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THE METHOD OF AVERAGES APPLIED TO THE KS DIFFERENTIAL EQUATIONS

ANALYTICAL AND COMPUTATIONAL MATHEMATICS, INC.

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OTIS F. GRAF, JR., ALAN MUELLER AND STEPHEN STARKE

ANALYTICAL AND COMPUTATIONAL MATHEMATICS, INC.

1275 SPACE PARK DRIVE, SUITE 114

HOUSTON, TEXAS 77058

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by

Otis F. Graf, Jr., Alan Mueller and Stephen Starke

1.0 INTRODUCTION

Solutions to ordinary differential equations are usually obtained through the application of either analytical or numerical solution methods. Each approach has its own set of advantages and disadvantages. A particular method is usually chosen according to how it may satisfy foreseen applications.

Analytical methods require extensive development of mathematical formulas. Once they have been obtained, the formulas can be used to study certain global properties of the solution. Also, the analytical formulas can be used in computing machines to provide extremely rapid numerical calculations.

There is relatively little apriori mathematical development of numerical methods. They allow extremely accurate numerical calculations of the solution and are usually not difficult to program on the computer. However, the solution must be developed step-by-step, which can take large amounts of computer time. Also, very little qualitative or global information on the solution is available.

For some applications, it may be advantageous to develop new specialized methods that combine features of both the analytical and the numerical methods. This is the suggested approach for those applications that require repetitive solutions of the differential equations, but for which no analytical solutions are available. Therefore, the advantage in speed of

an analytical solution might be partially realized by using a limited application of an analytical method. Numerical step-by-step calculations still need to be done, but usually in a more efficient manner.

This report concerns a new approach for the solution of artifical satellite trajectory problems. The basic idea is to apply an analytical solution method (the Method of Averages) to an appropriate formulation of the orbital mechanics equations of motion (the KS-element differential equations). The result is a set of transformed equations of motion that are more amenable to numerical solution.

The following three subsections in this Introduction give an overview of the Method of Averages, a review of the KS-elements, a short discussion of orbit perturbations, and finally, a statement of the overall objectives of this work.

1.1 Overview on the Method of Averages

The purpose of the Method of Averages is to eliminate "fast variables" from the differential equations. It is based on the following *Main Theorem of Averaging* (reference 1) for the periodic case:

Theorem

Consider the initial value problem

$$\frac{d\vec{x}}{dt} = \varepsilon \vec{f}(\vec{x},t,\varepsilon) \quad , \quad \vec{x}(0,\varepsilon) = \vec{x}_o \qquad (1)$$

with $t \in [0,\infty]$, $\varepsilon \in [0,\varepsilon_o]$, $\vec{x} \in G$, an open bounded set in R^n , and the following conditions are satisfied:

(1) $\vec{f}$ is defined in a connected set G ;

(2) $\vec{f}$ is continuous and uniformally bounded in G ;

(3) $\vec{f}$ is Lipschitz-continuous with respect to $\vec{x}$ in G[†] ;

(4) the limit

$$\lim_{\varepsilon \to 0} \vec{f}(\vec{x},t,\varepsilon) = f(\vec{x},t,0)$$

exists uniformally in G ;

(5) $\vec{f}$ possesses a bounded derivative of ε in G ;

(6) $\vec{f}$ is periodic in t with period T .

Consider the associated initial value problem

$$\frac{d\vec{y}}{dt} = \varepsilon \vec{f}_o(\vec{y}) \quad , \quad \vec{y}(0,\varepsilon) = \vec{x}_o \tag{2}$$

where

$$\vec{f}_o(\vec{y}) = \frac{1}{T} \int_0^T f(\vec{y},\tau,0)d\tau$$

If it is also assumed that $\vec{y}(t,\varepsilon)$ exists, then

$$\vec{x}(t,\varepsilon) - \vec{y}(t,\varepsilon) = O(\varepsilon)$$

on the time scale $\frac{1}{\varepsilon}$.

Comments

(1) Equation (1) is the "standard form" for the averaging method. Not all problems can be reduced to the standard form.

(2) The conditions expressed above are generally satisfied by orbital mechanics problems.

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† The first three conditions ensure that the solution to (1) exists and is unique.

(3) The theorem establishes that the solution of the Averaged Differential Equations (2) is always close to the true solution, in the time interval of interest.

(4) Notice that the right side of (2) contains only slowly varying variables. Therefore, these equations should be easier to solve.

The Generalized Method of Averaging is discussed in reference 2, and concerns the system of equations:

$$\frac{dx_i}{dt} = \varepsilon\, f_i(\vec{x},\vec{y},\varepsilon) \quad , \qquad i = 1,\ 2,\ \cdots,\ n$$

$$\frac{dy_j}{dt} = w_j(\vec{x}) + \varepsilon\, g_j(\vec{x},\vec{y},\varepsilon) \qquad j = 1,\ 2,\ \cdots,\ m$$

where f_i and g_j are periodic in each component of $\vec{y}$ with period 2π. The y_j are called "rapidly rotating" phases. The purpose of the method is to eliminate $\vec{y}$ from the differential equations.

The Method of Averages has found widespread use in applied mathematics and engineering. Different formulations, examples, as well as rigorous developments are given in references 3, 4 and 5.

In this report, a Modified Method of Averages is developed for application to equations of the form

$$\frac{d\vec{x}}{dt} = \varepsilon\, \vec{f}(\vec{x},y,t,\varepsilon)$$

$$\frac{dy}{dt} = w(\vec{x}) + \varepsilon\, g(\vec{x},y,t,\varepsilon)$$

Note that there is only one rapidly rotating phase, but that the independent variable appears explicitly on the right hand side. It will be found that the KS equations can be put in the above form. For the applications presented in this report, it is convenient to eliminate only y from the differential equations.

1.2 KS Total Energy Orbital Elements

A very powerful method for the numerical solution of the differential equations of satellite motion is the total energy formulation of the KS element equations. The theoretical details of the KS method are developed in reference 6. References 7 and 8 give evaluations of the KS elements when applied to numerical computation of satellite orbits. It has been found that the KS formulation offers the following advantages:

(1) Instabilities associated with solving the two-body (conic) equations are eliminated.

(2) An orbital frequency based on the total energy gives more accuracy to calculations of the in-orbit (downrange) position.

(3) The differential equations are "smoothed" for eccentric orbits because the eccentric anomaly is the independent variable.

(4) The equations are less sensitive to roundoff and truncation errors in the numerical integration algorithm.

It will be shown in Section 2 that the total energy feature can be used to eliminate the tesseral geopotential terms from the numerical orbit computation. This is an additional advantage of the KS formulation.

1.3 Orbit Perturbations Due to a Nonspherical, Nonhomogenous Earth

Let the geopotential function be denoted by V . It contains two types of terms

$$V = V_Z + V_T$$

where V_Z are the zonal terms and V_T the tesseral terms. The zonal terms represent a potential function that is independent of the rotation of the earth. Thus, V_T contains the longitude dependent part of the geopotential, and therefore, contains explicitly the time.

