Euler Angles, Quaternions, and Transformation Matrices

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EULER ANGLES, QUATERNIONS, AND TRANSFORMATION MATRICES.

WORKING RELATIONSHIPS

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EULER ANGLES, QUATERNIONS, AND TRANSFORMATION MATRICES - WORKING RELATIONSHIPS

By D. M. Henderson
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1.0 INTRODUCTION

Due to the extensive use of the quaternion in the onboard Space Shuttle Computer System, considerable analysis is being performed using relationships between the quaternion, the transformation matrix, and the Euler angles. This Internal Note offers a brief mathematical development of the relationships between the Euler angles and the transformation matrix, the quaternion and the transformation matrix, and the Euler angles and the quaternion. The analysis and equations presented here apply directly to current Space Shuttle problems. Appendix A presents the twelve three-axis Euler transformation matrices as functions of the Euler angles, the equations for the quaternion as a function of the Euler angles, and the Euler angles as a function of the transformation matrix elements.

The equations of Appendix A are a valuable reference in Shuttle analysis work and this Internal Note is the only known document where each of the twelve Euler angle to quaternion relationships are given. Appendix B presents a group of utility subroutines to accomplish the Euler-matrix, quaternion-matrix, and Euler-quaternion relationships of Appendix A.
2.0 DISCUSSION

2.1 Euler Angle Transformation Matrices

The following analysis and utility subroutines are offered to simplify computer programs when working with coordinate transformation matrices and their relationships with the Euler Angles and the Quaternions. The coordinate transformation matrices discussed here are defined using the following figure.

![Diagram of coordinate system and Euler angles]

Figure 1.- Coordinate system and Euler angles.
The transformation matrix \( M \), is defined to transform vectors in the \( \bar{x} \)-system \((\bar{x}, \bar{y}, \bar{z})\) into the original \( x \)-system \((x, y, z)\) and is given by the equation,

\[
x = M \bar{x}
\]

where

\[
x = (x, y, z) \quad \text{and} \quad \bar{x} = (\bar{x}, \bar{y}, \bar{z}).
\]

Using the right-hand rule for positive rotations, the \( M \) matrix in (1) above is constructed by the following analysis. The first rotation in Figure 1 above is about the \( x \)-axis by the amount \( \theta_1 \). The single rotation about the \( x \)-axis results in the following transformation,

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & -\sin \theta_1 \\
0 & \sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\bar{x}' \\
\bar{y}' \\
\bar{z}'
\end{pmatrix}
\]

or \( x = \bar{x}' \) in matrix form. Rotation about the \( \bar{y}' \)-axis by the amount \( \theta_2 \) yields the intermediate transformation matrix:

\[
\begin{pmatrix}
\bar{x}' \\
\bar{y}' \\
\bar{z}'
\end{pmatrix}
= 
\begin{pmatrix}
\cos \theta_2 & 0 & \sin \theta_2 \\
0 & 1 & 0 \\
-\sin \theta_2 & 0 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix}
\bar{x}'' \\
\bar{y}'' \\
\bar{z}''
\end{pmatrix}
\]

or \( \bar{x}' = Y \bar{x}'' \) in matrix form. Finally rotation about the \( \bar{z}'' \)-axis by the amount \( \theta_3 \) yields the intermediate transformation matrix,
\[
\begin{pmatrix}
\bar{x}' \\
\bar{y}' \\
\bar{z}'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_3 & -\sin \theta_3 & 0 \\
\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix}
\]

and in matrix form \( \bar{x}' = Z \bar{x} \). Now using the three equations,

\[
\begin{align*}
x &= \bar{x}' \\
\bar{x}' &= Y \bar{x}'' \\
\bar{x}'' &= Z \bar{x}
\end{align*}
\]

by substitution

\[
x = (X Y Z) \bar{x}\]

Then from equation 1,

\[
M = (X Y Z)
\]

Computation for the \( M \) matrix from the indicated matrix multiplication in equation (7) yields,

\[
M = \begin{pmatrix}
\cos \theta_2 \cos \theta_3 & -\cos \theta_2 \sin \theta_3 \\
\sin \theta_2 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \sin \theta_2 \sin \theta_3 - \sin \theta_1 \cos \theta_2 \\
\sin \theta_1 \sin \theta_3 - \cos \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \cos \theta_3 + \cos \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_2
\end{pmatrix}
\]

The matrix \( M \) in equation (8) is a function of:

1. The three Euler angles \( \theta_1, \theta_2 \), and \( \theta_3 \) and
2. The sequence of rotations used to generate the matrix.

By examination of equation (7) it is possible to show that there are twelve possible Euler rotational sequences. If the \( (X Y Z) \) notation in equation (7) represents a rotation about the \( X \) axis, then the \( Y \) axis and finally the \( Z \) axis, then the following per-
mutations of the rotational order represents the twelve possible Euler angle sets using three rotations,

\[
\begin{align*}
X Y Z & \quad Y X Z & \quad Z X Y \\
X Z Y & \quad Y Z X & \quad Z Y X \\
X Y X & \quad Y X Y & \quad Z X Z \\
X Z X & \quad Y Z Y & \quad Z Y Z
\end{align*}
\]

(9)

Any repeated axis rotation such as XXY does not represent a three axis rotation but reduces to the two axis rotation XY. Hence the rotations described in (9) above represent all twelve possible sets of Euler angle defining sequences. Conversely then, for a given transformation matrix there are twelve Euler angle sets which can be extracted from the matrix. The Euler matrices corresponding to all possible rotational sequences of (9) above are presented in Appendix A.

