(NASA-CR-155589) DESIGN OF NONINTERACTING FLIGHT CONTROL SYSTEMS IN THE PRESENCE OF LARGE PARAMETER VARIATIONS Final Report, 1 Jul. 1975 - 30 Nov. 1977 (Colorado Univ.) 86 p HC A05/MF A01 CSCL 01C G3/08

N78-16064

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DESIGN OF NONINTERACTING FLIGHT CONTROL SYSTEMS IN THE PRESENCE OF LARGE PARAMETER VARIATIONS

Final Report
July 1, 1975 - November 30, 1977

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NASA Grant: NSG 1213

ABSTRACT

This report summarizes the progress under NASA Grant NSG 1213 from July 1, 1975 to November 30, 1977. Section I provides a presentation of the problem under consideration. In Section II, we consider the effects of system parameter variations on the over-all system stability. Section III contains results from the application of numerical minimization to reduce system coupling. In Section V, some future research directions are outlined.

I. Problem description

Consider a given linear time-invarient system described by the following equations:

$$\dot{x}(t) = Ax(t) + Bu(t) + Lw_g(t) \tag{1a}$$

$$y_m(t) = Cx(t) + w_s(t) \tag{1b}$$

$$y_c(t) = Dx(t) \tag{1c}$$

where $x(t)$ is the state vector, $u(t)$ is the input vector, $w_g(t)$ is the gust disturbance, $y_m(t)$ is the measurable output, $y_c(t)$ is the output vector to be controlled. The gust $w_g(t)$ is assumed to satisfy the following equation:

$$\dot{w}_g(t) = F_1 w_g(t) + F_2 \eta(t).$$

η and w_s are white noises.

In the above system, we assume that the matrices A B, L, F_1 and F_2 may vary under different operating conditions. Let A_0 and B_0 denote the nominal values of A and B at some chosen operating condition.

Since the full state vector $x(t)$ is not directly accessible, the following full-order state observer is used to estimate $x(t)$:

$$\dot{\hat{x}}(t) = (A_0-GC)\hat{x}(t) + B_0 u(t) + G[Cx(t)+w_s(t)] \tag{2}$$

where G matrix is chosen to assign a set of desired poles for (A_0-GC) in order to guarantee $\hat{x}(t) \to x(t)$ as $t \to \infty$.

The above estimated state $\hat{x}(t)$ is used in the following feedback control law:

$$u(t) = Fv(t) + K\hat{x}(t)$$

where $v(t)$ is the external command input. By using decoupling theory, one can find appropriate matrices F and K such that the transfer function from v to y_c is diagonal.

The complete system is described by the following equations:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\begin{bmatrix} v \\ w_s \\ \eta \end{bmatrix} \tag{3a}$$

$$y_c(t) = \bar{C}\bar{x}(t) \tag{3b}$$

where

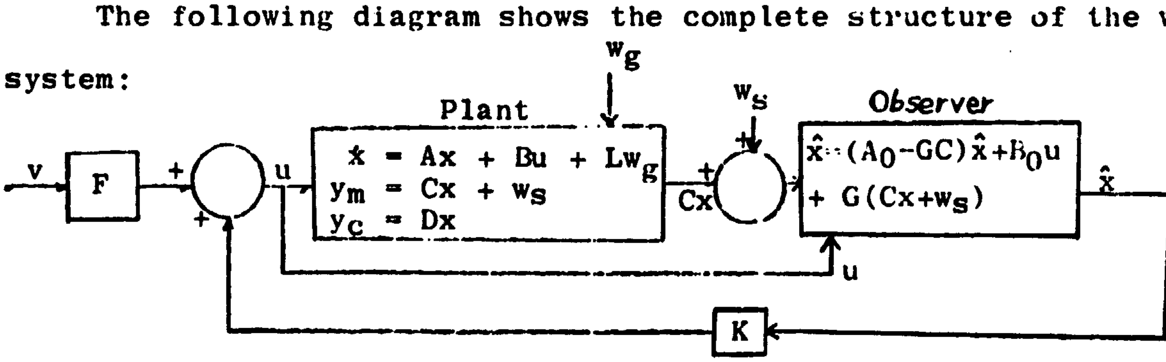
$$\bar{A} = \begin{bmatrix} A & BK & L \\ GC & A_0-GC+B_0K & 0 \\ 0 & 0 & F_1 \end{bmatrix} \tag{4a}$$

$$\bar{B} = \begin{bmatrix} BF & 0 & 0 \\ B_0F & G & 0 \\ 0 & 0 & F_2 \end{bmatrix} \tag{4b}$$

$$\bar{C} = [\, D \quad 0 \quad 0 \,] \tag{4c}$$

$$\bar{x} = \begin{bmatrix} x \\ \hat{x} \\ w_g \end{bmatrix} \tag{4d}$$

The following diagram shows the complete structure of the whole system:



The aircraft model studied is based on the report by Miller, Deal and Champine[1]. The system input consists of throttle deflection, horizontal-tail deflection and flap deflection. The system state consists of pitch angle, pitch rate, angle of attack and forward speed. The system outputs to be controlled are flight path angle, pitch angle and forward speed. The measurable outputs are pitch rate and forward speed. For different values of angle of attack, the system parameters A and B vary. We choose $\alpha = 10^\circ$ as the trim flight condition and its parameters as the nominal A_o and B_o. From the decoupling theory, the decoupled system transfer function has the following form:

$$\begin{bmatrix} \gamma \\ \hat{\theta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} \dfrac{\lambda_1}{s - \sigma_{11}} & 0 & 0 \\ 0 & \dfrac{\lambda_2}{s^2 - \sigma_{21} s - \sigma_{22}} & 0 \\ 0 & 0 & \dfrac{\lambda_3}{s - \sigma_{31}} \end{bmatrix} \begin{bmatrix} V_\gamma \\ \hat{V}_\theta \\ \hat{V}_u \end{bmatrix}$$

where λ_1, λ_2, λ_3, σ_{11}, σ_{21}, σ_{22}, σ_{31} can be arbitrarily assigned by choosing appropriate feedback and feedforward matrices F and K.

The external disturbance α_g on the angle of attack α is due to the vertical wind of gust w_g,

$$\alpha_g = \frac{w_g}{u}$$

u is the forward velocity and w_g is assumed to have Dryden power spectrum density of the form:

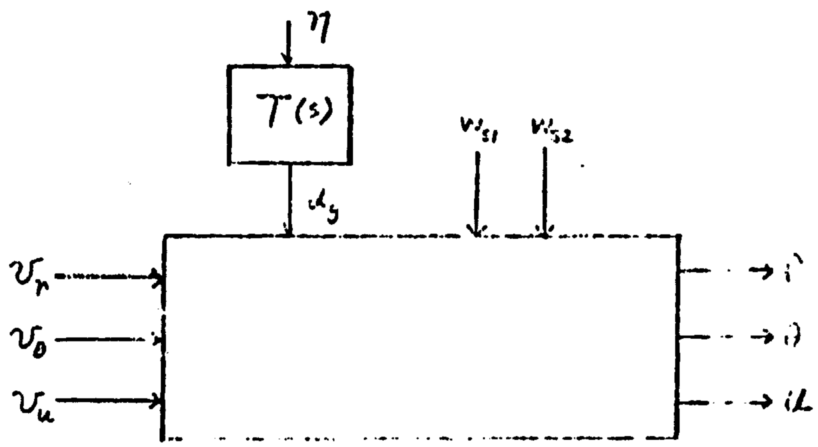
$$\Phi(\Omega) = \sigma_w^2 \frac{\ell w}{\pi} \frac{1 + 3\,(\ell w\Omega)^2}{(1 + (\ell w\Omega)^2)^2}$$

ℓw is the scale of turbulence, σ_w is the variance of the gust velocity. Hence α_g can be modeled as the output of a second-order filter

$$T(s) = \sigma_w \sqrt{\frac{\ell w}{\pi}} \frac{1 + \sqrt{3}\,\ell w s}{(1 + \ell w s)^2}$$

driven by white noise η. The sensor noises in the measurement of pitch rate and forward speed are assumed to be white noises w_{s1} and w_{s2} respectively.

The complete system has the following configuration:



$$\begin{bmatrix} \hat{\gamma} \\ \hat{\theta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{15} & h_{16} \\ h_{21} & h_{22} & h_{23} & h_{24} & h_{25} & h_{26} \\ h_{31} & h_{32} & h_{33} & h_{34} & h_{35} & h_{36} \end{bmatrix} \begin{bmatrix} \hat{v}_\gamma \\ \hat{v}_\theta \\ \hat{v}_u \\ \hat{w}_{s1} \\ \hat{w}_{s2} \\ \hat{\eta} \end{bmatrix}$$

II. Parameter variations and systems stability

The design of observer and feedback control law is based on the nominal system parameters A_o and B_o corresponding to 10 degrees of angle of attack. We have assumed that the angle of attack varies between 5 degrees and 15 degrees. Different sets of poles have been assigned to the observer and the system based on the nominal system parameters A_o and B_o. It turns out that the overall system poles are very sensitive to system parameter variations. In Figure 1 and Figure 2, the overall system root locus are plotted as the angle of attack varies from 5 degrees to 15 degrees. For the case in which the nominal system poles are -8, -5, -5, and -10, the system remains stable in the whole range of variations of α. However, if one assigns -8, $-9 \pm 7j$ and -10 as the nominal system poles, the overall system becomes unstable as α approaches 5 degrees. In the former case, the time history of the system response are shown in Figure 3-6. It can be seen that system parameter variations have caused significant coupling among system inputs and outputs. Hence the straightforward application of observer theory and decoupling theory to flight control system may not always be practical.

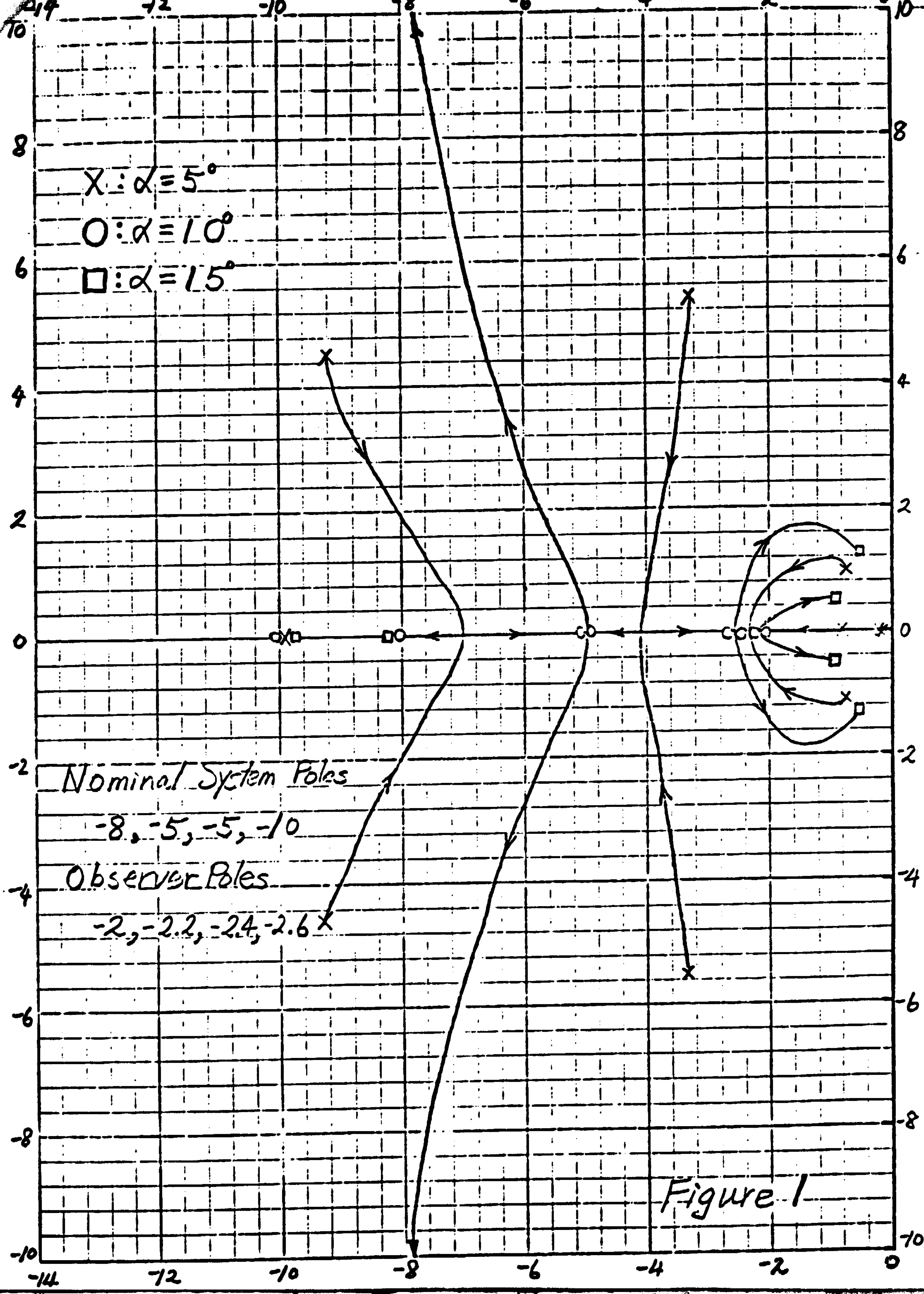

X : α = 5°
O : α = 10°
□ : α = 15°
Nominal System Poles
-8, -5, -5, -10
Observer Poles
-2, -2.2, -2.4, -2.6

Figure 1

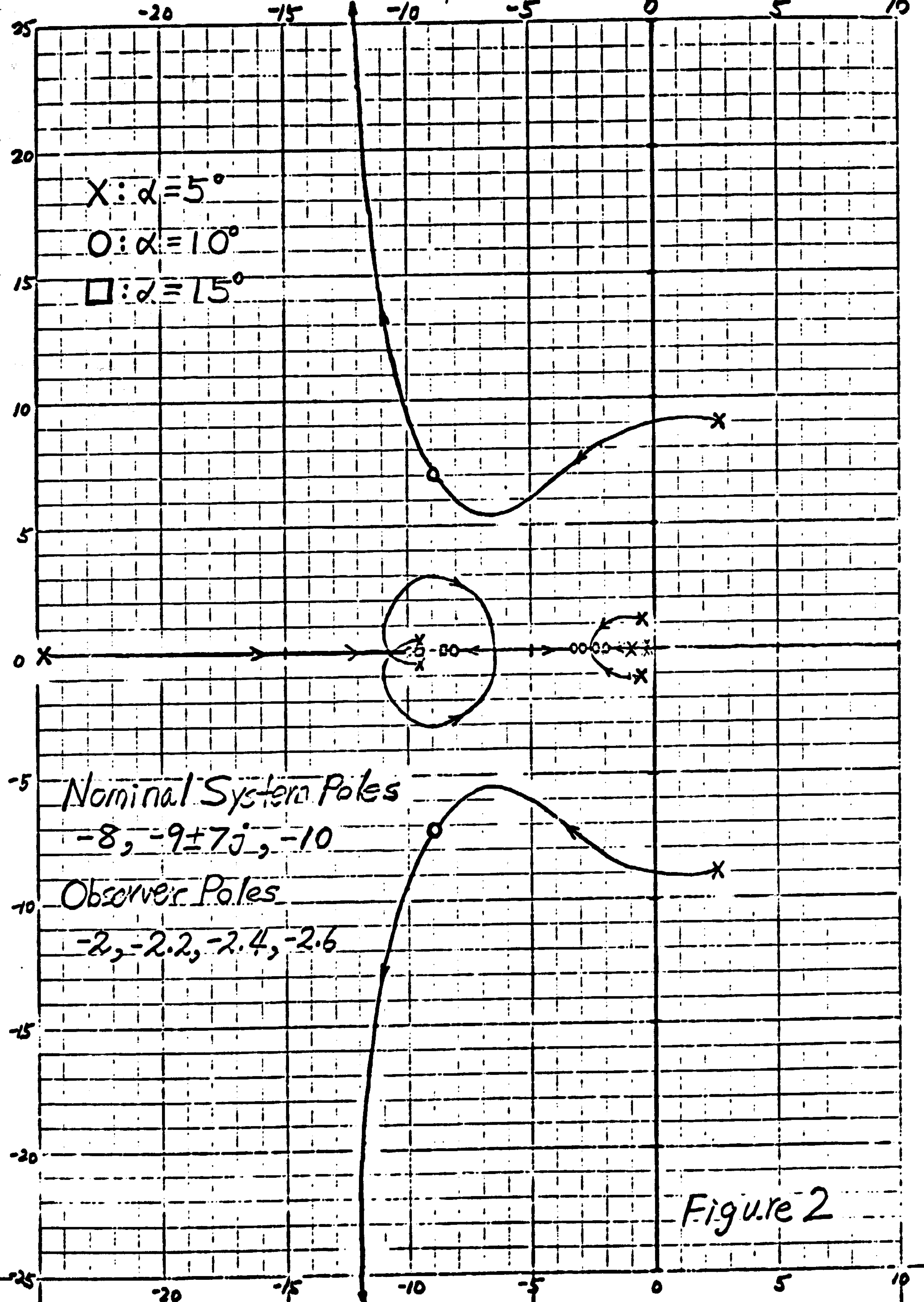
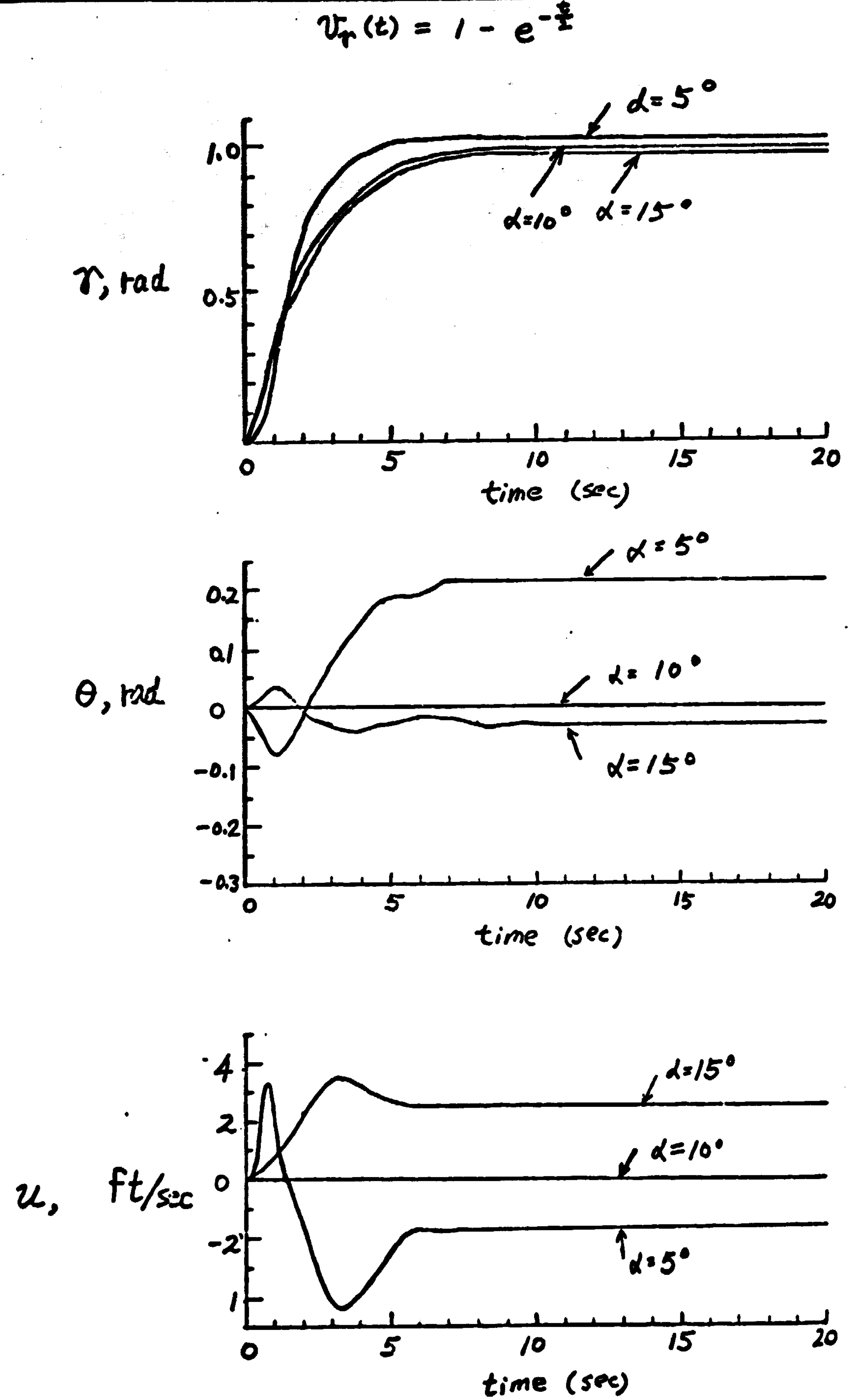

