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Qualitative Differential Game. with Two Targets

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Abstract

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Abstract

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1. INTRODUCTION

Differential games of kind, especially games of pursuit and evasion, were introduced by Isaacs, Ref. 1. Much of the subsequent literature on this subject deals with extensions and generalizations of problems discussed by Isaacs; a particular favorite is the "homicidal chauffeur game", Refs. 2,3. These games concern situations in which two players exert control over a system; one player (pursuer) wishes to steer the state of the system to a given set, while the other player (evader) desires to keep it out of that set. Thus, such games involve a single set which is the pursuer's "target" and the evader's "anti-target".

Differential games of kind (qualitative games) in which each of two players has his own target toward which he wishes to steer the system's state were considered by Blaquière et al., Ref. 4; such multi-target games encompass games with a single target, of course. A general discussion of two-target games may be found in Ref. 4, where there are also presented conditions sufficient to assure one player that a particular subset of the state space is one from which his opponent cannot guarantee himself a win (that is, termination on his target).

In this paper we consider qualitative two-target differential games. For such games we give conditions sufficient to assure one player that a particular subset of the state space is one from which he is guaranteed a win; that is, if he utilizes a strategy for which certain conditions are met and play begins in an appropriate subset of the state space, then he is assured of steering the state to his own target before his opponent can steer it to his.

2. PROBLEM STATEMENT

In order to admit a class of strategies sufficiently general to be of interest in many applications we allow the system under discussion to be a generalized dynamical system, Refs. 5, 6.

We suppose that n real numbers, x_i, $i = 1, 2, \ldots, n$, fully describe the system at a given instant of time, $t \in (-\infty, \infty)$. Thus, the system is described by a vector $x = (x_1, x_2, \ldots, x_n)^T \in D$, loosely called the "state" of the system, where D is a domain or the closure of a domain in R^n. The state evolves, that is, changes with time, as a trajectory of a generalized dynamical system

$$\dot{x}(t) \in F(x(t), t) \tag{1}$$

where $F(\cdot)$ is an appropriately defined vector-valued function from $D \times R^1$ into all nonempty subsets of R^n.

Since the system is under the control of two agents (players), we consider two prescribed sets, P^i, $i = 1, 2$, of set-valued functions of x and t. Let U^i, $i = 1, 2$, be given subsets of R^{d_i}, the players' control spaces.† The elements of P^i are the i-th player's admissible feedback controls (strategies) $p^i(\cdot) : D \times R^1 \to$ all nonempty subsets of U^i.

Next let there be given a function $f(\cdot) : D \times U^1 \times U^2 \to R^n$, and for given $p^i(\cdot) \in P^i$, $i = 1, 2$, define $F(\cdot)$ by

$$F(x, t) \triangleq \{z \in D \mid z = f(x, u^1, u^2), u^i \in p^i(x, t)\}$$
$$= f(x, p^1(x, t), p^2(x, t)).$$

For given $(x_0, t_0) \in D \times R^1$, a solution of (1) is a function

†One can allow for state and time-dependent constraint sets $U^i = U^i(x, t)$ by prescribing set-valued functions $U^i(\cdot)$.

$x(\cdot) : [t_0, t_f) \to D$, $x(t_0) = x_0$, that is absolutely continuous on all compact subintervals of $[t_0, t_f)$ and satisfies

$$\dot{x}(t) \in f(x(t), p^1(x(t), t), p^2(x(t), t)) \quad (3)$$

a.e. $[t_0, t_f)$.

To assure existence of solutions of (3) we make the following assumption.

Assumption 1. The sets of admissible strategies, P^i , $i = 1, 2$, are such that for all $p^i(\cdot) \in P^i$, $i = 1, 2$, and all $(x_0, t_0) \in D \times R^1$, there is at least one solution of (3).

