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THE USE OF PARABOLIC VARIATIONS AND THE DIRECT DETERMINATION OF STRESS INTENSITY FACTORS USING THE BIE METHOD

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THE USE OF PARABOLIC VARIATIONS AND THE DIRECT DETERMINATION OF STRESS INTENSITY FACTORS USING THE BIE METHOD

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Summary

Two advances in the numerical techniques of utilizing the BIE method are presented. The boundary unknowns are represented by parabolas over each interval which are integrated in closed form. These integrals are listed for easy use. For problems involving crack tip singularities, these singularities are included in the boundary integrals so that the stress intensity factor becomes just one more unknown in the set of boundary unknowns thus avoiding the uncertainties of plotting and extrapolating techniques. The method is applied to the problems of a notched beam in tension and bending, with excellent results.

1. Introduction

Knowledge of the stress distribution in the neighborhood of a singularity, such as the tip of a crack in a beam loaded in tension or bending, is of fundamental importance in evaluating the resistance to fracture of structural materials. Elastic solutions to various geometries have been obtained by a number of different methods. Among the more effective ones, are the complex variable method, collocation method, and finite element method. However, the first two of these methods are not general enough nor readily adaptable to three-dimensional or elastoplastic problems. And the finite element method requires solutions of large sets of equations and fails to give sufficiently fine resolution in the vicinity of crack tips.

The recently developed boundary integral methods, Mendelson [1], offer an attractive alternative to other methods of analysis. These methods have a number of advantages which may be listed as follows:

(1) They obviate the need for conformal mapping.

(2) Mixed boundary value problems are handled with ease.

(3) Stresses and displacements are obtained directly without need for numerical differentiation.

(4) No special considerations are needed for multiply connected regions.

(5) The internal stresses and/or displacements are calculated only where and when needed.

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(6) The extension to three-dimensional problems is direct.

(7) Nodal points are needed only on the boundary instead of throughout the interior as required by finite element methods.

The last point is probably the most important one. For the finite element method, the whole region must be covered by a grid producing a large number of nodal points and corresponding unknowns. Thus a large number of simultaneous equations must be solved. For the boundary integral methods, nodal points are taken only on the boundary, resulting in a much smaller number of unknowns. On the other hand, the resulting matrices are full, whereas in the finite-element methods, for example, the matrices are usually sparse and can be more or less banded.

The method generally used is to divide the boundary C into a number, e.g., n, intervals and replace the integrals over C by a sum of n integrals over the n boundary intervals. The unknowns are assumed constant over each interval. This technique gives good results with relatively few equations to solve for regions with smooth boundaries. Once we introduce geometric singularities, however, such as cracks or notches, then a very large number of intervals are required in the vicinity of the crack boundary to obtain reasonable accuracy. An improvement can be made by assuming linear variations of the unknowns along the boundary intervals, Riccardella [2]. However, this still results in a relatively large number of intervals for reasonable engineering accuracy.

The present paper presents two major advances in the numerical techniques of utilizing the BIE method. Firstly, the boundary unknowns are represented in terms of parabolas over each interval which are integrated in closed form. These integrals are listed for easy use. Secondly, for problems involving crack tip singularities, these singularities are included in the boundary integrals so that the stress intensity factor becomes just one more unknown in the set of boundary unknowns. When the set of linear algebraic equations is solved for the boundary unknowns, the stress intensity factor is obtained at the same time, thus avoiding the uncertainties of the usual plotting and extrapolating techniques. These innovations result in greater accuracy than was possible heretofore, using substantially fewer boundary intervals and consequently less computer time.

Results are presented for two practical fracture mechanics configurations of the edge-cracked plate in pure bending and in tension, and compared with the standard values quoted in the literature, excellent agreement being obtained.