X : α = 5°
O : α = 10°
□ : α = 15°
Nominal System Poles
-8, -9±7j, -10
Observer Poles
-2, -2.2, -2.4, -2.6

Figure 2



$v_r(t) = 1 - e^{-\frac{t}{2}}$
$\alpha = 5°$
$\alpha = 10°$
$\alpha = 15°$
γ, rad
1.0
0.5
0
5
10
15
20
time (sec)
$\alpha = 5°$
$\alpha = 10°$
$\alpha = 15°$
θ, rad
0.2
0.1
0
-0.1
-0.2
-0.3
0
5
10
15
20
time (sec)
$\alpha = 15°$
$\alpha = 10°$
$\alpha = 5°$
u, ft/sec
4
2
0
-2
1
0
5
10
15
20
time (sec)

Figure 3

$$v_r(t) = 1 - e^{-\frac{t}{2}}$$

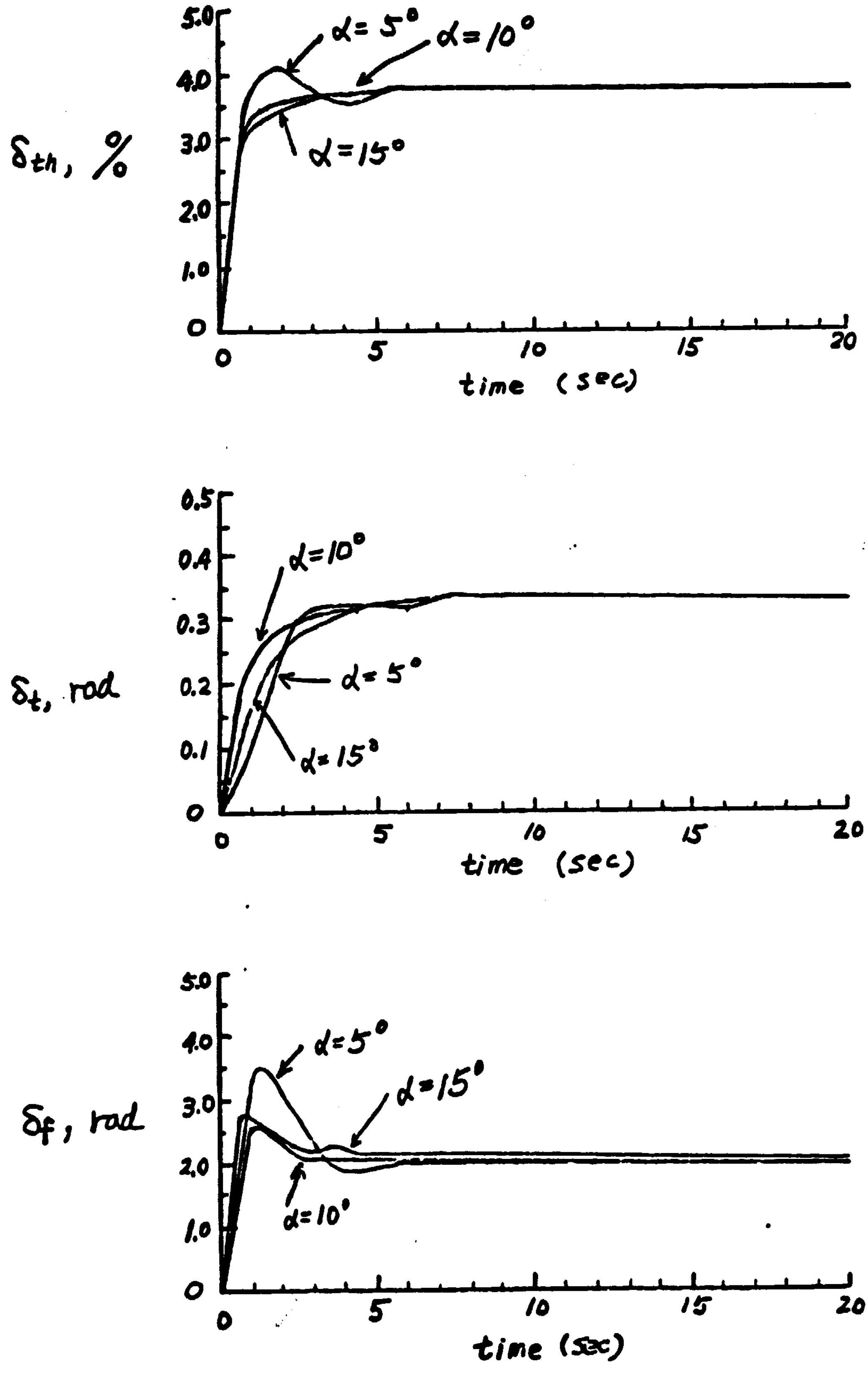


Figure 4

$$v_\theta = 1 - e^{-t}$$

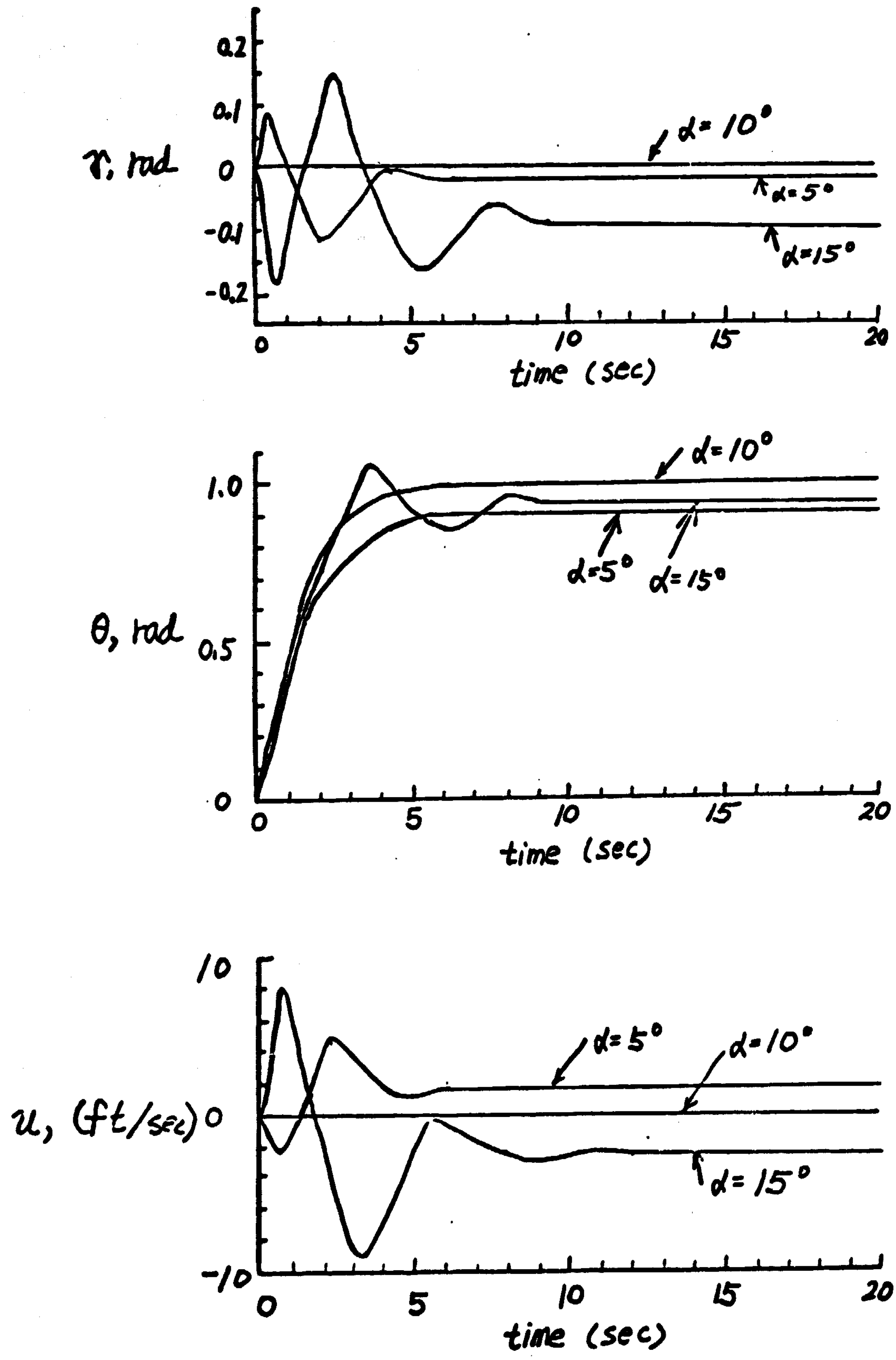
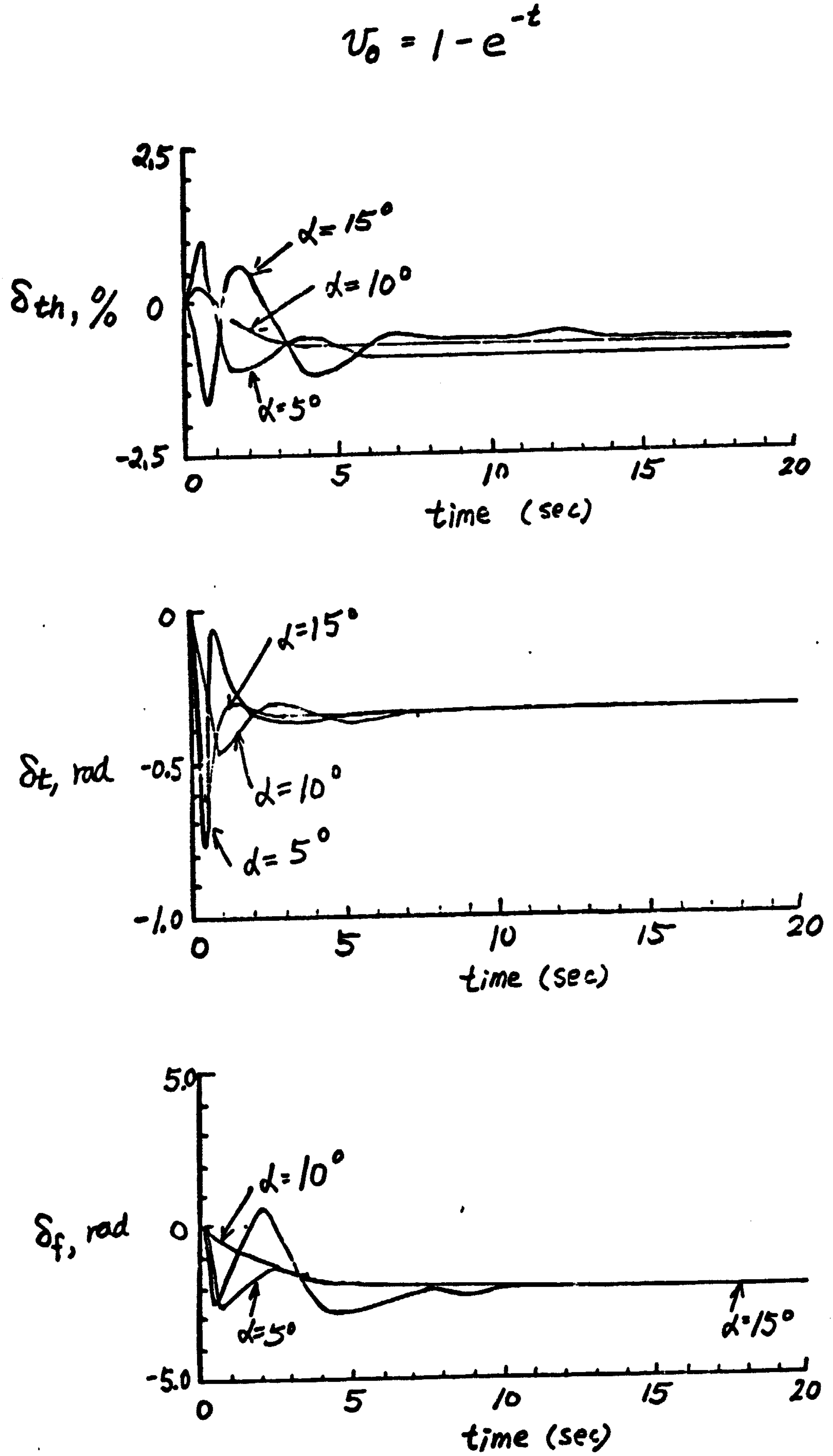


Figure 5



$v_0 = 1 - e^{-t}$
δ_{th}, %
2.5
0
-2.5
$\alpha = 15°$
$\alpha = 10°$
$\alpha = 5°$
0
5
10
15
20
time (sec)
δ_t, rad
0
-0.5
-1.0
$\alpha = 15°$
$\alpha = 10°$
$\alpha = 5°$
0
5
10
15
20
time (sec)
δ_f, rad
5.0
0
-5.0
$\alpha = 10°$
$\alpha = 5°$
$\alpha = 15°$
0
5
10
15
20
time (sec)

Figure 6

III. Reduction of coupling by using numerical minimization algorithm

In this section, we apply Box Complex Algorithm [2] to the adjustment of feedback and feedforward control law (F, K) in order to reduce the coupling of the system under parameter variations. Two different cases were tried; each with a different function to be minimized and different explicit and implicit constraints.

Case 1

The function to be minimized is

$$F_{OD} = \sum_{\alpha = 5^\circ, 10^\circ, 15^\circ} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \sum_{w=0}^{10} \left| h_{ij}(jw) \right|^2 ,$$

which is the sum of the magnitude squared of the off-diagonal elements of the over all system transfer function when alpha is 5°, 10° and 15°.

There are three constraints imposed when searching for optimal (F, K),

1) All elements of F and K, (totalling 21 in our case), can differ from the initial value of F and K (computed by Gilbert's decoupling program) by no more than ± 15.0. This provides an explicit feasible region for the search of F and K.

2) The determinat of F must be greater than certain value. This prevents the trival solution where F becomes a zero matrix.

3) The new pair (F,K) must produce a stable over-all system.

This requirement is guaranteed by using Routh criterion.

The results are summarized as follows:

Before optimization - F_{OD} = 13.52.

After optimization - F_{OD} = 0.94.

Let

$$F_{DG} = \sum_{\alpha=5^\circ,10^\circ,15^\circ} \sum_{i=1}^{3} \sum_{w=0}^{10} \left| h_{ii}(jw) \right|^2$$

be a measure of the magnitude of the diagonal elements of the over-all transfer function. Then we have the following:

Before optimization - F_{DG} = 57.40.

After optimization - F_{DG} = 33.90.

In the following table, we give details on the magnitudes of diagonal and off-diagonal elements of the over-all system transfer function when $\alpha = 5^\circ$, 10° and 15°.

	ALPHA (degrees)	Diagonal Elements	Off-Diagonal Elements
Before Opt.	5	21.54	9.07
	10	18.86	5.5×10^{-8}
	15	17.01	4.45
After Opt.	5	10.59	0.787
	10	11.76	0.0213
	15	11.54	0.130

Although there seems to be a substantial improvement in the reduction of coupling after optimization procedures, however, time histories of the optimized system show that there is not a significant difference. In fact, some steady state values of the optimized system actually increased as shown below:

Let H(s) be the transfer function of the decoupled system, i.e.,

$$\begin{bmatrix} \hat{\gamma} \\ \hat{\theta} \\ \hat{u} \end{bmatrix} = H(s) \begin{bmatrix} \hat{V}_T \\ \hat{V}_\theta \\ \hat{V}_u \end{bmatrix}.$$

The steady state values of the transfer function, i.e., $\lim_{s \to c} H(s)$, are shown below:

	Before Opt.			After Opt.		
$\alpha=5^\circ$	100.	2.7	6.7	100.	3.9	4.1
	19.	100.	8.8	24.2	100.	6.3
	1.64	2.1	100.	1.2	23.1	100.
$\alpha=10^\circ$	100.	0.007	0	100.	4.72	0.584
	0.001	100.	0	5.4	100.	2.9
	0.007	0.001	100.	3.34	21.96	100.
$\alpha=15^\circ$	100.	9.7	7.9	100.	3.8	9.1
	3.7	100.	2.7	0.04	100.	4.1
	2.6	2.8	100.	5.9	24.6	100.

Each column has been normalized so that the diagonal elements are 100. This result suggests that minimization of the steady state values of the coupling might improve the system performance. This has been carried out as follows:

Case 2

The function to be minimized is

$$F_{SOD} = \sum_{\alpha=5^\circ,10^\circ,15^\circ} \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \left| h_{ij}(0) \right|^2$$

which is the sum of the square of the steady-state value of the off-diagonal elements of transfer function, for alpha equals $5^\circ, 10^\circ$ and 15°.

The following constraints are imposed for the optimization algorithm:

1) All elements of F and K can differ from the initial values of F and K (computed by Gilbert's decoupling program for the nominal system, i.e, when $\alpha = 10^\circ$) by no more than $\pm$ 3.0.
2) The diagonal elements of H(0), i.e, $h_{11}(0)$, $h_{22}(0)$ and $h_{33}(0)$ are constrained to be greater than certain positive values.
3) Every pair (F,K) should result in a stable system.

The results are as follows:

Before optimization - F_{SDA} = 7.85×10^{-2}
After optimization - F_{SDA} = 6.73×10^{-2}.

Let

$$F_{SDG} = \sum_{\alpha=5^\circ,10^\circ,15^\circ} \sum_{i=1}^{3} \left| h_{ii}(0) \right|^2$$

Before optimization - F_{SDG} = 8.95

After optimization - F_{SDG} = 7.69

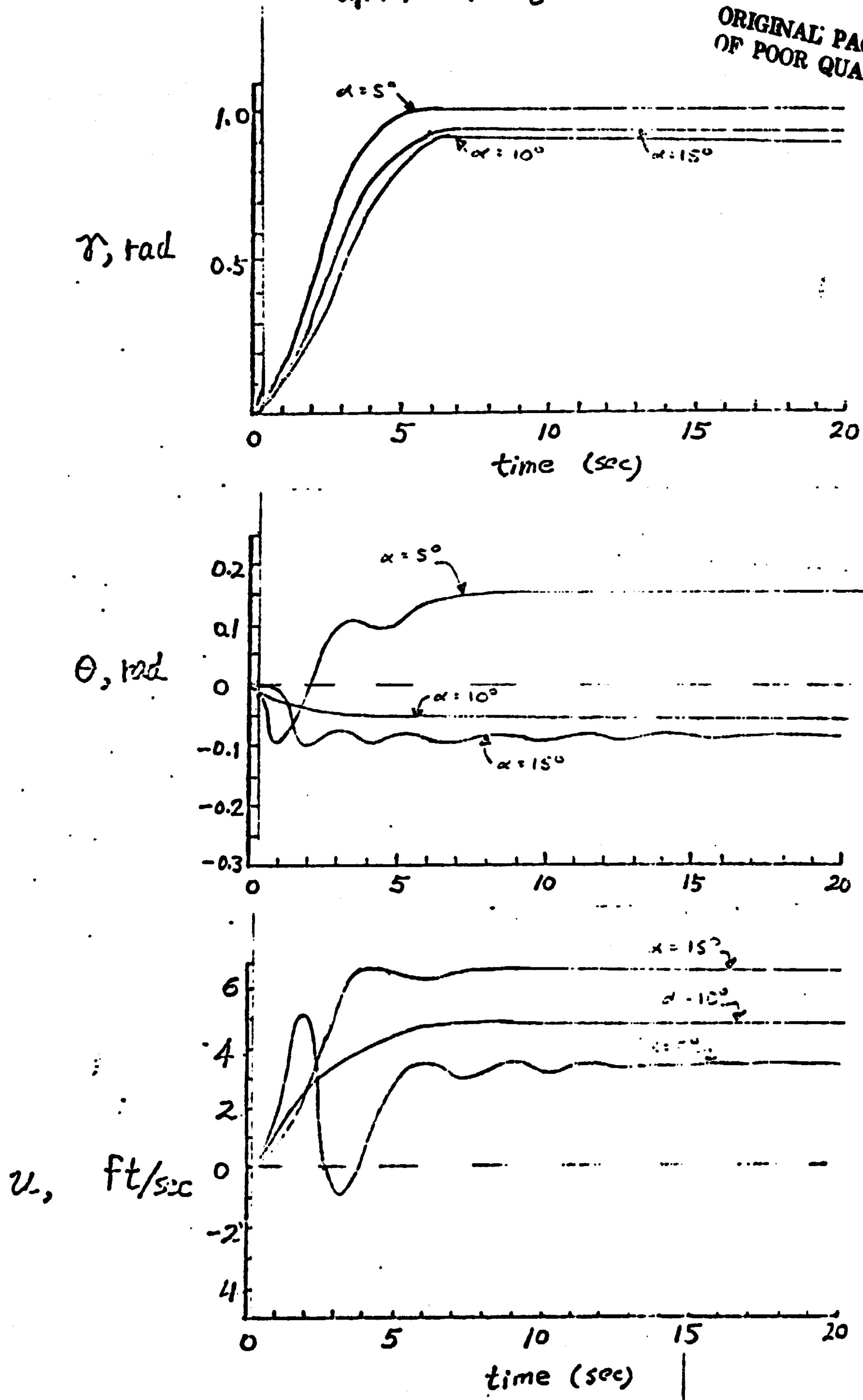
The following table shows the steady state values of the transfer function matrix before and after optimization, (normalized by columns with diagonal elements being 100)

	Before Optimization			After Optimization		
$\alpha = 5^{\circ}$	100.	2.7	6.7	100.	1.14	9.9
	19.	100.	8.8	14.8	100.	4.4
	1.64	2.1	100.	3.5	0.69	100.
$\alpha = 10^{\circ}$	100.	0.007	0	100.	2.8	3.1
	0.001	100.	0	6.1	100.	3.9
	0.007	0.001	100.	5.6	2.3	100.
$\alpha = 15^{\circ}$	100.	9.7	7.5	100.	2.0	4.3
	3.7	100.	3.7	9.4	100.	0.5
	2.6	2.8	100.	8.02	5.0	100.