The utility subroutines "EULMAT" generates the transformation matrix from a given Euler sequence and the Euler angles. The utility subroutine "MATEUL" extracts the Euler angles from a given Euler rotational sequence. The convention is established that the Euler angles occur in the same sequences as the axis rotations. Using this concept and functional notation, equation (7) could be expressed as

\[
M = X Y Z = M(\theta_x, \theta_y, \theta_z)
\]

(10)

and from (9)
\[ M = XZX = M(\theta_x, \theta_y, \theta_z) \text{ etc.} \] (11)

This convention is assumed in both subroutines and also used throughout this design note when Euler angles are used. A brief explanation of the use of these two utility subroutines is given in Appendix B.

It is interesting to note that a negative rotation in the single axis rotation matrices of equations (2), (3) and (4) will result in the formation of the transpose of the matrix. However the transpose of \( M \) in equation (7) is formed from reversing the order of multiplication and transposing the individual axis rotation equations, i.e.

\[ M^T = (X Y Z)^T = (Y Z)^T X^T = Z^T Y^T X^T. \] (12)

Hence the transposes of the matrices of (9) are easily formed by reversing the order of multiplication and transposing each single axis rotation equation. Using the notation in equations (10) and (11) above equation (12) could be written,

\[ M^T(\theta_x, \theta_y, \theta_z) = M(-\theta_z, -\theta_y, -\theta_x). \] (13)

It is recommended to avoid confusion that the forward transformation be computed and simply transposed to yield the reverse transformation matrix. All matrices of Appendix A are in the forward form, i.e. \( X = M\bar{x} \) and formed from (9).
2.2 Transformation Matrices Using the Hamilton Quaternion

The transformation matrix of equation (1) can be written as a function of the Hamilton Quaternion:

\[
\begin{align*}
q_1 &= \cos \omega/2 \\
q_2 &= \cos \alpha \sin \omega/2 \\
q_3 &= \cos \beta \sin \omega/2 \\
q_4 &= \cos \gamma \sin \omega/2 ,
\end{align*}
\]

(14)

where \( \omega \) is the rotation angle about the rotation axis with \( \alpha, \beta, \) and \( \gamma \) direction angles with the x, y and z axes respectively. Notice also that \( q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \), since \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \). The rotation angle, \( \omega \), is assumed positive according to the right-hand rule of axis rotation.

The matrix \( M \) becomes

\[
M = \begin{pmatrix}
(q_1^2 + q_2^2 - q_3^2 - q_4^2) & 2(q_2q_3 - q_1q_4) & 2(q_2q_4 + q_1q_3) \\
2(q_2q_3 + q_1q_4) & (q_1^2 - q_2^2 + q_3^2 - q_4^2) & 2(q_3q_4 - q_1q_2) \\
2(q_2q_4 - q_1q_3) & 2(q_3q_4 + q_1q_2) & (q_1^2 - q_2^2 - q_3^2 + q_4^2)
\end{pmatrix}
\]

(15)

For a more detailed discussion of the derivation of equation (15), see Reference 1. Using functional notation, equation (15) can be written,

\[
M = M(q_1, q_2, q_3, q_4).
\]

(16)

Unlike the Euler angle rotational sequences to describe the transformation matrix of equation (1), only two quaternions can be found from equation (15). The two quaternions are:
These two quaternions represent a positive rotation about
the rotation axis pointing in one direction and a positive
rotation about the same line of rotation pointing in the
opposite direction. Both quaternions of (17) satisfy equation
(15).

The utility subroutine "QMAT" generates the transformation
matrix from a given quaternion. The "QMAT" algorithm generates
the matrix as given in equation (15) without duplicating any
arithmetic operations. The subroutine "MATQ" extracts the positive
quaternion, i.e., \( q_1 > 0 \), from the transformation matrix and
normalizes the results to guarantee an orthogonal matrix. In
order to avoid any discontinuity in extracting the quaternion
from the transformation matrix, the procedure as described in
Reference 2 is used.

Early works by Hamilton (Référence 3) presented the quaternion
as having a scalar and a vector part, i.e.,

\[
q_1 = S \quad \mathbf{v} = (q_2, q_3, q_4) \tag{18}
\]

and equation (16) could be expressed as,

\[
M = M(q_1, q_2, q_3, q_4) = M(S, \mathbf{v}). \tag{19}
\]
For a given quaternion the following relationship is true (from (17) above),

\[ M(S, V) = M(-S, -V). \] (20)

The transpose of the transformation matrix is given by,

\[ M^T(S, V) = M(-S, V) = M(S, -V). \] (21)

