NASA TM-78,457

(NASA-TM-78457) NONLINEAR SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEMS WITH SINGULAR ARCS (NASA) 10 p HC A02/MF A01 CSCL 12B

N78-15751

Unclas
G3/63 55570

Nonlinear Singularly Perturbed Optimal Control Problems with Singular Arcs

Mark D. Ardema

December 1977

NASA

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field California 94035

NONLINEAR SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEMS WITH SINGULAR ARCS

Mark D. Ardema

Research Scientist, NASA Ames Research Center

Moffett Field, CA 94035

Abstract. A third order, nonlinear, singularly perturbed optimal control problem is considered under assumptions which assure that the full problem is singular and the reduced problem is nonsingular. The separation between the singular arc of the full problem and the optimal control law of the reduced one, both of which are hypersurfaces in state space, is of the same order as the small parameter of the problem. Boundary layer solutions are constructed which are stable and reach the outer solution in a finite time. A uniformly valid composite solution is then formed from the reduced and boundary layer solutions. The value of the approximate solution is that it is relatively easy to obtain and does not involve singular arcs. To illustrate the utility of the results, the technique is used to obtain an approximate solution of a simplified version of the aircraft minimum time-to-climb problem. A numerical example is included.

Keywords. Singular perturbations, optimal control, singular arcs, flight mechanics.

INTRODUCTION

Singular arcs have been found to occur in optimal control problems from many fields. They are particularly prevalent in flight mechanics because in many problem formulations vehicle thrust is modeled as a bounded control variable which appears linearly in the state equations and performance index. Well-known examples are the problems of maximizing the range of a rocket in vacuo (Leitmann, 1966), maximizing the range of a lifting rocket in horizontal flight (Leitmann, 1966), minimizing the fuel required for orbit rendezvous of a rocket (Breakwell and Dixon, 1975), Goddard's problem of maximizing the final altitude of a sounding rocket (Bryson and Ho, 1969), and minimizing the time-to-climb of an airplane under certain approximations (Ardema, 1975). Current interest in flight mechanics problems with possible singular arcs is evidenced by many recent papers concerning the optimality of cruise for airplanes (Schultz, 1974; Speyer, 1976). In this paper, we investigate the usefulness of singular perturbation techniques for dealing with singular arc problems by analyzing a relatively low-order but otherwise general system. This system encompasses many flight mechanics problems including Goddard's problem and a version of the minimum time-to-climb problem.

Consider the third order system, defined on $0 \leq t \leq t_f$,

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x,y,z) + f'(x,y,z)u \\ \frac{dy}{dt} &= g(x,y,z) + g'(x,y,z)u \\ \varepsilon\frac{dz}{dt} &= h(x,y,z) + h'(x,y,z)u \end{aligned}\right\} \quad (1)$$

subject to suitable boundary conditions, where x, y, and z are scalars, $\varepsilon > 0$ is a parameter, and t_f is unspecified. It is desired to

$$\text{Min } \phi\left[x(t_f),y(t_f)\right] \quad (2)$$

The scalar control is restricted to

$$u_m \leq u \leq u_M \quad (3)$$

Since the Hamiltonian for this problem will be linear in u, a singular arc is possible. It is assumed that a unique optimal control exists and that in the region of state space of interest:[1]

A. f_x, f_y, f_z, f'_x, f'_y, f'_z, g_x, g_y, g_z, g'_x, g'_y, g'_z, h_x, h_y, h_z, h'_x, h'_y, h'_z are defined and continuous.

B. $h + h'u_m \neq 0$, $h + h'u_M \neq 0$, $\text{sgn}\,(h + h'u_m) = -\,\text{sgn}\,(h + h'u_M)$.

[1] Functional dependence will be omitted when this does not result in confusion.

C. Either $\frac{\partial}{\partial z}\left(f - f' \frac{h}{h'}\right) \neq 0$ or $\frac{\partial}{\partial z}\left(g - g' \frac{h}{h'}\right) \neq 0$ or both.

D. The singular arc satisfies (3) and the generalized convexity condition (Bryson and Ho, 1969; Kelley, Kopp, and Moyer, 1967).

Thus we have a singularly perturbed optimal control problem for which the full problem is singular but the reduced problem is not.

Assumption B assures that 1) $h' \neq 0$, 2) z can be either decreased or increased, and 3) the alternative procedure may be used for the reduced problem (Ardema, 1975). Assumption C then implies that the reduced problem is not itself singular.

Singular perturbation problems involving singular arcs have been considered previously, for example by O'Malley (1975) and O'Malley and Jameson (1975). However, these references consider linear systems for which the full problem is nonsingular but the reduced problem is singular; therefore they bear little connection with the present work. Rather, our approach follows Calise (1973) who first proposed using singular perturbation techniques to solve nonlinear singularly perturbed optimal control problems. This was given further consideration by Ardema (1975).