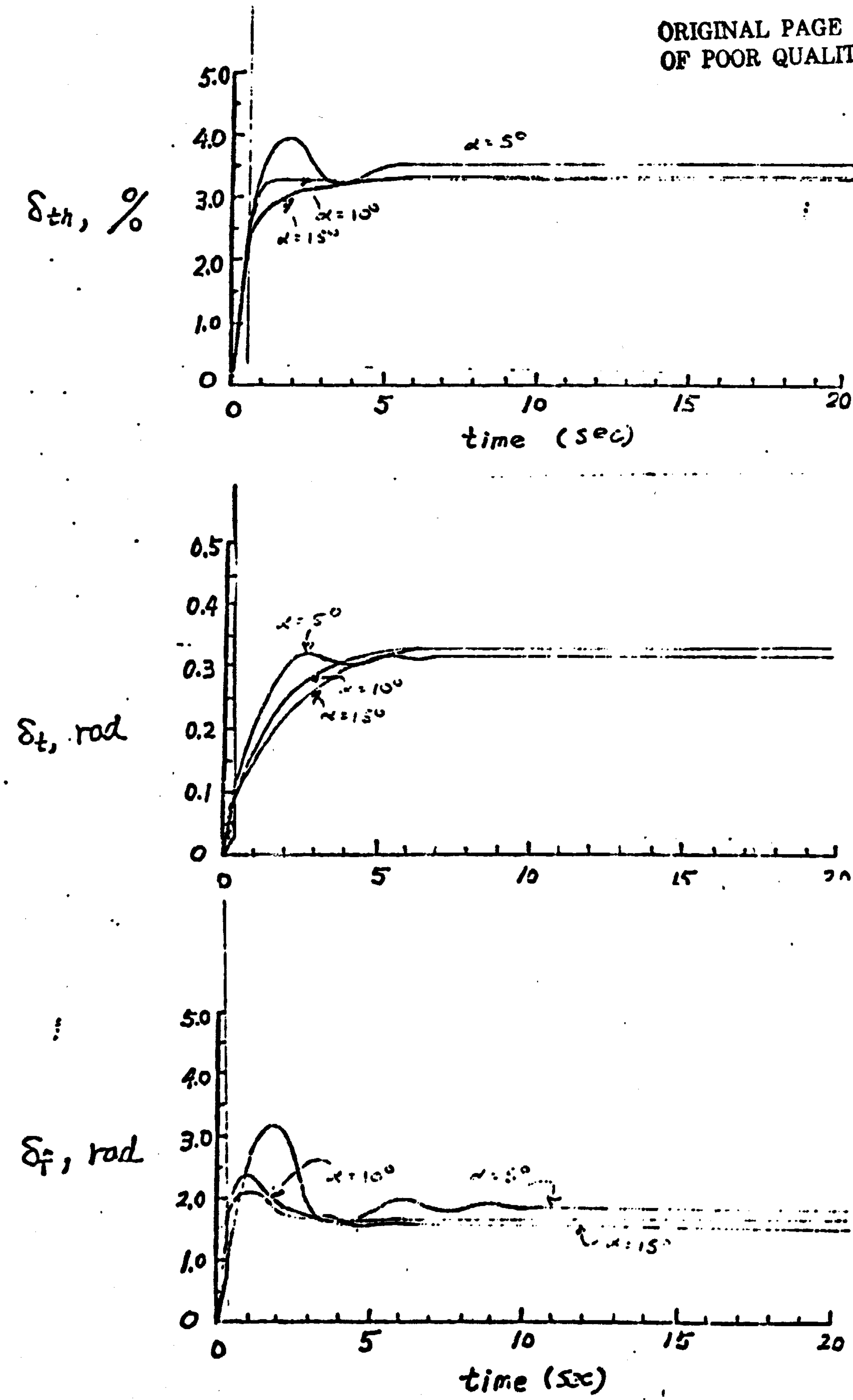
Time histories for the decoupled system with optimized (F,K) are given in the next four pages. It can be seen that there is an improvement in the steady-state coupling for the (2,1) element, which has been reduced from 19 to 14.8 for the $\alpha = 5^{\circ}$ case. For other elements such as the (3,1) element in the $\alpha = 15^{\circ}$ case, the steady state value has been increased about three times.

Based on these results, it seems that the decoupling matrices (F, K) has to be finely adjusted for the parameter variations in (A,B,C). Although we are able to find (F,K) through optimization procedure so that some couplings are reduced, however, it seems to be very difficult to achieve significant reductions for all couplings. Therefore, an alternative approach will be adopted. Details are given in Section IV.

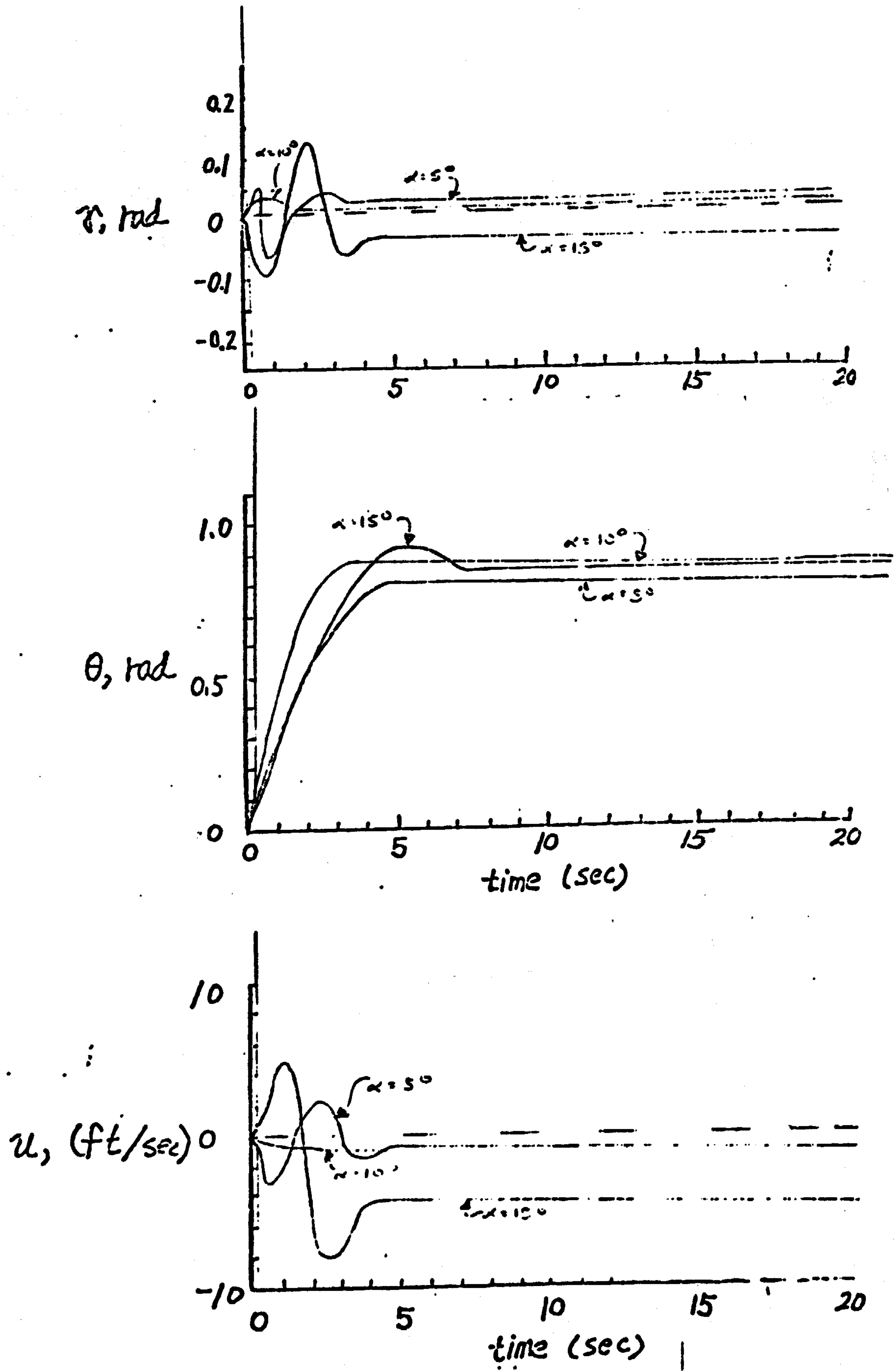
$$v_r(t) = 1 - e^{-\frac{t}{2}}$$



$v_r(t) = 1 - e^{-\frac{t}{2}}$

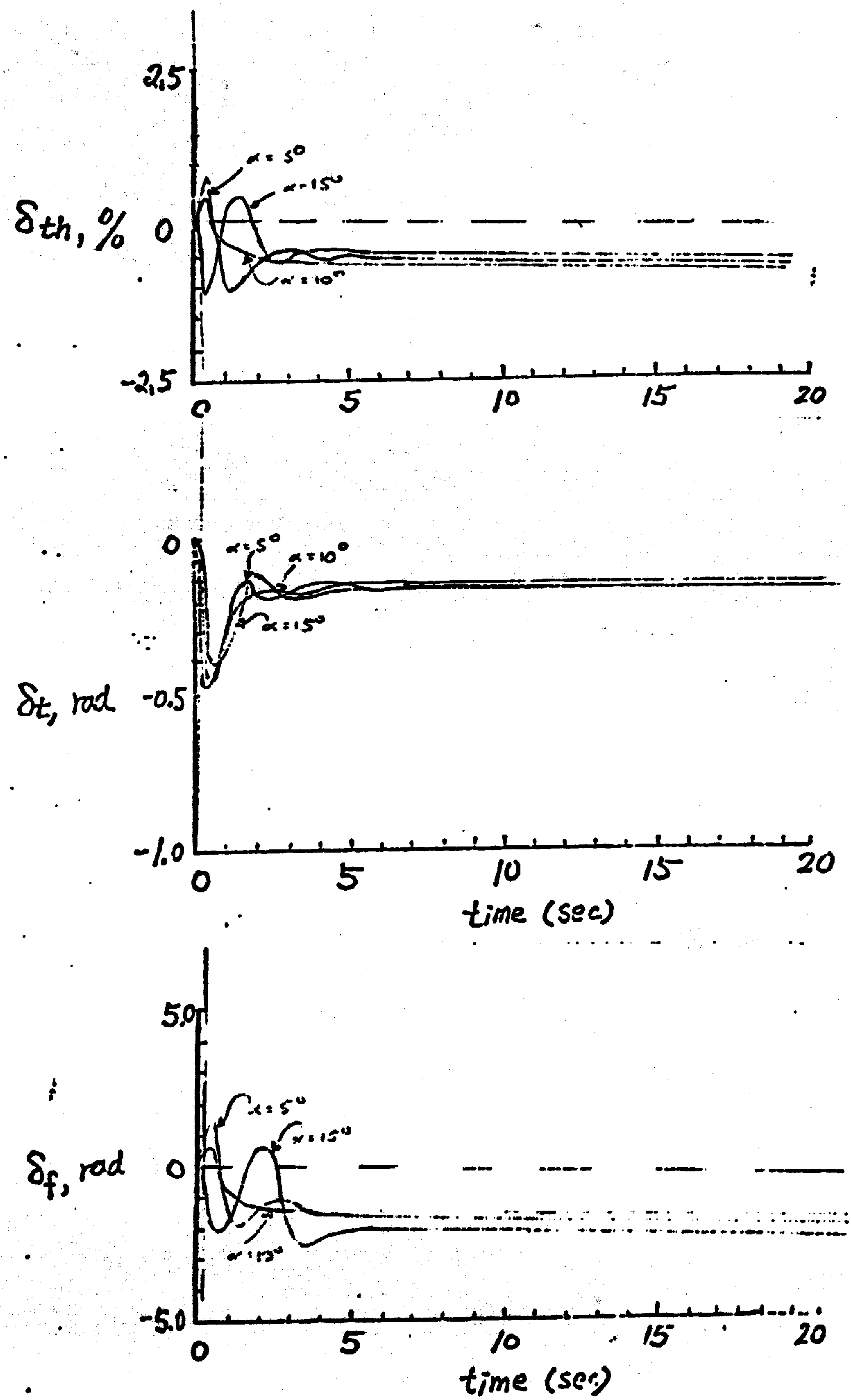


$$v_0 = 1 - e^{-t}$$



γ, rad
0.2
0.1
0
-0.1
-0.2
α=10°
α=5°
α=15°
θ, rad
1.0
0.5
0
α=15°
α=10°
α=5°
time (sec)
u, (ft/sec)
10
0
-10
α=5°
α=10°
α=15°
0
5
10
15
20
time (sec)

$v_0 = 1 - e^{-t}$

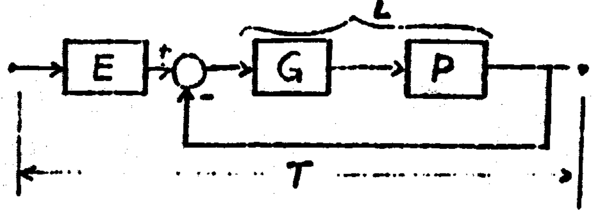


2.5
0
-2.5
δ_{th}, %
$\alpha = 5°$
$\alpha = 15°$
$\alpha = 10°$
0
5
10
15
20
0
-0.5
-1.0
δ_t, rad
$\alpha = 5°$
$\alpha = 10°$
$\alpha = 15°$
0
5
10
15
20
time (sec)
5.0
0
-5.0
δ_f, rad
$\alpha = 5°$
$\alpha = 15°$
$\alpha = 10°$
0
5
10
15
20
time (sec)

IV. Future research directions

The following derivation is based mainly on [3] (see Chapter 10).

Let the system plant transfer function be denoted by $P(s)$, which has m inputs and m outputs. Let $E(s)$ and $G(s)$ both be square rational matrices which are the compensators to be designed.



Due to system parameter variations, the plant transfer function may become $P'(s)$. Let $L \triangleq PG$ and $L' \triangleq P'G$. The over-all transfer function $T = (I + L)^{-1} LE$. It is easy to see that

$$T^{-1} = E^{-1}L^{-1} (I + L).$$

$$E = L^{-1} (I + L) T. \tag{5}$$

Let T' denote the over-all transfer function when the plant changes to P', *i.e.*,

$$T' = (I + L')^{-1} L'E$$

$$T'^{-1} = E^{-1}L'^{-1} (I + L')$$

Hence

$$T'^{-1} - T^{-1} = E^{-1}L'^{-1} (I + L')$$

$$-E^{-1}L^{-1} (I + L)$$

$$= E^{-1} (L'^{-1} - L^{-1}).$$

$$TT'^{-1} - I = TE^{-1} (L'^{-1} - L^{-1})$$

$$T - T' = TE^{-1} (L'^{-1} - L^{-1})T'$$

Let $\Delta T \triangleq T' - T$, then from the above equation and (5),

$$\Delta T = T \left\{L^{-1}(I + L)T\right\}^{-1} (L^{-1} - L'^{-1})T'$$

$$= (I + L)^{-1} (I - LL'^{-1})T'.$$

$$= (I + L)^{-1} (I - PP'^{-1})T'.$$

Suppose ΔT is relatively small, *i.e.*, $T \approx T'$, then the above equation can be further reduced to the following,

$$\Delta T \approx (I + L)^{-1} (I - PP'^{-1})T \tag{6}$$

This equation is very useful in the case that the nominal plant transfer function P is diagonal. It is natural to assume L is also diagonal, then $G \triangleq P^{-1}L$ is also diagonal. Since both P and P' are known, the matrix $W \triangleq (I - PP'^{-1})T$ is also known. Consider the simple case where $m = 2$, then equation (6) can be rewritten as

$$\begin{bmatrix} \Delta t_{11} & \Delta t_{12} \\ \Delta t_{21} & \Delta t_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{1 + l_1} & 0 \\ 0 & \frac{1}{1 + l_2} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

Suppose that the variations of T(s) are required to satisfy

$|\Delta t_{ij}(jw)| \leq k_{ij}(w)$, where $k_{ij}(w)$ are some given bounds on the elements of $\Delta t_{ij}(jw)$, then

$$\left|\frac{w_{ij}(jw)}{1 + l_i(jw)}\right| \leq k_{ij}(w),$$

or

$$\left|\frac{w_{ij}(jw)}{k_{ij}(w)}\right| \leq \left|1 + l_i(jw)\right|. \tag{7}$$

The above equation shows that in order to keep the variations of T(s) within certain bounds, (i.e., $k_{ij}(wj)$), then the loop transmission $L \triangleq \text{diag}\ (l_1, l_2)$ has to be above certain value.

Our next task is to find $G \triangleq \text{diag}\ (g_1, g_2)$ such that $L = PG$ satisfies the above requirement. Furthermore, all zeros of $(1 + l_i)$ must be stable, i.e., the over-all system must be stable. This will be done on a cut and try basis on Bode plot. Finally, E(s) is choosen to realize T(s) as close as possible.

In summary, the given system (A, B, C) will first be decoupled by using Gilberts' decoupling program, the decoupled system transfer function is called P. Then we design compensators E and G to reduce the effect due to parameter variations of (A, B).

New results on robust servomechanisms [4] should be very useful in the design of compensators E and G.

References

[1] G. K. Miller, Jr., P. L. Deal and R. A. Champine, "Fixed-base simulation study of decoupled controls during approach and landing of a STOL transport airplane," NASA Technical Note, NASA TN D-7363, February 1964.

[2] J. L. Kuester and J. H. Mize, "Optimization techniques with Fortran," McGraw-Hill Book Company, 1973.

[3] I. M. Horowitz, "Synthesis of feedback systems," Academic Press, 1963.

[4] C. A. Desoer and Y. T. Wang, "Linear time-invariant robust servomechanism problem: A self-contained exposition," ERL report M77/50, College of Engineering, University of California, Berkeley.

APPENDIX

This appendix contains copies of publications supported by this grant.

Submitted to IEEE Trans. on Automatic Control

An Example in Decentralized Control Systems

by

Shih-Ho Wang

Abstract - This note provides an example of a system which cannot be stabilized by decentralized state feedback, even though the over-all system and each local subsystem are completely controllable.

In [1], Aoki and Li have made some interesting studies on the controllability and stabilizability of decentralized dynamic systems. They have suggested the following result:

Consider a linear dynamic system described by

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_{11} u_1$$

$$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_{22} u_2 .$$

Assume that the pairs (A_{11}, B_{11}), (A_{22}, B_{22}) and $\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}\right)$ are completely controllable, then one can find a set of decentralized (local) state feedback $u_i = C_i x_i$, $(i = 1, 2)$ such that the resulting system matrix

$$\begin{bmatrix} A_{11} + B_{11} C_1 & A_{12} \\ A_{21} & A_{22} + B_{22} C_2 \end{bmatrix}$$

is stable, (see Proposition 7 of [1]).