2. Analysis

Although boundary integral methods can be formulated in many ways, for elasticity and elastoplatic problems the most natural formulation is in terms of the Navier equations of equilibrium. A solution to these equations can be obtained by making use of Kelvin's singular solution of the Navier equations due to a point load and also making use of Betti's reciprocal theorem, Rizzo [3]. We then arrive at a solution that is known as Somigliana's identity, namely:

$$\lambda u_i(P) = \int_C (U_{ij}P_j - T_{ij}u_j)ds, \qquad i = 1,2 \tag{1}$$

where u_j and P_j are the boundary displacements and boundary loads, respectively, and the usual tensor notation is used. The tensors U_{ij}, T_{ij}, are given by

$$\left.\begin{aligned} U_{ij} &= C_1(\delta_{ij}C_2 \ln r - r_{,i}r_{,j}) \\ T_{ij} &= \frac{C_3}{r}\left[\frac{\partial r}{\partial n}(\delta_{ij}C_4 + 2r_{,i}r_{,j}) + C_4(r_{,j}n_i - r_{,i}n_j)\right] \end{aligned}\right\} \tag{2}$$

with

$$\left.\begin{aligned} C_1 &= -\frac{1}{8\pi G(1-\mu)}, \qquad C_2 = 3 - 4\mu \\ C_3 &= -\frac{1}{4\pi(1-\mu)}, \qquad C_4 = 1 - 2\mu \end{aligned}\right\} \tag{3}$$

and r is the distance from the fixed point P to the variable point of integration, s. The above equations are for the case of plane strain. For plane stress one replaces Poisson's ratio μ by $\mu/(1 + \mu)$. The coefficient λ is equal to 1, if P is an interior point and is equal to 1/2, if $P = p$ is a boundary point.

For λ equal to 1/2 Eqs. (1) become a set of 2 or 3 (plane problem or 3-D problem) Fredholm equations for the boundary unknowns. These may be, boundary tractions, displacements or combinations of the two. Thus the first, second or mixed boundary-value problems of elasticity can be solved with equal ease. Once the appropriate unknowns are determined on the boundary, the displacement at any interior point P, can be obtained from Eqs. (1) with λ equal to one. The stresses can be obtained by appropriate differentiation under the integral sign of the tensors U_{ij} and T_{ij}. No numerical differentiation is required.

The problem then resolves itself to the solution of Eqs. (1). The method generally used is to divide the boundary C into a number, e.g., n intervals and to replace the integral over the boundary C by a sum of n integrals (for each equation) over the n boundary intervals. The unknowns u_j and P_j being assumed constant over each interval. Improved results can be obtained by assuming the unknowns to vary linearly over each interval, Riccardella [2]. In the present investigation it was assumed that the unknowns varied parabolically over each boundary interval (see appendix).

When Eqs. (1) are replaced by a sum of integrals over the n intervals and the integrations carried out, a set of $2n$ equations are obtained in $2n$ unknowns which can be written in matrix form as

$$\begin{bmatrix} \alpha & \beta \\ \alpha' & \beta' \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \tag{4}$$

where α, β, α', β' are $n \times n$ matrices, whose elements are described in the appendix, u and v are $n \times 1$ matrices of the x and y boundary displacements, respectively, and P_1, P_2 are $n \times 1$ matrices determined from the boundary loads as described in the appendix.

Equation (4) has been written in a form applicable to the first boundary value problem of elasticity, i.e., the tractions are assumed known over the complete boundary. For the mixed boundary value problem, i.e., the displacements are given over part of the boundary, the roles of those displacements and the corresponding boundary tractions are interchanged in Eq. (4). This merely involves replacing the appropriate columns.

2.2 Stress Intensity Factor

The usual procedure for determining stress intensity factors for specimens with cracks is to calculate the stresses just ahead of the crack tip, or the displacements behind the crack tip, or both, and then make use of appropriate graphical procedures. For reasonable accuracy it is necessary, when using numerical methods, to calculate the stresses at a sufficient number of points lying approximately between 0.01 a and 0.1 a ahead of the crack tip, where a is the crack length. This will usually involve a very fine grid or close interval spacing in the crack tip vicinity.

A method has been developed which avoids both the necessity for a large number of intervals in the crack tip vicinity as well as the need for graphical procedures with their inherent inaccuracy. The technique will be illustrated for the specific problem of an edge-cracked plate loaded either in tension or bending (or both).

Consider the edge-cracked plate shown in Fig. 1. A distribution of end loads producing pure bending is shown, but if desired a tensile load could be applied instead. If the complete boundary CDEFAF'E'D'C is used, we have a first boundary value problem. However, since in this case the problem is symmetric, half the plate can be used if desired, involving the boundary ACDEFA. We then have a mixed boundary value problem, the tractions being unknown on the boundary segment AC.