It has been shown by analytical satellite theories that, whereas the zonal terms contribute long period and secular terms to the solution, the tesseral terms contribute primarily periodic terms. For solutions that are valid over several hundred revolutions, tesseral resonance terms must be included. However, for many applications, resonance effects are not important and they will be neglected in this report.

Even though the tesseral terms in the solution are small and periodic, it has been shown in references 9 and 10 that they can contribute large errors if neglected from numerical integrations. Downrange errors of 20 to 30 km are typical for near earth satellites (reference 9). These errors can be attributed to the fact that an incorrect "mean" mean motion results when tesseral terms are arbitrarily dropped from the differential equations.

It was shown in reference 10 that neglecting a small periodic perturbation from the differential equations, does not necessarily mean that the error in the solution will remain small. Consider the following differential equations

$$\frac{dx}{dt} = \varepsilon a \sin\omega t \qquad x(t=o) = x_o$$

$$\frac{dy}{dt} = x + \varepsilon b \sin\omega t \qquad y(t=o) = y_o$$

where ε is some small parameter. This situation is similar to the geopotential tesseral perturbations. The x is analogous to the mean motion and y is analogous to a fast variable such as the mean anomaly. Both are being perturbed by a periodic function. The solution for x and y is

$$x = \frac{\varepsilon a}{\omega}(1-\cos\omega t) + x_o$$

$$y = \left(x_o + \frac{\varepsilon a}{\omega}\right) t - \frac{\varepsilon a}{\omega^2}\sin\omega t + \frac{\varepsilon b}{\omega}(1-\cos\omega t) + y_o$$

Suppose that the small perturbation is neglected. This is equivalent to "crude" averaging, mentioned in reference 2. The solution is

$$x' = x_o \quad ,$$

$$y' = x_o t + y_o$$

The difference in the two solutions is

$$x-x' = \frac{\varepsilon}{a}(1-\cos\omega t)$$

$$y-y' = \frac{\varepsilon}{a} t - \frac{\varepsilon a}{\omega^2}\sin\omega t + \frac{\varepsilon b}{\omega}(1-\cos\omega t)$$

Note, that disregarding the perturbation results in a periodic error in x. However, the error in y is linear.

Suppose, instead, that an average value for x

$$x'' = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{\varepsilon a}{\omega} (1 - \cos\theta) + x_o \right\} d\theta$$

is inserted into the differential equation for y. Then, neglecting the perturbation, the solution for y is

$$y'' = \int_0^t x'' \, dt = \int_0^t \left(\frac{\varepsilon a}{\omega} + x_o \right) dt$$

or

$$y'' = \left(\frac{\varepsilon a}{\omega} + x_o \right) t + y_o$$

The error incurred by this solution is periodic.

This example and the numerical experiments of reference 9 illustrate that crude averaging will lead to incorrect results. It will be shown in this report that with a proper initialization of the mean motion, the numerical integration can proceed without including the tesseral geopotential terms.

1.4 Objectives of this Work

An important problem that is found when attempting the numerical solution of the satellite differential equations, is that the maximum size of the numerical integration steps is limited by the high frequency (short period) terms contained in the geopotential. This problem appears even with

the KS formulation. Even though their amplitudes are small and may be considered negligible for many applications, the short period terms cause the following practical problems:

(i) Since the tesseral terms depend on time, the orbital frequency based on the total energy is no longer constant. If small steps are not taken to "track" the high frequency oscillations, large down range errors can result.

(ii) If the tesseral terms were simply neglected in the numerical integration, larger steps could be taken but unacceptable intrack errors would result (reference 9).

(iii) Evaluation of all tesseral terms in the numerical integration force model consumes a major part of the computation time.

An approach to solving the three above problems is discussed in this report. Basically, the idea is to carry out a numerical integration without the tesseral terms.† In order to avoid the second of the above problems, the KS elements are initialized with a mean frequency (total energy). The mean values are obtained via a specialized application of the Method of Averages. Since the force model contains only zonal geopotential terms plus, possibly, atmospheric drag and external bodies, computer run time and stability problems associated with the tesseral terms can be avoided.

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† It will be assumed here that there are no important resonance effects.

2.0 MODIFIED METHOD OF AVERAGES FOR KS EQUATIONS

One of the reasons for applying the Method of Averages to differential equations is to eliminate terms that do not make important contributions to the solution, but that can, nevertheless, cause difficulties in obtaining a numerical or analytical solution. That will be the objective in this section with regard to the KS differential equations. It was mentioned in Section 1.3 that neglecting the tesseral geopotential terms would cause a linear growth in the error of predicted satellite positions. In reference 10 it was shown experimentally that this linear error could be eliminated if a mean KS frequency could be found. In this section, a modified method of averages is developed and applied to the KS equations. This results in a mean KS frequency $\bar{\omega}$ that is defined by a partial differential equation. When this value of $\bar{\omega}$ is used in the numerical integration, the tesseral geopotential terms can be safely neglected for many applications.

2.1 The KS Differential Equations

The KS total energy element differential equations that will be used here are given in reference 6, section 23:

$$\frac{d\omega}{dE} = -\frac{r}{8\omega^2}\frac{\partial V}{\partial t} - \frac{1}{\partial \omega}(\vec{u}^*,\ L^T \vec{P}) \tag{3}$$

$$\frac{d\tau}{dE} = \frac{1}{8\omega^3}\left[\mu - rV + 2(\vec{u},\ \vec{Q})\right] - \frac{2}{\omega^2}\frac{d\omega}{dE}(\vec{u},u^*) \quad , \tag{4}$$

$$\frac{d\vec{\alpha}}{dE} = -\left[\frac{1}{\partial\omega^2}\vec{Q} - \frac{2}{\omega}\frac{d\omega}{dE}\vec{u}^*\right] sin\ \frac{E}{2} \tag{5}$$

$$\frac{d\vec{\beta}}{dE} = \left[\frac{1}{\partial\omega^2}\vec{Q} - \frac{2}{\omega}\frac{d\omega}{dE}\vec{u}^*\right] cos\ \frac{E}{2} \tag{6}$$

where

$$\vec{Q} = -\tfrac{1}{2}\, L^T\left[\frac{\partial}{\partial x}(rV)\right] + \frac{r}{2}\, L^T\, \vec{P} \tag{7}$$

$\vec{u}$, $\vec{u}^*$, $\vec{\alpha}$, $\vec{\beta}$ and $\vec{Q}$ are 4-vectors.

$\vec{P}$ is a 3-vector of perturbing accelerations that may or may not be derivable from a potential. V is the disturbing potential function.