This work has been supported by the National Aeronautics and Space Administration, Langley Research Center, under Grant No.: NSG 1213. The author is with the Department of Electrical Engineering, University of Maryland, College Park, Md. 20742.

In the following example, we consider a system with three local control stations and the pairs (A, B), (A_{ii}, B_{ii}), $(i = 1, 2, 3)$ are all controllable. Yet the system cannot be stabilized by local state feedback.

$$
\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \hline \dot{x}_{21} \\ \dot{x}_{22} \\ \hline \dot{x}_{31} \\ \dot{x}_{32} \end{bmatrix}
=
\left[\begin{array}{cc|cc|cc}
0 & 1 & & & & \\
0 & 0 & & & & \\ \hline
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & 0 & 1 & 0 & 1 \\
 & & 0 & 0 & 0 & 0
\end{array}\right]
\begin{bmatrix} x_{11} \\ x_{12} \\ \hline x_{21} \\ x_{22} \\ \hline x_{31} \\ x_{32} \end{bmatrix}
+
\left[\begin{array}{c|c|c}
0 & & \\
1 & & \\ \hline
 & 0 & \\
 & 1 & \\ \hline
 & & 0 \\
 & & 1
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

where u_i is the control input at station i, $(i = 1, 2, 3)$. The set of decentralized local state feedback is of the form

$$
u_i = [k_{i1}, k_{i2}] \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \qquad (i = 1, 2, 3).
$$

Applying the above set of feedback, the over-all system matrix is as follows:

$$
\left[\begin{array}{cc|cc|cc}
0 & 1 & & & & \\
k_{11} & k_{12} & & & & \\ \hline
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & k_{21} & k_{22} & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & k_{31} & k_{32}
\end{array}\right]
$$

It can be easily seen that the first, second, third and fifth rows of the above matrix are linearly dependent for arbitrary k_{ij}'s. In other words, the above system has a fixed mode [2] at the origin. Hence the closed-loop system cannot be asymptotically stable.

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Presented at Conference on Information Science and Systems,
Baltimore, MD., March 1977

RELIABILITY OF SELF-PURGING REDUNDANCY WITH MULTIPLE VOTERS

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ABSTRACT

For the design of highly reliable computer hardware systems, various redundancy techniques have been proposed. Due to the simplicity of its switching mechanisms, self-purging redundancy is a useful scheme for certain applications. Various aspects of self-purging redundancy with single voter have been studied in [1] - [3]. In this paper, a self-purging redundancy scheme with multiple voters is proposed to further improve the overall system performance. This configuration is carefully studied and compared with several other redundancy schemes.

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This research has been supported by the National Aeronautics and Space Administration under Grant No. NSG-1213.

Presented at 1977 Asilomar Conference

An Approach to the Design of Decoupled Systems in the Presence of Large Parameter Variations*

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Summary

In this study we outline a design approach for developing control systems in the presence of large parameter variations. The decoupling problem is of practical interest in flight control problems, especially in the landing mode where it is desireable to independently control certain output variables, e.g. flight path angle, foward speed, etc. A common source of parameter variations in flight control problems is the result of linearization about a number of different operating points. It is assumed in this study that parameter variations may be represented by a discrete finite set of possible values, corresponding to different flight conditions and that the complete system state is not available.

The theory of decoupling with no parameter variations is well developed. See, for example, references [1] and [2]. Some work has been done by Fabian and Wonham, [3], on decoupling

*This research has been supported by NASA Langley Research Center under grant NSG 1213 at The University of Colorado.

with small parameter variations. However, very little work has been done on decoupling with large parameter variations. In reference [4], Dorato, Wang, and Asher develop conditions for perfect decoupling with a discrete finite parameter set. However, in most practical cases the conditions for perfect decoupling are not satisfied. Thus in general, with parameter variations it is necessary to minimize some measure of interaction in any decoupling design.

A classical approach to the problem posed here is to use large loop gain aroung the plant to reduce the matrix transfer function to a good approximation of the unit matrix and then precede the feedback system with a known diagonal precompensator. An outline of this approach may be found in chapter ten of reference [5]. This approach has considerable merit, however, it is a trial and error approach which in addition requires a large number of graphical displays.

The design approach suggested here involves the following steps.

1. A full order observer, [6], is designed, based on a set of nominal parameter values, to generate a state estimated, $\hat{x}$.

2. A feedback control law of the form $u = Fv + K\hat{x}$ is selected with design parameters consisting of all or selected entries of F and K.

3. An interaction function is minimized with respect to F and K, subject to constraints on the diagonal elements of the closed-loop transfer function. The minimization problem is then solved via a nonlinear programming algorithim.

The interaction function is selected to be of the form

$$F_{OD} = \sum_{\alpha} \sum_{\substack{i,j \\ i \neq j}} \sum_{\omega} |h_{ij}(j\omega)|^2 .$$

Where α denotes the parameter set, ω the frequency variable, and h_{ij} the i-j entry of the closed-loop transfer function.

The above procedure involves a considerable amount of computation for any practical problem. However, all the steps are based on well documented algorithim which have been computerized. Some preliminary computational results indicate that the interaction measure can be reduced compared to a design based entirely on nominal values.

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Perfect Decoupling of Linear Systems with Discrete Parameter Uncertainties

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Abstract—A simple design procedure is presented, based on Gilbert's decoupling parameters [1]–[2] for perfect scaler decoupling via state feedback of a linear time-invariant system with parameter uncertainties defined over a discrete set of possible values.

Manuscript received June 28, 1976. This work was supported in part by the National Aeronautics and Space Administration under Grant NSG 1213.

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I. Introduction

The design problem is to determine a fixed state feedback control law of the form

$$u(t)=Fx(t)+Gv(t) \tag{1}$$

which decouples the linear dynamical system, with input $v(t)$ and output $y(t)$,

$$\begin{cases} \dot{x}(t)=A_jx(t)+B_ju(t) \\ y(t)=C_jx(t) \end{cases} \tag{2}$$

for all admissible system parameters (A_j,B_j,C_j). The index j ranges over a finite set of integers J, and represents a discrete uncertainty set for the system matrices A, B, and C. This type of uncertainty appears naturally in systems which are linearized about a finite number of operating points, such as flight control systems and process control systems. In addition, this discrete model may be used as an approximation to problems with continuous parameter uncertainties. The literature on decoupling of systems with no uncertainty is rather extensive. See, for example [1], [2], [6], [8], and [9]. The literature on decoupling with parameter uncertainties is much more limited. In [3] the problem of decoupling in the presence of continuous parameter uncertainties is considered from the geometric point of view. Necessary and sufficient conditions are presented for the existence of a fixed state feedback law which preserves input-output properties of a system, including decoupling and stability, for sufficiently small parameter variations. The present study differs from [3] in the following respects.

1) The uncertainty set is discrete rather than continuous.

2) No restrictions are placed on the size of the parameter variations.

3) The diagonal elements of the decoupled system transfer matrix are not required to remain invariant or stable with respect to parameter variations.

4) The output matrix C also may be variable.

In [3] the invariance with respect to parameter variations of the input-output properties is referred to as data insensitivity. The relaxation of the conditions of data sensitivity and stability for the diagonal elements in the closed-loop system permit the inclusion of a wider class of problems. Problems of invariance and stability of the diagonal elements can be treated, if necessary, by single-loop techniques such as in [4]. However, even with relaxed conditions on diagonal elements many practical systems cannot be perfectly decoupled, as illustrated in Example 2 in Section III.

In Section II conditions are given for perfect decoupling. These conditions lead directly to the required control matrices F and G.

II. Decoupling Design

As shown by Gilbert [1], any control matrices F and g which decouple the system (2), for fixed i, must be of the form

$$F=-D^{-1}A^{*}+\sum_{i=1}^{m}\sum_{k=1}^{p_i}(\sigma_{ik}-\pi_{ik})J_{ik}+\sum_{i=1}^{m}\sum_{k=1}^{r_{m+2}}\rho_{ik}K_{ik} \tag{3}$$

$$G=\sum_{i=1}^{m}\lambda_iG_i \tag{4}$$

where D, A^*, r_{m+2}, p_i, π_{ik}, J_{ik}, K_{ik}, and G_i are parameters fixed by (A_j,B_j,C_j). We refer to these parameters as the Gilbert decoupling parameters. The parameters λ_i, σ_{ik}, and ρ_{ik} are free parameters which may take on arbitrary real values. To stress the dependence of the control matrices and Gilbert decoupling parameters on (A_j,B_j,C_j) we denote F as $F(j)$, J_{ik} as $J_{ik}(j)$, etc. As further shown in [1], the free parameters σ_{ik} determine the poles of the closed-loop diagonal elements and λ_i determine the gain factors for the diagonal elements. A detailed program for the numerical computation of the Gilbert decoupling parameters is described in [2]. We can now state the condition and design procedure for decoupling, given any index set $J=(j_1,j_2,\cdots j_m)$.

The system (2) can be decoupled for all j belonging to J if and only if there exists a set of free parameters $\lambda_i(j)$, $\sigma_{ik}(j)$, and $\rho_{ik}(j)$ for every j, such that

$$F(j_1)=F(j_2)=\cdots=F(j_m) \tag{5}$$

$$G(j_1)=G(j_2)=\cdots=G(j_m). \tag{6}$$

If a suitable set of free parameters exists, the required control matrices are given by the common value of F and of G in (5) and (6). If the same free parameters can be used in each $F(j)$ and $G(j)$ to satisfy (5) and (6), then the system is data insensitive in addition to decoupled. Finally if the free parameters can be selected so that the poles of the diagonal elements all have negative real parts, the decoupled system also will be input-output stable.

It should be noted that (5) and (6) represent a system of linear equations in the variables $\lambda_i(j)$, $\sigma_{ik}(j)$, and $\rho_{ik}(j)$.

III. Examples

Example 1: Consider the system

$$\dot{x}=A_jx+Bu,\quad y=Cx,\quad J=\{1,2,3\}$$

where

$$A_1=\begin{pmatrix}1 & -3\\ 1 & -2\end{pmatrix},\quad A_2=\begin{pmatrix}0 & -2\\ 1 & -1\end{pmatrix},\quad A_3=\begin{pmatrix}1 & -4\\ 1 & -2\end{pmatrix}$$

$$B=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad C=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}.$$

The control matrices for this example are

$$\begin{cases} F(1)=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}+\sigma_{11}(1)\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}+\sigma_{21}(1)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\ G(1)=\lambda_1(1)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\lambda_2(1)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \end{cases}$$

$$\begin{cases} F(2)=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}+\sigma_{11}(2)\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}+\sigma_{21}(2)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\ G(2)=\lambda_1(2)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\lambda_2(2)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \end{cases}$$

$$\begin{cases} F(3)=\begin{pmatrix}0 & 2\\ -1 & 2\end{pmatrix}+\sigma_{11}(3)\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}+\sigma_{21}(3)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\ G(3)=\lambda_1(3)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\lambda_2(3)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \end{cases}$$

The following values of free parameters satisfy (5) and (6).

$$\sigma_{11}(1)=-1,\quad \sigma_{11}(2)=-1,\quad \sigma_{11}(3)=0$$

$$\sigma_{21}(1)=-1,\quad \sigma_{21}(2)=0,\quad \sigma_{21}(3)=-1$$

$$\lambda_1(1)=\lambda_1(2)=\lambda_1(3)=1,\quad \lambda_2(1)=\lambda_2(2)=\lambda_2(3)=1.$$

The resulting control matrices are

$$F=\begin{pmatrix}0 & 2\\ -1 & 1\end{pmatrix},\quad G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

In this problem the diagonal elements are given by

$$h_i(s)=\frac{\lambda_i(j)}{s-\sigma_{i1}(j)},\qquad i=1,2.$$

Note that the closed-loop poles of the diagonal elements vary with the parameter variations. The condition for data insensitivity required in [3] is not met by this example. However, it can be perfectly decoupled for $j=1,2,3$.

Example 2: The numerical values in this example are obtained from the linearized longitudinal dynamics of a STOL aircraft [7]. The uncertainty corresponds to two possible operating conditions, one corresponding to an angle of attack equal to 5° and the other to an angle of attack equal to 10°. For convenience we select $J=(5,10)$.

$$\dot{x} = A_j x + B_j u, \quad y = Cx$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.285 & -0.6725 & 0.209 \\ 0 & 1 & -0.403 & -0.570 \\ -0.286 & 0 & 0.1105 & -0.0785 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2.515 & 0.127 \\ 0 & -0.0646 & -0.190 \\ 0.1047 & -0.00383 & -0.1247 \end{bmatrix}$$

$$A_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.23 & -0.520 & 0.225 \\ 0 & 1 & -0.368 & -0.640 \\ -0.3195 & 0 & 0.157 & -0.1018 \end{bmatrix}$$

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2.38 & 0.1487 \\ 0 & -0.0676 & -0.1712 \\ 0.1047 & -0.01406 & -0.1190 \end{bmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The control matrices $F(j)$ are in this case

$$F(5) = \begin{pmatrix} 2.73161 & 0.18503 & -3.4599 & -2.79854 \\ 0 & -0.50231 & -0.36818 & -0.06724 \\ 0 & 0.17079 & -1.99587 & -2.97714 \end{pmatrix}$$

$$+\sigma_{11}(5) \begin{pmatrix} 6.17229 & 0 & -6.17229 & 0 \\ 0.26129 & 0 & -0.26129 & 0 \\ 5.17432 & 0 & -5.17432 & 0 \end{pmatrix}$$

$$+\sigma_{21}(5) \begin{pmatrix} 0 & 0.1440 & 0 & 0 \\ 0 & -0.3909 & 0 & 0 \\ 0 & 0.13291 & 0 & 0 \end{pmatrix}$$

$$+\sigma_{22}(5) \begin{pmatrix} 0.144 & 0 & 0 & 0 \\ -0.3909 & 0 & 0 & 0 \\ 0.13291 & 0 & 0 & 0 \end{pmatrix}$$

$$+\sigma_{31}(5) \begin{pmatrix} 0 & 0 & 0 & 9.5511 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$F(10) = \begin{pmatrix} 3.05158 & 0.15862 & -3.83436 & -3.23393 \\ 0 & -0.50436 & -0.34429 & -0.13568 \\ 0 & 0.19915 & -2.01358 & -3.68474 \end{pmatrix}$$

$$+\sigma_{11}(10) \begin{pmatrix} 6.52689 & 0 & -6.52689 & 0 \\ 0.35616 & 0 & -0.35616 & 0 \\ 5.70049 & 0 & -5.70049 & 0 \end{pmatrix}$$

$$+\sigma_{21}(10) \begin{pmatrix} 0 & 0.12896 & 0 & 0 \\ 0 & -0.41005 & 0 & 0 \\ 0 & 0.16191 & 0 & 0 \end{pmatrix}$$

$$+\sigma_{22}(10) \begin{pmatrix} 0.12896 & 0 & 0 & 0 \\ -0.41005 & 0 & 0 & 0 \\ 0.16191 & 0 & 0 & 0 \end{pmatrix}$$

$$+\sigma_{31}(10) \begin{pmatrix} 0 & 0 & 0 & 9.5511 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the free parameters are missing from the f_{24} and f_{34} entries of the $F(5)$ and $F(10)$ matrices; hence no choice of free parameters can satisfy $F(5) = F(10)$. Thus, this system cannot be perfectly decoupled for all admissible parameter variations. The only reasonable decoupling design criterion in this case is to minimize some measure of interaction as in [5].

IV. Conclusions

Conditions are presented for perfect decoupling of linear systems with discrete parameter uncertainties. The conditions involve a test for the existence of a solution to a system of linear equations. An actual solution of the linear equations yields the fixed decoupling control law. While there exists systems which can be perfectly decoupled in the presence of parameter variations, e.g., Example 1, many practical systems cannot be perfectly decoupled, e.g., Example 2. However, it is of some interest to know when perfect decoupling is possible.

Acknowledgment

The authors are grateful to A. A. Schy and H. A. Hamer, of the NASA Langley Research Center, for supplying a practical example and pointing out the limitations of perfect decoupling for systems with uncertain parameters, and to O. Lopez for the numerical computation of the Gilbert decoupling parameters in Section III.

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Invited Paper for 20th Midwest Symposium on Circuits and Systems, August 15-16, Lubbock, Texas 1977

ON THE IMPLEMENTATION OF A POLE ASSIGNMENT ALGORITHM FOR DECENTRALIZED CONTROL SYSTEMS

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Abstract

This paper provides some preliminary results on the implementation of a pole-assignment algorithm for decentralized control systems. Some possible improvements on existing algorithm and some numerical examples are given.

1. Introduction

The problem of stabilizing a large scale decentralized control system has been an interesting topic recently [1-3]. In [2] a necessary and sufficient condition for the existence of local dynamic controllers was given. An algorithm for the construction of such controllers was also given as part of the proof of the main theorem in [2]. This paper suggests some possible improvements on this algorithm and some numerical examples are given.

2. Problem Statements

Consider a linear time-invariant multivariable system with ν local control stations described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{\nu} B_i u_i(t) \qquad (1a)$$

$$y_i(t) = C_i x(t) , \qquad (i = 1,\ldots,\nu) . \qquad (1b)$$

where $x(t) \in \mathcal{R}^n$ is the state, $u_i(t) \in \mathcal{R}^{m_i}$ and $y_i(t) \in \mathcal{R}^{p_i}$ are the input and output, respectively, of the ith local control station ($i = 1,\ldots,\nu$). The problem is to find ν local output feedback control laws with dynamic compensation in order to stabilize the whole system. The set of local feedback laws are assumed to be generated by the following feedback controllers:

$$\dot{z}_i(t) = S_i z_i(t) + R_i y_i(t) \qquad (2a)$$

$$u_i(t) = Q_i z_i(t) + K_i y_i(t) + v_i(t) ,$$

$$i = 1,\ldots,\nu \qquad (2b)$$

where $z_i(t) \in \mathcal{R}^{n_i}$ is the state of the ith feedback controller, $v_i(t) \in \mathcal{R}^{m_i}$ is the ith local external input and S_i, R_i, Q_i, and K_i are real constant matrices of appropriate size. Equation (2) can be written more compactly as follows:

$$\dot{z}(t) = Sz(t) + Ry(t) \qquad (3a)$$

$$u(t) = Qz(t) + Ky(t) + v(t) \qquad (3b)$$

where

$$S \triangleq \text{block diag } [S_1,\ldots,S_\nu]$$

$$R \triangleq \text{block diag } [R_1,\ldots,R_\nu]$$

$$Q \triangleq \text{block diag } [Q_1,\ldots,Q_\nu]$$

$$K \triangleq \text{block diag } [K_1,\ldots,K_\nu]$$

$$z^T(t) \triangleq [z_1^T(t) \ldots z_\nu^T(t)]$$

$$y^T(t) \triangleq [y_1^T(t) \ldots y_\nu^T(t)]$$

$$u^T(t) \triangleq [u_1^T(t) \ldots u_\nu^T(t)]$$

and

$$v^T(t) \triangleq [v_1^T(t) \ldots v_\nu^T(t)] . \qquad (4)$$

When the feedback control law generated by (3) is applied to system (1), the closed-loop system is described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + BKC & BQ \\ RC & S \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t) \qquad (5)$$

where

$$B = [\underbrace{B_1}_{m_1} \ldots \underbrace{B_\nu}_{m_\nu}] \text{ and } C = \begin{bmatrix} C_1 \\ \vdots \\ C_\nu \end{bmatrix} \begin{matrix} \}p_1 \\ \\ \}p_\nu \end{matrix} \qquad (6)$$

The set of local feedback control laws of (3) is to be chosen so that the overall system (5) is asymptotically stable, i.e., all poles of system (5) are

This research has been supported by the National Aeronautics and Space Administration, Langley Research Center, under Grant [illegible] and the National Research Council of [illegible] under Grant No. [illegible].

in the open left-half complex plane ($\mathbb{C}^-$). This then is the problem of stabilizing a decentralized control system via local output feedback with dynamic compensation.