Considering the case of the half boundary, we exclude two small intervals ε and ε_1 in the vicinity of the crack tip. The rest of boundary is divided into intervals with nodal points taken at their centers as shown. Equations (1) and (4) are applied to this boundary with the ε and ε_1 intervals excluded. For the intervals ε and ε_1, use is made of the known relations defining the mode 1 stress intensity factor, K_I, i.e.,

$$\left.\begin{aligned} P_y &= -\sigma_y = -\frac{K_I}{\sqrt{2\pi r}} \\ u &= 2(1+\mu)(1-2\mu)\left(\frac{r}{2\pi}\right)^{1/2} K_I \\ v &= 0 \end{aligned}\right\} \quad 0 \le r \le \varepsilon \qquad (5)$$

$$\left.\begin{aligned} v &= 4(1-\mu^2)\left(\frac{r}{2\pi}\right)^{1/2} K_I \\ P_y &= u = 0 \end{aligned}\right\} \quad 0 \le r \le \varepsilon_1 \tag{6}$$

where P_y is the traction on the boundary AC, and r is measured along the boundary from the crack tip. Substituting into Eqs. (1) and assuming T_{ij} and U_{ij} to be constant at their midpoint values over the intervals ε_1 and ε, respectively, gives

$$\left.\begin{aligned} \int_0^{\varepsilon_1} T_{i2} v \, dr &\cong a_1 T_{i2} K_I \\ \int_0^{\varepsilon} T_{i1} u \, dr &\cong a_2 T_{i1} K_I \\ \int_0^{\varepsilon} U_{i2} P_y \, dr &\cong -2\left(\frac{\varepsilon}{2\pi}\right)^{1/2} U_{i2} K_I \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} a_1 &= \frac{8}{3}(1-\mu^2)\left(\frac{\varepsilon_1}{2\pi}\right)^{1/2} \varepsilon_1 \\ a_2 &= \frac{4}{3}(1+\mu)(1-2\mu)\left(\frac{\varepsilon}{2\pi}\right)^{1/2} \varepsilon \end{aligned}\right\} \tag{8}$$

T_{i2} are evaluated at $\varepsilon/2$, and T_{i1}, U_{i2} are evaluated at $\varepsilon_1/2$.

Adding Eqs. (7) to the rest of the integrals in (1) gives the integrals over the complete boundary and also adds one more unknown, namely K_I, to the $2n$ unknowns appearing in Eq. (4). Since the number of unknowns is now one greater than the number of equations, an additional equation is required. This can be obtained either from the conditions of continuity of the boundary tractions at $r = \varepsilon$, or the continuity of the crack opening v at $r = \varepsilon_1$, or both. Using the parabolic assumption gives the equations

$$\left.\begin{aligned} 15P_{y1} - 10P_{y2} + 3P_{y3} + \frac{8}{\sqrt{2\pi\varepsilon}} K_I &= 0 \\ -3v_{n-2} + 10v_{n-1} - 15v_n + 32(1-\mu^2)\sqrt{\frac{\varepsilon_1}{2\pi}}\, K_I &= 0 \end{aligned}\right\} \tag{9}$$

It was found that the best results were obtained by either using both equations giving a slightly overdetermined system, or by adding the two equations together to give a single equation.

3. Results and Conclusions

Calculations were carried out for the plate shown in Fig. 1 for both bending and tension. The relative dimensions of the plate were taken as $W = 1$, $a = 0.5$, $L = 1.2$, with $\mu = 0.33$. In all the calculations each of the segments CD, DE, and EF was divided into seven intervals. The numbers of intervals along FA and AC (NFA and NAC) were varied to determine the effect of increasing the number of these intervals.

Some of the results are shown in Figs. 2 to 5. The crack opening displacements and the normal stresses ahead of the crack are plotted and compared with results obtained by Gross [4] using boundary collocation. Excellent agreement is obtained using relatively few nodal points. As a matter of fact taking NAC = NFA = 10, changes the K_I values to 1.76 and 3.46 for the cases of bending and tension, respectively. This is still sufficiently accurate from an engineering viewpoint and makes use of a total of 41 boundary intervals resulting in just 83 unknowns (including K_I).

It can be concluded from the results obtained that the Boundary Integral Equation method utilizing the techniques described herein can be an efficient and accurate tool in fracture mechanics analysis.