For the purposes of this section, we will assume that $\vec{P}$ is zero and that V is composed only of geopotential terms. In this case, equations (3) and (7) become

$$\frac{d\omega}{dE} = -\frac{r}{8\omega^2}\frac{\partial V}{\partial t} \tag{8}$$

$$\vec{Q} = -\tfrac{1}{2}\, L^T\left[\frac{\partial}{\partial x}(rV)\right] \tag{9}$$

The gravitational acceleration on a satellite is $\frac{\partial U}{\partial \vec{x}}$ where

$$U = \frac{\mu}{r} - V$$

and

$$V = V_Z + V_T \tag{10}$$

Thus the geopotential can be separated into zonal (time independent) terms and tesseral (time dependent) terms, and

$$V_Z = \frac{\mu}{r} \sum_{n=2}^{\infty} J_n \left(\frac{R_\oplus}{r}\right)^n P_n (\sin \phi) \tag{11}$$

$$V_T = -\frac{\mu}{r} \sum_{n=2}^{\infty} \left(\frac{R_\oplus}{r}\right)^n \sum_{m=1}^{n} P_{nm} (\sin \phi) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right] \tag{12}$$

where

ϕ = geocentric latitude,

λ = geographic longitude,

P_n = ordinary legendre function,

P_{nm} = associated legendre function,

$R_\oplus$ = equatorial mean radius of the earth,

J_n, C_{nm}, S_{nm} = geopotential coefficients.

The time appears explicitly in V_T through λ :

$$\lambda = \psi - (\dot{\theta}_E t + \theta_{E0}) \tag{13}$$

where

ψ = inertial longitude of the satellite

θ_E = Greenwich longitude.

The time is given in terms of KS-elements by

$$t = \tau - \frac{1}{\omega} (\vec{u}, \vec{u}^*) \tag{14}$$

Making use of equations (13), (14) and the transformation equations in section 19 of reference 6, it can be shown that V_T may be written as

$$V_T = \frac{\mu}{a} J_2^2 \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left[A_{nm} (\vec{\alpha}, \vec{\beta}, \omega, E) \cos m\Theta + B_{nm} (\vec{\alpha}, \vec{\beta}, \omega, E) \sin m\Theta \right] , \tag{15}$$

where

$$\Theta = \dot{\theta}_E \tau + \theta_{EO} \tag{16}$$

and

$$a = \frac{1}{2} \sum_{i=1}^{4} (\alpha_i^2 + \beta_i^2)$$

Comments

(1) Notice that J_2^2 has been brought outside the summation. Therefore, the magnitudes of the functions A_{nm} and B_{nm} are of order 1.

(2) The functions A_{nm} and B_{nm} are periodic in E with period 2π .

(3) V_T is "purely" periodic in Θ with period 2π . That means that there are no terms in V_T which are independent of Θ .

Making use of the expressions for V_z , V_T , assuming that $\vec{P} \equiv 0$, and using the new time element Θ , the KS differential equations can be formally expressed as

$$\frac{d\omega}{dE} = \varepsilon^2 \, f_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \tag{17}$$

$$\frac{d\Theta}{dE} = \frac{\mu}{8\omega^3} \, \dot{\theta}_E + \varepsilon \, g_1 \, (\omega, \vec{\alpha}, \vec{\beta}, E) + \varepsilon^2 \, g_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, , \tag{18}$$

$$\frac{d\vec{\alpha}}{dE} = \varepsilon \, \vec{h}_1 \, (\omega, \vec{\alpha}, \vec{\beta}, E) + \varepsilon^2 \, \vec{h}_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, , \tag{19}$$

$$\frac{d\vec{\beta}}{dE} = \varepsilon \, \vec{k}_1 \, (\omega, \vec{\alpha}, \vec{\beta}, E) + \varepsilon^2 \, \vec{k}_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, . \tag{20}$$

The parameter $\varepsilon = J_2$ has been introduced. The function $\vec{h}_1$ and $\vec{k}_1$ depend only on V_z and the functions $\vec{h}_2$ and $\vec{k}_2$ depend only on V_T . Based on the above comments concerning V_T , it can be concluded that the functions f_2 , g_2 , $\vec{h}_2$ and $\vec{k}_2$ are purely periodic in Θ , i.e.

$$\int_0^{2\pi} f_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, d\Theta = 0 \tag{21}$$

$$\int_0^{2\pi} g_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, d\Theta = 0 \tag{22}$$

$$\int_0^{2\pi} \vec{h}_2 \, (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \, d\Theta = \vec{0} \tag{23}$$

$$\int_0^{2\pi} \vec{k}_2 \ (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \ d\Theta = \vec{0} \tag{24}$$

These results will be important in the theoretical developments with the Method of Averages.

2.2 The Modified Method of Averages

For the developments that follow, there will be no loss of generality if the order of the KS differential equations (17) through (20) is reduced. Thus, the equations will be represented as

$$\frac{d\omega}{dE} = \varepsilon^2 \ f_2 \ (\omega, \Theta, \alpha, E) \tag{25a}$$

$$\frac{d\Theta}{dE} = u(\omega) + \varepsilon \ g_1 \ (\omega, \alpha, E) + \varepsilon^2 \ g_2 \ (\omega, \Theta, \alpha, E) \tag{25b}$$

$$\frac{d\alpha}{dE} = \varepsilon \ h_1 \ (\omega, \alpha, E) + \varepsilon^3 \ h_2 \ (\omega, \Theta, \alpha, E) \tag{25c}$$

where

$$u(\omega) = \frac{\mu}{8\omega^3} \dot{\theta}_E \tag{26}$$

The functions on the right side of equations (25) have the properties:

(i) f_2, g_1, g_2, h_1, h_2 are periodic functions of E .

(ii) f_2, g_2 and h_2 are purely periodic functions of Θ . They represent the tesseral terms of the geopotential.

(iii) g_1 and h_1 do not depend on θ . They represent the zonal terms of the geopotential.

Since the right sides of equations (25) are periodic in the angular variables, it is appropriate to apply averaging for their solution. A Modified Method of Averages (MMOA) will be applied in this section to equations (25). The MMOA is a modification of the averaging method that is developed in reference 2.

The purpose of the MMOA is to eliminate only those periodic terms that depend on a *dependent* angular variable (i.e. time). This approach will eliminate the tesseral geopotential terms from the problem. The averaged differential equations will still depend on the independent variable E and the right sides will contain short period terms. However, numerical solution of the averaged equations will be enhanced because they will have the following properties:

(i) The frequency of the fast variable will be a constant. This means that the numerical solution will have increased stability, i.e. larger steps can be taken.

(ii) Evaluation of the right sides will be much quicker since the time consuming tesseral terms need not be computed.