3. Summary of Previous Results

Consider the following definitions:

<u>Definition 1</u> [2]. Consider the triple $(C,A,B) \in \mathcal{R}^{p \times n} \times \mathcal{R}^{n \times n} \times \mathcal{R}^{n \times m}$ and the two sets of integers $m_1,\ldots,m_v$ and $p_1,\ldots,p_v$, with $m = \sum_{i=1}^{v} m_i$ and $p = \sum_{i=1}^{v} p_i$, which specifies system (1). Let K be the set of block diagonal matrices as follows:

$$\Sigma = \{K \mid K = \text{block diag}\,[K_1,\ldots,K_v],$$

$$K_i \in \mathcal{R}^{m_i \times p_i}, \quad i = 1,\ldots,v\} . \qquad (7)$$

Then the greatest common divisor of the set of characteristic polynomials of $(A + BKC)$, for all $K \in \Sigma$, is called the fixed polynomial of the triple (C,A,B) with respect to Σ, denoted by $\psi(\lambda;C,A,B,\Sigma)$, i.e.,

$$\psi(\lambda;C,A,B,K) = \gcd_{K \in \Sigma}\{\det(\lambda I - A - BKC)\} .$$

<u>Comment</u>: The map $\psi(\lambda;C,A,B,\cdot):\Sigma \to \gcd_{K \in \Sigma} \det(\lambda I - A - BKC)$ is well defined for any subset $K \subset \mathcal{R}^{m \times p}$ and not just for the set Σ of block diagonal matrices in (7).

<u>Definition 2</u>. For a given triple $(C,A,B) \in \mathcal{R}^{p \times n} \times \mathcal{R}^{n \times n} \times \mathcal{R}^{n \times m}$ and the given set Σ of block diagonal matrices in (7), the set of fixed modes of (C,A,B) with respect to Σ is defined as follows:

$$\Lambda(C,A,B,\Sigma) = \bigcap_{K \in \Sigma} \sigma[A + BKC] ,$$

where $\sigma[A + BKC]$ denotes the set of eigenvalues of $(A + BKC)$. Equivalently, $\Lambda(C,A,B,\Sigma)$ can be defined as follows:

$$\Lambda(C,A,B,K) = \{\lambda \mid \lambda \in \mathbb{C} \text{ and } \psi(\lambda;C,A,B,\Sigma) = 0\} .$$

The following theorem provides the main results on the stabilization of decentralized systems.

<u>Theorem 1</u> [2]. Consider the given system (C,A,B) of (1), with [illegible] and [illegible] defined [illegible]. Let Σ be the set of [illegible] matrices defined in (7). Then a necessary and sufficient condition for the existence of a set of local feedback control laws (3) such that the closed-loop system (5) is asymptotically stable is that

$$\Lambda(C,A,B,K) \subset \mathbb{C}^- .$$

where $\mathbb{C}^-$ denotes the open left-half complex plane. For a proof of theorem 1 and further details, please refer to [2].

4. A Simplified Algorithm

In this section we propose a simplified al[illegible] [illegible] the [illegible] of dynamic controllers, as foll[illegible]:

Step 1. For given triple (C,A,B), find an arbitrary matrix $K \in \Sigma$ (by using a pseudo-random number generator), so that $\|A\| \doteq \|BKC\|$, then compute $\hat{A} \doteq A + BKC$.

Step 2. Find the set of eigenvalues of A and $\hat{A}$. The intersection of these two sets of eignevalues is the <u>fixed modes</u> of (C,A,B).

Step 3. Transform $\hat{A}$ into its Jordan Canonical Form $\bar{A} \doteq TAT^{-1}$. Also compute $\bar{B} \doteq TB$ and $\bar{C} \doteq CT^{-1}$.

Step 4. Find a set of local stations so that all eigenvalues of $\bar{A}$, except the set of fixed modes, will be completely controllable and completely observable from at least one of these stations.

Step 5. Use any pole assignment algorithm for centralized control systems to assign poles to the system by applying dynamic compensators at these selected local stations.

Consider the following examples:

<u>Example 1</u>.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u_2$$

$$y_1 = [1 \;\; 1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y_2 = [0 \;\; 1 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Randomly choosing $K = \begin{bmatrix} 1.5 & 0 \\ 0 & 2.5 \end{bmatrix}$, one has

$$\hat{A} = A + BKC = \begin{bmatrix} 2.5 & 1.5 & 0 \\ 1.5 & 6.0 & 2.5 \\ 0 & 2.5 & 5.5 \end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} [illegible].47 & 0 & 0 \\ 0 & [illegible].85 & 0 \\ 0 & 0 & 1.67 \end{bmatrix}$$

$$\bar{B} = TB = \begin{bmatrix} [illegible].594 & 0.[illegible] \\ -0.706 & [illegible].175 \\ 0.111 & -0.945 \end{bmatrix}$$

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1.[illegible]9 & -1.38 & 1.[illegible] \\ [illegible].1[illegible] & 0.343 & -0.[illegible]33 \end{bmatrix}$$

One can easily see that the system is completely controllable and completely observable from both station 1 and station 2. Hence it can be easily stabilized by using any pole assignment algorithm, e.g., [4].

Example 2.

Consider the following system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u_2$$

$$y_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Applying a feedback $K = \begin{bmatrix} 1.5 & 0 \\ 0 & 2.5 \end{bmatrix}$, where K is chosen randomly, we have

$$\hat{A} = A + BKC = \begin{bmatrix} 2.5 & 1.5 & 0 \\ 0 & 2.0 & 2.5 \\ 0 & 0 & 5.5 \end{bmatrix}.$$

One can show that

$$T = \begin{bmatrix} 1.0 & 3.0 & -2.499 \\ 0 & 1.0 & -0.714 \\ 0 & 0 & 1.0 \end{bmatrix}$$

$$\bar{A} = T\hat{A}T^{-1} = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.0 & 0 \\ 0 & 0 & 5.5 \end{bmatrix}$$

$$\bar{B} = TB = \begin{bmatrix} 1 & 0.501 \\ 0 & 0.286 \\ 0 & 1 \end{bmatrix}$$

and

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1 & -2 & 1.97 \\ 0 & 0 & 1 \end{bmatrix}$$

One can see that $\lambda_2 = 2$ is a fixed mode of the system. It is interesting to note that by applying local feedback randomly, the system becomes completely controllable from station 2 and completely observable from station 1. However, in order to stabilize a mode by local feedback, this mode must be both completely controllable and completely observable from a single station. Hence, one has to utilize dynamic feedback at station 1 and station 2 to stabilize the first and third mode of the system, respectively. The problem of stabilizing the fixed modes for a decentralized system has been treated in a separate paper [5].

5. Conclusions

Some examples on the stabilization of decentralized control systems are considered in this paper. The problem of controllability and observability properties for a decentralized control system has not been fully investigated. Results from those studies should be very useful for the stabilization problem as considered in this paper.

6. References

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Presented at 1977 Conference on Decision and Control

An Algorithm for Obtaining the Minimal Realization of a Linear Time-Invariant System and Determining if a System is Stabilizable-Detectable

by

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ABSTRACT

A definition of centralized fixed modes of a system, which is an adoption of the definition of fixed modes for a decentralized system [1], is made. It is shown that the centralized fixed modes of a system can be easily and efficiently calculated in terms of the eigenvalues of the system, and that the calculation of such fixed modes enables one to determine, in a numerically efficient may, whether a system is controllable, observable, stabilizable, detectable. An efficient algorithm for reducing a system to minimal realization form is then given. The algorithms have been effectively used on systems of order up to 100.

1. INTRODUCTION

Although the concepts of controllability, stabilizability, observability, detectability, etc. are well known for linear time-invariant systems, the actual determination whether a given system has one of the above properties is non trivial especially when the order n of the system is high (e.g. $n>>10$); this is because (i) most methods usually require the calculation of the rank of a certain matrix (which gets larger as the order of the system increases), and this always involves a decision, more or less arbitrary, as to what value to choose for zero (i.e. zero $\doteq 10^{-8}$ or $\doteq 10^{-20}$). (ii) often the rank of a large number (n) of matrices of order n must be found or the rank of a single matrix, (of order n), but ill-conditioned, must be found.

For example, it is well known that the pair $(A, B) \in R^{n\times n} \times R^{n\times m}$ is controllable if:

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This work has been supported by the National Research Council of Canada under Grant A4396, and the National Aeronautics and Space Administration, Langley Research Center, under Grant No: NSG 1213.

$$\text{rank}\,(B, AB, \ldots, A^{n-1}B) = n \tag{1}$$

In this case, however, the determination whether (1) is true will be extremely difficult for large n (e.g. $n \gg 10$) due to the ill-conditioning of the controllability matrix $(B, AB, \ldots, A^{n-1}B)$ (i.e. the columns of the controllability matrix tend to become linearly dependent for large n). Alternately, it is well known that the pair (A, B) is controllable if:

$$\text{rank}\,(A-\lambda_i I, B) = n, \quad i = 1,2,\ldots,n \tag{2}$$

where λ_i, $i = 1,2,\ldots,n$ are the eigenvalues of A [2]. In this case, however, the rank of a $n\times(n+m)$ matrix must be determined n times, and this is non trivial to do for large n. It will be shown that the above problems can be eliminated by calculating a set of p scalars for the system $(0 \le p \le n)$ (called the centralized fixed modes of the system), which only requires the determination of the eigenvalues of two $n\times n$ matrices for the system.

Similarly, although the concept of the minimal realization of a linear, time-invariant system is well-known, the actual determination of a minimal realization for a linear system is non-trivial, especially when the order n of the system is high (e.g. $n \gg 10$). This is because most existing algorithms, e.g. [3]-[8], involve a branching process, which depends on whether a certain number is zero, and subjective decisions must be made as to what value to choose for zero. Different realizations may then be obtained depending on what value to choose for zero (i.e. zero $\doteqdot 10^{-8}$ or $\doteqdot 10^{-16}$). It will be shown that the above problem can be eliminated if the centralized fixed modes of the system are known.

2. DEVELOPMENT

The following definition is required in the development to follow; it is adopted from Wang and Davison [1], who defined fixed modes of a decentralized system.

Definition

Consider the triple $(C,A,B) \in R^{r\times n} \times R^{n\times n} \times R^{n\times m}$; then the set of centralized fixed modes of (C,A,B), denoted by $\Lambda(C,A,B)$ is defined as follows:

$$\Lambda(C,A,B) = \bigcap_{K\in R^{m\times r}} \sigma(A+BKC) \tag{3}$$

where $\sigma(A+BKC)$ denotes the list of eigenvalues of $(A+BKC)$.

Remark 1

The set of centralized modes of a given triple (C,A,B) can be calculated as follows. Let $k(i, j)$ denote the $(i, j)^{th}$ element of K. Since K contains the null matrix, then $\Lambda(C,A,B) \subseteq \sigma(A) = \{\lambda_1,\ldots,\lambda_n\}$. For each $\lambda_i \in \sigma(A)$, $\det(\lambda_i I - A - BKC)$ is a polynomial in $k(i, j)$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,r$, and if $\det(\lambda_i I - A - BKC)$ is identically zero, then clearly $\lambda_i \in \Lambda(C,A,B)$. On the other hand, if $\det(\lambda_i I - A - BKC)$ is not identically zero, then there exists $\tilde{K} \in R^{m\times r}$ so that $\det(\lambda_i I - A - B\tilde{K}C) \neq 0$, and hence $\lambda_i \notin \Lambda(C,A,B)$. This procedure determines the fixed modes of $\Lambda(C,A,B)$.

The following lemma is obtained by using identical arguments as used in the proof of theorems 1-3 of [9].

Lemma 1

Given $(C,A,B) \in R^{r\times n} \times R^{n\times n} \times R^{n\times m}$, then the class of matrices $K \in R^{m\times r}$, which does not result in the centralized fixed modes of (C,A,B) being equal to those eigenvalues of $A+BKC$ which are common with the eigenvalues of A, is either empty or lies on a hypersurface [9] in the parameter space of K.

The above lemma forms the basis of the following algorithm to determine the centralized fixed modes of (C,A,B).

Algorithm I: to Determine the Centralized Fixed Modes of (C,A,B)

1. Find the eigenvalues of A.

2. Select an arbitrary matrix $K \in R^{m\times r}$ (by using a pseudo-random number generator, say) so that $\|A\| \doteq \|BKC\|$.

3. Find the eigenvalues of $A+BKC$.

4. Then, the centralized fixed modes of (C,A,B) are contained in those eigenvalues of $A+BKC$ which are common with the eigenvalues of A (if there are any). Moreover for "almost any" K chosen, the centralized fixed modes will be identically equal to those eigenvalues of $A+BKC$ which are common with the eigenvalues of A, i.e. the class of matrices K which does not result in the fixed modes being identically equal to the eigenvalues of A which are common with the eigenvalues of $A+BKC$ is either empty or lies on a hypersurface [9] in the parameter space of K.

5. Repeat steps 2 to 4 using a different K if there is any doubt which eigenvalues of A are the fixed modes of (C,A,B).

Proof

The proof of statement 4 in the above algorithm follows directly from lemma 1.

Remark 2

The only possible source of difficulty which may occur in the above algorithm is in making a decision as to what eigenvalues of A are common with the eigenvalues of $A+BKC$. For example, it may occur that an eigenvalue of A only differs from an eigenvalue of $A+BKC$ in the 7th significant figure. Such cases can normally be decided upon by utilizing the physical properties of the plant associated with the mathematical model (C,A,B). For example, if the plant data is only accurate to 3 significant figures, then in the above case, it can safely be assumed that the eigenvalue in question is a fixed mode.

3. APPLICATION OF CENTRALIZED FIXED MODES

An application of the above algorithm can immediately be made to determine certain properties of (C,A,B) and to find the minimal realization of (C,A,B). The following results follow directly from the definition of centralized fixed modes:

To Determine if (A, B) is Controllable

(A, B) is controllable if and only if (I_n, A, B) has no centralized fixed modes.

To Determine if (A, B) is Stabilizable

(A, B) is stabilizable if and only if (I_n, A, B) has no centralized fixed modes which lie in the closed right half part of the complex plane.

To Determine if (C, A) is Observable

(C, A) is observable if and only if (C, A, I_n) has no centralized fixed modes.

To Determine if (C, A) is Detectable

(C, A) is detectable if and only if (C, A, I_n) has no centralized fixed modes which lie in the closed right half part of the complex plane.

To Determine if (C, A, B) is Controllable-Observable

(C, A, B) is controllable and observable if and only if (C, A, B) has no centralized fixed modes.

To Determine if (C, A, B) is Stabilizable-Detectable

(C, A, B) is stabilizable and detectable if and only if (C, A, B) has no centralized fixed modes which lie in the closed right half part of the complex plane.

Remark 3

The centralized fixed modes of (I_n, A, B), (C, A, I_n) correspond to the input-decoupling zeros, output-decoupling zeros of (C, A, B) respectively, defined by Rosenbrock [2].

To Determine the Minimal Realization of (C, A, B)

The following algorithm is obtained:

Algorithm II

1. Determine the centralized fixed modes of (C, A, B); let these modes be denoted by $\{\lambda_1,\lambda_2,\ldots,\lambda_p\}$, $0 \le p \le n$. For non-triviality, assume $p < n$.

2. Apply a arbitrary output gain matrix K to the system (by using a pseudo-random number generator, say so that $\|A\| \doteq \|BKC\|$), so that the controllable-observable modes of $(C, \tilde{A}, B)$, $\tilde{A} \triangleq A+BKC$ are all distinct and disjoint from the fixed modes of (C, A, B). This will be true for "almost all" gains K from [9].

3. Determine the eigenvalues of $\tilde{A}$. Let these eigenvalues be denoted by $\{\lambda_1,\lambda_2,\ldots,\lambda_p, \lambda_{p+1},\lambda_{p+2},\ldots,\lambda_n\}$.

4. Determine the eigenvectors of $\tilde{A}$ for the eigenvalues λ_i, $i = p+1,p+2,\ldots,n$. Let these eigenvectors be denoted by x_i, $i = p+1,p+2,\ldots,n$ respectively.

5. Determine the eigenvectors of $\tilde{A}'$ for the eigenvalues λ_i, $i = p+1,p+2,\ldots,n$ Let these eigenvectors be denoted by y_i, $i = p+1,p+2,\ldots,n$ respectively.

6. Normalize the eigenvectors y_i, $i = p+1,p+2,\ldots,n$ so that $\tilde{y}'_i x_i = 1$, $i = p+1,p+2,\ldots,n$.

7. Then the minimal realization of $(C, \tilde{A}, B)$ has order $n-p$ and is given by $(\bar{C}, \bar{A}, \bar{B})$ where:

$$\bar{C} \triangleq C(x_{p+1}, x_{p+2}, \ldots, x_n)$$

$$\bar{A} \triangleq \mathrm{diag}(\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n) \tag{4}$$

$$\bar{B} \triangleq \begin{pmatrix} \tilde{y}'_{p+1} \\ \tilde{y}'_{p+2} \\ \vdots \\ \tilde{y}'_n \end{pmatrix} B$$

and the minimal realization of (C, A, B) is given by $(\bar{C}, \bar{A}-\bar{B}K\bar{C}, \bar{B})$.

The above algorithm produces a minimal realization which in general has complex elements appearing in (C, A, B); the following algorithm gives a minimal realization which has only real elements appearing in (C, A, B).

<u>Algorithm III (to Determine Real Minimal Realization of (C, A, B))</u>

Assume that steps 1 to 5 in the above algorithm have been carried out, and assume that the ordering of the eigenvalues of $\tilde{A}$ has been carried out so that:

$$\{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n\} = \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+t}; \lambda_{p+t+1}, \tilde{\lambda}_{p+t+1}, \lambda_{p+t+2}, \tilde{\lambda}_{p+t+2}, \ldots$$
$$\ldots, \lambda_{p+t+q}, \tilde{\lambda}_{p+t+q}\} \tag{5}$$

where $\lambda_{p+1}, \ldots, \lambda_{p+t}$ are all real eigenvalues, $\lambda_{p+t+1}, \lambda_{p+t+2}, \ldots, \lambda_{p+t+q}$ are all complex eigenvalues and $t+2q = n-p$. Steps 6, 7 of the previous algorithm now become:

6' For $i = p+1, p+2, \ldots, p+t$, let $y_i^* \triangleq \dfrac{y_i}{y_i^T x_i}$.

For $i = p+t+1, p+t+2, \ldots, p+t+q$, let $y_i^* \triangleq [\frac{a}{a^2+b^2} \mathrm{Re}(y_i) - \frac{b}{a^2+b^2} \mathrm{Im}(y_i)] +$

$j[-\frac{b}{a^2+b^2} \mathrm{Re}(y_i) - \frac{a}{a^2+b^2} \mathrm{Im}(y_i)]$, where $a \triangleq \mathrm{Re}'(y_i)\mathrm{Re}(x_i) + \mathrm{Im}'(y_i)\mathrm{Im}(x_i)$,

$b \triangleq \mathrm{Re}'(y_i)\mathrm{Im}(x_i) - \mathrm{Im}'(x_i)\mathrm{Re}(x_i)$.

7' Let $\Gamma_1 \in R^{n\times(n-p)}$, $\Gamma_2 \in R^{(n-p)\times m}$, $\bar{A} \in {}^{(n-p)\times(n-p)}$ be defined as follows:

$$\Gamma_1 \triangleq \{x_{p+1}, x_{p+2}, \ldots, x_{p+t}; \sqrt{2}\mathrm{Re}(x_{p+t+1}), \sqrt{2}\mathrm{Im}(x_{p+t+1}), \sqrt{2}\mathrm{Re}(x_{p+t+2}), \sqrt{2}\mathrm{Im}(x_{p+t+2}), \ldots, \sqrt{2}\mathrm{Re}(x_{p+t+q}), \sqrt{2}\mathrm{Im}(x_{p+t+q})\} \tag{6}$$

$$\Gamma_2' \triangleq \{y^*_{p+1}, y^*_{p+2}, \ldots, y^*_{p+t}; \sqrt{2}\mathrm{Re}(y^*_{p+t+1}), \sqrt{2}\mathrm{Im}(y^*_{p+t+1}), \sqrt{2}\mathrm{Re}(y^*_{p+t+2}), \sqrt{2}\mathrm{Im}(y^*_{p+t+2}), \ldots, \sqrt{2}\mathrm{Re}(y^*_{p+t+q}), \sqrt{2}\mathrm{Im}(y^*_{p+t+q})\} \tag{7}$$

$$\bar{A} \triangleq \text{block diag}\left\{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+t}, \begin{bmatrix} \mathrm{Re}(\lambda_{p+t+1}), & \mathrm{Im}(\lambda_{p+t+1}) \\ -\mathrm{Im}(\lambda_{p+t+1}), & \mathrm{Re}(\lambda_{p+t+1}) \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{Re}(\lambda_{p+t+q}), & \mathrm{Im}(\lambda_{p+t+q}) \\ -\mathrm{Im}(\lambda_{p+t+q}), & \mathrm{Re}(\lambda_{p+t+q}) \end{bmatrix}\right\} \tag{8}$$

Then the minimal realization of $(C, \tilde{A}, B)$ has order n-p and is given by $(\bar{C}, \bar{A}, \bar{B})$ where $\bar{C} \triangleq C\Gamma_1$, $\bar{B} \triangleq \Gamma_2 B$, and the minimal realization of (C, A, B) is given by $(\bar{C}, \bar{A}-\bar{B}K\bar{C}, \bar{B})$.