Of course, the choice of P^i satisfying Assumption 1 depends on function $f(\cdot)$. Conditions assuring existence of a solution at (x_0, t_0) are of the following kind (e.g., see Refs. 5,6,7):

i) $F(x_0, t_0)$ is compact and convex, and

ii) $F(\cdot)$ is upper semicontinuous on a compact set containing (x_0, t_0) .

Now let T_1 and T_2 be two prescribed closed sets contained in D and such that $D \nsubseteq T_1 \cup T_2$. These are the "targets" of players 1 and 2, respectively.

Before we can define a "winning strategy" and a "winning set" for a player we need to introduce the concept of "play".

Definition 1. A play is a quadruple $\{(x_0, t_0) , p^1(\cdot), p^2(\cdot) , x(\cdot)\}$ such that

1.[†] $(x_0, t_0) \in [D \setminus (T_1 \cup T_2)] \times R^1$,

2. $p^i(\cdot) \in P^i$, $i = 1, 2$,

[†]Given two sets, A and B , $A \setminus B \triangleq \{a \in A \mid a \notin B\}$.

3. $x(\cdot) : [t_0, t_f] \to D$, or $x(\cdot) : [t_0, t_f) \to D$ if $x(t_f)$ is not defined, with $x(t_0) = x_0$, is a solution of (3) generated by $p^i(\cdot)$, $i = 1, 2$,

4. $x(t) \notin T_1 \cup T_2$ for $t \in [t_0, t_f)$,

5. either i) $x(t_f) \in T_1 \cup T_2$ for $t_f < \infty$, or

   ii) $t_f = \infty$, or

   iii) t_f is a finite escape time for $x(\cdot)$;

   that is, $x(t) \to x \in \partial D$ or $\|x(t)\| \to \infty$ as $t \to t_f$.

Now we can define the concepts of winning strategy at a point and then of a winning set.

Definition 2. A strategy $\hat{p}^1(\cdot) \in P^1$ is winning for player 1 at $(x_0, t_0) \in [D \setminus (T_1 \cup T_2)] \times R^1$ iff the set of all plays $\{(x_0, t_0) , \hat{p}^1(\cdot) , p^2(\cdot) , x(\cdot)\}$ for all $p^2(\cdot) \in P^2$ is nonempty and contains no members satisfying 5i) with $x(t_f) \in T_2$, or 5ii) or 5iii) of Definition 1. A completely analogous definition, with 1 and 2 interchanged, holds for a winning strategy $\hat{p}^2(\cdot) \in P^2$ at (x_0, t_0) .

Simply stated, a winning strategy for a player at (x_0, t_0) guarantees termination on his, and only his, target no matter what the strategy of the other player.

Definition 3. A set $W \subset D \setminus (T_1 \cup T_2)$ is 1-winning (or 2-winning) iff player 1 (or 2) has a strategy $\tilde{p}^1(\cdot)$ (or $\tilde{p}^2(\cdot)$) that is winning for him at all $(x_0, t_0) \in W \times R^1$.

An essential part of the solution of a game is the mapping of D into its 1-winning and 2-winning subsets, along with the characterization of the associated winning strategies, $\tilde{p}^1(\cdot)$ and $\tilde{p}^2(\cdot)$. In the terminology of Ref. 4, the union of all 1-winning (or 2-winning) sets together with T_1 (or T_2) is S_E (or S_P).

In the next section we consider the problem of finding winning strategies by giving a theorem of conditions sufficient for a set to be 1-winning (2-winning).

3. WINNING SETS

The game as defined in Section 2 is completely symmetrical with respect to players 1 and 2. In this section we give conditions sufficient for a set to be 1-winning. Of course, the theorem with 1 and 2 interchanged holds equally for a 2-winning set.

The conditions in the theorem below depend on an a priori chosen quadruple $\{V_1(\cdot), V_2(\cdot), k_1, k_2\}$ where $V_i(\cdot) : D \to R^1$, $i = 1, 2,$ are C^1 functions for which there exist constants C_1 and C_2 such that

i) $T_1 \supset \Delta_1 \triangleq \{x \in D \mid V_1(x) \leq C_1\}$

ii) $T_2 \subset \Delta_2 \triangleq \{x \in D \mid V_2(x) \leq C_2\}$

and

iii) $k_1 > 0$, k_2 are scalar constants.