Given the differential equations (25), assume asymptotic expansions of the form

$$\omega = \bar{\omega} + \varepsilon\eta_1\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \varepsilon^2\eta_2\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \cdots \quad , \qquad (27a)$$

$$\theta = \bar{\theta} + \varepsilon\phi_1\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \varepsilon^2\phi_2\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \cdots \quad , \qquad (27b)$$

$$\alpha = \bar{\alpha} + \varepsilon\chi_1\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \varepsilon^2\chi_2\,(\bar{\omega},\bar{\theta},\bar{\alpha},E) + \cdots \quad . \qquad (27c)$$

The functions η_i, ϕ_i, χ_i, $(i = 1,2, \cdots)$ are required to be periodic in $\bar{\theta}$ and E with period 2π . The new (averaged)

elements are defined by the differential equations,

$$\frac{d\bar{\omega}}{dE} = \varepsilon F_1\ (\bar{\omega},\bar{\alpha},E) + \varepsilon^2 F_2\ (\bar{\omega},\bar{\alpha},E) + \cdots \quad , \tag{28a}$$

$$\frac{d\bar{\theta}}{dE} = U(\bar{\omega}) + \varepsilon G_1\ (\bar{\omega},\bar{\alpha},E) + \varepsilon^2 G_2\ (\bar{\omega},\bar{\alpha},E) + \cdots \quad , \tag{28b}$$

$$\frac{d\bar{\alpha}}{dE} = \varepsilon H_1\ (\bar{\omega},\bar{\alpha},E) + \varepsilon^2 H_2\ (\bar{\omega},\bar{\alpha},E) + \cdots \tag{28c}$$

Equations (28) are referred to as the "averaged" equations. It will be shown that they have the properties that were previously discussed.

The functions η_i, ϕ_i, χ_i, F_i, G_i, H_i $(i = 1,2, \cdots)$ are not yet known and will now be determined by making expansions in powers of ε. The coefficients of the powers of ε will provide a set of partial differential equations. The solution of these equations will give the required functions.

Power series expansions in ε are obtained by the following algorithm:

(i) Compute the total derivative of both sides of equations (27), using the chain rule for the functions on the right side.

(ii) Insert equations (28) into the right sides of the equations produced in step *(i)*. This will give expressions for $\frac{d\omega}{dE}$, $\frac{d\theta}{dE}$, and $\frac{d\alpha}{dE}$ in terms the averaged variables.

(iii) Insert equations (27) into equation (25) and expand in powers of ε.

(iv) Equate the expressions for the total derivatives of ω, θ, and α that were produced in step *(ii)* to those that were produced in step *(iii)*.

(v) Collect terms of like powers of ε. The coefficients are the required partial differential equations.

The defining equations are:

ε^0 Terms

$$U(\bar{\omega}) = u(\bar{\omega}) \tag{29}$$

ε^1 Terms

$$F_1 + U\frac{\partial \eta_1}{\partial \bar{\theta}} + \frac{\partial \eta_1}{\partial E} = 0 \tag{30}$$

$$G_1 + U\frac{\partial \phi_1}{\partial \bar{\theta}} + \frac{\partial \phi_1}{\partial E} = \eta_1\frac{\partial u}{\partial \bar{\omega}} + g_1 \tag{31}$$

$$H_1 + U\frac{\partial \chi_1}{\partial \bar{\theta}} + \frac{\partial \chi_1}{\partial E} = h_1 \tag{32}$$

ε^2 Terms

$$F_2 + F_1\frac{\partial \eta_1}{\partial \bar{\omega}} + G_1\frac{\partial \eta_1}{\partial \bar{\theta}} + H_1\frac{\partial \eta_1}{\partial \bar{\alpha}} + U\frac{\partial \eta_2}{\partial \bar{\theta}} + \frac{\partial \eta_2}{\partial E} = f_2 \tag{33}$$

$$G_2 + F_1 \frac{\partial \phi_1}{\partial \bar{\omega}} + G_1 \frac{\partial \phi_1}{\partial \bar{\theta}} + H_1 \frac{\partial \phi_1}{\partial \bar{\alpha}} + U \frac{\partial \phi_2}{\partial \bar{\theta}} + \frac{\partial \phi_2}{\partial E} =$$

(34)

$$= \eta_2 \frac{\partial u}{\partial \bar{\omega}} + \frac{1}{2}\eta_1^2 \frac{\partial^2 u}{\partial \bar{\omega}^2} + \frac{1}{2}\eta_1 \frac{\partial g_1}{\partial \bar{\omega}} + \chi_1 \frac{\partial g_1}{\partial \bar{\alpha}} + g_2$$

$$H_2 + F_1 \frac{\partial \chi_1}{\partial \bar{\omega}} + G_1 \frac{\partial \chi_1}{\partial \bar{\theta}} + U \frac{\partial \chi_2}{\partial \bar{\theta}} + \frac{\partial \chi_2}{\partial E} =$$

(35)

$$= \frac{1}{2}\eta_1 \frac{\partial h_1}{\partial \bar{\omega}} + \frac{1}{2}\chi_1 \frac{\partial h_1}{\partial \bar{\alpha}} + h_2$$

Note that all dependent variables in equations (29) through (35) are barred.

Equation (30) has two unknown functions F_1 and η_1. Thus, define

$$F_1 = 0 \tag{36}$$

Then, η_1 can also be set to zero, i.e.

$$\eta_1 = 0 \tag{37}$$

In equation (31), make use of equation (37) and define

$$G_1 = \frac{1}{2\pi} \int_0^{2\pi} g_1 \, d\,\bar{\theta}$$

Since g_1 does not depend on $\bar{\theta}$,

$$G_1\ (\bar{\omega},\bar{\alpha},E) = g_1(\bar{\omega},\bar{\alpha},E) \tag{38}$$

The defining equation for ϕ_1 is therefore

$$U\ \frac{\partial \phi_1}{\partial \bar{\theta}} + \frac{\partial \phi_1}{\partial E} = 0$$

so that

$$\phi_1 = 0 \tag{39}$$

Similarly, from equation (32),

$$H_1\ (\bar{\omega},\bar{\alpha},E) = h_1\ (\bar{\omega},\bar{\alpha},E) \tag{40}$$

and

$$\chi_1 = 0 \tag{41}$$

There has been established the following important results:

(i) All first order terms in the transformation equations (27) are zero.

(ii) The averaged KS frequency $\bar{\omega}$ is constant to first order.

(iii) The first order terms in the derivatives of the remaining averaged KS elements are unchanged from the original equations.