ANALYSIS

The Hamiltonian for the problem just stated is

$$H = (f + f'u)\lambda_x + (g + g'u)\lambda_y + (h + h'u)\lambda_z \tag{4}$$

and the adjoint equations are

$$\left.\begin{aligned}
\frac{d\lambda_x}{dt} &= -(f_x + f_x'u)\lambda_x - (g_x + g_x'u)\lambda_y - (h_x + h_x'u)\lambda_z \\
\frac{d\lambda_y}{dt} &= -(f_y + f_y'u)\lambda_x - (g_y + g_y'u)\lambda_y - (h_y + h'_yu)\lambda_z \\
\epsilon\frac{d\lambda_z}{dt} &= -(f_z + f_z'u)\lambda_x - (g_z + g'_zu)\lambda_y - (h_z + h'_zu)\lambda_z
\end{aligned}\right\} \tag{5}$$

The "switching function" (coefficient of u) in (4) is

$$S = f'\lambda_x + g'\lambda_y + h'\lambda_z \tag{6}$$

The part of H not involving S is

$$\bar{H} = f\lambda_x + g\lambda_y + h\lambda_z \tag{7}$$

Using (1) and (5), the time derivative of (6) is, after multiplying by ϵ,

$$\begin{aligned}
\epsilon\frac{dS}{dt} = {} & [f_z'h - h'f_z + \epsilon(f_x'f + f_y'g - f'f_x - g'f_y)]\lambda_x + [g_z'h - h'g_z + \epsilon(g_x'f \\
& + g_y'g - f'g_x - g'g_y)]\lambda_y + [h'_zh - h'h_z + \epsilon(h'_xf + h'_yg - f'h_x - g'h_y)]\lambda_z
\end{aligned} \tag{8}$$

The terms involving u cancel out. On a singular arc of an optimal trajectory, $H = 0$ and $S = 0$, so that we have the following system of homogeneous equations in the adjoint variables:

$$\left.\begin{aligned}
H &= 0 \\
S &= 0 \\
\frac{dS}{dt} &= 0
\end{aligned}\right\} \tag{9}$$

For compatibility (that is, existence of nontrivial solutions)[2] the determinant of the coefficient matrix of (9) must be zero which gives

$$\begin{aligned}
(fg' - f'g)&\left[h'_zh - h'h_z + \epsilon(h'_xf + h_y'g - f'h_x - g'h_y)\right] + (gh' - hg')\left[f_z'h - h'f_z \right. \\
&\left. + \epsilon(f_x'f + f_y'g - f'f_x - g'f_y)\right] + (hf' - fh')\left[g_z'h - h'g_z + \epsilon(g_x'f + g_y'g - f'g_x - g'g_y)\right] = 0
\end{aligned} \tag{10}$$

which is the singular arc.

We next show directly that the control law for the reduced problem is (10) with ϵ set to zero. We adopt the alternative procedure and set $\epsilon = 0$ in (1) and eliminate u to get the reduced system of equations

$$\left.\begin{aligned}
\frac{dx_o}{dt} &= f_o - f_o'\frac{h_o}{h_o'} \\
\frac{dy_o}{dt} &= g_o - g_o'\frac{h_o}{h_o'}
\end{aligned}\right\} \tag{11}$$

where, for example, $f_o = f(x_o, y_o, z_o)$. The performance index becomes $\phi\left[x_o(t_{f_o}), y_o(t_{f_o})\right]$. This problem has control variable z and is nonsingular. For optimal control

$$H_o = \left(f_o - f_o'\frac{h_o}{h_o'}\right)\lambda_{x_o} + \left(g_o - g_o'\frac{h_o}{h_o'}\right)\lambda_{y_o} = 0 \tag{12}$$

$$H_{z_o} = \left[f_{z_o} - f'_{z_o}\frac{h_o}{h_o'} - f_o'\frac{\partial}{\partial z_o}\left(\frac{h_o}{h_o'}\right)\right]\lambda_{x_o} + \left[g_{z_o} - g'_{z_o}\frac{h_o}{h_o'} - g_o'\frac{\partial}{\partial z_o}\left(\frac{h_o}{h_o'}\right)\right]\lambda_{y_o} = 0 \tag{13}$$

For compatibility of these equations,[3]

---

[2]Since for some combinations of performance index and terminal boundary conditions it is possible that $\lambda_o \neq 0$ even if $\lambda_x = \lambda_y = \lambda_z = 0$ (Leitmann, 1966), the trivial solution of (9) may in fact satisfy the Maximum Principle. The subsequent development is tacitly restricted to problems for which this does not occur.

[3]A comment similar to the one in footnote[2] also applied here.

$$\left(h'_{z_0}h_0 - h_0'h_{z_0}\right)(f_0g_0' - f_0'g_0) + \left(f'_{z_0}h_0 - h_0'f_{z_0}\right)(g_0h_0' - h_0g_0') + \left(g'_{z_0}h_0 - h_0'g_{z_0}\right)(h_0f_0' - f_0h_0') = 0 \qquad (14)$$

which is (10) with $\varepsilon = 0$. Thus the singular arc of the full problem and the control law of the reduced problem differ by at most terms of order ε.