Given a quadruple $\{V_1(\cdot), V_2(\cdot), k_1, k_2\}$, let

$$W(V_1, V_2, k_1, k_2) \triangleq \{x \in D \setminus (T_1 \cup \Delta_2) \mid \frac{k_1}{V_1(x) - C_1} > \frac{k_2}{V_2(x) - C_2}\} . \qquad (4)$$

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Before stating a theorem, we introduce another assumption.

Assumption 2. If $D = R^n$, given $p^i(\cdot) \in P^i$, $i = 1, 2$, and a quadruple $\{V_1(\cdot), V_2(\cdot), k_1, k_2\}$, no solution of (3) with $(x_0, t_0) \in W(V_1, V_2, k_1, k_2) \times R^1$ has a finite escape time.

This assumption is satisfied if

i) equation (3) satisfies a linear growth condition, or

ii) function $V_1(\cdot)$ in Theorem 1 below is radially unbounded; that is, $V_1(x) \to \infty$ as $\|x\| \to \infty$.

Now we are ready to state the main theorem.

THEOREM 1. If there exist a quadruple $\{V_1(\cdot), V_2(\cdot), k_1, k_2\}$, a strategy $\tilde{p}^1(\cdot) \in P^1$ and a function $\bar{p}^1(\cdot) : D \to$ all nonempty subsets of U^1, such that

i) $\tilde{p}^1(x, t) = \bar{p}^1(x) \quad \forall (x, t) \in D \times R^1$,

ii) $\forall\, x \in W(V_1, V_2, k_1, k_2)$ and $\forall\, u^1 \in \bar{p}^1(x)$,

$$\sup_{u^2 \in U^2} \nabla V_1(x) f(x, u^1, u^2) \leqslant -k_1 , \quad \inf_{u^2 \in U^2} \nabla V_2(x) f(x, u^1, u^2) \geqslant -k_2$$

iii) Assumption 1 is met,

iv) D is an invariant set of (3) with $p^1(\cdot) = \tilde{p}^1(\cdot)$ and all $p^2(\cdot) \in P^2$, or else $D = R^n$ and Assumption 2 is satisfied,

then $W(V_1, V_2, k_1, k_2)$ is 1-winning.

Proof. First we show that if a trajectory of (3), with $p^1(\cdot) = \tilde{p}^1(\cdot)$, any $p^2(\cdot) \in P^2$ and $(x_0, t_0) \in W(V_1, V_2, k_1, k_2) \times R^1$, remains in $W(V_1, V_2, k_1, k_2)$ for $t < t_f$, then it terminates on $T_1 \cup T_2$; that is, 5ii) of Definition 1 cannot occur. In view of condition iv), no such trajectory has a finite escape time and so may be extended over any interval $[t_0, t_f)$, including $t_f = \infty$ in case of nontermination. Consider a trajectory corresponding to solution $x(\cdot) : [t_0, t_f] \to D$, or $x(\cdot) : [t_0, t_f) \to D$ if $t_f = \infty$, with $x(t_0) = x_0 \in W(V_1, V_2, k_1, k_2)$. In view of ii) and the supposition that $x(t) \in W(V_1, V_2, k_1, k_2)$ for $t \in [t_0, t_f)$

$$\nabla V_1(x(t))\, \dot{x}(t) \leqslant - k_1$$

$$\nabla V_2(x(t))\, \dot{x}(t) \geqslant - k_2$$

a.e. $[t_0, t_f)$. Upon integration, e.g., see Ref. 7, we obtain for $t \in [t_0, t_f)$