4. Appendix

4.1 Parabolic Assumption

It is assumed that the unknowns vary parabolically over each boundary interval. Thus for the jth boundary interval

$$\left.\begin{aligned} u_j &= a_{0j} + a_{1j}s + a_{2j}s^2 \\ v_j &= b_{0j} + b_{1j}s + b_{2j}s^2 \end{aligned}\right\} \qquad (10)$$

where u_j and v_j are the displacements in the x and y directions, respectively, and s is measured from the beginning of the interval under consideration. Similar formulas apply to the boundary tractions.

The coefficients in Eq. (10) are given by

$$\left.\begin{aligned} a_{0j} &= \frac{1}{8}(3u_{j-1} + 6u_j - u_{j+1}) \\ a_{1j} &= \frac{1}{h}(u_j - u_{j-1}) \\ a_{2j} &= \frac{1}{2h^2}(u_{j-1} - 2u_j + u_{j+1}) \end{aligned}\right\} \qquad (11)$$

with similar relations for b_{0j}, b_{1j}, and b_{2j}. h is the interval length which in the above formulas, is assumed to be the same for the jth interval and the adjacent $j - 1$ and $j + 1$ intervals. Similar type formulas can be obtained if desired for unequal intervals.

If the interval under consideration is an end interval such as, for example, the first interval of segment AC and the last interval of the segment CD of Fig. 1, then Eqs. (11) are modified as follows. For a beginning interval such as interval 1,

$$\left.\begin{aligned} a_{0,1} &= \frac{1}{8}\left(15u_1 - 10u_2 + 3u_3\right) \\ a_{1,1} &= -\frac{1}{h}\left(2u_1 - 3u_2 + u_3\right) \\ a_{1,2} &= \frac{1}{2h^2}\left(u_1 - 2u_2 + u_3\right) \end{aligned}\right\} \tag{12}$$

For an end interval such as interval ND

$$\left.\begin{aligned} a_{0,ND} &= -\frac{1}{8}\left(u_{ND-2} - 6u_{ND-1} - 3u_{ND}\right) \\ a_{1,ND} &= \frac{1}{h}\left(u_{ND} - u_{ND-1}\right) \\ a_{2,ND} &= \frac{1}{2h^2}\left(u_{ND-2} - 2u_{ND-1} + u_{ND}\right) \end{aligned}\right\} \tag{13}$$

4.2 Matrix Coefficients

The coefficients entering into the matrices of Eq. (4) are obtained as follows. The boundary is divided into n intervals and Eqs. (1) replaced by sums of integrals over the boundary intervals. Equations (2) and (11) through (13) are substituted in and the integrals evaluated in closed form. Let i designate a fixed nodal point on the boundary and j a boundary interval whose length is Δs_j as shown in Fig. 6. r_j, r_{j+1}, θ_j, θ_{j+1}, and D are defined as shown in the figure. Note that $D = r_j \cos\theta_j = r_{j+1}\cos\theta_{j+1}$. Using this notation we define a set of E matrices whose elements of the ith row are given by

$$\left.\begin{aligned} &E_1 = \theta\Big|_j^{j+1} \qquad E_2 = \frac{1}{2}\sin 2\theta\Big|_j^{j+1} \qquad E_3 = \sin^2\theta\Big|_j^{j+1} \\ &E_5 = \ln r\Big|_j^{j+1} \qquad E_8 = \tan\theta\,\ln r\Big|_j^{j+1} \qquad E_9 = \tan\theta\Big|_j^{j+1} \\ &E_{10} = \frac{1}{2}\tan^2\theta\Big|_j^{j+1} \end{aligned}\right\} \tag{14}$$

Also let ℓ_j and m_j be the direction cosines of the normal $\overline{n}_j$ to the interval, and define

$$z_1 = \ell_j^2 - m_j^2, \qquad z_2 = 2\ell_j m_j$$

Then the following matrix coefficients are computed.

$$A_{ij} = \int_{\Delta s_j} T_{11}\, ds = C_3[(C_4 + 1)E_1 + z_1E_2 - z_2E_3]$$

$$B_{ij} = \int_{\Delta s_j} T_{12}\, ds = C_3(C_4E_5 + z_1E_3 + z_2E_2)$$

$$A'_{ij} = \int_{\Delta s_j} T_{21}\, ds = B_{ij} - 2C_3C_4E_5$$

$$B'_{ij} = \int_{\Delta s_j} T_{22}\, ds = -A_{ij} + 2C_3(C_4 + 1)E_1$$

$$C_{ij} = \int_{\Delta s_j} T_{11}s\, ds = C_3D\Big\{(C_4 + 2m^2)E_5 + z_1E_3 - z_2(E_1 - E_2)$$
$$- \tan\theta_j[(C_4 + 1)E_1 + z_1E_2 - z_2E_3]\Big\}$$