The transformation approach to singular arcs developed by Kelley, Kopp, and Moyer (1967) is particularly advantageous for singularly perturbed problems. Under fairly mild assumptions, a transformation of state variables may be found such that in the new variables the state equations are of the form

$$\left.\begin{aligned} \frac{d\hat{x}}{dt} &= \hat{f}(\hat{x},\hat{y},\hat{z}) \\ \frac{d\hat{y}}{dt} &= \hat{g}(\hat{x},\hat{y},\hat{z}) \\ \varepsilon\frac{d\hat{z}}{dt} &= \hat{h}(\hat{x},\hat{y},\hat{z}) + \hat{h}'(\hat{x},\hat{y},\hat{z})u \end{aligned}\right\} \qquad (15)$$

This transformation (Kelley, Kopp, and Moyer, 1967) may be obtained as follows: Let the solution of

$$\frac{d\hat{x}}{d\hat{z}} = \frac{f'(\hat{x},\hat{y},\hat{z})}{h'(\hat{x},\hat{y},\hat{z})} \; ; \; \frac{d\hat{y}}{d\hat{z}} = \frac{g'(\hat{x},\hat{y},\hat{z})}{h'(\hat{x},\hat{y},\hat{z})} \qquad (16)$$

be denoted by

$$\left.\begin{aligned} C_x &= \psi_x(\hat{x},\hat{y},\hat{z}) \\ C_y &= \psi_y(\hat{x},\hat{y},\hat{z}) \end{aligned}\right\} \qquad (17)$$

where C_x and C_y are constants of integration and ψ_x and ψ_y are two independent integrals. Then the transformation of variables

$$\left.\begin{aligned} \hat{x} &= \psi_x(x,y,z) \\ \hat{y} &= \psi_y(x,y,z) \\ \hat{z} &= z \end{aligned}\right\} \qquad (18)$$

transforms (1) into (15). The advantage of this approach for singularly perturbed problems is that from (10) and (14) we see that for (15) the singular arc of the full problem and the control law of the reduced problem are identical and are given by

$$\hat{f}\hat{g}_{\hat{z}} - \hat{g}\hat{f}_{\hat{z}} = 0 \qquad (19)$$

The principal difficulty with implementing this approach lies in solving the generally nonlinear equations (16).

In many applications, a small parameter must be inserted artificially to create a singularly perturbed problem (Aiken and Lapidus, 1974; Ardema, 1975. 1976; Calise, 1973; and Kelley, 1973). In this case, the transformation approach has the further advantage of clarifying where to insert the parameter. The transformation results in a system in which variables x and y are controlled by a relatively faster variable z which is, in turn, controlled by a still faster variable u. Thus it is clear that ε is to be inserted as a multiplier of the derivative term of the fast state variable z.

The above results are also valid for a slightly different problem. Instead of (1) and (2) consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x,z) + f'(x,z)u \\ \frac{dz}{dt} &= h(x,z) + h'(x,z)u \end{aligned}\right\} \qquad (20)$$

where it is desired to

$$\text{Min}\ \left\{\phi[x(t_f)] + \int_0^{t_f}[L(x,z) + L'(x,z)u]dt\right\} \qquad (21)$$

subject to (3). Then the Hamiltonian is

$$H = (f + f'u)\lambda_x + (h + h'u)\lambda_z + (L + L'u)\lambda_0 \qquad (22)$$

Comparing (4) with (22) we see that the same results are valid as before provided L, L', and λ_0 are identified with g, g' and λ_y, respectively.[4]

Returning to the problem defined by (1)-(3), the initial zeroth order boundary layer equation (ZOBLE) is formed from (1) by transforming $\tau_1 = \frac{t}{\varepsilon}$ and setting $\varepsilon = 0$. The result is

$$\frac{dz_1}{d\tau_1} = h_1 + h_1'u_1, \qquad (23)$$

where, for example, $h_1 = h(x(0),y(0),z_1)$. Similarly for the terminal ZOBLE set

$\tau_2 = (t_f - t)/\varepsilon$ to get

$$\frac{dz_2}{d\tau_2} = -h_2 - h_2'u_2 \qquad (24)$$

where, for example, $h_2 = h(x(t_f),y(t_f),z_2)$.

CONSTRUCTION OF APPROXIMATE SOLUTION

We will now construct an approximate solution to the problem (1)-(3) under the stated assumptions. To be specific, consider the case for which[5]

1. $x(0)$, $y(0)$, $z(0)$, $x(t_f)$ $z(t_f)$ are given.

2. $\phi = \phi\,[y(t_f)]$.

[4] In this case the trivial solution of (9) cannot satisfy the Maximum Principle.

[5] These conditions were chosen only for the purpose of discussion; the method is not limited to this case.

