(NASA-CR-155578) POST-NEWTONIAN GRAVITATIONAL BREMSSTRAHLUNG (Stanford Univ.) 28 p HC A03/MF A01 CSCL 08N

N78-16520

Unclas
G3/46 01831

POST-NEWTONIAN GRAVITATIONAL BREMSSTRAHLUNG*

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ABSTRACT

We present formulae and numerical results for the gravitational radiation emitted during a low-deflection encounter between two massive bodies ("gravitational bremsstrahlung"). Our results are valid through post-Newtonian order within general relativity. We discuss in detail the gravitational waveform (transverse-traceless part of the metric perturbation tensor), the total luminosity and total emitted energy, the angular distribution of emitted energy (antenna pattern), and the frequency spectrum. We also present a method of "boosting" the accuracy of these quantities to post-3/2-Newtonian order. A numerical comparison of our results with those of Peters and of Kovács and Thorne shows that the post-Newtonian method is reliable to better than 0.1 percent at v = 0.1 c, to a few percent at v ≈ 0.35 c, and to 10-20 percent at v = 0.5 c. We also compare our results with those of Smarr.

*Supported in part by the National Aeronautics and Space Administration [NSG 7204 S1] and the National Science Foundation [PHY 76-21454].

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ITP-580 7/77

I. INTRODUCTION AND SUMMARY

The emission of gravitational radiation by massive bodies moving in high-velocity, low-deflection unbound orbits — gravitational bremsstrahlung — has recently been suggested as a possible source of radiation, detectable in principle by doppler tracking of interplanetary spacecraft (Estabrook and Wahlquist 1975, Thorne and Braginsky 1976). Several studies have been made of the nature of gravitational bremsstrahlung, each starting from a different point of view and making different assumptions. The pioneering analysis is that of Peters (1970), who studied perturbations of the Schwarzschild geometry by test particles moving with arbitrary velocities in large-impact-parameter orbits. Matzner and Nutku (1974) focussed on the high-frequency behavior of the bremsstrahlung spectrum of ultrarelativistic test particles in the Scharzschild geometry using the method or virtual quanta. Smarr (1977) calculated the zero frequency limit of gravitational bremsstrahlung produced by a test particle moving with arbitrary velocity in the field of a massive body. Perhaps the definitive analysis of gravitational bremsstrahlung has been performed by Kovács and Thorne (1977a, b, hereafter KTa, KTb) based on the post-linear formalism developed by Thorne and Kovács (1975) and Crowley and Thorne (1977). Their analysis treats bodies with arbitrary masses and arbitrary relative velocities, while maintaining the condition of small-angle deflections. Their formal results may be written very elegantly (albeit in very lengthy and complex form) in terms of frame-invariant quantities. In the special case of test-particle bremsstrahlung their results are in complete numerical agreement with Peters (see also §IX). Unfortunately, because of the complexity of both sets of formulae, a direct analytic comparison was not possible (KTb).

Our goal in this paper is somewhat more modest. We present and discuss a detailed analysis of the semi-relativistic limit of gravitational bremsstrahlung, using the post-Newtonian methods developed by Epstein and Wagoner (1975) and by Wagoner and Will (1976, WW hereafter). The post-Newtonian technique has a variety of advantages. Like the KT method, it allows particles of arbitrary mass. Unlike the KT method (or Peters' method), it is very simple; the form for the transverse traceless components of the metric perturbation h^{ij}_{TT} ("gravitational waveform") requires only half a dozen full lines of Astrophysical Journal type (cf. eq. [16]), and makes use of familiar Newtonian variables. Its limitation, the restriction to "post-Newtonian" velocities, is only one of principle. For most astrophysical purposes, a restriction to velocities smaller than about one-half the speed of light is undoubtedly adequate, since potential bremsstrahlung orbits inside bound systems (such as globular clusters or galactic nuclei) satisfy a virial relation of the form $v^2 \sim GM/R$. On the other hand, stability arguments indicate that the most relativistic of such systems must satisfy $GM/Rc^2 \lesssim 0.5$. Thus we feel the post-Newtonian technique gives more reliable estimates of the nature of gravitational bremsstrahlung in astrophysical contexts than can be obtained from a Newtonian analysis, and yet avoids the unnecessary complications of ultrarelativistic formulae.