$$E_{ij} = \int_{\Delta s_j} T_{12}s\, ds = C_3D[z_1(E_1 - E_2) + z_2(E_3 - E_5) - C_4E_1 \tag{15}$$
$$- \tan\theta_j(z_1E_3 + z_2E_2)] + C_3C_4[r_{j+1}\sin\theta_{j+1} - r_j\sin\theta_j(1 + E_5)]$$

$$C'_{ij} = \int_{\Delta s_j} T_{21}s\, ds = \text{same as } E_{ij} \text{ with } C_4 \text{ replaced by } -C_4$$

$$E'_{ij} = \int_{\Delta s_j} T_{22}s\, ds = C_3D\Big\{(C_4 + 2\ell^2)E_5 - z_1E_3 + z_2(E_1 - E_2)$$
$$- \tan\theta_j[(C_4 + 1)E_1 - z_1E_2 + z_2E_3]\Big\}$$

$$D_{ij} = \int_{\Delta s_j} T_{11} s^2 \, ds = C_3 D^2 \Big\{ (C_4 + 2m^2)(E_9 - E_1) + z_1(E_1 - E_2) - z_2(2E_5 - E_3)$$

$$- 2 \tan \theta_j [(C_4 + 2m^2)E_5 + z_1 E_3 - z_2(E_1 - E_2)]$$

$$+ \tan^2\theta_j [(C_4 + 2m^2)E_1 + z_1(E_1 + E_2) - z_2 E_3] \Big\}$$

$$F_{ij} = \int_{\Delta s_j} T_{12} s^2 \, ds = C_3 D^2 \Big\{ z_1(2E_5 - E_3) + z_2(2E_1 - E_2 - E_9) - C_4 E_5$$

$$- 2 \tan \theta_j [z_1(E_1 - E_2) + z_2(E_3 - E_5) - C_4 E_1] + \tan^2\theta_j (z_1 E_3 + z_2 E_2) \Big\}$$

$$+ C_3 C_4 \left[\frac{1}{2} r_{j+1}^2 \sin^2\theta_{j+1} - 2 r_j r_{j+1} \sin \theta_j \sin \theta_{j+1} + r_j^2 \sin^2\theta_j \left(\frac{3}{2} + E_5 \right) \right]$$

$$D'_{ij} = \int_{\Delta s_j} T_{21} s^2 \, ds = \text{obtained from } F_{ij} \text{ by replacing } C_4 \text{ by } -C_4$$

$$F'_{ij} = \int_{\Delta s_j} T_{22} s^2 \, ds = \text{obtained from } D_{ij} \text{ by replacing } \ell \text{ by } m \text{ and } m \text{ by } -\ell$$

$$G_{ij} = \int_{\Delta s_j} U_{12} s \, ds = C_1 D^2 (E_9 - E_1 - \tan \theta_j E_5)$$

$$G'_{ij} = \int_{\Delta s_j} U_{22} s \, ds = C_1 D^2 \left[\frac{1}{2} (C_2 - 2)E_5 + (1 - C_2) \tan \theta_j E_1 \right]$$

$$+ C_1 C_2 \Big\{ \frac{1}{2} r_{j+1}^2 \sin^2\theta_{j+1} \left(\ln r_{j+1} - \frac{1}{2} \right) - \frac{1}{2} r_j^2 \sin^2\theta_j \left(\ln r_j - \frac{1}{2} \right)$$

$$+ r_j \sin \theta_j [r_{j+1} \sin \theta_{j+1} (1 - \ln r_{j+1}) - r_j \sin \theta_j (1 - \ln r_j)] \Big\}$$

$$H_{ij} = \int_{\Delta s_j} U_{12} s^2 \, ds = -C_1 D^3 [E_5 - E_{10} + 2 \tan \theta_j (E_9 - E_1) - \tan^2\theta_j E_5]$$

$$H'_{ij} = \int_{\Delta s_j} U_{22}s^2\, ds = C_1D^3\left[\frac{1}{3}(C_2 - 3)(E_9 - E_1) - (C_2 - 2)\tan\theta_j E_5\right.$$