3. The optimal control sequence is $\{u_M, u_{SINGULAR}, u_M\}$.

By assumption C, (14) may be solved for z; the solution is written as $z_o = \psi(x_o, y_o)$. First, integrate (11) from $[x(0), y(0)]$, to $x(t_f)$. This gives the zeroth order outer (reduced) solution, denoted $x_o(t)$, $y_o(t)$, and $z_o(t)$, as well as an estimate of t_f, denoted t_{f_o}, and of $y(t_f)$, denoted $y_o(t_{f_o})$. Second, solve the boundary layer equations. The initial ZOBLE is

$$\frac{dz_1}{d\tau_1} = h_1 + h_1' u_M \tag{25}$$

This is integrated until the reduced solution is reached:

$$\tau_1^* = \int_{z(0)}^{z_o(0)} \frac{dz_1}{h[x(0), y(0), z_1] + h'[x(0), y(0), z_1] u_M} \tag{26}$$

At $\tau_1 = \tau_1^*$, u is set to $-h_1/h_1'$ which holds z_1 at $z_o(0)$. A similar procedure is used for the terminal ZOBLE. This construction obviously produces stable boundary layer solutions which not only approach the outer solution as $\tau_1 \to \infty$ but reach it in finite time.

A possible representation of the exact solution may be obtained by "patching" the unsteady portion of the boundary layer solutions onto the ends of the outer solution. The trajectory time is estimated as $t_{f_p} = t_{f_o} + \tau_1^* \varepsilon + \tau_2^* \varepsilon$ by this solution and the performance index as $\phi[y_o(t_{f_o})]$. All state variables generally will have points of non-differentiability.

A more satisfying approximate solution is available from the method of matched asymptotic expansions. In this method, "composite" solutions are constructed from the outer and boundary layer solutions (Ardema, 1975, 1976; Van Dyke, 1964). The most common of these is the additive composite, the zeroth order of which takes the following form:

$$\left.\begin{aligned} x_a(t) &= x_o(t) \\ y_a(t) &= y_o(t) \\ z_a(t) &= z_o(t) + z_1\left(\frac{t}{\varepsilon}\right) + z_2\left(\frac{t_{f_o} - t}{\varepsilon}\right) \\ &\quad - z_o(0) - z_o\left(t_{f_o}\right) \end{aligned}\right\} \tag{27}$$

Such a solution will satisfy all boundary conditions. Because "time stands still" in the boundary layer for the slow variables x and y, there are no boundary layer corrections to these variables to zeroth order. The slow variables are approximated by $x_o(t)$ and $y_o(t)$, the trajectory time by t_{f_o}, the final value of $y(t)$ by $y\left(t_{f_o}\right)$ and the performance index by $\phi\left[y_o\left(t_{f_o}\right)\right]$. Thus, if a knowledge of $z(t)$ is of no particular interest, the boundary layer equations need not be solved.

The exact, patched, and zeroth order additive composite solutions for a typical situation are shown in Fig. 1. It is clear from the construction that for ε sufficiently small the latter is generally a uniformly-valid, zeroth-order approximation, that is, that

$$\left.\begin{aligned} x(t) &= x_a(t) + \mathcal{O}(\varepsilon) \\ y(t) &= y_a(t) + \mathcal{O}(\varepsilon) \\ z(t) &= z_a(t) + \mathcal{O}(\varepsilon) \end{aligned}\right\} \tag{28}$$

on $0 \le t \le t_{f_o}$. Higher order approximations may be obtained by expanding all variables in asymptotic power series, matching outer and boundary layer expansions, and forming composite solutions (Ardema, 1975, 1976; Van Dyke, 1964).

A procedure for solving the exact system would be as follows: Equations (1) are integrated from $x(0)$, $y(0)$, $z(0)$, with $u = u_M$, until the singular arc (10) is reached. The singular arc portion of the trajectory is then obtained by integrating (1) subject to (10). At some time t', the control is set to u_M and (1) is integrated until $x(t_f)$ is reached. If the value of z is $z(t_f)$ when the value of x is $x(t_f)$, then the solution has been obtained; if it isn't, repetitive solutions of the terminal arc must be made by varying t', the time of departure from the singular arc, until both the terminal conditions are met. Since this iteration is not required for the zeroth order additive composite solution, obtaining such an approximation requires considerably less computational effort than does obtaining the exact solution.