$$V_1(x_0) \geqslant V_1(x(t)) + k_1(t - t_0) \tag{5}$$

$$V_2(x_0) \leqslant V_2(x(t)) + k_2(t - t_0) \quad . \tag{6}$$

Let

$$\tilde{t} \triangleq t_0 + \frac{V_1(x_0) - C_1}{k_1} \tag{7}$$

and suppose that $\tilde{t} < t_f$. Then it follows from (5) and (7) that

$$V_1(x(\tilde{t})) \leqslant C_1 \ .$$

By the definition of $V_1(\cdot)$, namely, $T_1 \supset \Delta_1$, this implies that $x(\tilde{t}) \in T_1$, contradicting the supposition of termination at $t = t_f$ or else non-termination, $t_f = \infty$. Thus, $\tilde{t} \geqslant t_f$ and termination must occur on $T_1 \cup T_2$ at $t = t_f < \infty$.

Next we demonstrate that termination takes place on $T_1 \setminus T_2$. Two cases must be considered, $k_2 \leqslant 0$ and $k_2 > 0$.

For $k_2 \leqslant 0$, since

$$V_1(x) - C_1 > 0 \quad \text{for} \quad x \in D \setminus \mathcal{T}_1 \tag{8}$$

$$V_2(x) - C_2 > 0 \quad \text{for} \quad x \in D \setminus \Delta_2 \tag{9}$$

it follows from (4) that

$$W(V_1, V_2, k_1, k_2) = D \setminus (\mathcal{T}_1 \cup \Delta_2) \ . \tag{10}$$

Hence, by condition iv), $x(t) \in W(V_1, V_2, k_1, k_2)$ for $t \in [t_0, t_f)$.

In view of (9), for each $x_0 \in W(V_1, V_2, k_1, k_2)$ there exists an $\varepsilon > 0$ such that

$$V_2(x_0) = C_2 + \varepsilon \ . \tag{11}$$

Since $x(t) \in W(V_1, V_2, k_1, k_2)$ for all $t \in [t_0, t_f)$, it follows from (6) that

$$V_2(x_0) \leq V_2(x(t)) \quad \forall\ t \in [t_0, t_f) \tag{12}$$

whence, by (11),

$$V_2(x(t)) \geq C_2 + \varepsilon \quad \forall\ t \in [t_0, t_f)$$

which implies that $x(t_f) \notin \Delta_2 \supset \mathcal{T}_2$. But $x(t_f) \in \mathcal{T}_1 \cup \mathcal{T}_2$ so that $x(t_f) \in \mathcal{T}_1 \setminus \mathcal{T}_2$.

For $k_2 > 0$, suppose there is a $t < t_f$ for which $x(t) \notin W(V_1, V_2, k_1, k_2)$. Then, by continuity of $V_1(\cdot)$, $V_2(\cdot)$ and $x(\cdot)$, there exists a $\bar{t} < t_f$ such that

$$x(t) \in W(V_1, V_2, k_1, k_2) \quad \forall\ t \in [t_0, \bar{t})$$

and

$$\frac{k_1}{V_1(x(\bar{t}))-C_1} = \frac{k_2}{V_2(x(\bar{t}))-C_2} \ . \tag{13}$$

Now, (4) and (5) hold for $x(t) \in W(V_1, V_2, k_1, k_2)$ and so, by continuity of $V_1(\cdot)$ and $V_2(\cdot)$ and by positivity of k_1 and k_2,

$$\frac{V_1(x_0)-C_1}{k_1} \geq \frac{V_1(x(\bar{t}))-C_1}{k_1} + (\bar{t} - t_0)$$
$$\frac{V_2(x_0)-C_2}{k_2} \leq \frac{V_2(x(\bar{t}))-C_2}{k_2} + (\bar{t} - t_0) \; . \qquad (14)$$

Since $x_0 \in W(V_1, V_2, k_1, k_2)$, it follows from (4) and (14) that

$$\frac{k_1}{V_1(x(\bar{t}))-C_1} > \frac{k_2}{V_2(x(\bar{t}))-C_2} \; . \qquad (15)$$

This contradicts (13), and so

$x(t) \in W(V_1, V_2, k_1, k_2) \quad \forall \; t \in [t_0, t_f)$.