Proof

The proof of statements 7, 7' in the above algorithms follow directly by applying a Jordan decomposition to $(C, \tilde{A}, B)$ and using the definition of fixed modes of (C, A, B). The details are omitted.

Discussion

In step 7' of algorithm III, the matrices Γ_1, Γ_2, $\bar{A}$ are all real, so that the minimal realization $(\bar{C}, \bar{A}, \bar{B})$ obtained contains only real elements.

In steps 3-5 of algorithm II and III, the calculation of the eigenvalues and eigenvectors of a matrix with only distinct eigenvalues is required. It should be noted that many extremely effective algorithms exist to do this (e.g. the QR family of algorithms) and are readily available (e.g. the EISPACK set of subroutines). It should also be noted that the computation time required to carry out all steps of algorithm III only varies as n^3 (the computation time required to carry out steps 3-5 of algorithm II, III only varying as $(n-p)^3$.)

4. NUMERICAL EXAMPLES

Example 1

It is desired to find a state space minimal realization of a boiler furnace model [10] whose transfer function matrix is given by:

$$G(s) = \begin{bmatrix} \frac{1}{1+4s} & \frac{0.7}{1+5s} & \frac{0.3}{1+5s} & \frac{0.2}{1+5s} \\ \frac{0.6}{1+5s} & \frac{1}{1+4s} & \frac{0.4}{1+5s} & \frac{0.3}{1+5s} \\ \frac{0.35}{1+5s} & \frac{0.4}{1+5s} & \frac{1.0}{1+4s} & \frac{0.6}{1+5s} \\ \frac{0.2}{1+5s} & \frac{0.3}{1+5s} & \frac{0.7}{1+5s} & \frac{1.0}{1+4s} \end{bmatrix} \tag{9}$$

In this case, a 16^{th} order nonminimal state space realization of (9) is directly obtained as:

$$\begin{aligned}
\dot{\eta}_1 &= -0.25\eta_1 + 0.25u_1 & \dot{\eta}_9 &= -0.20\eta_9 + 0.07u_1 \\
\dot{\eta}_2 &= -0.20\eta_2 + 0.14u_2 & \dot{\eta}_{10} &= -0.20\eta_{10} + 0.08u_2 \\
\dot{\eta}_3 &= -0.20\eta_3 + 0.06u_3 & \dot{\eta}_{11} &= -0.25\eta_{11} + 0.25u_3 \\
\dot{\eta}_4 &= -0.20\eta_4 + 0.04u_4 & \dot{\eta}_{12} &= -0.20\eta_{12} + 0.12u_4 \\
\dot{\eta}_5 &= -0.20\eta_5 + 0.12u_1 & \dot{\eta}_{13} &= -0.20\eta_{13} + 0.04u_1 \\
\dot{\eta}_6 &= -0.25\eta_6 + 0.25u_2 & \dot{\eta}_{14} &= -0.20\eta_{14} + 0.06u_2 \\
\dot{\eta}_7 &= -0.20\eta_7 + 0.08u_3 & \dot{\eta}_{15} &= -0.20\eta_{15} + 0.14u_3 \\
\dot{\eta}_8 &= -0.20\eta_8 + 0.06u_4 & \dot{\eta}_{16} &= -0.20\eta_{16} + 0.25u_4
\end{aligned} \tag{10}$$

$$\begin{aligned}
y_1 &= \eta_1 + \eta_2 + \eta_3 + \eta_4 \\
y_2 &= \eta_5 + \eta_6 + \eta_7 + \eta_8 \\
y_3 &= \eta_9 + \eta_{10} + \eta_{11} + \eta_{12} \\
y_4 &= \eta_{13} + \eta_{14} + \eta_{15} + \eta_{16}
\end{aligned}$$

and on applying the feedback $u = Ky$ to (10), where K is given by (arbitrarily chosen):

$$K = \begin{bmatrix} -.368462 & .255606 & -.041349 & .032768 \\ -.28104 & -.452955 & .178865 & .179297 \\ .434693 & -.116497 & .019417 & .330966 \\ -.465427 & -.446538 & .029701 & .17115 \end{bmatrix}$$

the following open loop and closed loop eigenvalues of (10) were obtained (see Table 1):

Table 1 Open and Closed Loop Eigenvalues of (10) with u = Ky

Eigenvalues of A	Eigenvalues of A+BKC
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	-0.343009 ± j0.0571058
-0.200000	-0.236782 ± j0.0630918
-0.200000	-0.225945
-0.200000	-0.164543
-0.250000	-0.170378
-0.250000	-0.201203
-0.250000	
-0.250000	

*Fixed modes of (10)

which implies that the fixed modes of (C,A,B) for (10) are given by {-0.2, -0.2, -0.2, -0.2, -0.2, -0.2, -0.2, -0.2}, and thus that the minimal realization of (10) has order 8.

Remark 4

It should be noted that the eigenvalues of the open loop and closed loop system corresponding to the fixed modes agree with each other to 15 significant figures (using double precision arithmetic on an IBM 370 digital computer).

On applying algorithm III to the system (10), the minimal realization of (C, $\tilde{A}$, B) is directly obtained and is given (truncated to 6 significant figures) in Table 2. The minimal realization of (C, A, B) is then immediately obtained and is given in Table 3 (truncated to 6 significant figures).

Remark 5

It should be noted that the transfer function representation obtained for (C, A, B) agrees with (9) to 15 significant figures.

Example 2

It is desired to determine if the system (10) is controllable. In this case, on applying the feedback $u = K^*x$ to (10), where K^* is given by (arbitrarily chosen):

$$K^* = \begin{bmatrix} -0.368 & -0.492 & -0.452 & -0.063 \\ 0.256 & -0.116 & 0.237 & 0.267 \\ -0.041 & -0.433 & -0.171 & -0.022 \\ 0.033 & -0.062 & 0.133 & -0.262 \\ -0.261 & 0.167 & 0.257 & 0.225 \\ -0.452 & 0.089 & 0.492 & -0.140 \\ 0.179 & 0.431 & -0.134 & -0.333 \\ 0.180 & 0.347 & -0.252 & -0.013 \\ 0.435 & 0.027 & 0.483 & 0.398 \\ -0.116 & -0.408 & 0.223 & 0.410 \\ 0.020 & 0.154 & 0.254 & 0.439 \\ 0.331 & 0.064 & 0.152 & 0.405 \\ -0.465 & 0.202 & -0.427 & 0.005 \\ -0.446 & 0.411 & 0.132 & 0.017 \\ 0.030 & 0.263 & 0.385 & 0.160 \\ 0.172 & -0.237 & -0.227 & 0.487 \end{bmatrix}'$$

the following open loop and closed loop eigenvalues of (10) were obtained (Table 4):

Table 4 Open and Closed Loop Eigenvalues of (10) with $u = K^*x$

Eigenvalues of A	Eigenvalues of $A+BK^*$
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	*-0.200000
-0.200000	0.042654
-0.200000	-0.426849
-0.200000	-0.227648 ± j0.072282
-0.200000	0.166892
-0.250000	-0.220722 ± j0.004844
-0.250000	-0.212503
-0.250000	
-0.250000	

*Fixed modes of (10)

which implies that the fixed modes of (I_n, A, B) for (10) are given by {-0.2, -0.2, -0.2, -0.2, -0.2, -0.2, -0.2, -0.2}, and thus that the system (10) is not controllable.

5. CONCLUSIONS

A definition of centralized fixed modes of a linear, time-invariant multivariable system is made, and an algorithm (algorithm I) for calculating the fixed modes of a system is given. The calculation of the fixed modes only requires the calculation of the eigenvalues of two $n \times n$ matrices for a n^{th} order system. It is then shown that the determination of whether a system is controllable, observable, stabilizable, detectable etc. can be easily carried out in a numerically efficient way by calculating the centralized fixed modes of the system. A direct application of fixed modes of a system is then made to obtain an extremely efficient algorithm (algorithms II, III) for obtaining the minimal realization of a linear time invariant system. The algorithms have been successfully used on systems up to 100^{th} order.

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Table 2 Minimal Realization of (C, $\tilde{A}$, B) Obtained For Example

$$\tilde{A} = \begin{bmatrix} -.343009 & -.0571058 & 0 & 0 & 0 & 0 & 0 & 0 \\ .0571058 & -.343009 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.236782 & -.0630918 & 0 & 0 & 0 & 0 \\ 0 & 0 & .0630918 & -.236782 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.225945 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.164543 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -.170378 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.201203 \end{bmatrix}$$

$$B = \begin{bmatrix} .27667 & .550161 & -.0783017 & -.0686386 \\ .734103 & .102412 & -.0332067 & -.0851425 \\ -.0948559 & -.352436 & .263659 & .57111 \\ .212113 & .465087 & .124411 & -.223794 \\ -.0280578 & -.058489 & -.00590984 & .0748105 \\ -.36691 & .057034 & .433436 & .350038 \\ -.413059 & .124359 & .215827 & .178217 \\ .0268964 & .059385 & .00925714 & -.0262933 \end{bmatrix}$$

$$C = \begin{bmatrix} .190429 & .266941 & .12385 & .123835 & .0553281 & .123578 & -.0813695 & .00434014 \\ .431295 & -.0556493 & .17273 & .20436 & -.012404 & .201877 & -.215097 & -.0209016 \\ -.180345 & .0613934 & .36793 & .665509 & .037844 & .181602 & -.0917767 & -.0614214 \\ .436308 & -.0539819 & .347657 & -.0890121 & -.199607 & .386107 & -.355563 & -.00109043 \end{bmatrix}$$

Table 3 Minimal Realization of (C, A, B) Obtained For Example

$$A = \begin{bmatrix} -0.24945 & 0.0028601 & 0.000953795 & 0.0111942 & 0.0234153 & 0.0177737 & -0.0183217 & 0.00342252 \\ 0.0202134 & -0.257011 & 0.00274719 & 0.0301366 & 0.0213759 & -0.00684762 & 0.0216006 & 0.00419163 \\ 0.00964616 & 0.00490643 & -0.236326 & 0.00697603 & 0.0241034 & -0.00663203 & 0.00597105 & -0.00521307 \\ 0.0122532 & 0.0102847 & 0.023397 & -0.254414 & 0.0178632 & -0.0281454 & 0.020118 & 0.00314811 \\ 0.00674467 & 0.00249873 & 0.00646932 & 0.0217891 & -0.225363 & 0.00553786 & -0.00469373 & -0.00676436 \\ 0.0332595 & -0.0488118 & -0.0470661 & 0.037323 & 0.0265031 & -0.202373 & 0.0172933 & -0.00451077 \\ 0.0521047 & -0.0460500 & -0.0262534 & 0.0151354 & 0.00205266 & -0.0136037 & -0.173467 & -0.00301061 \\ 0.00320933 & 0.00223664 & -0.00300911 & -0.00183704 & 0.0021333 & -0.00234723 & 0.00155134 & -0.200795 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.27667 & 0.550161 & -0.0783017 & -0.0686386 \\ 0.734103 & 0.102412 & -0.0332067 & -0.0851425 \\ -0.0948559 & -0.352436 & 0.263659 & 0.57111 \\ 0.212113 & 0.465087 & 0.124411 & -0.223794 \\ -0.0280578 & -0.058489 & -0.00590984 & 0.0748105 \\ -0.36691 & 0.057034 & 0.433436 & 0.350038 \\ -0.413059 & 0.124359 & 0.215827 & 0.178217 \\ 0.0268964 & 0.059385 & 0.00925714 & -0.0262933 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.190429 & 0.266941 & 0.12385 & 0.123835 & 0.0553281 & 0.123578 & -0.0813695 & 0.00434014 \\ 0.431295 & -0.0556493 & 0.17273 & 0.20436 & -0.012404 & 0.201877 & -0.215097 & -0.0209016 \\ -0.180345 & 0.0613934 & 0.36793 & 0.665509 & 0.037844 & 0.181602 & -0.0917767 & -0.0614214 \\ 0.436308 & -0.0539819 & 0.347657 & -0.0890121 & -0.199607 & 0.386107 & -0.355563 & -0.00109043 \end{bmatrix}$$

To be published in IEEE Trans. on Automatic Control

An Algorithm for the Calculation of Transmission Zeros of the System (C,A,B,D) Using High Gain Output Feedback

by

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ABSTRACT

A new algorithm, which is an extension of algorithm II given in [1], is given to determine the transmission zeros of the system $\dot{x} = Ax + Bu$, $y = Cx + Du$ denoted by (C,A,B,D). The algorithm is based on the observation that for nondegenerate (C,A,B,D) systems, the transmission zeros of (C,A,B,D) are contained in the finite eigenvalues of the closed loop system matrix $\{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$, where K is any arbitrary matrix of full rank, $y \in R^r$ and $\rho \to \infty$.

1. INTRODUCTION

This paper is concerned about the efficient calculation of the transmission zeros of the following linear, time-invariant system denoted by (C,A,B,D):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input and $y \in R^r$ is the output. It is not necessarily assumed that (1) is controllable or observable nor that rank $B = m$, rank $C = r$ necessarily.

In [1]--[3] the transmission zeros of (1) were defined to be the set of complex numbers λ which satisfy the following inequality:

$$\text{rank} \begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} < n + \min(r,m) \tag{2}$$

in particular, <u>the transmission zeros are the zeros (multiplicities included)</u>

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This work has been supported by the National Research Council of Canada under Grant A4396 and by the National Aeronautics and Space Administration under Grant No. NSG 1215

of the greatest common divisor of all $(n + \min(r,m))\times(n + \min(r,m))$ minors of $\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}$. Two algorithms for calculating the transmission zeros of (1) were given in [1]. The purpose of this paper is to give a new algorithm for finding the transmission zeros of (1), which is an extension of algorithm II given in [1]*.

2. DEVELOPMENT

The following definition is required in the development to follow:

Definition [1]

(C,A,B,D) is degenerate if for any specified $n+1$ distinct real scalars λ_i^*, $i = 1,2,\ldots,n+1$, the following is true:

$$\text{rank}\begin{pmatrix} A-\lambda_i^* I, & B \\ C, & D \end{pmatrix} < n + \min(r,m), \quad i = 1,2,\ldots,n+1 \qquad (3)$$

A system which is not degenerate is called a non-degenerate system. A degenerate system has transmission zeros which include the whole complex plane.

The following lemmas follow by observation:

Lemma 1

Assume that $(A-\lambda I)$ is nonsingular; then

$$\text{rank}\begin{pmatrix} A-\lambda I, & B \\ C, & D \end{pmatrix} = n + \text{rank}\,[D - C(A-\lambda I)^{-1}B] \qquad (4)$$

Lemma 2

Assume there exists a scalar $\lambda \in \mathbb{C}$ so that:

$$\text{rank}\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = n + \min(r,m) ;$$

then (C,A,B,D) is non-degenerate.

Lemma 3

Given $D \in R^{r\times m}$, $K \in R^{m\times r}$ with $\text{rank}\, K = \min(r,m)$, then there exists a constant $\rho^* > 0$ so that $(\frac{I_r}{\rho} - DK)$ is nonsingular $\forall\rho > \rho^*$.

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* It should be noted that an interesting algorithm which is quite different from the one presented here has recently been suggested in [6].

The following lemma is required in the development to follow:

<u>Definition</u>

Given the polynomial equation in λ:

$$c(\lambda^p + \sum_{i=0}^{p-1} c_i\lambda^i) + \varepsilon(\sum_{i=0}^{q} \overline{c}_i(\varepsilon)\lambda^i) = 0 \ , \ q \geq p \tag{5}$$

with real coefficients $c, c_i, \overline{c}_i$ with $c \neq 0$ such that:

$\overline{c}_q(\varepsilon) \neq 0$, $|\overline{c}_i(\varepsilon)| < M$, $i = 1,2,\ldots,q$, $\forall\varepsilon \in [0,\infty)$, let the roots of (5) be denoted by λ_j, $j = 1,2,\ldots,q$. Let the roots of the polynomial equation in λ:

$$c(\lambda^p + \sum_{i=0}^{p-1} c_i\lambda^i) = 0$$

be denoted by σ_j, $j = 1,2,\ldots,p$.

<u>Lemma 4</u> [1]

For any $(\delta,\overline{\delta})$, $\delta > 0$, $\overline{\delta} > 0$, there exists an $\epsilon^*(\delta,\overline{\delta}) > 0$ so that $\forall\varepsilon \in [0,\epsilon^*)$, p roots of (5) suitably ordered, have the property that $|\lambda_j - \sigma_j| < \delta$, $j = 1,2,\ldots,p$ and $q - p$ roots of (5) suitably ordered, have the property that $|\lambda_j| > \overline{\delta}$, $j = p+1,p+2,\ldots,q$.

The following lemma follows by observation from the definition of transmission zeros given in (2).

<u>Lemma 5</u>

Assume that (C,A,B,D) is nondegenerate and that $K \in R^{m\times r}$ has rank $K = \min(r,m)$. Then:

(1) If $m > r$ for "almost all" K, the system (C,A,BK,DK) is nondegenerate.

(2) If $m < r$ for "almost all" K, the system (KC,A,B,KD) is nondegenerate.

i.e., the class of K matrices in which the above statements are not true is either empty or lies on a hypersurface [4] in the parameter space of K.

<u>Lemma 6</u>

Assume that (C,A,B,D) is nondegenerate and that $K \in R^{m\times r}$ has rank $K = \min(r,m)$. Then:

(1) If $m > r$, for "almost all K", the transmission zeros of (C,A,B,D) are contained in the transmission zeros of (C,A,BK,DK).

(2) If $r < m$, for "almost all K", the transmission zeros of (C,A,B,D) are contained in the transmission zeros of (KC,A,B,KD).

i.e., the class of K matrices in which the above statements are not true is either empty or lies on a hypersurface in the parameter space of K.

Proof

The proof of this result follows from lemma 5 on using the definition of transmission zeros given by (2).

3. MAIN RESULTS

The following initial result is obtained:

Theorem 1

Assume that (C,A,B,D) is nondegenerate; then for "almost all" $\lambda \in \mathbb{C}$, rank $[D-C(A-\lambda I)^{-1}B] = \min(r,m)$, i.e. the class of λ which results in rank $[D-C(A-\lambda I)^{-1}B] < \min(r,m)$ or in $(A-\lambda I)$ being singular is either empty or lies on a hypersurface [2] in the parameter space $\lambda \in \mathbb{C}$.

Proof

The proof follows directly from lemmas 1, 2 on using identical arguments as used in the proof of theorems 1, 2 of [4].

Definition

Given the system (1), let $\sigma(\rho)$ denote an eigenvalue of

$$M(\rho) \triangleq \{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$$

where $K \in R^{m\times r}$ with rank $K = \min(r,m)$ and $\rho \in R$. Assume now that there exists a scalar $\lambda^* \in \mathbb{C}$ with the property that for any $\epsilon > 0$ there exists $\bar{\rho}(\epsilon) > 0$ so that $|\sigma(\rho)-\lambda^*| < \epsilon \quad \forall \rho > \bar{\rho}(\epsilon)$; then λ^* is called a finite eigenvalue of $M(\rho)$.

Remark 1

From lemma 3, the matrix $M(\rho)$ in the above definition is well defined if ρ is sufficiently large.

The following main result is now obtained:

Theorem 2 (High Gain Output Feedback Theorem)

Given the nondegenerate system (C,A,B,D), let $K \in R^{m\times r}$ be an arbitrary matrix with rank $K = \min(r,m)$. Then if $m = r$, the finite eigenvalues of $\{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$ as $\rho \to \infty$ coincide with the transmission zeros of (C,A,B,D): if $m \neq r$ then for "almost all" K, the transmission zeros of (C,A,B,D) are contained in the finite eigenvalues of $\{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$, i.e. the class of K matrices in which the above statement is not true is either empty or lies on a hypersurfurce in the parameter space of K.

Remark 2

This theorem is an extension of theorem 8 of [1] and is a generalization of the classical result for single-input, single-output systems, in which it is well known that for high gain output feedback, the poles of the closed loop system approach the zeros of the system.

