266	2.282	-1.016	.382
4	2.6	.697	3.297	2.941	.356	.341
5	3.5	-.686	2.814	3.768	-.954	.268
6	4.6	1.375	5.975	4.762	1.213	.162
7	5.9	.785	6.685	5.924	.761	.024
8	7.4	-.963	6.437	7.254	-.817	-.146
9	9.1	.759	9.859	8.751	1.108	-.349
10	11	-1.782	9.218	10.415	-1.197	-.585
11	13.1	-.600	12.500	12.247	.253	-.853

We will fit

$$\hat{y}_n = \hat{a}_o + \hat{a}_1 n + \hat{a}_2 n^2$$

to the noisy y^* data, using the equations of Section 4.

From Eq. 27

$$P^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 \end{bmatrix}$$

Using the values of y^* in Table 3, we get

$$P^T\underline{y}^* = \begin{bmatrix} 61.600 \\ 488.203 \\ 4328.695 \end{bmatrix}$$

Eq. 28 gives us the value of $(P^TP)^{-1}$ for N = 11 of

$$(P^TP)^{-1} = \frac{1}{4290} \begin{bmatrix} 5174 & -1794 & 130 \\ -1794 & 759 & -60 \\ 130 & -60 & 5 \end{bmatrix}$$

The estimated state vector is given by

$$\hat{\underline{x}} = \begin{bmatrix} \hat{a}_o \\ \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = (P^TP)^{-1}P^T\underline{y}^* = \begin{bmatrix} 1.3083 \pm 1.1 \\ .0732 \pm .42 \\ .08375 \pm .034 \end{bmatrix}$$

where the accuracy estimates were obtained by taking the square roots of the diagonal elements of the state error covariance matrix,

$$C = \sigma_q^2 (P^T P)^{-1}$$

$$\sigma_q^2 = 1$$

The residuals are now estimated from

$$y_n^* - \hat{y}_n = y_n^* - 1.3083 - .0732n - .08375n^2$$

and their values are shown in Table 3. Note that

$$\sum_{n=1}^{11} (y_n^* - \hat{y}_n) = .001$$

very commendable.

If we have a large number of measurements, $\hat{y}_n$ will approach y_n, making $y_n^* - \hat{y}_n$ an estimate of q. Thus

$$\hat{\sigma}_q = \sqrt{\frac{1}{11} \sum_{n=1}^{11} (y_n^* - \hat{y}_n)^2} = .91$$

a very good estimate for just 11 points, only off by 9% from the true value of 1. While $\hat{\sigma}_q$ appears to be quite accurate, the individual values of $\hat{q}_n = y_n^* - \hat{y}_n$ are not nearly so accurate. It is desirable to have an equation to express the accuracy of estimating σ_q in the above manner, since it is frequently used to obtain the noise statistics for measurement data. Perhaps such a relationship will be found with the desirable accuracy.

The last column of Table 3 shows $\hat{y}$-y, the error in the estimate of $\hat{y}$. As predicted by our previously derived equations, y is more accurate near the middle of the fit interval than it is near the end points of the interval. That is, from Eqs. 30 and 31

$$\sqrt{E[(\hat{y}_1-y_1)^2]} = \sqrt{E[(\hat{y}_{11}-y_{11})^2]} = \pm .76$$

$$\sqrt{E[(\hat{y}_6-y_6)^2]} = \pm .46$$

8. CONCLUSIONS

One final thought to close this report. A lot of data are needed to reduce the measurement error standard deviation by a significant amount, say by a factor of 10. In this example with 11 points of data, we saw that at the end points, σ_q was reduced by a factor of .76, not much of a reduction. To get a factor of 10 reduction would have required about 900 data points, a lot of points.