We begin with a brief summary of the important results. Our

starting point is the post-Newtonian gravitational waveform given by WW (eq. [97]); in a frame in which the center of mass of the system is at rest at the origin, it has the general form[1]

$$h^{ij}_{TT} = (2m_1m_2/Rb)\left[N^{ij} + \tfrac{1}{4}v_\infty(\delta m/m)(P_{1/2}N)^{ij} + \tfrac{1}{4}v_\infty^{\,2}(PN)^{ij}\right] , \quad (1)$$

where m_1, m_2 and $m = m_1 + m_2$ are the masses of the bodies and the total mass of the system, R is the distance between the observer and the center of mass of the system, $b \gg m$ is the impact parameter of the two-body orbit, v_∞ is the relative velocity of the two bodies at infinite separation, and $\delta m = m_1 - m_2$. The terms N^{ij}, $(P_{1/2}N)^{ij}$ and $(PN)^{ij}$ represent respectively the Newtonian, post-1/2-Newtonian, and post-Newtonian contributions to the waveform. They depend in general on time, on the reduced mass $\mu = m_1m_2/m$ and on the orientation of the orbit relative to the observer's direction. An important goal of most bremsstrahlung computations is the derivation of such gravitational waveforms since it is these waveforms that are detected in spacecraft tracking and in other broad-band detection schemes. Thus we have evaluated explicitly the two independent components of h^{ij}_{TT} detected by distant observers, the "plus-polarization", h_+, and the "cross-polarization", $h_\times$, as functions of time (see §II for detailed definitions) for a variety of bremsstrahlung situations. Figure 1 shows the results for the case $m_2 \gg m_1$ (see §IX for an additional interpretation of this case) for $v_\infty \approx 0$ (the

[1]We use units in which the speed of light c and the Newtonian gravitational constant G are unity.

Newtonian limit) and for $v_\infty = 0.5$. The waveforms h_+ and $h_\times$ are plotted in units of $(2m_1m_2/Rb)$ for various observer locations relative to the orbit. Time is plotted in units of the interaction time $\tau = b/v_\infty$, with $t = 0$ coincident with the moment of closest approach. Figure 1 indicates clearly the front-back asymmetry of the waveforms that develops when $v_\infty = 0.5$; this is a manifestation of relativistic "beaming" of the radiation. More detailed discussion of these curves is given in §§II and IX.

We have also calculated the gravitational luminosity of the system; it should be noted that the luminosity has a clear meaning only when averaged over several wavelengths of the radiation, whereas the bremsstrahlung process emits a single burst of length $\sim\tau$ making such an average impossible. Nevertheless, many interesting features of the radiation are revealed by a study of the formal luminosity, given by

$$\frac{dL}{d\Omega} = \frac{R^2}{32\pi} \left(\frac{\partial h^{ij}_{TT}}{\partial t} \frac{\partial h^{ij}_{TT}}{\partial t} \right) . \qquad (2)$$

From this luminosity, one can calculate the angular distribution of the total radiated energy ("antenna pattern")

$$\frac{d(\delta E)}{d\Omega} = \int_{-\infty}^{\infty} \frac{dL}{d\Omega}\, dt \, . \qquad (3)$$

Figure 2 shows the resulting antenna patterns for various velocities ($v_\infty \approx 0$ [the Newtonian contribution], $v_\infty = 0.5$) and masses ($m_2 \gg m_1$, $m_2 = m_1$) and compares those with the corresponding electromagnetic bremsstrahlung antenna pattern for $v_\infty = 0.5$, $m_2 \gg m_1$ [Fig. 2(d)] (see §§IX and X for further interpretation of these patterns). For $m_1 \neq m_2$, as v_∞ increases there is beaming in the direction of travel of the less massive ojbect [Fig. 2(c)]. If $m_1 = m_2$ as v_∞ increases there is "beaming in both directions", i.e. a broadening of the lobes in the forward and backward directions [Fig. 2(b)]. In all cases (and in the electromagnetic case) most of the energy is radiated out of the orbital plane and nearly normal to it.

The post-Newtonian method makes errors of order v_∞^3 (post-3/2-Newtonian terms) in the waveforms $h_{+(\times)}$. We have made direct numerical comparisons between our results and the Kovács and Thorne results (an analytic comparison was not possible), and have found complete agreement within errors of order 0.1 percent at $v_\infty = 0.1$, a few percent at $v_\infty = 0.35$ and 10 to 20 percent at $v_\infty = 0.5$, in other words errors of $O(v_\infty^3)$. Peters (1977) has examined the post-Newtonian limit of his formulae and has found complete analytic agreement with us for the case $m_2 \gg m_1$.