$$\left. + (C_2 - D\tan^2\theta_j E_1\right] + C_1C_2\left[\frac{1}{3} r_{j+1}^3 \sin^3\theta_{j+1}\left(\ln r_{j+1} - \frac{1}{3}\right)\right.$$

$$- \frac{1}{3} r_j^3 \sin^3\theta_j\left(\ln r_j - \frac{1}{3}\right) - r_j \sin\theta_j\, r_{j+1}^2 \sin^2\theta_{j+1}\left(\ln r_{j+1} - \frac{1}{2}\right)$$

$$\left. + r_j^2 \sin^2\theta_j\, r_{j+1} \sin\theta_{j+1}(\ln r_{j+1} - 1) + \frac{1}{2} r_j^3 \sin^3\theta_j\right]$$

$$K_{ij} = \int_{\Delta s_j} U_{12}\, ds = DC_1E_5$$

$$K'_{ij} = \int_{\Delta s_j} U_{22}\, ds = DC_1(C_2 - 1)E_1$$

$$+ C_1C_2\left[r_{j+1}\sin\theta_{j+1}(\ln r_{j+1} - 1) - r_j \sin\theta_j(\ln r_j - 1)\right]$$

Note that for the problems under consideration, the last six integrals involve intervals on side AC for which $\ell = 0$ and $m = -1$.

The vectors P_1 and P_2 appearing in Eq. (4) are given by

$$\left.\begin{aligned} P_1 &= \int_D^E U_{12}P_y\, ds \\ P_2 &= \int_D^E U_{22}P_y\, ds \end{aligned}\right\} \qquad (16)$$

where for the bending case: $P_y = -2\sigma_o x = -2x$

and for the tensile case: $P_y = \sigma_o = 1$, σ_o being taken as 1.

Carrying out the integrations gives

For bending:

$$P_1 = 2C_1(y_D - y_i)\left[1 + X_i \ln\frac{r_D}{r_E} + (y_D - y_i)\sin^{-1}\frac{y_i - y_D}{r_D r_E}\right] \qquad (17a)$$

$$P_2 = 2C_1\Big\{C_2x_i[(x_E - x_i)\ln r_E - (x_D - x_i)\ln r_D - (x_E - x_D) + (y_D - y_i)T]$$

$$+ \frac{1}{2}C_2\left[r_E^2 \ln r_E - r_D^2 \ln r_D + \frac{1}{2}(x_D - x_E)^2 - \frac{1}{2}(x_E - x_i)^2\right]$$

$$- (y_D - y_i)^2 \ln \frac{r_E}{r_D} - x_i(y_D - y_i)T\Big\} \qquad (17b)$$

where: $T = \sin^{-1} \frac{y_i - y_D}{r_D r_E}$

For tension:

$$\left.\begin{aligned} P_1 &= (y_D - y_i)C_1 \ln \frac{r_E}{r_D} \\ P_2 &= -C_1C_2[1 - (x_i - x_E)\ln r_E + (x_i - x_D)\ln r_D - (y_D - y_i)T] - C_1(y_D - y_i)T \end{aligned}\right\} \qquad (18)$$

The matrices α, α', β, β' of Eq. (4) are now assembled from the matrices A through K' using the parabolic coefficients of Eqs. (11) to (13). As an example, consider the line AC having NC nodal points with interval length HAC. Then for $3 < j < NC - 3$, the central formulas are used and we write

$$\alpha_{ij} = -\frac{1}{8}(A_{i,j-1} - 3A_{i,j+1} - 6A_{ij}) + \frac{1}{HAC}(C_{ij} - C_{i,j+1})$$

$$+ \frac{1}{2(HAC)^2}(D_{i,j-1} + D_{i,j+1} - 2D_{ij})$$

For the ends we use beginning and end formulas. Thus

$$\alpha_{i1} = \frac{1}{8}(15A_{i1} + 3A_{i2}) - \frac{1}{HAC}(C_{i2} + 2C_{i1}) + \frac{1}{2(HAC)^2}(D_{i1} + D_{i2})$$

$$\alpha_{i2} = \frac{1}{8}(-10A_{i1} + 6A_{i2} + 3A_{i3}) + \frac{1}{HAC}(3C_{i1} + C_{i2} - C_{i3})$$