Now the second order equations (33), (34) and (35) will be used. Making use of $\eta_1 = 0$, equation (33) becomes

$$F_2 + U \frac{\partial \eta_2}{\partial \bar{\theta}} + \frac{\partial \eta_2}{\partial E} = f_2$$

The above equation has two unknown functions, so one can be arbitrarily defined. Remember η_2 is required to be purely periodic in $\bar{\theta}$ and/or E. Therefore, set

$$F_2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f_2 \, d\bar{\theta} dE$$

But recall that f_2 is purely periodic in $\bar{\theta}$ (since it contains only tesseral terms). Thus, F_2 is zero and η_2 is defined by the equation

$$U \frac{\partial \eta_2}{\partial \bar{\theta}} + \frac{\partial \eta_2}{\partial E} = f_2 \qquad (42)$$

Making use of similar arguments, it is found that

$$G_2 = 0 \quad , \qquad (43)$$

$$H_2 = 0 \quad , \qquad (44)$$

and ϕ_2 and χ_2 are defined, respectively, by the equations

$$U \frac{\partial \phi_2}{\partial \bar{\theta}} + \frac{\partial \phi_2}{\partial E} = \eta_2 \cdot \frac{\partial u}{\partial \bar{\omega}} + g_2 \qquad (45)$$

$$U \frac{\partial \chi_2}{\partial \bar{\theta}} + \frac{\partial \chi_2}{\partial E} = h_2 \qquad (46)$$

The additional important results have been established:

(i) The derivative of $\bar{\omega}$ is zero through <u>second</u> order in ε .

(ii) The averaged differential equations do not depend on the tesseral terms, through second order in ε

(iii) Transformation between osculating and mean elements is given by the solution of partial differential equations.

If the power series expansion method is carried to third order, the following results involving $\bar{\omega}$ are obtained:

$$F_3 = 0 \quad , \tag{47}$$

$$U \frac{\partial \eta_3}{\partial \bar{\theta}} + \frac{\partial \eta_3}{\partial E} = - \left[G_1 \frac{\partial \eta_2}{\partial \bar{\theta}} + H_1 \frac{\partial \eta_2}{\partial \bar{\alpha}} \right] \quad . \tag{48}$$

Thus, $\bar{\omega}$ is constant through <u>third</u> order in ε . The function η_3 is defined by a partial differential equation, the right hand side of which contains coupling effects between tesseral and zonal geopotential terms.

Finally, the averaged differential equations are:

$$\frac{d\bar{\omega}}{dE} = O(\varepsilon^4) \tag{49a}$$

$$\frac{d\bar{\theta}}{dE} = U(\bar{\omega}) + \varepsilon\, g_1\, (\bar{\omega}, \bar{\alpha}, E) + O(\varepsilon^3) \quad , \tag{49b}$$

$$\frac{d\bar{\alpha}}{dE} = \varepsilon\, h_1\, (\bar{\omega}, \bar{\alpha}, E) + O(\varepsilon^3) \tag{49c}$$

The transformation equations between mean and instantaneous elements are:

$$\omega = \bar{\omega} + \varepsilon^2 \eta_2 (\bar{\omega},\bar{\theta},\bar{\alpha},E) + O(\varepsilon^3) \qquad (50a)$$

$$\theta = \bar{\theta} + \varepsilon^2 \phi_2 (\bar{\omega},\bar{\theta},\bar{\alpha},E) + O(\varepsilon^3) \quad , \qquad (50b)$$

$$\alpha = \bar{\alpha} + \varepsilon^2 \chi_2 (\bar{\omega},\bar{\theta},\bar{\alpha},E) + O(\varepsilon^3) \qquad (50c)$$

Based on the averaging theorem discussed in Section 1.1, it is expected that the averaged solution will differ from the exact solution by order ε .

2.3 Computational Algorithm

It will be necessary to express the averaged elements in terms of the instantaneous elements. The required transformation can be obtained from equations (50) by observing that

$$\eta_2 (\omega,\theta,\alpha,E) = \eta_2 (\bar{\omega},\bar{\theta},\bar{\alpha},E) + O(\varepsilon^2) \quad ,$$

with similar expressions for ϕ_2 and χ_2 . Therefore, the required transformation equations are obtained by "reversing" (50):

$$\bar{\omega} = \omega + \varepsilon^2 \eta_2 (\omega,\theta,\alpha,E) + O(\varepsilon^3) \quad , \qquad (51a)$$

$$\bar{\theta} = \theta + \varepsilon^2 \phi_2 (\omega,\theta,\alpha,E) + O(\varepsilon^3) \quad , \qquad (51b)$$

$$\bar{\alpha} = \alpha + \varepsilon^2 \chi_2 (\omega,\theta,\alpha,E) + O(\varepsilon^3) \qquad (51c)$$

The computational algoritam for solving the differential equations is as follows:

(1) Solve the partial differential equations (42), (45) and (46). Notice that the equations depend only on the tesseral geopotential terms and that the osculating values of the elements ω , Θ , α are used. This is allowed because of equations (51).

(2) Use the values of η_2 , ϕ_2 and χ_2 obtained in step (1) to obtain $\bar{\omega}$, $\bar{\Theta}$ and $\bar{\alpha}$ by equations (51).

(3) Solve the averaged differential equations (49) by using the initial conditions obtained from step (2).

3.0 APPLICATIONS OF AVERAGED KS ELEMENTS

In this section, the averaged KS equations will be applied to satellite orbit prediction problems. By making use of the theory developed in Section 2, it will be found that the objectives stated in Section 1.4 are realized.

3.1 Numerical Solution of the Partial Differential Equation

It has been shown that the averaged elements were determined by functions that are solutions of partial differential equations. Nothing has been said, so far, about the solution of these equations. A numerical solution algorithm will be developed in this section.

First, an important simplification can be made. Consider the averaged differential equations (49) and make the following observations:

(i) The right sides of equations (49) do not depend on $\bar{\theta}$. Therefore, any small initial error in $\bar{\theta}$ (i.e. in time) will remain constant and will not affect the solution.

(ii) If $\bar{\alpha}$ is replaced by α in g_1 and h_1 , the error is of order ε^3 . Since this error is of the same order as neglected terms in equations (49b) and (49c), it is permissible to insert α into g_1 and h_1 .

(iii) The function U is a secular term depending only on $\bar{\omega}$. Therefore it is important that $\bar{\omega}$ be accurate, otherwise there would be a secular (linearly increasing) error. This is the source of the error demonstrated in reference 9.

(iv) Since $\bar{\omega}$ is a constant, it is necessary only to <u>initialize</u> $\bar{\omega}$, using equation (51a).

From the above considerations, it is concluded that only the partial differential equation (42) for η_2 needs to be solved.

Consider the equation for η_2 :

$$U \frac{\partial \eta_2}{\partial \Theta} + \frac{\partial \eta_2}{\partial E} = f_2 \ (\omega, \Theta, \vec{\alpha}, \vec{\beta}, E) \tag{52}$$

where the bars have been removed from the dependent variables since, according to Section 2.3 and equation (51a), it is allowed to express η_2 in terms of the unaveraged elements. Also, the dependence of f_2 on all the KS elements is shown in equation (52).