Thus, as shown earlier, $x(t_f) \in T_1 \cup T_2$.

Now suppose there is a $\bar{t} \leq t_f$ such that $x(\bar{t}) \in \Delta_2 \setminus T_1 \supset T_2 \setminus T_1$ and $x(t) \notin T_1 \cup \Delta_2 \supset T_1 \cup T_2$ for all $t \in [t_0, \bar{t})$. Since $x(\bar{t}) \in \Delta_2 \setminus T_1$,

$$V_1(x(\bar{t})) > C_1 \qquad (16)$$

and

$$V_2(x(\bar{t})) \leq C_2 \; . \qquad (17)$$

Substitution of (16) and (17) in (14) yields

$$\frac{k_1}{V_1(x_0)-C_1} < \frac{1}{\bar{t}-t_0}$$

and

$$\frac{k_2}{V_2(x_0)-C_2} \geqslant \frac{1}{\bar{t}-t_0}$$

so that

$$\frac{k_1}{V_1(x_0)-C_1} < \frac{k_2}{V_2(x_0)-C_2}$$

which contradicts $x_0 \in W(V_1, V_2, k_1, k_2)$. Hence, $x(t) \notin \Delta_2 \setminus T_1 \supset T_2 \setminus T_1$ for $t \leqslant t_f$ so that $x(t_f) \in T_1 \setminus \Delta_2 \subset T_1 \setminus T_2$.

Since these conclusions hold for all $x_0 \in W(V_1, V_2, k_1, k_2)$, that set is a 1-winning set.

<u>Remarks</u>

1. Since a winning strategy $\tilde{p}^1(\cdot)$ with $(x_0, t_0) \in W(V_1, V_2, k_1, k_2) \times R^1$ guarantees that $x(t) \in W(V_1, V_2, k_1, k_2)$ for all $t \in [t_0, t_f)$, only the restriction of the strategy to the 1-winning set need be determined.
2. There may exist 1-winning strategies at $(x_0, t_0) \notin W(V_1, V_2, k_1, k_2) \times R^1$. In general, however, such strategies will not be 1-winning at other initial points.
3. If $k_2 \leqslant 0$, region Δ_2 is an avoidance set in the sense of Ref. 8.
4. As $k_1 \to 0$, $\tilde{t}$ given in (7) tends to ∞ . Thus, if $k_1 = 0$, termination cannot be guaranteed; in particular $W(V_1, V_2, k_1, k_2)$ need not be 1-winning. However, the inequality in (4) then requires $k_2 < 0$ so that, by the second of conditions ii) of the Theorem, $W(V_1, V_2, k_1, k_2)$ cannot be 2-winning.
5. If $T_2 = \phi$, let $V_2(x) = \text{constant} \neq C_2$ for all $x \in D \setminus T_1$ and let $k_2 = 0$. The inequality in (4) is then satisfied trivially.

The set $W(V_1, V_2, k_1, k_2)$ being 1-winning is then equivalent to target T_1 being capturable in finite time. Theorem 1 is thus related to the target capture theorem of Ref. 9.

6. Let

$$S(\ell) \triangleq \{x \in W(V_1, V_2, k_1, k_2) \mid V_1(x) \leq \ell\} \supset T_1 \cup T_2$$

and suppose that it is required to accomplish termination on $T_1 \setminus T_2$ for all $x_0 \in S(\ell)$ in a time interval not exceeding $t^* - t_0$. This can be assured by the conditions of Theorem 1 with

$$k_1 \geq \frac{\ell - C_1}{t^* - t_0} \quad . \tag{18}$$

For, as shown in the proof of Theorem 1, for given $x_0 \in W(V_1, V_2, k_1, k_2)$

$$\tilde{t} = t_0 + \frac{V_1(x_0) - C_1}{k_1} \geq t_f$$

and $x(t_f) \in T_1 \setminus T_2$. Thus we impose $\sup_{x_0 \in S(\ell)} \tilde{t} \leq t^*$, whence

$$t_f - t_0 \leq \frac{\ell - C_1}{k_1} \leq t^* - t_0$$

giving the condition (18).