### 2.3 Euler Angle and Quaternion Relationships

By examination of equations (10) and (16) the equality,

\[ M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = M(\theta_1, \theta_2, \theta_3) = M(q_1, q_2, q_3, q_4) \] (22)

can be written. Based on an equality for each element of the matrix the following nine equations must be true;

\[
\begin{align*}
\cos \theta_2 \cos \theta_3 &= q_1^2 + q_2^2 - q_3^2 - q_4^2 \\
-\cos \theta_2 \sin \theta_3 &= 2(q_2q_3 - q_1q_4) \\
\sin \theta_2 &= 2(q_2q_4 + q_1q_3) \\
\cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 &= 2(q_2q_3 + q_1q_4) \\
\cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_2 \sin \theta_3 &= q_1^2 - q_2^2 + q_3^2 - q_4^2 \\
-\sin \theta_1 \cos \theta_2 &= 2(q_3q_4 - q_1q_2) \\
\sin \theta_1 \sin \theta_3 - \cos \theta_1 \sin \theta_2 \cos \theta_3 &= 2(q_3q_4 - q_1q_3) \\
\sin \theta_1 \cos \theta_3 + \cos \theta_1 \sin \theta_2 \sin \theta_3 &= 2(q_3q_4 + q_1q_2) \\
\cos \theta_1 \cos \theta_2 &= q_1^2 - q_2^2 - q_3^2 + q_4^2.
\end{align*}
\] (23)

It is possible to solve for the values of the quaternion using the trigonometric half angle identities. For this Euler sequence, i.e. \( X(\theta_1) Y(\theta_2) Z(\theta_3) \), the following quaternion results;
\[ q_1 = -\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3 \]
\[ q_2 = +\sin \theta_1 \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \theta_1 \]
\[ q_3 = -\sin \theta_1 \sin \theta_3 \cos \theta_2 + \sin \theta_2 \cos \theta_1 \cos \theta_3 \]
\[ q_4 = +\sin \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_3 \cos \theta_1 \cos \theta_2 \]

(24)

Appendix A gives the quaternion as a function of the Euler angles for each of the twelve Euler rotational sequence presented in Section 2.1. The special equations for the quaternion as functions of the Euler angles, like equations (24) above, are sometimes cumbersome to use, especially when multiple Euler sequences are utilized. Less coding, but perhaps more computer operations, are required by using the more general method of first generating the matrix M from the given Euler angle set and then simply extracting the quaternion from the matrix. This method is effected by first a call to "EULMAT" and then a call to "MATQ".
3.0 REFERENCES


APPENDIX A

RELATIONSHIPS FOR THE THREE-AXIS EULER ANGLE ROTATION SEQUENCES

The twelve Euler matrices for each rotation sequence are given based on the single axis rotation equations (2), (3) and (4). The Euler matrices transform vectors from the system that has been rotated into vectors in the stationary system. Also presented here are the equations for the quaternion as a function of the Euler angles and the Euler angles as a function of the matrix elements for each rotation sequence.
(1) \( M = M(X(\theta_1), Y(\theta_2), Z(\theta_3)) = XYZ \)

**Axis Rotation Sequence:** 1, 2, 3

\[
M = \begin{bmatrix}
\cos \theta_1 \cos \theta_3 & -\cos \theta_1 \sin \theta_3 & \sin \theta_1 \\
\sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 & -\sin \theta_1 \cos \theta_2 \\
-\cos \theta_1 \sin \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 & \cos \theta_1 \cos \theta_2 \\
\end{bmatrix}
\]

\[q_1 = -\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3\]
\[q_2 = \sin \theta_1 \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \theta_1\]
\[q_3 = -\sin \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_2 \cos \theta_1 \cos \theta_3\]
\[q_4 = \sin \theta_1 \sin \theta_2 \sin \theta_3 + \sin \theta_3 \cos \theta_1 \cos \theta_2\]

\[\theta_1 = \tan^{-1} \left( \frac{-m_{23}}{m_{33}} \right)\]
\[\theta_2 = \tan^{-1} \left( \frac{m_{13}}{\sqrt{1 - m_{13}^2}} \right)\]
\[\theta_3 = \tan^{-1} \left( \frac{-m_{12}}{m_{11}} \right)\]
(2) \( M = M(X(\theta_1), Z(\theta_2), Y(\theta_3)) = XYZ \)

Axis Rotation Sequence: 1, 3, 2

\[
M = \begin{bmatrix}
\cos \theta_2 \cos \theta_3 & -\sin \theta_2 & \cos \theta_2 \sin \theta_3 \\
\cos \theta_1 \sin \theta_2 \cos \theta_3 & \cos \theta_1 \cos \theta_2 & \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
+\sin \theta_1 \sin \theta_3 & \cos \theta_1 \cos \theta_3 & -\sin \theta_1 \cos \theta_3 \\
\sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \cos \theta_2 & \sin \theta_1 \sin \theta_2 \sin \theta_3 \\
\cos \theta_1 \sin \theta_3 & -\sin \theta_1 \cos \theta_3 & +\cos \theta_1 \cos \theta_3
\end{bmatrix}
\]

\( q_1 = +\sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_1 \cos \theta_2 \cos \theta_3 \)
\( q_2 = +\sin \theta_1 \cos \theta_2 \sin \theta_3 \cos \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_2 \)
\( q_3 = -\sin \theta_2 \sin \theta_2 \cos \theta_3 \cos \theta_1 \sin \theta_3 \cos \theta_1 \sin \theta_2 \cos \theta_2 \)
\( q_4 = +\sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_1 \cos \theta_2 \sin \theta_3 \cos \theta_1 \sin \theta_2 \cos \theta_3 \)