Proof

The proof of this result follows very closely the proof of theorem 8 in [1]. In particular following [1], assume with no loss of generality (theorem 7 [1]), that (A,B) is controllable and that (A,B) is in Brunosky's canonical form. Let n_i, $i = 1,2,\ldots,m$ be the controllability indices [5] of (A,B) with $\sum_{i=1}^{m} n_i = n$. Let $C \triangleq (C_1, C_2, \ldots, C_m)$ be the output matrix (not necessarily unique) which results from transforming (A,B) into this canonical form and let $D \triangleq (D_1, D_2, \ldots, D_m)$ where $C_i \in R^{r \times n_i}$ and $D_i \in R^{r \times 1}$. Then the transmission zeros of (C,A,B,D) are given by the set $\{\lambda\}$ which satisfy the condition

$$\text{rank} \begin{pmatrix} A-\lambda I, & B \\ C, & D \end{pmatrix} < n + \min(r,m)$$

which after substitution and simplification may be written as

$$\text{rank } \Gamma(\lambda) < \min(r,m) \tag{6}$$

where

$$\Gamma(\lambda) \triangleq \left[(C_1, D_1)\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n_1} \end{pmatrix}, (C_2, D_2)\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n_2} \end{pmatrix}, \ldots, (C_m, D_m)\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n_m} \end{pmatrix} \right] \tag{7}$$

Assume now that $m = r$ and that ρ is sufficiently large so that $(\frac{I_r}{\rho} - DK)$ is nonsingular (lemma 3). Consider the eigenvalues of the matrix $M(\rho) \triangleq \{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$; they are given by the solution for λ of the following equation:

$$\det (M(\rho) - \lambda I) = 0 \tag{8}$$

which can be simplified to the following equation (see Appendix 1):

$$\det\{\rho\ K\ \Gamma(\lambda) - \mathrm{diag}(\lambda^{n_1}, \lambda^{n_2}, \ldots, \lambda^{n_m})\} = 0 \tag{9}$$

and on expanding the determinant of the left hand side of (9), the following is obtained:

$$\rho^m \det [K\ \Gamma(\lambda)] + \{\lambda^n + \sum_{i=0}^{n-1} c_i(\rho)\lambda^i\} = 0 \tag{10}$$

where the coefficients $c_i(\rho)$ have the property that

$$\lim_{\rho\to\infty} \frac{c_i(\rho)}{\rho^m} = 0\ ,\ i = 0,1,\ldots,n-1. \tag{11}$$

Since K is nonsingular, (10) may be written as:

$$\det [\Gamma(\lambda)] + \frac{1}{\rho^m \det K} \{\lambda^n + \sum_{i=0}^{n-1} c_i(\rho)\lambda^i\} = 0. \tag{12}$$

Now since it has been assumed that (C,A,B,D) is nondegenerate, then $\det [\Gamma(\lambda)]$ is a nonzero polynomial in λ and hence from lemma 4, it follows that the finite eigenvalues of the matrix $M(\rho)$ are given by the solution of the equation:

$$\det [\Gamma(\lambda)] = 0 \tag{13}$$

for λ. This means, from (6), that the finite eigenvalues of the matrix $M(\rho)$ are equal to the transmission zeros of (C,A,B,D) for the case $m = r$. The proof of the case $m \neq r$ follows directly from this result on using lemmas 5,6.

4. AN ALGORITHM FOR FINDING TRANSMISSION ZEROS OF A SYSTEM

Theorems 1, 2 form the basis of a new algorithm for finding transmission zeros of (1). The following algorithm is obtained from theorem 1.

Algorithm to Determine if (C,A,B,D) is Non-degenerate

1. Choose an arbitrary real λ^* (by using a pseudo-random number generator say) so that $\|\lambda^* I_n\| \doteq \|A\|$) so that $(A - \lambda^* I_n)$ is nonsingular. This will be true for "almost all" λ^* chosen.
2. Calculate rank $(D - C(A-\lambda^* I)^{-1}B)$. If rank $(D - C(A-\lambda^* I)^{-1}B) = \min(r,m)$, the system (C,A,B,D) is nondegenerate; if rank $(D - C(A-\lambda^* I)^{-1}B) < \min(r,m)$, then (C,A,B,D) is "almost always" degenerate, i.e. (C,A,B,D) can only be nondegenerate provided λ^* lies on a hypersurface in the parameter space $\lambda^* \in \mathbb{C}$.

3. If in doubt as to whether (C,A,B,D) is degenerate, repeat step 2 using a different real λ^* ; if $\text{rank }(D - C(A-\lambda^*I)^{-1}B) < \min(r,m)$ for $\min(r,m)+1$ distinct real values of λ^* , the system (C,A,B,D) is degenerate.

The following algorithm follows from theorem 2. In this algorithm, it is assumed that (C,A,B,D) is nondegenerate, but it is not necessarily assumed that (C,A,B) is controllable or observable or that $\text{rank } B = m$, $\text{rank } C = r$ or that $m = r$.

Algorithm to Find Transmission Zeros of a Nondegenerate (C,A,B,D) System

1. Choose an arbitrary matrix $K \in R^{m\times r}$ (by using a pseudo random number generator say) so that $\text{rank } K = \min(r,m)$.
2. Calculate the eigenvalues of the matrix $M(\rho) \triangleq \{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$ for large ρ (e.g. $\rho = 10^{15}$ *) ; then if $m = r$, the transmission zeros of (C,A,B,D) are equal to the finite eigenvalues of $M(\rho)$, and if $m \neq r$, then for "almost all" K, the transmission zeros of (C,A,B,D) are contained in the finite eigenvalues of $M(\rho)$.
3. If in doubt as to what the finite eigenvalues are, repeat steps 1, 2 with a different K and different ρ (e.g. $\rho = 10^{16}$).

Remark 3

In case the condition $m = r$ is not satisfied, those finite eigenvalues of $M(\rho)$ which do not satisfy (2) should be discarded; the remaining finite eigenvalues are then the transmission zeros of the system.

Remark 4

Note that it is not necessary to actually calculate the rank of matrices in the above algorithms, e.g. let $N \triangleq (D - C(A-\lambda^*I)^{-1}B)$ — then if $r \le m$, $\text{rank } N = r$ if and only if NN' is nonsingular, and if $r > m$, $\text{rank } N = m$ if and only if $N'N$ is nonsingular.

---

* More precisely, $\rho = 10^{15} \frac{\|A\|}{\|BKC\|}$. Here, it is assumed that double precision arithmetic is used with this value of ρ . If single precision arithmetic is used, a smaller value of ρ is suggested, say $\rho = 10^8$, to minimize the effect of rounding errors in the digital computer.

Remark 5

Note that many effective algorithms exist to find the eigenvalues of a matrix (e.g. the QR family of algorithms); because of the large value of ρ occurring in $M(\rho)$, it is essential however that balancing techniques [7] be used in finding the eigenvalues of the matrix $M(\rho)$ to prevent problems of ill-conditioning arising in the eigenvalue calculation. Most existing eigenvalues packages have this feature (e.g. the EISPACK set of subroutines).

5. NUMERICAL EXAMPLE

It is desired to find the transmission zeros of (C,A,B,D) where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

In this case, it is observed that $r = m = 2$ and that for a choice of $\lambda^* = 2.17$ (arbitrarily chosen) rank $[D-C(A-\lambda^* I)^{-1}B] = 2$, so that the system is nondegenerate. On applying the algorithm of the previous section, the following eigenvalues of $\{A + BK(\frac{I_r}{\rho} - DK)^{-1}C\}$ were obtained, for the case $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, on an IBM 370 digital computer using double precision arithmetic; the eigenvalue algorithm used was a QR algorithm with balancing features (taken from the EISPACK set of subroutines).

Eigenvalues of $A + BK(\frac{I}{\rho} - DK)^{-1}C$]

ρ	Eigenvalues
10^8	0.9999999899 0.3411639019 ± j1.161541399 -0.6823278038 -0.4999999949 ± j0.1000000004 × 10^5
10^{12}	0.9999999999 0.3411639019 ± j1.161541399 -0.6823278038 -0.4999999999 ± j0.1000000000 × 10^7
10^{14}	* 0.9999999999 * 0.3411639019 ± j1.161541399 * -0.6823278038 -0.4999999999 ± j0.1000000000 × 10^8

*finite eigenvalues of $A + BK(\frac{I}{\rho} - DK)^{-1}C$

In this case, it can be seen that the transmission zeros of (C,A,B,D) are given by (to 10 significant figures): 1.000000000, 0.3411639019 ± j1.161541399, -0.6823278038. The transmission zeros obtained in this case agree with the exact values (given by the solution of $(\lambda-1)(\lambda^3+\lambda+1) = 0$) to at least 12 significant figures.

6. CONCLUSIONS

This paper has shown that for a general linear multivariable system described by (1), the poles of the closed loop system approach the transmission zeros of the system when high gain output feedback is used (theorem 2). This result has formed the basis of a new algorithm for the numerical calculation of the transmission zeros of a general system. In the proposed algorithm, the typical computation time/storage space required corresponds to that of finding the eigenvalues of a single $n \times n$ matrix, where n is the order of the system. The algorithm has been successfully used on systems of up to 50th order.

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APPENDIX 1

Simplification of Equation (8)

It is desired to show that (8) can be simplified to (9). It is assumed that $m = r$ and that $(I_r - DK\rho)$, K are nonsingular matrices. Let $A_i \in R^{n_i \times n_i}$, $b_i \in R^{n_i}$, $i = 1,2,\dots,m$, H be defined as follows:

$$A_i \triangleq \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad b_i \triangleq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad H \triangleq K\rho(I_r - DK\rho)^{-1}C \qquad (1a)$$

Then (8) can be written as:

$$\det\,[A + BK\rho(I_r - DK\rho)^{-1}C - \lambda I] = 0 \qquad (2a)$$

which can be written as:

$$\det\left[\text{block diag}(A_1 - \lambda I,\dots,A_m - \lambda I) + \text{block diag}(b_1,\dots,b_m)\,H\right] = 0 \qquad (3a)$$

which can be simplified to:

$$\det\left\{ -\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m}) + \Theta \text{ block diag}\left[\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_1-1}\end{pmatrix},\ldots,\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_m-1}\end{pmatrix}\right]\right\} = 0 \tag{4a}$$

which can be written as:

$$\det\left\{ -\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m}) + (\rho^{-1}K^{-1}-D)^{-1}C \text{ block diag}\left[\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_1-1}\end{pmatrix},\ldots,\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_m-1}\end{pmatrix}\right]\right\} = 0 \tag{5a}$$

Equation (5a) now can be written as:

$$\det\left\{ (\rho^{-1}K^{-1}-D)\cdot -\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m}) + C \text{ block diag}\left[\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_1-1}\end{pmatrix},\ldots,\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_m-1}\end{pmatrix}\right]\right\} = 0 \tag{6a}$$

or alternately as:

$$\det\left\{ -\rho^{-1}K^{-1}\,\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m}) + C \text{ block diag}\left[\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_1-1}\end{pmatrix},\ldots,\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_m-1}\end{pmatrix}\right] + D\,\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m})\right\} = 0 \tag{7a}$$

which can be written as:

$$\det\left\{ -\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m}) + \rho K\left[C \text{ block diag}\left[\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_1-1}\end{pmatrix},\ldots,\begin{pmatrix}1\\ \lambda\\ \vdots\\ \lambda^{n_m-1}\end{pmatrix}\right] + D\,\mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m})\right]\right\} = 0 \tag{8a}$$

or as

$$\det\,[\rho\, K\, \Gamma(\lambda) - \mathrm{diag}(\lambda^{n_1},\ldots,\lambda^{n_m})] = 0 \tag{9a}$$

which is the desired result.

Presented at 1977 International Forum on Alternatives for Multivariable Control. Also submitted to IEEE Trans. on Automatic Control.

OBSERVING PARTIAL STATES FOR SYSTEMS WITH UNMEASURABLE DISTURBANCES

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ABSTRACT

This paper considers the problem of observing a set of linear functionals of states for systems with unmeasurable disturbances. A transfer function approach is adopted to characterize the set of observers for a given system. The result also has direct application to state estimation in decentralized control systems.

1. INTRODUCTION

The problem of observing the state of a linear time-invariant multivariable system has been investigated by Luenberger in his early papers [1], [2]. Luenberger has considered the design of minimal-order observers which estimate the whole state as well as the minimal-order observers which only estimate a given linear functional of the state. The problem of constructing a minimal-order observer which observes a given set of linear functionals of the state has been investigated in [3] - [6]. In [7] - [9], different approaches are used to construct observers which can estimate the states of systems with unmeasurable disturbances. This paper considers the general problem of observing a given set of linear functionals of the state for systems with unmeasurable disturbances. In section II, a transfer function approach is adopted to characterize observers for a given system see [10] [11]. In Section III, it is shown that based on this transfer function characterization, one can then apply the minimization algorithm in [11] - [15] to obtain a lower bound for the order of dynamic observers required. For relatively low order systems, one can usually construct such observers with arbitrary poles without difficulty. However, the general problem of constructing minimal-order observers with arbitrary poles for systems with unmeasurable disturbances has not been fully resolved. In [6] Tarski's decision method has been applied to the design of minimal-order observers. For more details, see [16], [17]. The result in this paper also has direct application to state estimation in decentralized control systems [18].

II. A TRANSFER FUNCTION CHARACTERIZATION OF OBSERVERS

In this section we adopt the transfer function approach to the design of dynamic observers. This approach has been used by Retallack [10]. In the following, Theorem 1 gives a complete characterization of dynamic observers which observes a specified set of linear functionals of the state in terms of transfer function matrices for systems with unmeasurable inputs.

Theorem 1

Consider a completely controllable linear system specified by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $u(t) \in R^m$ is the input, $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the output and A, B and C are real constant matrices of appropriate size.

Let $\Lambda = \text{diag}\{\lambda_{ii}\}$ be a mxm diagonal matrix with $\lambda_{ii} = 1$ if $u_i(t)$ is measurable and $\lambda_{ii} = 0$ if $u_i(t)$ is unmeasurable. Define $\tilde{u}(t) = \Lambda u(t)$.
Consider another linear system with inputs $\tilde{u}(t) \in R^m$ and $y(t) \in R^p$, and output $w(t) \in R^q$,

$$\dot{z}(t) = Fz(t) + G\tilde{u}(t) + Jy(t) \tag{3}$$

$$w(t) = Lz(t) + M\tilde{u}(t) + Ny(t) \tag{4}$$

where $z(t) \in R^{n'}$ is the state, and F, G, J, L, M and N are real matrices of appropriate size, and the pair (L, F) is completely observable. (See Figure 1). Let K be an arbitrary real matrix of size $q \times n$. Then the following two conditions are equivalent:

(I) There exists a real number $\gamma > 0$ and a map $g: R^n \times R^{n'} \to [0, \infty)$ with $g(0, 0) = 0$, such that $\forall x(o) \in R^n$ $z(o) \in R^{n'}$, and for any piecewise continuous control $u: [0, \infty) \to R^m$, we have

$$\|w(t) - Kx(t)\| \leq g(x(0), z(0)) \times \exp(-\gamma t) \quad \forall t \geq 0.$$

(II) The transfer function from u to w is equal to

$$H_2(s)H_1(s) = K(sI - A)^{-1}B, \tag{5}$$

where

$$H_1(s) = \left[\begin{array}{c} C(sI - A)^{-1}B \\ \hdashline \Lambda \end{array}\right]$$

$$H_2(s) = L(sI - F)^{-1} [J \mid G] + [N \mid M]$$

and

$\mathrm{Re}\,\lambda(F) \leq -\gamma$ if $\lambda(F)$ is a simple root of the minimal polynomial of F

$\mathrm{Re}\,\lambda(F) < -\gamma$ if $\lambda(F)$ is a multiple root of the minimal polynomial of F.

Remark

Theorem 1 states that the class of linear multivariable systems with proper rational transfer functions, whose outputs $w(t)$ asymptotically approach $Kx(t)$ as $t \to \infty$, is characterized by the conditions of (II).

In order to prove Theorem 1, we first state the following lemma.

Lemma 1

Consider a completely observable system specified by

$$\dot{x}(t) = Ax(t)$$

$$y(t) = Cx(t)$$

where $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the output, and A, C are real constant matrices of appropriate size. The following two conditions are equivalent:

(i) There exists a function $g: R^n \to [0,\infty)$ with $g(0) = 0$, and a real number $\gamma > 0$, such that

$$\|C \exp(At). x_o\| \le g(x_o) \exp(-\gamma t), \ \forall x_o \in R^n, \ \forall t \ge 0.$$

(ii) $\operatorname{Re} \lambda(A) \le -\gamma$ if $\lambda(A)$ is a simple root of the minimal polynomial of A,

$\operatorname{Re} \lambda(A) < -\gamma$ if $\lambda(A)$ is a multiple root of the minimal polynomial of A.

The simple proof of Lemma 1 is omitted.

Proof of Theorem 1

(I) $\Rightarrow$ (II)

Let $x(0) = 0$ and $z(0) = 0$, then for any input $u(.)$, we have $w(t) = Kx(t)$. Hence the transfer function from u to w is equal to $K(sI - A)^{-1}B$. On setting

$x(0) = 0$ and $u(t) = 0 \ \forall t \ge 0$, then

$$w(t) = L \exp(Ft)\, z(0)$$

and

$$x(t) = 0 \quad \forall t \geq 0.$$

Hence

$$\|w(t)\| = \|L \exp(Ft) x(0)\| \leq g\,(0,\, z(0)) \exp(-\gamma t), \forall t \geq 0.$$

Since (L, F) is completely observable and $z(0) \in R^{n'}$ is arbitrary, then from Lemma 1 the result follows.

(II) $\Rightarrow$ (I)

Let $x(0) = 0$ and $z(0) = 0$. Then since the transfer function from u to w is $K(sI - A)^{-1}B$, we have

$$0 = w(t) - Kx(t) \qquad \forall t \geq 0. \tag{6}$$

From (4), (6), and $\tilde{u} = \Lambda u$,

$$0 = Lz(t) + M\Lambda u(t) + (NC - K)\, x(t)\ . \tag{7}$$

Assume that $u(.)$ has a discontinuity at time t and that both $x(\cdot)$ and $z(.)$ are continuous functions at time t. In order to satisfy (7) , $M\Lambda$ must be a zero matrix.

Taking derivatives of (7).

$$0 = LF\, z(t) + [LG\Lambda + NCB - KB]\, u(t) + [LJC + NCA - KA]x(t) \tag{8}$$

Using the same arguments as above, we conclude that the matrix $[LG\Lambda + NCB - KB] = 0$. Continuing this process, we have

$$0 = LF^{i-1}z(t) + W_i x(t),\ (I = 1, 2, \ldots, n') \tag{9}$$

where $W_1 = NC - K$, $W_2 = LJC + NCA - KA$, etc.

Eq. (9) can be written as

$$Qz(t) = Wx(t) , \tag{10}$$

where $Q = \begin{bmatrix} L \\ \hdashline LF \\ \hdashline \\ \hdashline LF^{n'-1} \end{bmatrix}$ and $W = - \begin{bmatrix} W_1 \\ \hdashline W_2 \\ \hdashline \\ \hdashline W_{n'} \end{bmatrix}$

From the observability property of (L, F), the matrix Q^TQ is nonsingular. Hence from (10), there follows

$$z(t) = (Q^TQ)^{-1} Q^T Wx(t)$$

$$= Tx(t), \qquad \forall t \geq 0, \tag{11}$$

where $T \triangleq (Q^TQ)^{-1}Q^TW$.

Next we calculate

$$\frac{d}{dt}[z(t) - Tx(t)] = F[z(t) - Tx(t)] + [JC - TA + FT]x(t) + [G\Lambda - TB]u(t) \tag{12}$$

Assume that $x(0) = 0$ and $z(0) = 0$, from (11) and (12),

$$[JC - TA + FT]x(t) + [G\Lambda - TB]u(t) = 0 \qquad \forall t \geq 0.$$

If $u(.)$ has a discontinuity at t, $x(.)$ is a continuous function at time t, hence $[G\Lambda - TB] = 0$. Furthermore, from the controllability property of (A, B), $x(t) \in R^n$ can be any vector at time $t > 0$. Therefore $[JC - TA + FT] = 0$. From (12), there follows

$$\frac{d}{dt}[z(t) - Tx(t)] = F[z(t) - Tx(t)] \tag{13}$$

for any $x(0)$, $z(0)$, $u(.)$ and $t \geq 0$.