7. If more than one set, say $T_2, T_3, \ldots, T_r$, is to be avoided before terminating on $T_1 \setminus (T_2 \cup T_3 \cup \ldots \cup T_r)$, the conditions of Theorem 1 are augmented by introducing for $i \in \{2, 3, \ldots, r\}$

$$\Delta_i \triangleq \{x \in D \mid V_i(x) \leq C_i\} \supset T_i \ ,$$

replacing $W(V_1, V_2, k_1, k_2)$ by

$$W(V_1, V_2, \ldots, V_r, k_1, k_2, \ldots, k_r) \triangleq \{x \in D \setminus (T_1 \cup \Delta_2 \cup \ldots \cup \Delta_r) \mid \frac{k_1}{V_1(x) - C_1} > \frac{k_i}{V_i(x) - C_i} \ , \ i = 2, \ldots, r\}$$

and requiring $\forall\, u^1 \in \bar{p}^1(x)$

$$\inf_{u^2 \in U^2} \nabla V_i(x)\; f(x, u^1, u^2) \geq -k_i$$

4. EXAMPLE

Here we illustrate the use of Theorem 1 by means of a very simple example, a pursuit-evasion game between inertialess objects P and E with constant speeds v_P and v_E, respectively. The equations of motion (see Figure 1) are

$$\begin{aligned} \dot{R} &= v_E \cos(\alpha + \theta_E) \\ \dot{\Theta} &= \frac{v_E}{R} \sin(\alpha + \theta_E) \\ \dot{r} &= v_P \cos\theta_p - v_E \cos\theta_E \\ \dot{\theta} &= \frac{1}{r}(v_P \sin\theta_p - v_E \sin\theta_E) \end{aligned} \qquad (19)$$

The targets of E and P, respectively, are

$$\begin{aligned} T_1 &= \{(R, \Theta, r, \theta) \in R^4 \mid R \leq \rho_E = \text{constant} > 0\} \\ T_2 &= \{(R, \Theta, r, \theta) \in R^4 \mid r \leq \rho_P = \text{constant} > 0\}\,. \end{aligned} \qquad (20)$$

Thus, the evader, E, wishes to choose the values of θ_E so that he reaches $T_1 (R = \rho_E)$ before he is intercepted $(r = \rho_P)$, no matter how the pursuer, P, chooses the values of θ_P.

To apply Theorem 1, let

$$V_1 = R \quad , \quad V_2 \doteq r$$

so that $\Delta_i = T_i$, $i = 1, 2$, and

$$\nabla V_1 = (1, 0, 0, 0)$$

$$\nabla V_2 = (0, 0, 1, 0) \quad .$$

Conditions ii) of Theorem 1 become

$$v_E \cos(\alpha + \theta_E) \leqslant -k_1 < 0 \tag{21}$$

$$-v_P - v_e \cos\theta_E \geqslant -k_2 \tag{22}$$

and $W(V_1, V_2, k_1, k_2)$ is defined by

$$r - \rho_P > \frac{k_2}{k_1}(R - \rho_E) \quad . \tag{23}$$

Condition (21) implies

$$-1 \leqslant \cos(\alpha + \theta_E) \leqslant -\frac{k_1}{v_E} < 0 \tag{24}$$

so that

$$k_1 = \frac{v_E}{1+\delta} \quad \text{for} \quad \delta \in [0, \infty) \ .$$

For given k_1 , region W is maximized by choosing the smallest k_2 such that (22) is met for all possible $\alpha = \Theta - \theta$. This results in

$$k_2 = v_P + \frac{v_E}{1+\delta}$$

whence

$$\frac{k_2}{k_1} = 1 + (1+\delta)\frac{v_P}{v_E} \quad .$$

Hence, the largest W results from $\delta = 0$, namely,

$$r - \rho_P > \frac{v_E+v_P}{v_P}(R - \rho_E) \quad . \tag{25}$$

The corresponding escape strategy, according to (24), is given by $\cos(\alpha + \theta_E) = -1$; that is, E moves radially inward and is assured termination without interception by P provided the initial conditions satisfy (25).