We have also tested the validity of our results by showing that the post-Newtonian waveforms embody the rather remarkable "universality" property discovered by Kovács and Thorne (1977a). They found that they could write the gravitational waveforms h^{ij}_{TT} in a form that depended upon the masses m_1 and m_2 only via the overall multiplicative factor m_1m_2 that appears in equation (1). In other words the amplitudes $\tilde{h}^{ij}_{TT} = (Rb/2m_1m_2)h^{ij}_{TT}$ were universal, i.e., independent of the masses. When specialized to a chosen frame of reference for explicit computation, they found the waveforms depended only upon the velocity of the chosen frame relative to the frame located at the mid-point of the separation of the two bodies (center-of-velocity frame).[2] We verified this property by

[2] Throughout this discussion, statements such as rest-frame, center-of-velocity frame and so on refer to the undeflected, straight-line orbits of the bodies.

rederiving the gravitational waveform in a frame in which the center-of-velocity moves with velocity $\underset{\sim}{V}$ and found that the waveform $\tilde{h}^{ij}_{TT}$ depends only on $\underset{\sim}{V}$, not on m_1 and m_2. The apparent mass dependence seen in the center-of-mass (CM) system (eq. [1]), is merely a consequence of the fact that in that frame, the center of velocity moves with mass-dependent velocity $|\underset{\sim}{V}| = \frac{1}{2}(\delta m/m)v_\infty$.

Universality has several important consequences. Because $\tilde{h}$ depends only on the chosen frame, then the waveforms shown in Figure 1 can also be interpreted as waveforms observed in the rest frame of body 2 (since the center-of-mass frame ≡ rest frame of m_2 for $m_2 \gg m_1$) for arbitrary masses m_1 and m_2. This additional interpretation also applies to the antenna patterns shown in Figure 2(c). The equal-mass antenna pattern shown in Figure 2(b) can be reinterpreted as a pattern observed in the center-of-velocity frame (since the center-of-mass coincides with the center of velocity). The Newtonian pattern shown in Figure 2(a) is automatically universal. Although most of the results of this paper will be expressed in the center-of-mass frame for specific mass ratios m_1/m_2, they all have equivalent interpretations as results valid for arbitrary masses, but viewed from suitably chosen frames (see Table I).

Another consequence of universality is that it provides a method of improving the accuracy of the post-Newtonian results from post-Newtonian order to post-3/2-Newtonian order. In §X we apply this "trick" to the luminosity $dL/d\Omega$ and to the antenna patterns; the patterns shown in Figure 2 were in fact computed using this trick, and are accurate through $O(v_\infty^3)$.

The rest of the paper is devoted to details. In §II we discuss the assumptions and notation of WW and derive formulae for the gravitational wave forms $h_+(t)$ and $h_\times(t)$. We also outline the method for performing a multipole analysis of the waveforms. Sections III-VIII contain detailed discussions of the gravitational luminosity $dL/d\Omega$, the power P, the antenna pattern $d(\delta E)/d\Omega$, the frequency spectrum of radiated energy $d(\delta E)/d\sigma$ and its zero-frequency limit $d[\delta E(0)]/d\sigma$ (Smarr 1977), and the total radiated energy δE. In §IX we analyze the property of universality in detail, and in §X we use this property to boost the accuracy of the post-Newtonian method by means of exact Lorentz boosts. We give concluding remarks in §XI. Explicit formulae for the tensor spherical harmonics necessary for carrying out a multipole analysis of post-Newtonian bremsstrahlung are given in the appendix.

II. GRAVITATIONAL WAVEFORMS

Gravitational bremsstrahlung is the high velocity, low-deflection encounter between two massive, unbound objects. Specifically, this means that $v_\infty^2 \gg m/r_{min}$, where v_∞ is their relative velocity at large separations, m is their total mass and r_{min} their minimum separation. To Newtonian order the energy per reduced mass, $\tilde{E}$, is given by

$$v_\infty^2 = 2\tilde{E} = m(e-1)/r_{min} \ , \tag{4}$$

so that the bremsstrahlung assumption is satisfied by systems with large orbital eccentricity ($e \gg 1$). In usual astrophysical situations $U \sim m/r$ and v^2 are related by virial theorems that imply $U \sim v^2$. That is not the case here as $U \sim m/r_{min} \ll v^2 \sim v_\infty^2$. In the standard post-Newtonian analysis of h^{ij}_{TT} (Epstein and Wagoner 1975) quantities are expanded to the same order in U and v^2. For the bremsstrahlung process we expand h^{ij}_{TT} to one order in v^2 past Newtonian order, to Newtonian order in U (as $U \ll v^2$) and to first order in 1/e. Wagoner and Will (1976) have derived the formula for h^{ij}_{TT} in the bremsstrahlung process under these assumptions.