\( \theta_1 = \tan^{-1}\left(\frac{m_{32}}{m_{22}}\right) \)

\( \theta_2 = \tan^{-1}\left(\frac{-m_{12}}{\sqrt{1-m_{12}^2}}\right) \)

\( \theta_3 = \tan^{-1}\left(\frac{m_{13}}{m_{11}}\right) \)
Axis Rotation Sequence: 1, 2, 1

\[
M = \begin{bmatrix}
    \cos \theta_2 & \sin \theta_2 \sin \theta_3 & \sin \theta_2 \cos \theta_3 \\
    \sin \theta_1 \sin \theta_2 & \cos \theta_1 \cos \theta_3 & -\cos \theta_1 \sin \theta_3 \\
    -\cos \theta_1 \sin \theta_2 & +\sin \theta_1 \cos \theta_3 & -\sin \theta_1 \cos \theta_2 \cos \theta_3
\end{bmatrix}
\]

\[q_1 = \cos \theta_2 \cos (\frac{1}{2}(\theta_1 + \theta_3))\]

\[q_2 = \cos \theta_2 \sin (\frac{1}{2}(\theta_1 + \theta_3))\]

\[q_3 = \sin \theta_2 \cos (\frac{1}{2}(\theta_1 - \theta_3))\]

\[q_4 = \sin \theta_2 \sin (\frac{1}{2}(\theta_1 - \theta_3))\]

\[\theta_1 = \tan^{-1}(\frac{m_{21}}{-m_{31}})\]

\[\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)\]

\[\theta_3 = \tan^{-1}(\frac{m_{12}}{m_{13}})\]
(4) $M = M(x(\theta_1), z(\theta_2), x(\theta_3)) = XZX$

Axis Rotation Sequence: 1, 3, 1

$$
M = \begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 \cos \theta_3 & \sin \theta_2 \sin \theta_3 \\
\cos \theta_1 \sin \theta_2 & \cos \theta_1 \cos \theta_2 \cos \theta_3 & -\cos \theta_1 \cos \theta_2 \sin \theta_3 \\
\sin \theta_1 \sin \theta_2 & \sin \theta_1 \cos \theta_2 \cos \theta_3 & -\sin \theta_1 \cos \theta_2 \sin \theta_3 \\
\end{bmatrix}
$$

$q_1 = \cos \frac{\theta_2}{2} \cos \left(\frac{1}{2}(\theta_1 + \theta_3)\right)$

$q_2 = \cos \frac{\theta_2}{2} \sin \left(\frac{1}{2}(\theta_1 + \theta_3)\right)$

$q_3 = -\sin \frac{\theta_2}{2} \sin \left(\frac{1}{2}(\theta_1 - \theta_3)\right)$

$q_4 = \sin \frac{\theta_2}{2} \cos \left(\frac{1}{2}(\theta_1 - \theta_3)\right)$

$\theta_1 = \tan^{-1}\left(\frac{m_{31}}{m_{21}}\right)$

$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1 - m_{11}^2}}{m_{11}}\right)$

$\theta_3 = \tan^{-1}\left(\frac{m_{13}}{-m_{12}}\right)$
(5) \( M = M(\theta_1, \theta_2, \theta_3) = YXZ \)

Axis Rotation Sequence: 2, 1, 3

\[
M = \begin{bmatrix}
\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_3 & \sin \theta_1 \sin \theta_2 \cos \theta_3 & \sin \theta_1 \cos \theta_2 \\
\cos \theta_2 \sin \theta_3 & \cos \theta_2 \cos \theta_3 & -\sin \theta_2 \\
\cos \theta_1 \sin \theta_2 \sin \theta_3 & \cos \theta_1 \sin \theta_2 \cos \theta_3 & \cos \theta_1 \cos \theta_2 \\
-\sin \theta_1 \cos \theta_3 & +\sin \theta_1 \sin \theta_3 & 0
\end{bmatrix}
\]

\( q_1 = \sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3 \)
\( q_2 = \sin \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_2 \cos \theta_1 \cos \theta_3 \)
\( q_3 = \sin \theta_1 \cos \theta_2 \cos \theta_3 - \sin \theta_2 \sin \theta_3 \cos \theta_1 \)
\( q_4 = -\sin \theta_1 \sin \theta_2 \sin \theta_3 + \sin \theta_2 \cos \theta_1 \cos \theta_3 \)

\( \theta_1 = \tan^{-1} \left( \frac{m_{31}}{m_{33}} \right) \)

\( \theta_2 = \tan^{-1} \left( \frac{-m_{23}}{\sqrt{1-m_{23}^2}} \right) \)