Strategy $\theta_E = \pi - (\theta - \Theta)$ is continuous on $D = R^4$. Hence, for sufficiently well-behaved θ_P , system (19) possesses a solution at every initial state. Furthermore, (19) satisfies a linear growth condition. Thus Assumptions 1 and 2 are met and W given by (25) is indeed 1-winning, that is, winning for the evader, E .

Since v_P and v_E are positive, (25) implies that

$r - \rho_P > R - \rho_E$ no matter how much faster the evader is than the pursuer. Thus, $W(V_1, V_2, k_1, k_2)$ does not include initial configurations for which P lies between E and his target, T_1 . This restriction is not surprising since Theorem 1 relates to all initial positions for which the evader's strategy is winning (see Remark 2).

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List of Symbols

x	lower case "eggs"
×	"multiplication" symbol
0	zero
$\mathcal{T}$	script upper case "tee"
∈	"belongs to" symbol
ε	lower case epsilon
ρ	lower case rho
θ	lower case theta
Θ	upper case theta
α	lower case alpha
δ	lower case delta
Δ	upper case delta
π	lower case pi
ℓ	script lower case "ell"
∞	"infinity" symbol
ϕ	"empty" symbol

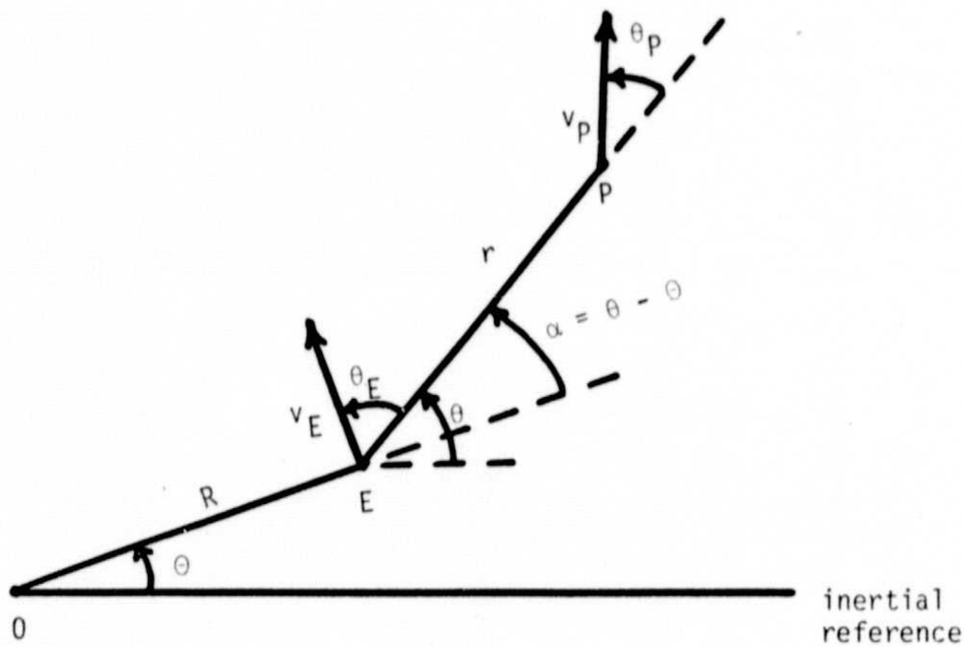

Figure 1, Pursuit-Evasion