MINIMUM TIME-TO-CLIMB PROBLEM

A well-known problem of flight mechanics is that of determining the flight path which gives the minimum time-to-climb between a given speed and altitude and another given speed and altitude for an aircraft in the atmosphere. In one simplified version of the problem, the two-state approximation, the rotational dynamics are neglected. This leads to the state equations

$$\left.\begin{aligned} \frac{de}{dt} &= v(e,h)F(e,h) \\ \varepsilon \frac{dh}{dt} &= v(e,h)u \end{aligned}\right\} \tag{29}$$

where

$$\left.\begin{aligned} u &= \sin\gamma \\ v(e,h) &= \sqrt{2G(e-h)} \\ F(e,h) &= \frac{1}{mG}\Big[T(e,h) - D_o(e,h) \\ &\qquad - D_L(e,h,L)\big|_{L = mG}\Big] \end{aligned}\right\} \tag{30}$$

and,

- t = time
- h = altitude[6]
- m = mass
- v = velocity
- T = thrust
- D_o = zero lift drag
- D_L = drag due to lift
- G = gravity force per unit mass
- e = energy per unit weight
- γ = flight path angle
- L = lift

In this model, m and G are constants. For the derivation of these equations and a complete discussion of the assumptions they imply, the reader is referred to Chapter 2 of Ardema (1975). It is desired to find the control u which gives

$$\text{Min} \int_0^{t_f} dt$$

subject to $-1 \leq u \leq 1$ and

$$e(0) = E_o, h(0) = H_o, e(t_f) = E_f, h(t_f) = H_f \tag{31}$$

The problem just stated is of the type (20)-(21) with

$$x = e,\ z = h,\ f = vf,\ f' = 0,\ h = 0,\ h' = v$$
$$\phi = 0,\ g = L = 1,\ g' = L' = 0 \tag{32}$$

All four assumptions listed in the introduction will be satisfied for physically meaningful problems. The problem is already in the form (15) and ϵ has been inserted in front of the derivative of the fast variable h. From (19),

$$\frac{\partial(vF)}{\partial h} = 0 \tag{33}$$

which is both the singular arc of the full problem and the control law of the reduced problem. The reduced problem is

$$\frac{de_o}{dt} = v(e_o, h_o)F(e_o, h_o) \tag{34}$$

subject to $e_o(0) = E_o$ and $e_o\left(t_{f_o}\right) = E_f$ where $u_o = 0$ and h_o is the control, determined from (33). The initial and terminal ZOBLE's are

$$\left.\begin{aligned} \frac{dh_1}{d\tau_1} &= v(E_o, h_1)u_1 \\ \frac{dh_2}{d\tau_2} &= -v(E_f, h_2)u_2 \end{aligned}\right\} \tag{35}$$

Assuming that the reduced solution lies on the "right-hand side" of both boundary conditions, we have $u_1 = -1$ and $u_2 = 1$. Using (30), (35) are then easily integrated to give

$$\left.\begin{aligned} h_1(\tau_1) &= E_o - \left(\sqrt{\tfrac{G}{2}}\,\tau_1 + \sqrt{E_o - H_o}\right)^2 \\ h_2(\tau_2) &= E_f - \left(\sqrt{\tfrac{G}{2}}\,\tau_2 + \sqrt{E_f - H_f}\right)^2 \end{aligned}\right\} \tag{36}$$

In particular, the times to get to the outer solution are given by

$$\left.\begin{aligned} \tau_1^* &= \sqrt{\tfrac{2}{G}}\left[\sqrt{E_o - h_o(0)} - \sqrt{E_o - H_o}\right] \\ \tau_2^* &= \sqrt{\tfrac{2}{G}}\left[\sqrt{E_f - h_o(t_f)} - \sqrt{E_f - H_f}\right] \end{aligned}\right\} \tag{37}$$

The zeroth order additive composite approximate solution (27) may be written as

$$\left.\begin{aligned} e_a(t) &= e_o(t) \\ h_a(t) &= h_o(t) + \sigma(\tau_1^* - t)[h_1(t) - h_o(0)] \\ &\quad + \sigma\left(t + \tau_2^* - t_{f_o}\right)\left[h_2\left(t_{f_o} - t\right) - h_o\left(t_{f_o}\right)\right] \end{aligned}\right\} \tag{38}$$

for this problem, where $\sigma(t)$ is the unit step function and ϵ has been set to 1, its proper value.

It is interesting to compare this approximate solution with solutions obtained from different dynamic models. One advantage of the singular perturbation viewpoint is that it provides a convenient means for making such a comparison. Many of the approximations commonly used in flight mechanics are critcally discussed in Leitmann (1962). The most widely used approximate formulation of the minimum time-to-climb problem is that of energy-state (Ardema, 1975; Bryson, Desai and Hoffman, 1969). In this model, there is only one state variable, e; the state equation is the first of (29),

$$\frac{de}{dt} = v(e,h)F(e,h) \tag{39}$$

subject to $e(0) = E_o$ and $e(t_f) = E_f$ where h is the control variable. The optimal control law is

$$\frac{\partial(vF)}{\partial h} = 0 \tag{40}$$

The path defined by (40) is often called the "energy climb path" for obvious reasons. Boundary conditions on h are met by adjoining constant energy arcs (traversed in zero time in this approximation) to the solution of (39) with (40). Because 1) the outer problem of the two-state approximation is identical to (39) and (40), and 2) boundary layer solutions (36) are constant energy arcs, the path in (e, h) space obtained by "patching" together the outer and boundary layer solutions of the two-state approximation is the same as that obtained from the energy-state approximation. The only difference is that the constant energy arcs are not traversed in zero time in the patched two-state solution.