\( \theta_3 = \tan^{-1} \left( \frac{m_{21}}{m_{22}} \right) \)
Axis Rotation Sequence: 2, 3, 1

\[ M = \begin{bmatrix}
\cos \theta_1 \cos \theta_2 & -\cos \theta_1 \sin \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
\sin \theta_2 & \cos \theta_2 \cos \theta_3 & -\cos \theta_2 \sin \theta_3 \\
-\sin \theta_1 \cos \theta_2 & \sin \theta_1 \sin \theta_2 \cos \theta_3 & \cos \theta_1 \sin \theta_3
\end{bmatrix} \]

\[ q_1 = -\sin \theta_1 \sin \theta_2 \sin \theta_3 + \cos \theta_1 \cos \theta_2 \cos \theta_3 \]
\[ q_2 = +\sin \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_3 \cos \theta_1 \sin \theta_2 \cos \theta_2 \]
\[ q_3 = +\sin \theta_1 \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \theta_1 \]
\[ q_4 = -\sin \theta_1 \sin \theta_3 \cos \theta_2 + \sin \theta_2 \cos \theta_1 \cos \theta_3 \]

\[ \theta_1 = \tan^{-1} \left( \frac{-m_{31}}{m_{11}} \right) \]
\[ \theta_2 = \tan^{-1} \left( \frac{m_{21}}{\sqrt{1-m_{21}^2}} \right) \]
\[ \theta_3 = \tan^{-1} \left( \frac{-m_{23}}{m_{22}} \right) \]
(7) $M = M(Y(\theta_1), X(\theta_2), Y(\theta_3)) = YXY$

Axis Rotation Sequence: 2, 1, 2

\[
M = \begin{bmatrix}
-sin\theta_1cos\theta_2sin\theta_3 & sin\theta_1sin\theta_2 & sin\theta_1cos\theta_2cos\theta_3 \\
+cos\theta_1cos\theta_3 & cos\theta_2 & +cos\theta_1sin\theta_3 \\
-sin\theta_2sin\theta_3 & cos\theta_2 & -sin\theta_2cos\theta_3 \\
+cos\theta_1cos\theta_3 & cos\theta_1sin\theta_2 & cos\theta_1cos\theta_2cos\theta_3 \\
-sin\theta_1cos\theta_3 & -sin\theta_1sin\theta_2 & -sin\theta_1sin\theta_3 \\
\end{bmatrix}
\]

$q_1 = +cos\theta_2cos(\frac{1}{2}(\theta_1 + \theta_3))$

$q_2 = +sin\theta_2cos(\frac{1}{2}(\theta_1 - \theta_3))$

$q_3 = +cos\theta_2sin(\frac{1}{2}(\theta_1 + \theta_3))$

$q_4 = -sin\theta_2sin(\frac{1}{2}(\theta_1 - \theta_3))$

$\theta_1 = \tan^{-1}(\frac{m_{12}}{m_{32}})$

$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right)$

$\theta_3 = \tan^{-1}(\frac{m_{21}}{m_{23}})$
(8) \( M = M(Y(\theta_1), Z(\theta_2), Y(\theta_3)) = YZY \)

Axis Rotation Sequence: 2, 3, 2

\[
M = \begin{bmatrix}
\cos \theta_1 \cos \theta_2 \cos \theta_3 & -\cos \theta_1 \sin \theta_2 & \cos \theta_1 \cos \theta_2 \sin \theta_3 \\
-\sin \theta_1 \sin \theta_3 & \cos \theta_2 & \sin \theta_1 \cos \theta_2 \sin \theta_3 \\
\sin \theta_2 \cos \theta_3 & \sin \theta_1 \sin \theta_2 & -\sin \theta_1 \cos \theta_2 \sin \theta_3 \\
\end{bmatrix}
\]

\( q_1 = \cos \theta_2 \cos (\frac{1}{2}(\theta_1 + \theta_3)) \)
\( q_2 = \sin \theta_2 \sin (\frac{1}{2}(\theta_1 - \theta_3)) \)
\( q_3 = \cos \theta_2 \sin (\frac{1}{2}(\theta_1 + \theta_3)) \)
\( q_4 = \sin \theta_2 \cos (\frac{1}{2}(\theta_1 - \theta_3)) \)

\( \theta_1 = \tan^{-1}\left(\frac{m_{32}}{-m_{12}}\right) \)
\( \theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{22}^2}}{m_{22}}\right) \)
\( \theta_3 = \tan^{-1}\left(\frac{m_{23}}{m_{21}}\right) \)
(9) \( M = M(Z(\theta_1), X(\theta_2), Y(\theta_3)) = ZXY \)

Axis Rotation Sequence: 3, 1, 2

\[
M = \begin{bmatrix}
-sin\theta_1 sin\theta_2 sin\theta_3 & -sin\theta_1 cos\theta_2 & sin\theta_1 sin\theta_2 cos\theta_3 \\
+cos\theta_1 cos\theta_3 & cos\theta_1 cos\theta_2 & -cos\theta_1 sin\theta_2 cos\theta_3 \\
+sin\theta_1 cos\theta_3 & cos\theta_1 sin\theta_2 & +sin\theta_1 sin\theta_3 \\
-cos\theta_2 sin\theta_3 & sin\theta_2 & cos\theta_2 cos\theta_3
\end{bmatrix}
\]

\[
q_1 = -sin\theta_1 sin\theta_2 sin\theta_3 + cos\theta_1 cos\theta_2 cos\theta_3
\]