A method of solving equation (52) is to express the right side as a truncated double fourier series in the variables Θ and E . This idea is presented in section 28 of reference 6 for a similar problem. The expression for f_2 is then

$$\begin{aligned} f_2 \ (\Theta, E) \cong & \frac{1}{4} a_{oo} + \frac{1}{2} \sum_{i=1}^{L} (a_{io} \cos i\Theta + \bar{a}_{io} \sin i\Theta) \\ & + \frac{1}{2} \sum_{j=1}^{M} (a_{oj} \cos jE + b_{oj} \sin jE) \\ & + \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{M} \Big[(a_{ij} - \bar{b}_{ij}) \cos (i\Theta + jE) \\ & + (a_{ij} + \bar{b}_{ij}) \cos (i\Theta - jE) \\ & + (b_{ij} + \bar{a}_{ij}) \sin (i\Theta + jE) \\ & - (b_{ij} - \bar{a}_{ij}) \sin (i\Theta - jE) \end{aligned} \tag{53}$$

The coefficients a_{ij} , $\bar{a}_{ij}$, b_{ij} , $\bar{b}_{ij}$ depend on $\vec{\alpha}$, $\vec{\beta}$ and ω only. Numerical algorithms for computing the coefficients are given in the Appendix. These algorithms make use of the fact that f_2 is a known function and can be computed by the routines in the KSFAST program (reference 11). See also the discussion given on page 163 of reference 6.

It is known that f_2 does not have a constant term or any terms independent of Θ . Therefore, some of the coefficients must be zero, i.e.

$$a_{oo} = 0 \tag{54}$$

$$a_{oj} = b_{oj} = 0 \quad , \qquad j = 1,2,3,\cdots.$$

The solution of equation (52) is then

$$\eta_2(\Theta,E) = \frac{1}{2U}\sum_{i=1}^{L}\frac{1}{i}(a_{io}\,sin\ i\Theta - \bar{a}_{io}\,cos\ i\Theta)$$

$$+\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{M}\left[\frac{a_{ij}-\bar{b}_{ij}}{(iU+j)}\,sin\ (i\Theta+jE)\right.$$

$$+\frac{(a_{ij}+\bar{b}_{ij})}{(iU-j)}\,sin\ (i\Theta-jE) \tag{55}$$

$$-\frac{(b_{ij}+\bar{a}_{ij})}{iU+j}\,cos\ (i\Theta+jE)$$

$$\left.+\frac{(b_{ij}-\bar{a}_{ij})}{iU-j}\,cos\ (i\Theta-jE)\right] \quad .$$

3.2 Computational Algorithm

According to the discussion in Section 3.1, it is only necessary to compute the initial value of $\bar{\omega}$:

$$\bar{\omega}_o = \omega_o + \eta_2 (\theta_o, 0) \quad , \tag{56}$$

where ω_o is the initial KS frequency computed from the initial position, velocity and time. It is assumed that $E_o = 0$. Also,

$$\theta_o = \dot{\theta}_E \tau_o + \theta_{Eo} \tag{57}$$

For a complete discussion of the initialization of the KS elements, refer to section 19 of reference 6.

Inserting $\theta = \theta_o$ and $E = 0$ into equation (55), the required expression is

$$\begin{aligned} \eta_2 (\theta_o,0) = & \frac{1}{2U} \sum_{i=1}^{L} \frac{1}{i} (a_{io} \sin i\theta_o - \bar{a}_{io} \cos i\theta_o) \\ & + \sum_{i=1}^{L} \sum_{j=1}^{M} (i^2U^2-j^2)^{-1} \Big[(j\bar{b}_{ij}+iUa_{ij}) \sin i\theta_o \\ & + (jb_{ij}-iU\bar{a}_{ij}) \cos i\theta_o \Big] \end{aligned} \tag{58}$$

The compucation of satellite orbits is carried out in the following steps:

(1) Compute the initial KS elements $\vec{\alpha}_o$, $\vec{\beta}_o$, ω_o ,

τ_o from the initial satellite state vector, by making use of the transformation equations (reference 6, section 19). Be sure that the disturbing potential function V_o that is used to compute ω_o includes all the tesseral and zonal terms in the geopotential.

(2) Compute the coefficients a_{ij} , b_{ij} , $\bar{a}_{ij}$, $\bar{b}_{ij}$ using the equations in the Appendix.

(3) Evaluate equation (58) for $\eta_2(\theta_o,0)$ and then compute

$$\bar{\omega}_o = \omega_o - \eta_2(\theta_o,0)$$

(4) Replace ω_o with $\bar{\omega}_o$ in the set of initial KS elements.

(5) Carry out the numerical integration of the KS differential equations (3), (4), (5), (6) with the condition that the function V contains only the zonal geopotential terms. Atmospheric drag and third body perturbing accelerations are included in $\underline{P}$.

A computed trajectory based on the above algorithm will agree to within a few hundred meters to the true trajectory.

3.3 Satellite Trajectory Prediction Experiments

A set of numerical experiments has been carried out using the averaging option (tesseral initialization) in the KSFAST program (reference 11, section 4.9). Selection of this option causes execution of the satellite orbit prediction algorithm that was described in Section 3.2. The experiments described in this section demonstrate the computation time savings that are available with this option.

Two sets of initial conditions are considered. Perturbing effects on the satellite orbit are atmospheric drag and

a nonspherical, nonhomogenous earth. The geopotential nominally includes all terms through eighth order and eighth degree. The purpose of the comparisions is to demonstrate that if the averaging algorithm is used, computation time can be reduced by a factor of about 4 . This is because the tesseral geopotential terms no longer need to be included in the numerical integration.

The initial conditions for the two trajectories are:

Case 1

Altitude	300 km
Eccentricity	0
Inclination	28 deg.
Longitude	136.93 deg.
Latitude	0
Epoch January 1, 1977	22 hrs.

Case 2

Perigee Altitude	300 km
Apogee Altitude	500 km
Eccentricity	.015
Inclination	28 deg.
Longitude	136.93 deg.
Latitude	0
Epoch January 1, 1977	22 hrs.

Figures 1 and 2 show how the error in position will increase if the tesseral terms are not included in the numerical integration and no averaging is done. (Compare with the experiments in reference 9.) This error is almost entirely in the along-track direction and is caused by an incorrect "mean" mean motion. Therefore, the error grows linearly. The purpose of the averaging method is to remove this linear error. A periodic error will still remain, but will be small (≈200 m.).