[6] Not to be confused with the function h of the preceding equations.

On the other hand, the times-to-climb, estimated by the zeroth order additive composite solution of the two-state model and by the solution of the energy-state model, will be identical, but the paths will be different.

A more realistic dynamic model than the two-state one is obtained by including the rotational dynamics as a small effect and allowing drag dependence on lift. This leads to the state equations

$$\left.\begin{aligned} \frac{de}{dt} &= v(e,h)[F(e,h) - \frac{1}{mG}D_L'(e,h,L)] \\ \epsilon \frac{dh}{dt} &= v(e,h)\sin\gamma \\ \epsilon \frac{d\gamma}{dt} &= \frac{1}{v(e,h)}(L - \cos\gamma) \end{aligned}\right\} \quad (41)$$

where $D_L'(e,h,L)$ is the increment in drag-due-to-lift over that at $L = mG$. The control variable in (41) is L, and the problem is nonsingular. This formulation of the problem is realistic enough for most purposes and was solved to first order in ϵ by Ardema (1975, 1976) by the method of matched asymptotic expansions. This solution was found to be in excellent agreement with a steepest descent solution, and it will be considered "exact" for the purposes of comparison.

There have been two other analyses of the minimum time-to-climb problem by singular perturbation methods. Calise (1975) considered the system

$$\left.\begin{aligned} \frac{de}{dt} &= v(e,h)[F(e,h) - \frac{1}{mG}D_L'(e,h,L)] \\ f_1(\epsilon)\frac{dh}{dt} &= v(e,h)\sin\gamma \\ f_2(\epsilon)\frac{d\gamma}{dt} &= \frac{1}{v(e,h)}(L - \cos\gamma) \end{aligned}\right\} \quad (42)$$

where

$$\lim_{\epsilon\to o} f_1(\epsilon) = 0$$

$$\lim_{\epsilon\to o} \frac{f_2(\epsilon)}{f_1(\epsilon)} = 0$$

This formulation leads to multiple boundary layers. Breakwell (1977) considered the system (41) with $\epsilon = 1$ and the reciptrocal of maximum lift-to-drag ratio treated as a small parameter. This formulation is appealing because of the physical significance of the small parameter.

The numerical example of Ardema (1975, 1976) will now be solved to illustrate and compare the various solutions. The boundary conditions are

$M(0) = 0.5$, $h(0) = 12{,}200$ meters, $\gamma(0) = 0$

$M(t_f) = 2.0$, $h(t_f) = 24{,}400$ meters, $\gamma(t_f)$ FREE

Here, M is the Mach number which is defined by $M = v/a(h)$ where a is the speed of sound, a known function of altitude. For formulations in which γ is not modeled as a state variable, the boundary conditions on γ do not apply; a similar comment applies to h. The data describing the aircraft are given in Ardema (1975) and Bryson, Desai, and Hoffman (1969). Figure 2 shows how the additive composite for altitude, $h_a(t)$, is formed from its constituent functions according to (38). This figure clearly shows the stability of the boundary layer solutions and the nature of the composite solution: near $t = 0$, $h_a(t)$ resembles $h_i(t)$; near $t = t_f$ it resembles $h_i(t_f - t)$; and away from the boundaries it is identical to $h_o(t)$. $\tau_1{}^*$ and $\tau_2{}^*$ are seen to be approximately 32 s and 20 s respectively.

Table 1 compares the times-to-climb as computed by the various methods. The additive composite solution (also energy-state) underestimates the time by a substantial amount while the patched solution overestimates it by an equally large amount. A similar result was found for the more exact model (41) in Ardema (1975). The good agreement between the patched and exact values is of no significance since the patched solution is an approximation of the two-state and not the exact solution.

Table 1. Comparison of Minimum Time-to-Climb by Various Methods

	Time-to-climb s
Energy-state (39) with (40)	105
Two-state (29) with $\epsilon = 1$	130
"Patched" approximation to two-state, (36) added to ends of solution of (34) with (33)	157
Zeroth order additive composite approximation to two-state (38)	105
Exact, first-order solution of (41) or steepest descent	162

Figure 3 compares the paths in the (h,M) plane. All the paths agree qualitatively in that they have the same characteristic shape: 1) an initial steep dive of nearly constant energy to the region of the energy-climb path, during which velocity is gained at the expense of a loss in altitude; 2) an intermediate portion spent in the vicinity of the energy climb path (locus along which energy can be gained most quickly) where velocity is accumulated with little change in altitude; and 3) a steep "zoom-climb" of nearly constant energy, during which the excess speed is traded for altitude gain. It is this characteristic shape which, in fact, makes the problem amenable to singular perturbation techniques. Quantitatively, the two-state solution appears to give a good representation of the exact solution. The zeroth-order additive composite is no better an approximation to the two-state solution than is the energy-state (reduced) solution. Again, a similar result has been found for the more exact model (Ardema, 1975). The usefulness of the zeroth-order solution is that it is relatively easy to obtain and can be used as the basis of a first-order solution. Based on Ardema's (1975) results, the first-order solution should have very good accuracy.