\[
q_2 = -sin\theta_1 sin\theta_3 cos\theta_2 + sin\theta_2 cos\theta_1 cos\theta_3
\]

\[
q_3 = +sin\theta_1 sin\theta_2 cos\theta_3 + sin\theta_3 cos\theta_1 cos\theta_2
\]

\[
q_4 = +sin\theta_1 cos\theta_2 cos\theta_3 + sin\theta_2 sin\theta_3 cos\theta_1
\]

\[
\theta_1 = \tan^{-1} \left( \frac{-m_{12}}{m_{22}} \right)
\]

\[
\theta_2 = \tan^{-1} \left( \frac{m_{32}}{\sqrt{1-m_{32}^2}} \right)
\]

\[
\theta_3 = \tan^{-1} \left( \frac{-m_{31}}{m_{33}} \right)
\]
(10) \( M = M(Z(\theta_1), Y(\theta_2), X(\theta_3)) = ZYX \)

Axis Rotation Sequence: 3, 2, 1

\[
M = \begin{bmatrix}
\cos\theta_1 \cos\theta_2 & \cos\theta_1 \sin\theta_2 \sin\theta_3 & \cos\theta_1 \sin\theta_2 \cos\theta_3 \\
\sin\theta_1 \cos\theta_2 & \sin\theta_1 \sin\theta_2 \sin\theta_3 & \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
-sin\theta_2 & \cos\theta_2 \sin\theta_3 & \cos\theta_2 \cos\theta_3
\end{bmatrix}
\]

\[
q_1 = +\sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 + \cos^2\theta_1 \cos^2\theta_2 \cos^2\theta_3
\]

\[
q_2 = -\sin^2\theta_1 \sin^2\theta_2 \cos^2\theta_3 + \sin^2\theta_2 \cos^2\theta_1 \cos^2\theta_2
\]

\[
q_3 = +\sin^2\theta_1 \sin^2\theta_3 \cos^2\theta_2 + \sin^2\theta_2 \cos^2\theta_1 \cos^2\theta_3
\]

\[
q_4 = +\sin^2\theta_1 \cos^2\theta_2 \cos^2\theta_3 - \sin^2\theta_2 \sin^2\theta_3 \cos^2\theta_1
\]

\[
\theta_1 = \tan^{-1}\left(\frac{m_{21}}{m_{11}}\right)
\]

\[
\theta_2 = \tan^{-1}\left(\frac{-m_{31}}{\sqrt{1-m_{31}^2}}\right)
\]

\[
\theta_3 = \tan^{-1}\left(\frac{m_{32}}{m_{33}}\right)
\]
(11) $M = M(Z(\theta_1), X(\theta_2), Z(\theta_3)) = ZXZ$

Axis Rotation Sequence: 3, 1, 3

$$M = \begin{bmatrix}
-sin\theta_1cos\theta_2sin\theta_3 & -sin\theta_1cos\theta_2cos\theta_3 & sin\theta_1sin\theta_2 \\
+cos\theta_1cos\theta_3 & -cos\theta_1cos\theta_3 & -cos\theta_1sin\theta_2 \\
cos\theta_1cos\theta_2sin\theta_3 & cos\theta_1cos\theta_2cos\theta_3 & -sin\theta_1sin\theta_3 \\
+sin\theta_1cos\theta_3 & -sin\theta_1sin\theta_3 & cos\theta_2 \\
sin\theta_2sin\theta_3 & sin\theta_2cos\theta_3 & cos\theta_2
\end{bmatrix}$$

$q_1 = +cos\theta_2cos(\frac{1}{2}(\theta_1 + \theta_3))$
$q_2 = +sin\theta_2cos(\frac{1}{2}(\theta_1 - \theta_3))$
$q_3 = +sin\theta_2sin(\frac{1}{2}(\theta_1 - \theta_3))$
$q_4 = +cos\theta_2sin(\frac{1}{2}(\theta_1 + \theta_3))$

$\theta_1 = \tan^{-1}\left(\frac{m_{13}}{-m_{23}}\right)$

$\theta_2 = \tan^{-1}\left(\frac{\sqrt{1-m_{33}^2}}{m_{33}}\right)$

$\theta_3 = \tan^{-1}\left(\frac{m_{31}}{m_{32}}\right)$
12) \( M = M(\theta_1, \gamma(\theta_2), Z(\theta_3)) = ZYZ \)

Axis Rotation Sequence: 3, 2, 3

\[
M = \begin{bmatrix}
    \cos \theta_1 \cos \theta_2 \cos \theta_3 & -\cos \theta_1 \cos \theta_2 \sin \theta_3 & \cos \theta_1 \sin \theta_2 \\
    -\sin \theta_1 \sin \theta_3 & -\sin \theta_1 \cos \theta_3 & \\
    \sin \theta_1 \cos \theta_2 \cos \theta_3 & -\sin \theta_1 \cos \theta_2 \sin \theta_3 & \sin \theta_1 \sin \theta_2 \\
    +\cos \theta_1 \sin \theta_3 & +\cos \theta_1 \cos \theta_3 & \\
    -\sin \theta_2 \cos \theta_3 & \sin \theta_2 \sin \theta_3 & \cos \theta_2
\end{bmatrix}
\]