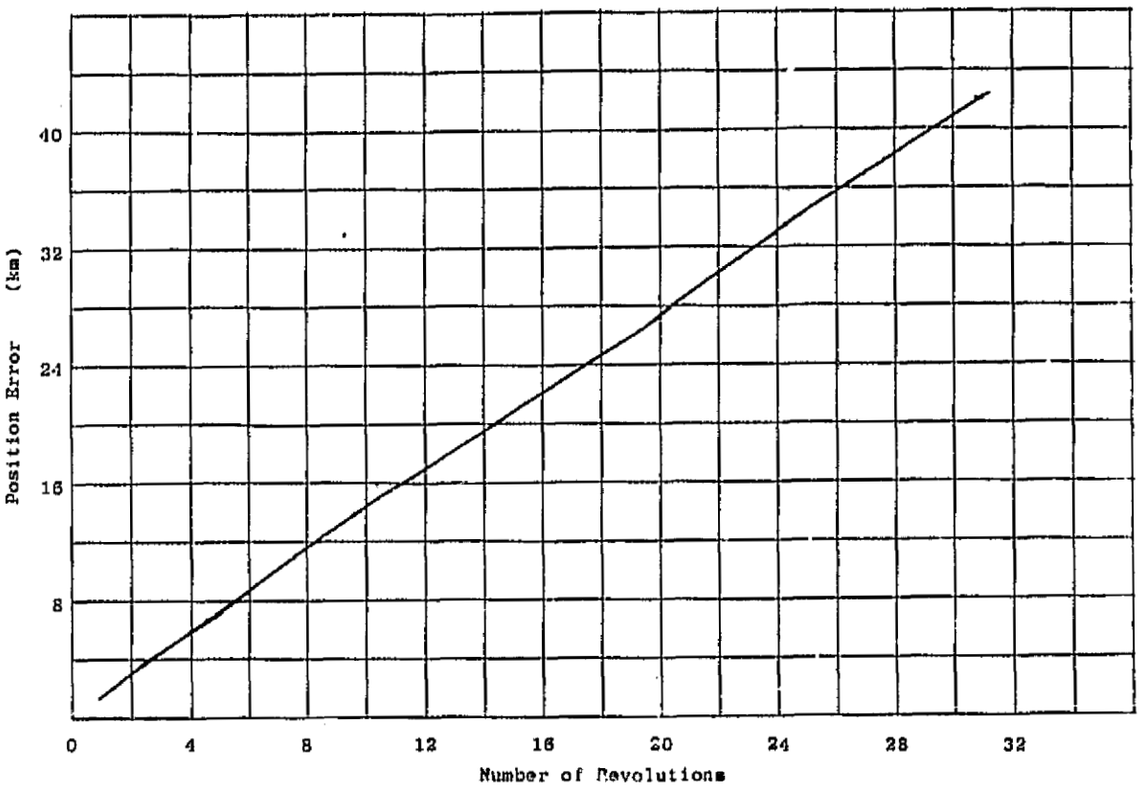

FIGURE 1: Position Error (No Average), Case 1

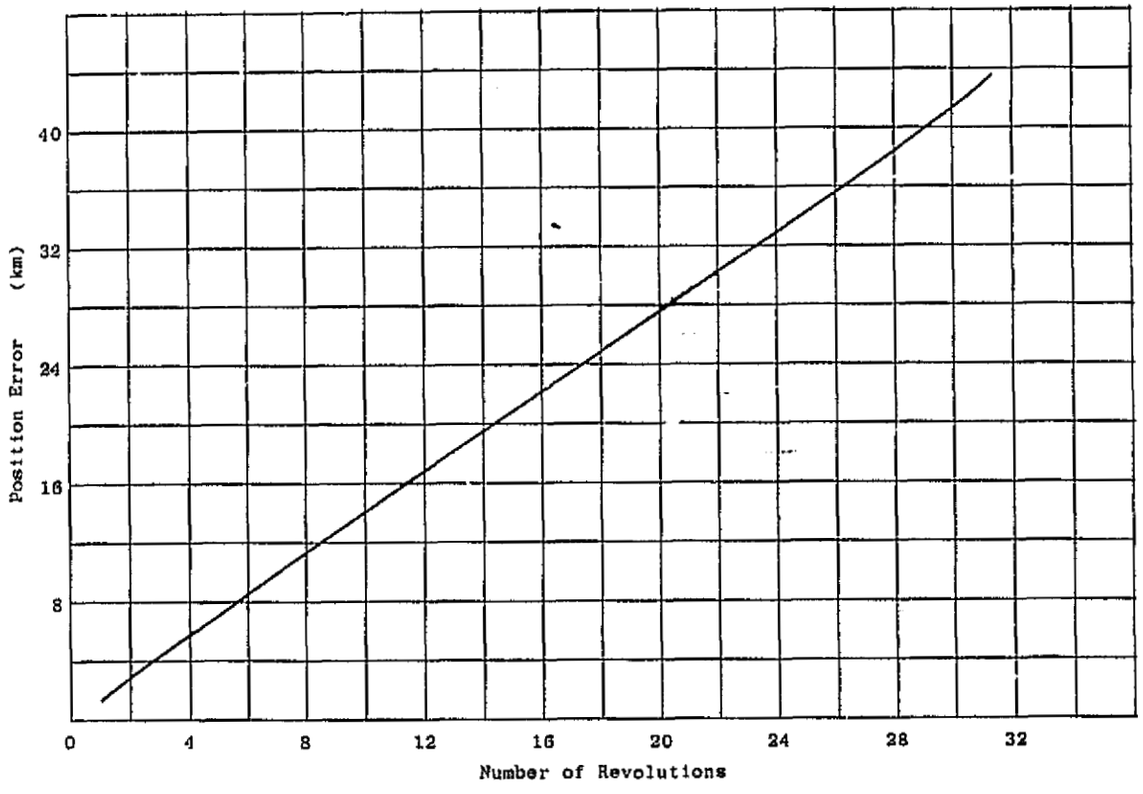

FIGURE 2: Position Error (No Average), Case 2

Results of satellite trajectory prediction experiments are displayed in Tables I and II. Two types of comparisons are made:

(1) Position error - The errors for the averaged method are shown, as well as the No Average case. In both cases, tesseral terms are neglected. However, in the No Average case, the averaging is not done. Almost all of the position error in each case is in the along-track direction.

(2) Computation time - The times are shown for the Averaged case (no tesseral terms) and a Precision case (all tesseral terms included in the numerical integration).

Discussion of Results

(1) The averaging method effectively reduces the linear growth of the along-track error in position. Periodic errors remain and are on the order of a few hundred meters.

(2) The larger errors found in Case 1 occur because of coupling between drag and tesseral terms. Case 1 is a rather 'ow circular orbit and the coupling effects are important. For Case 2, the satellite spends less time in the dense atmosphere so the coupling effect is not so important. Even so, the coupling is only important after about four days.

(3) The errors in the Averaged Method in both cases are so small that they would not show up on the scale of Figures 1 and 2.

(4) The run time comparisons of Tables I and II show that the Averaged Method is about three to four times more efficient than the Precision Method.

(5) About two seconds are required for the averaging initialization algorithm. Therefore, the Averaged Method is most efficient for prediction intervals of more than one day.