REFERENCES

Aiken, R. C. and Lapidus, L. (1974). An effective numerical integration method for typical stiff systems. AIChe J., 20.

Ardema, M.D. (1975). Singular perturbations in flight mechanics. NASA TM X-62,380.

Ardema, M.D. (1976). Solution of the minimum time-to-climb problem by matched asymptotic expansions. AIAA J., 14.

Breakwell, J.V. (1977). Optimal flight-path-angle transitions in minimum-time airplane climbs. J. of Aircraft, 14.

Breakwell, J.V., and Dixon, J.F. (1975). Minimum-fuel-rocket trajectories involving intermediate-thrust arcs. J. Optimiz. Theory & Appl., 17.

Bryson, A.E., Jr., and Ho, Y-C. (1969). Applied Optimal Control, Blaisdell.

Bryson, A.E., Jr., Desai, M.N., and Hoffman, W. C. (1969). Energy-state approximation in performance optimization of supersonic aircraft. J. of Aircraft, 6.

Calise, A.J. (1973). On the use of singular perturbation methods in variational problems. Paper No. 7-4, presented at the Joint Automatic Control Conference, Columbus, OH.

Calise, A.J. (1975). Extended energy management methods for flight performance optimization. AIAA Paper 75-30, presented at the 13th Aerospace Sciences Meeting, Pasadena, CA.

Kelley, H.J. (1973). Aircraft maneuver optimization by reduced-order approximation. In C.T. Leondes (Ed.), Control and Dynamic Systems, 10, Academic Press, NY.

Kelley, H.J., Kopp, R.E., and Moyer, H.G. Singular extremals. In G. Leitmann (Ed.), Topics in Optimization, Academic Press, NY.

Leitmann, G. (1962). On optimal control of lifting vehicles. Proc. of Fourth U.S. National Congress on Applied Mechanics, 1.

Leitmann, G. (1966). An Introduction to Optimal Control. McGraw-Hill.

O'Malley, R.E. (1975). The singular perturbation approach to singular arcs. In H.A. Antosiewicz (Ed.), Proc. of the International Conference of Differential Equations, Academic Press, NY.

O'Malley, R.E., and Jameson, A. (1975). Singular perturbations and singular arcs, I. IEEE Trans. Auto. Control, 20.

Schultz, R. (1974). Fuel optimality of cruise. J. of Aircraft, 11.

Speyer, J.L. (1976). Nonoptimality of the steady-state cruise for aircraft. AIAA J. 14.

Van Dyke, M. (1964). Perturbation Methods in Fluid Mechanics, Academic Press, NY.