\[
q_1 = +\cos \frac{\theta_2}{2} \cos \left( \frac{\theta_1 + \theta_3}{2} \right)
\]

\[
q_2 = -\sin \frac{\theta_2}{2} \sin \left( \frac{\theta_1 - \theta_3}{2} \right)
\]

\[
q_3 = +\sin \frac{\theta_2}{2} \cos \left( \frac{\theta_1 - \theta_3}{2} \right)
\]

\[
q_4 = +\cos \frac{\theta_2}{2} \sin \left( \frac{\theta_1 + \theta_3}{2} \right)
\]

\[
\theta_1 = \tan^{-1} \left( \frac{m_{23}}{m_{13}} \right)
\]

\[
\theta_2 = \tan^{-1} \left( \frac{\sqrt{1 - m_{33}^2}}{m_{33}} \right)
\]

\[
\theta_3 = \tan^{-1} \left( \frac{m_{32}}{m_{31}} \right)
\]
APPENDIX B
COMPUTER SUBROUTINES FOR THE RELATIONSHIPS

The following subroutines with a brief description of their use are presented in this appendix.

(1) "EULMAT" - Generates the transformation matrix from a given set of Euler angles and an axis rotation sequence.

(2) "MATEUL" - Extracts the Euler angles from the given transformation matrix and an axis rotation sequence.

(3) "QMAT" - Generates the transformation matrix from a given quaternion.

(4) "MATQ" - Extracts the quaternion from a given transformation matrix.

(5) "YPRQ" - Generates the quaternion directly from the yaw-pitch-roll Euler angles.

(6) "POSNOR" - Computes the positive-normalized quaternion from the given quaternion.
NAME: EULMAT

PURPOSE: Generates a 3 x 3 transformation matrix from a given sequence and Euler angle set.

INPUT: ISEQ - Rotation Sequence (Integer Array (3); i.e., 1, 2, 3)
       EUL - Euler Angles in radians, in "ISEQ" Order; ARRAY(3)

OUTPUT: A - The 3 x 3 transformation matrix

ALGORITHM REFERENCE: Appendix A; Euler Sequences (1) thru (12).
EULER ANGLES TO THE TRANSFORMATION MATRIX

SUBROUTINE EULMAT

ENTRY POINT: EULMAT

FORTRAN IS - EULMAT

FOR SEC3-02/19/77-06:24:23 L.01

SUBROUTINE EULMAT ENTRY POINT: EULMAT

STORAGE USEFUL: CODE(1) 003265; DATA(G) 003164; BLANK COMMON(2)

EXTERNAL REFERENCES (BLOCK, NAME)

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

CONTINUE

IF (ISEQ(I, KET) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE

IF (ISEQ(3) .GE. 1) GO TO 10

C

SUBROUTINE EULMAT(ISL, EUL, A)

DIMENSION ISL(3), EUL(3, A(3, 3))

DO 10 I=1, 3

CONTINUE
EULER ANGLES TO THE TRANSFORMATION MATRIX

(Continued)

196 CONTINUE
DO 194 L = 1, 2
M = 3 - L

194 CONTINUE
DO 192 I = 1, 3
DO 192 J = 1, 3

192 CONTINUE
IF (LEG.1) HOLD=X(K,L,J)
IF (LEG.2) HOLD=Y(K,L,J)
IF (ABS(HOLD) LT (1.1,YC-1)) GO TO 290
IF (ABS(X(L,X,M)*HOLD-1.E-12) GO TO 253
TEMP=TEMP*X(L,K,1)*HOLD

253 CONTINUE
IF (LEG.1) C(J)=TEMP
IF (LEG.2) A(J)=TEMP

290 CONTINUE
433 CONTINUE
RETURN
END

END OF COMPILATION: NO DIAGNOSTICS.

ORIGINAL PAGE IS OF POOR QUALITY
NAME: MATEUL

PURPOSE: Extracts the Euler angles from the given transformation matrix and the required Euler rotational sequence.

INPUT: ISEQ—Rotation sequence, (Integer Array (3), i.e., 1,2,3.)

A — The 3 x 3 transformation

OUTPUT: EUL — The Euler angles, in "ISEQ" order; ARRAY(3).

ALGORITHM REFERENCE: Appendix A; Euler angles as a function of the matrix elements, sequences (1) thru (12).
TRANSFORMATION MATRIX TO THE CULER ANGLES

SUBROUTINE MATEUL ENTRY POINT U*12335

STORAGE USED: CODE(11) O*2335; DATA(15) O*2352; BLANK COMMON(2)

EXTERNAL REFERENCES (BLOCK, NAME)

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

RESULT: MATEUL(15) USEFUL

DIMENSION A(15,15)

-resources-
TRANSFORMATION MATRIX TO THE EULER ANGLES

(Continued)