TABLE I.- NUMERICAL COMPARISONS, CASE 1

Prediction Interval (Days)	Position Error (km)		Computation Time (sec)	
	Averaged Method	No Average	Averaged Method	Precision Method
1	.10	21.4	3.5	8.8
2	.27	42.9	5.6	17.7
4	.80	86.3	9.9	35.4
6	1.58	130.3	14.1	53.1
8	2.74	174.8	18.4	70.7
10	4.40	220.0	22.6	88.4
15	11.06	337.0	33.2	132.6

TABLE II.- NUMERICAL COMPARISONS, CASE 2

Prediction Interval (Days)	Position Error (km)		Computation Time (sec)	
	Averaged Method	No Average	Averaged Method	Precision Method
1	.15	21.2	4.0	8.6
2	.29	43.2	6.1	17.3
4	.63	85.5	10.3	34.6
6	.41	126.7	14.4	51.9
8	.29	169.1	18.6	69.1
10	.48	214.4	22.7	86.4
15	.93	319.2	33.1	129.6

(6) The efficiency of the Averaged Method is because the tesseral terms do not need to be included in the step by step numerical integration. Also, a large stepsize can be taken because the averaged differential equations are more stable. This is because the KS frequency element $\bar{\omega}$ is nearly constant.

The errors of a few hundred meters in the solution by the Averaged Method are caused by the diurnal (daily) variations produced by the neglected tesseral terms. It would be possible to add these variations to the solution by converting back to osculating elements via equations (51). The position error would then be reduced to a few meters. This procedure could be added to the Averaged Method in case more accuracy is required. There would be a negligible increase in computation time since the back conversion would be done only for output.

A further refinement would eliminate much of the long term error growth of the Case 1 Averaged Method. As was mentioned above, this error is due to coupling between atmospheric drag and tesseral perturbations in the altitude. Since the density of the atmosphere increases exponentially with decreasing altitude, the small variations in altitude caused by tesseral perturbations become important over many revolutions. Equations (51) could be used to compute the actual altitude at each numerical integration step. This altitude would go into the density model. More accurate calculation of drag perturbations would result.

4.0 CONCLUSIONS

The Averaging Initialization Algorithm that is developed in this report can decrease the computation time required for predicting near earth satellite orbits. For prediction intervals of one or more days, the computation time will be reduced by a factor of three to four. The method is particularly suited to iterative applications where accurate in-orbit satellite position information is required.

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APPENDIX

TRIGONOMETRIC POLYNOMIAL APPROXIMATION IN TWO VARIABLES

Suppose that there exists a function $f(\theta,\phi)$ which is 2π periodic in both variables. It is required to establish a fourier approximation for f in the region

$$0 \le \theta \le 2\pi \qquad 0 \le \phi \le 2\pi$$

The approximation will be based on numerical evaluations of f at an odd number of values of the arguments. Therefore, let

$$\begin{aligned} x &= 0,\ 1,\ 2,\ \cdots,\ 2L \ , \\ y &= 0,\ 1,\ 2,\ \cdots,\ 2M \ , \end{aligned} \tag{A1}$$

where

$$x = \frac{2L+1}{2\pi}\,\theta \qquad y = \frac{2M+1}{2\pi}\,\phi \tag{A2}$$

The fourier approximation formula in one dimension is (reference 12)

$$f(x) \cong \frac{1}{2} A_o + \sum_{k=1}^{L}\left(A_k \cos \frac{2\pi}{2L+1} kx + B_k \sin \frac{2\pi}{2L+1} kx\right) \ , \tag{A3}$$

where the coefficients are

$$A_j = \frac{2}{2L+1} \sum_{x=0}^{2L} f(x) \cos \frac{2\pi}{2L+1} j\,x \quad , \quad j = 0,1,\cdots,\ L, \tag{A4}$$

$$B_j = \frac{2}{2L+1} \sum_{x=0}^{2L} f(x) \sin \frac{2\pi}{2L+1} j\, x \quad , \quad j = 1,2,\cdots, L. \tag{A5}$$

Assume that the coefficients A_k and B_k in (A3) are functions of y. Then they can also be expressed as fourier approximations:

$$A_i(y) \cong \frac{1}{2} a_{io} + \sum_{j=1}^{M} \left(a_{ij} \cos \frac{2\pi}{2M+1} j\, y + b_{ij} \sin \frac{2\pi}{2M+1} j\, y \right) , \quad i = 0, 1, 2, \cdots L, \tag{A6}$$

$$B_i(y) \cong \frac{1}{2} \bar{a}_{io} + \sum_{j=1}^{M} \left(\bar{a}_{ij} \cos \frac{2\pi}{2M+1} j\, y + \bar{b}_{ij} \sin \frac{2\pi}{2M+1} j\, y \right) , \quad i = 1, 2, \cdots, L. \tag{A7}$$

Expressions (A6) and (A7) are inserted into (A3). Using equation (A2) and trigonometric identities, the double fourier approximation is

$$f(\theta,\phi) \cong \frac{1}{4} a_{oo} + \frac{1}{2} \sum_{i=1}^{L} (a_{io} \cos i\,\theta + \bar{a}_{io} \sin i\,\theta) + \frac{1}{2} \sum_{j=1}^{M} (a_{oj} \cos j\,\phi + b_{oj} \sin j\,\phi) + \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{M} \Big[(a_{ij} - \bar{b}_{ij}) \cos (i\,\theta + j\,\phi) \tag{A8}$$

$$+ (a_{ij} + \bar{b}_{ij}) \cos (i\,\theta - j\,\phi)$$
$$+ (b_{ij} + \bar{a}_{ij}) \sin (i\,\theta + j\,\phi) \qquad \text{(A8)}$$
$$- (b_{ij} - \bar{a}_{ij}) \sin (i\,\theta - j\,\phi)\Big]$$

The coefficients are computed from the following expressions:

$$a_{ij} = \kappa \sum_{y=0}^{2M} \sum_{x=0}^{2L} f(x,y) \cos (ix\gamma) \cos (jy\delta) \quad ,$$
$$i = 0, 1, 2, \cdots, L,$$
$$j = 0, 1, 2, \cdots, M,$$

$$b_{ij} = \kappa \sum_{y=0}^{2M} \sum_{x=0}^{2L} f(x,y) \cos (ix\gamma) \sin (jy\delta) \quad ,$$
$$i = 0, 1, 2, \cdots, L.$$
$$j = 1, 2, \cdots, L,$$

$$\bar{a}_{ij} = \kappa \sum_{y=0}^{2M} \sum_{x=0}^{2L} f(x,y) \sin (ix\gamma) \cos (jy\delta) \quad ,$$
$$i = 1, 2, \cdots, L,$$
$$j = 0, 1, 2, \cdots, M,$$

$$\bar{b}_{ij} = \kappa \sum_{y=0}^{2M} \sum_{x=0}^{2L} f(x,y) \sin (ix\gamma) \cos (jy\delta) \quad ,$$
$$i = 1, 2, \cdots, L,$$
$$j = 1, 2, \cdots, M.$$

The following abbreviations were used in the above expressions:

$$\kappa = \frac{4}{(2M+1)\,(2L+1)}$$

$$\gamma = \frac{2\pi}{2L+1} \qquad \delta = \frac{2\pi}{2M+1}$$