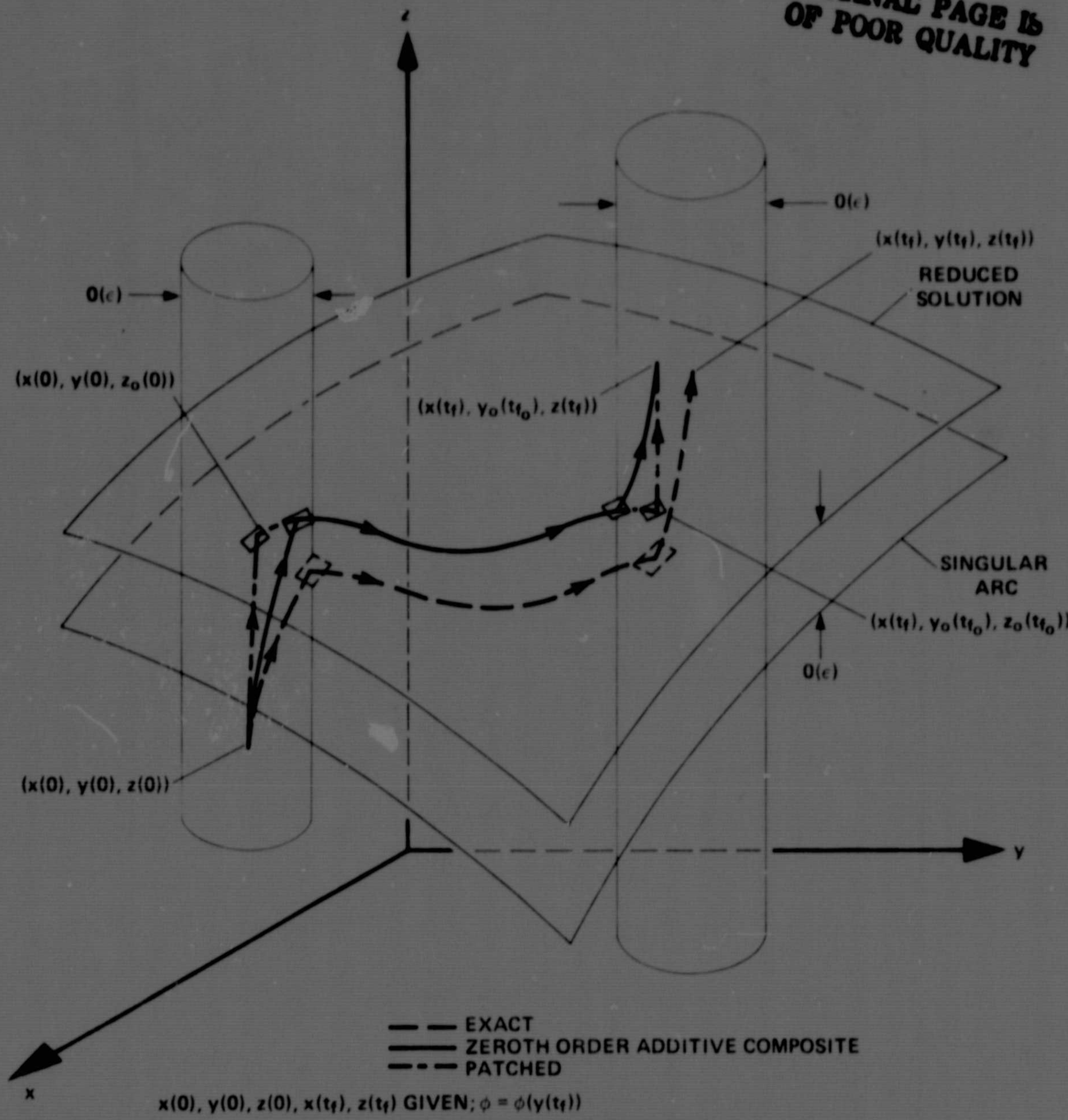

Fig. 1. Sketch of Exact and Approximate Solutions for a Typical Case

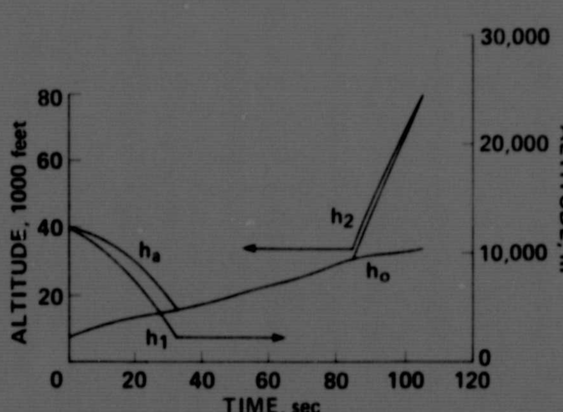

Fig. 2. Additive Composite for Altitude and Its Constituents

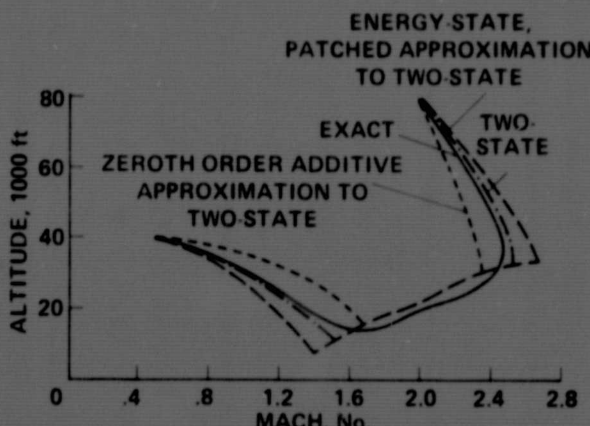

Fig. 3. Comparison of Paths

ORIGINAL PAGE IS
OF POOR QUALITY

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-78,457		
4. Title and Subtitle NONLINEAR SINGULARLY PERTURBED OPTIMAL CONTROL PROBLEMS WITH SINGULAR ARCS		5. Report Date 6. Performing Organization Code
7. Author(s) Mark D. Ardema		8. Performing Organization Report No. A-7289
9. Performing Organization Name and Address NASA Ames Research Center Moffett Field, Calif. 94035		10. Work Unit No. 791-40-13 11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Memorandum 14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A third order, nonlinear, singularly perturbed optimal control problem is considered under assumptions which assure that the full problem is singular and the reduced problem is nonsingular. The separation between the singular arc of the full problem and the optimal control law of the reduced one, both of which are hypersurfaces in state space, is of the same order as the small parameter of the problem. Boundary layer solutions are constructed which are stable and reach the outer solution in a finite time. A uniformly valid composite solution is then formed from the reduced and boundary layer solutions. The value of the approximate solution is that it is relatively easy to obtain and does not involve singular arcs. To illustrate the utility of the results, the technique is used to obtain an approximate solution of a simplified version of the aircraft minimum time-to-climb problem. A numerical example is included.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Singular perturbations Optimal control Singular arcs Flight mechanics	Unlimited STAR Category - 63

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	9	$3.50

*For sale by the National Technical Information Service, Springfield, Virginia 22161