$$+ \frac{1}{2(HAC)^2}(D_{i3} - 2D_{i1} - 2D_{i2})$$

$$\alpha_{i3} = \frac{1}{8}(6A_{i3} + 3A_{i1} + 3A_{i4} - A_{i2}) + \frac{1}{HAC}(C_{i3} - C_{i1} - C_{i4})$$

$$+ \frac{1}{2(HAC)^2}(D_{i1} + D_{i2} + D_{i4} - 2D_{i3})$$

At the end of the line we have

$$\alpha_{i,NC} = \frac{1}{8}(-A_{i,NC-1} + 3A_{i,NC}) + \frac{1}{HAC} C_{i,NC} + \frac{1}{2(HAC)^2}(D_{i,NC} + D_{i,NC-1})$$

$$\alpha_{i,NC-1} = \frac{1}{8}(-A_{i,NC-2} + 6A_{i,NC-1} + 6A_{i,NC}) + \frac{1}{HAC}(C_{i,NC-1} - C_{i,NC}) + \frac{1}{2(HAC)^2}(D_{i,NC-2} - 2D_{i,NC-1} - 2D_{i,NC})$$

$$\alpha_{i,NC-2} = \frac{1}{8}(-A_{i,NC-3} - A_{i,NC} + 3A_{i,NC-1} + 6A_{i,NC-2}) + \frac{1}{HAC}(C_{i,NC-2} - C_{i,NC-1}) + \frac{1}{2(HAC)^2}(D_{i,NC-3} + D_{i,NC} + D_{i,NC-1} - 2D_{i,NC-2})$$

Exactly similar relations hold for α', β, and β', replacing A, C, and D by (A', C', D'), (B, E, F) and (B', E', F'), respectively.

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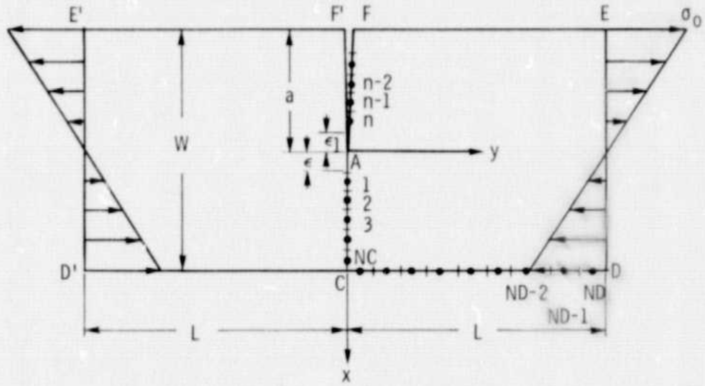

Figure 1. - Edge-cracked plate.

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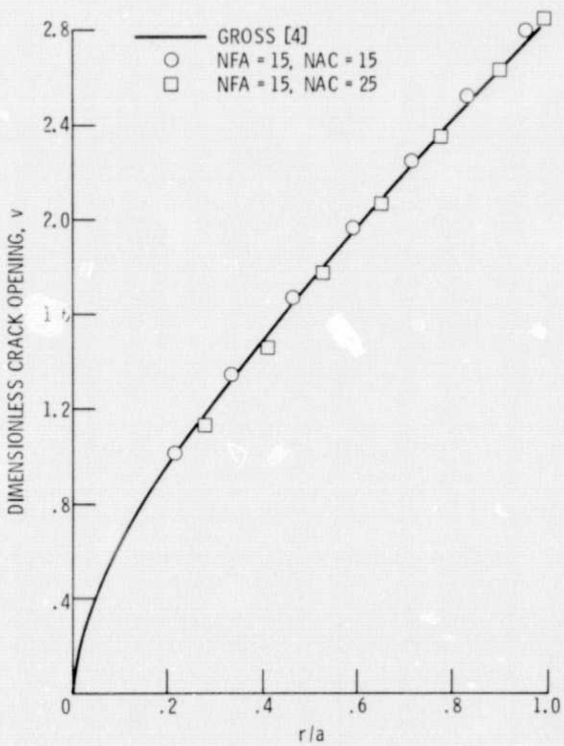

Figure 2. - Crack opening in pure bending. Plane strain.