Consider the following,

$$\begin{aligned} w(t) - Kx(t) &= Lz(t) + NCx(t) - Kx(t) \\ &= L[z(t) - Tx(t)] + [LT + NC - K]x(t) \\ &= L\cdot \exp(Ft)\,[z(0) - Tx(0)] + [LT + NC - K]x(t). \end{aligned} \tag{14}$$

From zero initial conditions $x(0) = 0$ and $z(0) = 0$, we know that $w(t) - Kx(t) = 0 \ \forall t \geq 0$. Then from the controllability property of (A, B), it follows that $LT + NC - K = 0$. Hence from (14), there exists a function $g: R^n \times R^{n'} \rightarrow [0,\infty)$ with $g(0,0) = 0$ such that

$$\|w(t) - Kx(t)\| = \|L \exp(Ft)\,[z(0) - Tx(0)]\| \leq g\,(x(0),\, z(0)) \exp\,(-\gamma t),$$

for all $x(0) \in R^n$, $z(0) \in R^{n'}$ and $u(.)$, and $t \geq 0$, where γ is as specified in condition (II) of Theorem 1. Q. E. D.

Remark: Part of the above proof was motivated by results in [4].

III. LOWER BOUND ON THE ORDER OF OBSERVERS

The minimization algorithm in [11] - [15] solves the following problem: let $P(s)$ and $Q(s)$ be given proper rational matrices of appropriate size: how does one find a proper rational matrix $H(s)$, (assuming it exists), which satisfies the following equation

$$P(s)H(s) = Q(s)$$

such that $H(s)$ is of least possible order (McMillan degree)?

In the following, we will apply the minimization algorithm to find a lower bound on the order of a dynamic observer which is required to estimate a specified set of linear functionals of the state of systems with unmeasurable inputs.

From (5), we have

$$H_2(s)\begin{bmatrix} C(sI - A)^{-1}B \\ \text{- - - - - - - - - -} \\ \Lambda \end{bmatrix} = K\,(sI - A)^{-1}B \tag{5'}$$

where $C(sI - A)^{-1}B$ is the transfer function of the given system in (1) and (2), $H_2(s)$ is the transfer function of a dynamic system whose output tends to $Kx(t)$. It is clear that if we take transpose of (5'),

$$[B^T(sI - A^T)^{-1}C^T \mid \Lambda]\, H_2^T(s) = B^T(s\,I - A^T)^{-1}K^T, \tag{15}$$

we can apply the minimization algorithm to the above equation to construct the set of solutions $H_2^T(s)$ of least McMillan degree. Moreover, as pointed out in [11], [12] the set of observability indices of all minimal-order solutions $H_2(s)$ are equal. Since the solutions $H_2(s)$ constructed by the minimization algorithm may not be stable only a lower bound on the order of the dynamic observer required is in general obtained. In the special case, however, when $q = 1$ or $p = 1$, one can rederive from (15) previous results on minimal-order observers [1] - [4]. For the general case $q > 1$ and $p > 1$, the problem of finding minimal-order dynamic observers with arbitrary pole locations is not yet fully resolved. As indicated in [6], Tarski's decision method seems to be suitable for this class of problems. For more details, see [16], [17].

IV. CONCLUSION

required is derived. The result in this paper has direct application to state estimation in decentralized control systems [18].

ACKNOWLEDGEMENT

This research has been supported by the National Aeronautics and Space Administration, Langley Research Center under Grant NSG 1213 and the National Research Council of Canada under Grant A4396.

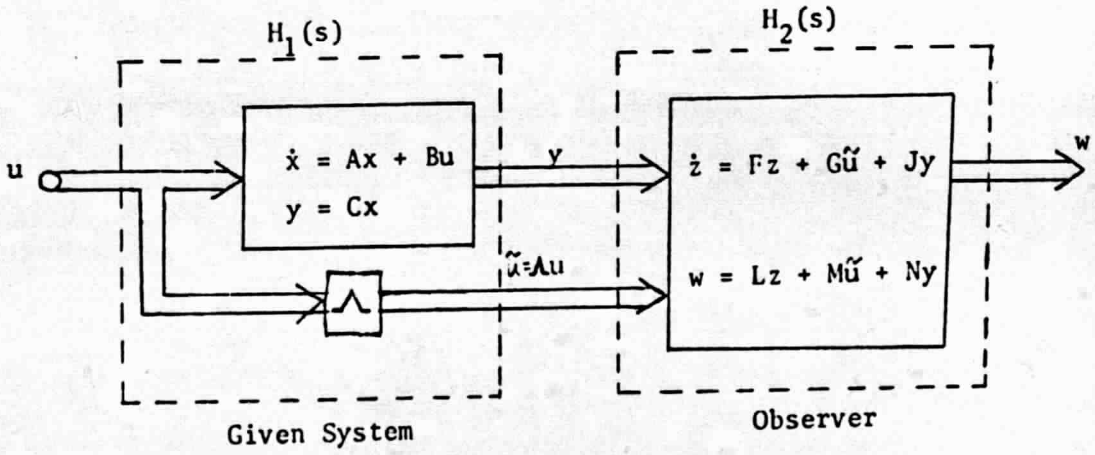

Figure 1.

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Presented at 1977 Conference on Information Sciences and Systems. Also to appear in International Journal of Control.

MINIMIZATION OF TRANSMISSION COST IN DECENTRALIZED CONTROL SYSTEMS

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ABSTRACT

This paper considers the problem of stabilizing a linear time-invariant multivariable system by using local feedback controllers and some limited information exchange among local stations. The problem of achieving a given degree of stability with minimum transmission cost is solved.

1. Introduction

When control theory is applied to solve problems of large systems, e.g., electric power systems, socioeconomic systems, etc., an important feature called decentralization often arises. Such systems have several local control stations; at each station, the controller observes only local system outputs and controls only local inputs. All the controllers are involved, however, in controlling the same large system. It is obviously important to know under what conditions there exists a set of appropriate local feedback control laws that will stabilize the complete system. Several different versions of this problem have been formulated and examined in [1]. A necessary and sufficient condition for the existence of local output feedback control laws with dynamic compensation to stabilize a given system is given in [2]. This condition is stated in terms of a new notion, called "fixed modes" of a decentralized control system. If some element of the set of fixed modes is unstable, then the system cannot be stabilized via local controllers only. Some transmission of output information among local

This work has been supported by the National Aeronautics and Space Administration, Langley Research Center under Grant No. NSG. 1213 and the National Research Council of Canada, under Grant No. A4396.

stations is necessary. An algorithm which minimizes the amount of transmission cost is given in Section 4. Section 2 defines the problem and Section 3 summarizes some previous results. An example is given in Section 5.

2. Problem Statement

Consider a linear time-invariant multivariable system with ν local control stations described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{\nu} B_i u_i(t) \tag{1a}$$

$$y_i(t) = C_i x(t), \quad (i = 1,\ldots,\nu), \tag{1b}$$

where $x(t)\varepsilon R^n$ is the state, $u_i(t)\varepsilon R^{m_i}$ and $y_i(t)\varepsilon R^{p_i}$ are the input and output, respectively, of the ith local control station $(i = 1,\ldots,\nu)$. The matrices A, B_i, and C_i, $(i = 1,\ldots,\nu)$, are real, constant, and of appropriate size. The problem is to find ν local output feedback control laws with dynamic compensation in order to stabilize the whole system. The set of local feedback laws are assumed to be generated by the following feedback controllers:

$$\dot{z}_i(t) = S_i z_i(t) + R_i y_i(t) \tag{2a}$$

$$u_i(t) = Q_i z_i(t) + K_i y_i(t) + v_i(t),$$

$$(i = 1,\ldots,\nu,) \tag{2b}$$

where $z_i(t)\varepsilon R^{n_i}$ is the state of the ith feedback controller, $v_i(t)\varepsilon R^{m_i}$ is the ith local external input, and S_i, R_i, Q_i, and K_i are real constant matrices of appropriate size. Equation (2) can be written more compactly as follows:

$$\dot{z}(t) = Sz(t) + Ry(t) \tag{3a}$$

$$u(t) = Qz(t) + Ky(t) + v(t), \tag{3b}$$

where

$$S \triangleq \text{block diag } [S_1,\ldots,S_\nu]$$
$$R \triangleq \text{block diag } [R_1,\ldots,R_\nu]$$
$$Q \triangleq \text{block diag } [Q_1,\ldots,Q_\nu]$$
$$K \triangleq \text{block diag } [K_1,\ldots,K_\nu]$$
$$z^T(t) \triangleq [z_1{}^T(t) \cdots z_\nu{}^T(t)]$$
$$y^T(t) \triangleq [y_1{}^T(t) \cdots y_\nu{}^T(t)]$$
$$u^T(t) \triangleq [u_1{}^T(t) \cdots u_\nu{}^T(t)]$$

and

$$v^T(t) \triangleq [v_1{}^T(t) \cdots v_\nu{}^T(t)]. \tag{4}$$

When the feedback control law generated by (3) is applied to system (1), the closed-loop system is described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + BKC & BQ \\ RC & S \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t), \tag{5}$$

where

$$B = [\underbrace{B_1}_{m_1} \vdots \cdots \vdots \underbrace{B_\nu}_{m_\nu}] \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ --- \\ \cdot \\ \cdot \\ \cdot \\ --- \\ C_\nu \end{bmatrix} \begin{matrix} \}\, p_1 \\ \\ \\ \\ \\ \\ \}\, p_\nu \end{matrix} \tag{6}$$

The set of local feedback control laws of (3) is to be chosen so that the overall system (5) is asymptotically stable, (i.e., all poles of system (5) are in the open left-half complex plane $\mathcal{C}^-$). This then is the problem of stabilizing a decentralized control system via local output feedback with dynamic compensation.

3. Solution to the Stabilization Problem.

The following definitions are required to state the necessary and sufficient condition for the existence of solutions to the stabilization problem.

Definition 1

Consider the triple $(C,A,B) \in R^{pxn} \times R^{nxn} \times R^{nxm}$ and the two sets of integers $m_1,\cdots,m_\nu$ and $p_1,\cdots,p_\nu$

$$m = \Sigma_{i=1}^{\nu} m_i \quad \text{and} \quad p = \Sigma_{i=1}^{\nu} p_i$$

which specifies system (1). Let $\mathcal{K}$ be the set of block diagonal matrices as follows:

$$\mathcal{K} = \{K | K = \text{block diag } [K_1,\cdots K_\nu], \quad K_i \in R^{m_i x p_i}, \quad i = 1,\cdots,\nu\}. \tag{7}$$

Then the greatest common divisor of the set of characteristic polynomials of $(A + BKC)$, for all $K \in \mathcal{K}$, is called the fixed polynomial of the triple (C,A,B) with respect to $\mathcal{K}$, denoted by $\psi(\lambda;C,A,B,\mathcal{K})$, i.e.,

$$\psi(\lambda;C,A,B,\mathcal{K}) = \underset{K\in\mathcal{K}}{\gcd}\{\det(\lambda I - A - BKC)\}.$$

Comment: The map $\psi(\lambda;C,A,B,\cdot):\mathcal{K} \mapsto \underset{K\in\mathcal{K}}{\gcd}\{\det(\lambda I - A - BKC)\}$ is well defined for any subset $\mathcal{K} \subset R^{mxp}$ and not just for the set $\mathcal{K}$ of block diagonal matrices in (7).

Definition 2

For a given triple $(C,A,B) \in R^{pxn} \times R^{nxn} \times R^{nxm}$ and the given set $\mathcal{K}$ of block diagonal matrices in (7), the set of fixed modes of (C,A,B) with respect to $\mathcal{K}$ is defined as follows:

$$\Lambda(C,A,B,\mathcal{K}) = \bigcap_{K\epsilon\mathcal{K}} \sigma[A + BKC]$$

where $\sigma[A + BKC]$ denotes the set of eigenvalues of $(A + BKC)$. Equivalently, $\Lambda(C,A,B,\mathcal{K})$ can be defined as follows:

$$\Lambda(C,A,B,\mathcal{K}) = \{\lambda | \lambda\epsilon\ \mathcal{C} \quad \text{and } \psi(\lambda;C,A,B,\mathcal{K}) = 0\} .$$

Theorem 1 [2]

Consider the given system (C,A,B) of (1), with B and C defined in (6). Let $\mathcal{K}$ be the set of block diagonal matrices defined in (7). Then a necessary and sufficient condition for the existence of a set of local feedback control laws (3) such that the closed-loop system (5) is asymptotically stable is that

$$\Lambda(C,A,B,\mathcal{K}) \subset \mathcal{C}^-$$

where $\mathcal{C}^-$ denotes the open left-half complex plane.

Corollary 1 [2]

Under the same assumptions as in Theorem 1, the necessary and sufficient condition fot the existence of a set of local feedback control laws in (3), such that all poles of the closed-loop system of (5) are in S is that

$$\Lambda(C,A,B,\mathcal{K}) \subset S$$

where S is any nonempty symmetric open subset on the complex plane $\mathcal{C}$(i.e., if $\lambda \ \epsilon\ S$ then its complex conjugate $\lambda^* \ \epsilon\ S$).

For the proof of theorem 1 and some further details, please refer to [2].

4. Minimum Transmission Cost

In the previous sections, the problem of stabilizing a linear system via local output feedback is investigated. In case that a given system cannot be stabilized via local output feedback then it is necessary to transmit some observations of local system outputs to other control stations in order to stabilize the whole system. It is natural to assume that there is a certain amount of transmission cost, say $\alpha_{ij} \geq 0$, of transmitting a scalar time function from station i to station j per unit time. The problem is to design a control scheme which stabilize a given system with minimum transmission cost.

Consider the following set of matrices

$$\mathcal{K}(\rho_{ij}, i,j=1,\cdots,\nu) = \left\{ K \middle| K = \text{block}\{K_{ij}\} \triangleq \left[\begin{array}{c:c:c} K_{11} & \cdots & K_{1\nu} \\ \hdashline \vdots & & \vdots \\ \hdashline K_{\nu 1} & \cdots & K_{\nu\nu} \end{array}\right], \; K_{ij} \in R^{m_i \times p_j}, \; \text{rank } K_{ij} = \rho_{ij} \right\} \tag{8}$$

The set of matrices $\mathcal{K}$ defined above corresponds to the set of constant output feedback laws $u = Ky$. Each feedback law $u = Ky$ consists of a set of (sub)feedback laws $u_i = K_{ij}y_j$ which requires the transmission of a time function with ρ_{ij} components from station j to station i. The total transmission cost required for the implementation of $K\epsilon\mathcal{K}(\rho_{ij}, i,j=1,\cdots,\nu)$ is

$$\mathcal{C}(\rho_{ij}, i,j=1,\ldots,\nu) = \sum_{j=1}^{\nu} \sum_{i=1}^{\nu} \rho_{ij}\alpha_{ji}$$

Hence the problem is to find a set of integers ρ_{ij}, $0 \leq \rho_{ij} \leq \min(m_i, p_j)$ in order to minimize

$$\mathcal{C}(\rho_{ij}, i, j=1,\ldots,\nu)$$

$$\text{subject to} \qquad \Lambda(C,A,B,\mathcal{K}(\rho_{ij}, i,j=1,\ldots,\nu)) \subset \mathcal{C}^{-}. \tag{9}$$

For a given set of $\rho_{ij}(i,j=1,\ldots,\nu)$, the condition of (9) can be checked as follows: Each submatrix K_{ij} of rank ρ_{ij} can be written as the product $L_{ij}M_{ij}$, where L_{ij} is of size $m_i x \rho_{ij}$ and M_{ij} is of size $\rho_{ij} x p_j$. Let $\{\lambda_\alpha, \ldots, \lambda_\delta\}$ be the set of eigenvalues of A with nonnegative real parts. Then condition (9) is satisfied if and only if

$$\prod_{i \in \{\alpha, \ldots, \delta\}} \det(\lambda_i I - A - B[\text{block}\{L_{ij}M_{ij}\}]C) \tag{10}$$

is not identically zero.

Remark

The above condition implies that all unstable eigenvalues of A have been shifted by the application of feedback $u = [\text{block}\{L_{ij}M_{ij}\}]\, y$, i.e., none of unstable eigenvalues of A is a fixed mode.

Since there is only a finite number of possibilities in choosing ρ_{ij}'s, the minimization problem of (9) can be solved in a finite number of steps. Once the set $(\rho_{ij}, i,j=1,\ldots,\nu)$ is determined, the actual construction of dynamic compensators is basically the same as in the proof of Theorem 1.

5. Example

Consider the following system with two control stations,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2^1$$

$$y_1^1 = [0 \;\; 0 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} y_2^1 \\ y_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where $\begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix}$ and y_1^1 are the input and output at station 1; u_2^1 and $\begin{bmatrix} y_2^1 \\ y_2^2 \end{bmatrix}$ are the input and output at station 2. Assume that the transmission cost $\alpha_{11} = \alpha_{22} = 0$, $\alpha_{12} = 1$ and $\alpha_{21} = 2$.

In order to minimize the transmission cost, first consider the decentralized feedback

$$\mathcal{K}(\rho_{11} = 1, \rho_{12} = \rho_{21} = 0, \rho_{22} = 1)$$

$$= \left\{ K \;\middle|\; K = \begin{bmatrix} k_{11} & 0 & 0 \\ k_{21} & 0 & 0 \\ 0 & k_{32} & k_{33} \end{bmatrix},\; k_{11}, k_{21}, k_{32}, k_{33} \in R \right\}$$

With the above set $\mathcal{K}$, the corresponding fixed polynomial of (C, A, B) is

$$\psi(\lambda; C, A, B, \mathcal{K}) = \underset{k_{11}, k_{21}, k_{32}, k_{33} \in R}{\text{g.c.d.}} \left\{ \det \begin{bmatrix} \lambda & 0 & -k_{11} \\ 0 & \lambda & -k_{21} \\ -k_{32} & -k_{33} & \lambda + 2 \end{bmatrix} \right\}$$

$$= \lambda .$$

This shows that with local feedback, there is a fixed mode at the origin. Hence the system can not be stabilized by local feedback. Now consider the case in which we transmit the output at station 1 to station 2. The corresponding transmission cost is

$$C(\rho_{11} = \rho_{21} = \rho_{22} = 1,\; \rho_{12} = 0) = 1.$$

And the set of feedback matrices is

$$\mathcal{K}(\rho_{11} = \rho_{21} = \rho_{22} = 1,\; \rho_{12} = 0)$$

$$= \left\{ K \;\middle|\; K = \begin{bmatrix} k_{11} & 0 & 0 \\ k_{21} & 0 & 0 \\ k_{31} & k_{32} & k_{33} \end{bmatrix},\ k_{11}, k_{21}, k_{31}, k_{32}, k_{33} \in R \right\}$$

It can be shown that with respect to the above set of $\mathcal{K}$, the fixed polynomial of (C, A, B) is λ. Hence, again the system can not be stabilized even with transmission of output from station 1 to station 2.

Next consider transmitting a single linear combination of the output at station 2 to station 1. The transmission cost is

$$C\ (\rho_{11} = \rho_{12} = \rho_{21} = 1,\ \rho_{21} = 0) = 2$$

The corresponding set of feedback matrices is

$$\mathcal{K}\ (\rho_{11} = \rho_{12} = \rho_{21} = 1,\ \rho_{21} = 0)$$

$$= \left\{ K \;\middle|\; K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ 0 & k_{32} & k_{33} \end{bmatrix},\ \text{all } k_{ij}\text{'s} \in R \text{ and rank} \begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix} = 1 \right\}$$

The additional rank constraint on the matrix $\begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix}$ can be expressed as

$$\begin{bmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}$$

where $\beta_1, \beta_2, \gamma_1$ and γ_2 are arbitrary real numbers.

The corresponding fixed polynominal is

$$\underset{\substack{\beta_i \gamma_i, \\ k_{ij}\text{'s}}}{\text{g.c.d.}} \left\{ \det \begin{bmatrix} \lambda - \beta_1\gamma_1 & -\beta_1\gamma_2 & -k_{11} \\ -\beta_2\gamma_1 & \lambda - \beta_2\gamma_2 & -k_{21} \\ -k_{32} & -k_{33} & \lambda + 2 \end{bmatrix} \right\} = 1$$

Hence there is no fixed mode in the system. By choosing $\beta_1 = \beta_2 = k_{11} = 1$, $\gamma_1 = k_{21} = k_{33} = 0$, and $\gamma_2 = k_{32} = -1$, the resulting system characteristic polynomial is $\lambda^3 + 3\lambda^2 + 3\lambda + 1$. Hence all three system poles have been assigned at -1.

6. Conclusions

In this paper, the problem of stabilizing a large scale decentralized system via local controllers and with minimum transmission cost for information exchange among local stations is discussed. Clearly, a more efficient algorithm is required if one would like to apply this method to practical design problems. Several related publications are included in the reference.

Acknowledgement

The authors would like to thank Professor W.M. Wonham and Dr. K.C. Chu for the suggestion of this topic.

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