36152 29  IF(IE0K.NE.0) L=1
36153 30  GO TO 30
36154 31  25 CSIGN=-1.C
36155 32  IF(IE0K.NE.0) L=3
36156 33  30  IF JL=1 N=1.3
36157 34  IF(JN.EQ.2) GO TO 70
36158 35  IF(JN.EQ.1) GO TO 50
36159 36  IF(IIE0K.NE.0) GO TO 40
36160 37  JN=1
36161 38  GO TO 45
36162 39  JJ=L
36163 40  IF(ISIGN.GT.0) FOSGNE=1.D
36164 41  FNUM=FSIGN*A(I,J)
36165 42  FDEN=FOSIGN*A(I,J)
36166 43  GO TO 90
36167 44  IF(IIE0K.NE.0) GO TO 55
36168 45  FNSGN=SIGN
36169 46  JJ=K
36170 47  IF(JN.EQ.2) GO TO 50
36171 48  IF(JN.EQ.1) GO TO 30
36172 49  FOSGNE=1.D
36173 50  FNUM=FSIGN*A(J,K)
36174 51  FDEN=FOSIGN*A(I,J)
36175 52  GO TO 90
36176 53  IF(IIE0K.NE.0) GO TO 60
36177 54  FNUM=FSIGN*A(I,K)
36178 55  FDEN=FOSIGN*A(I,J)
36179 56  GO TO 90
36180 57  IF(JN.EQ.2) GO TO 40
36181 58  IF(JN.EQ.1) GO TO 30
36182 59  IF(IIE0K.NE.0) GO TO 30
36183 60  FNUM=SQRT(I10-4(I1,K)**2)
36184 61  FDEN=ATAN2(FNUM,FDEN)
36185 62  GO TO 90
36186 63  FOM=SIGN
36187 64  10 CONTINUE
36188 65  RETURN
36189 66  END

END OF COMPILATION: NO DIAGNOSTICS.

ORIGINAL PAGE IS OF POOR QUALITY
NAME: QMAT

PURPOSE: Generates the transformation matrix from the given quaternion.

INPUT: Q - The quaternion; ARRAY(4).

OUTPUT: A - The 3 x 3 transformation matrix

ALGORITHM REFERENCE: Equation (15) from Section 2.2.
QUATERNION TO THE TRANSFORMATION MATRIX

SUBROUTINE OMAT, ENTRY POINT OMAT

STORAGE USED: CODE (11, CODE (13), DATA (1) INCLUDES BLANK COMMON (1))

EXTERNAL REFERENCES (BLOCK, NAME)

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

END OF COMPILATION: NO DIAGNOSTICS.
NAME: MATQ

PURPOSE: Extracts the positive quaternion from the given transformation matrix.

INPUT: A - The 3 x 3 transformation matrix

OUTPUT: Q - The positive quaternion; ARRAY(4).

ALGORITHM REFERENCE: See Reference 2.
TRANSFORMATION MATRIX TO THE QUATERNION OF POOR QUALITY

FOR SFRS-2/120/77-562421 (C)
NAME: YPRQ

PURPOSE: Generates the quaternion directly from the yaw-pitch-roll Euler angles, i.e., a 3, 2, 1 Euler sequence.

INPUT: YPR - The yaw-pitch-roll Euler angles; ARRAY (3).

OUTPUT: QO - The positive quaternion, ARRAY (4).

ALGORITHM REFERENCE: Appendix A, the quaternion equations for Euler sequence (10), a 3, 2, 1 sequence.

NOTE: This subroutine calls "POSNOR" to take the normal of the positive quaternion.
YAW-PITCH-ROLL EULER ANGLES TO THE QUATERNION

SUBROUTINE YPRO
ENTRY POINT D7C114

STORAGE USED: CODE(1) N00121: DATA(D) N00025: PLANK COMMON(2) CQ

EXTERNAL REFERENCES (BLOCK, NAME)

J003 POSNOR
J004 COS
J005 SIN
J006 NEPR31

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

J000 R 70-11: CP
J000 R 70-11: HY
J000 R 70-12: SY

J0101 1*
J0103 2*
J0104 3*
J0105 4*
J0106 5*
J0107 6*
J0110 9*
J0111 9*
J0113 13*
J0114 11*
J0115 12*
J0116 12*
J0117 14*
J0120 15*
J0121 16*
J0122 17*
J0123 15*

END OF COMPILATION: NO DIAGNOSTICS.
NAME: POSNOR

PURPOSE: To output the positive and normalized quaternion from the given quaternion.

INPUT: Q - The quaternion; ARRAY (4).

OUTPUT: QO - The positive-normalized quaternion; ARRAY (4).

ALGORITHM REFERENCE:
1. If the sign of Q(I) is negative:
   Set QO(I) = -Q(I) for I = 1, 2, 3, 4.
2. Set QO(I) = QO(I)/TEMP
   where TEMP = \sqrt{QO_1^2 + QO_2^2 + QO_3^2 + QO_4^2}
SELECTS THE POSITIVE QUATERNION AND NORMALIZES

FOR S03-02/19/77-06:124:14 (1, J)

SUBROUTINE POSN OR ENTRY POINT 0400:55

STORAGE USED: CODE(11) DOC(M4); DATA(11) OCO:17; BLOK COMMON(2)

EXTERNAL REFERENCES (BLOCK, NAME)

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2004 NLORIS

STORAGE ASSIGNMENT (BLOCK, TYPE, RELATIVE LOCATION, NAME)

3001 20916 1116 3001...02003 1216 3001 30015

END OF COMPILED: NO DIAGNOSTICS.

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