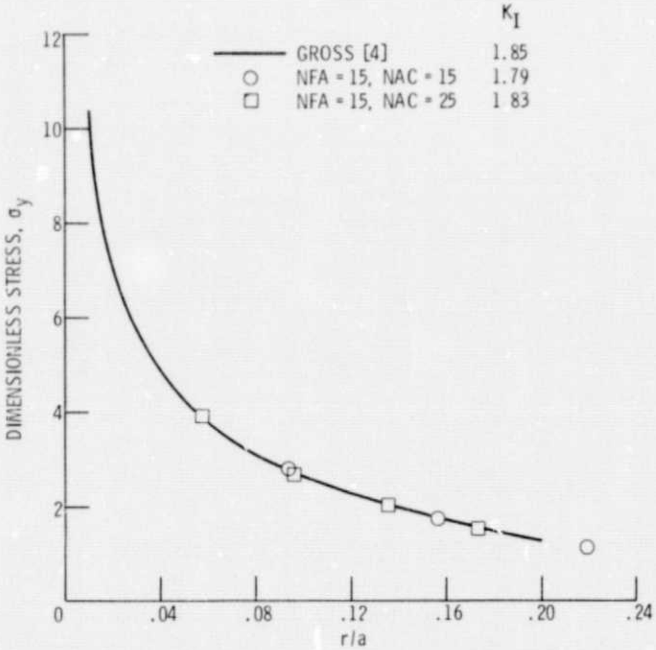

Figure 3. - Stress ahead of crack tip in pure bending.

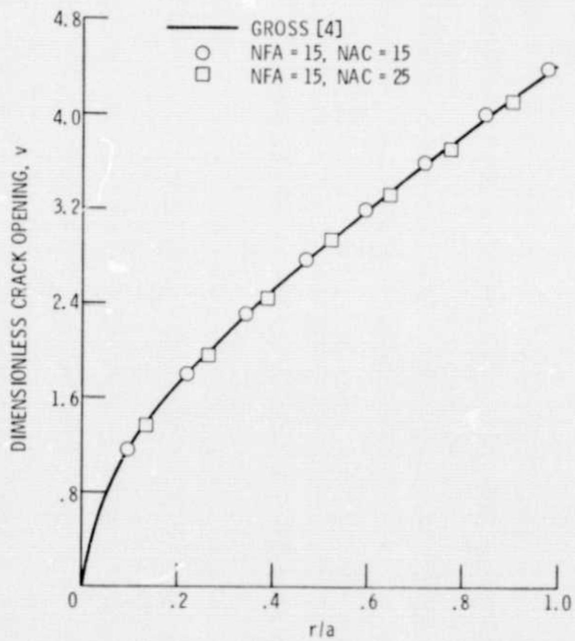

Figure 4. - Crack opening in tension. Plane strain

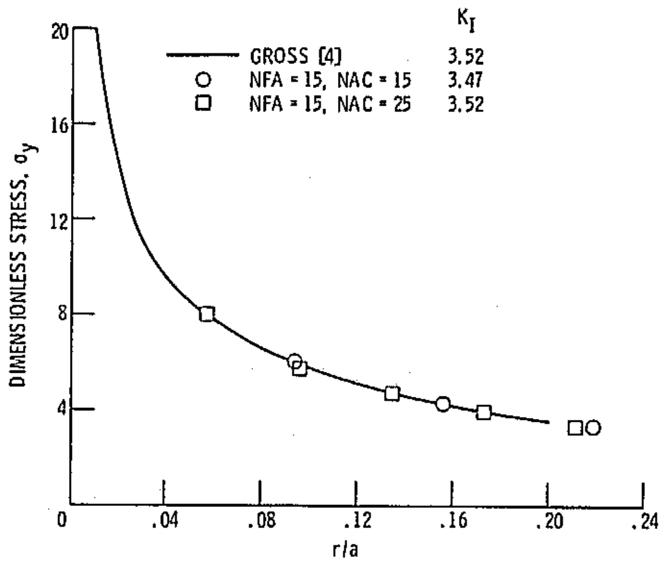

Figure 5. - Stress ahead of crack tip in tension.

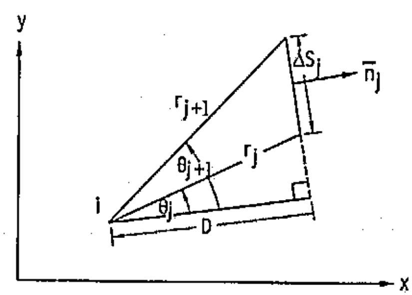

Figure 6. - Notation used in evaluating the matrix coefficients.

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