We shall summarize the main points of their work and then proceed from their expression for h^{ij}_{TT}. Let $\underset{\sim}{r}$ be the vector from m_2 to m_1, $\underset{\sim}{v}$ the relative velocity of m_1 with respect to m_2, b the impact parameter ($= r_{min}$ to lowest order in 1/e), and v_∞ the magnitude of $\underset{\sim}{v}$ at $r = \infty$ to lowest order in 1/e ($v_\infty^2 = em/b$), then the constants of the motion $\tilde{E}$ (= energy/μ) and $\tilde{J}$ (= angular momentum/μ) through post-Newtonian order and to first order in 1/e are

$$\tilde{E} = \frac{1}{2}v_\infty^2 + \frac{3}{8}(1-3\mu/m)v_\infty^4 + O(e^{-2}v_\infty^2) \ , \tag{5}$$

$$\tilde{J} = bv_\infty + \frac{1}{2}(1-3\mu/m)bv_\infty^3 + O(e^{-2}bv_\infty^3) \ . \tag{6}$$

The equations of motion through post-Newtonian order and to first order in 1/e in the center-of-mass system of m_1 and m_2 are

$$\begin{aligned}\underset{\sim}{v} = {} & v_\infty \underset{\sim}{e}_y + e^{-1}v_\infty(-\sin\chi\, \underset{\sim}{e}_x + \cos\chi\, \underset{\sim}{e}_y) \\ & + e^{-1}v_\infty^3\{\underset{\sim}{e}_x[-(1+21\mu/8m)\sin\chi - (\mu/8m)\sin 3\chi] \\ & + \underset{\sim}{e}_y[-(3-31\mu/8m)\cos\chi + (\mu/8m)\cos 3\chi]\} + O(e^{-2}v_\infty^3) \ ,\end{aligned} \tag{7}$$

$$m/r = e^{-1}v_\infty^2\cos\chi + O(e^{-2}v_\infty^2) \ , \tag{8}$$

$$dr/dt = v_\infty\sin\chi - e^{-1}v_\infty^3(4-\mu/m)\cos\chi + O(e^{-2}v_\infty^3) \ , \tag{9}$$

$$d\chi/dt = (v_\infty/b)\cos^2\chi + O(e^{-1}v_\infty/b) \ , \tag{10}$$

$$\underset{\sim}{\hat{r}} = \underset{\sim}{e}_x\cos\chi + \underset{\sim}{e}_y\sin\chi \ , \tag{11}$$

$$\underset{\sim}{\hat{\lambda}} = -\underset{\sim}{e}_x\sin\chi + \underset{\sim}{e}_y\cos\chi \ , \tag{12}$$

where χ parametrizes the orbit by

$$\cos\chi = b(b^2+v_\infty^2t^2)^{-1/2} \ , \tag{13}$$

$$\sin\chi = v_\infty t(b^2+v_\infty^2t^2)^{-1/2} \ . \tag{14}$$

Let the observer be at rest with respect to the center of mass and be at an observation point specified by the direction,

$$\underset{\sim}{n} = \underset{\sim}{e}_x \sin\theta\cos\phi + \underset{\sim}{e}_y \sin\theta\sin\phi + \underset{\sim}{e}_z \cos\theta \ , \tag{15}$$

and by R, the distance from the center-of-mass.

The coordinate system used throughout the calculation is shown in Figure 3. Note: extreme care should be taken when comparing bremsstrahlung calculations of different authors as almost every author employs a different coordinate system.

The formula for h^{ij}_{TT} to post-Newtonian order in $v_\infty{}^2$, to Newtonian order in m/b, and to first order in 1/e is (WW eq. [97])

$$
\begin{aligned}
h^{ij}_{TT}(t) = (2\mu/R)\Big[& 2v_\infty{}^2\delta^{iy}\delta^{jy}[1-(\delta m/m)v_\infty n_y + \tfrac{1}{2}(1-3\mu/m)v_\infty{}^2(1+2n_y{}^2)] \\
& + (e^{-1}v_\infty{}^2)(4\hat{\lambda}^{(i}\delta^{j)y} - 2\cos\chi\,\hat{r}^i\hat{r}^j) + (e^{-1}v_\infty{}^3\delta m/m) \\
& \times [3\cos\chi(\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})(2\hat{r}^{(i}\delta^{j)y} - \sin\chi\,\hat{r}^i\hat{r}^j) - 2\delta^{iy}\delta^{jy}(\underset{\sim}{n}\cdot\hat{\underset{\sim}{\lambda}}) \\
& - n_y(4\hat{\lambda}^{(i}\delta^{j)y} - \cos\chi\,\hat{r}^i\hat{r}^j)] + (e^{-1}v_\infty{}^4)\{2[3+(3-\sin^2\chi)\mu/m]\hat{\lambda}^{(i}\delta^{j)y} \\
& - 4(4-\mu/m)\cos\chi\,\delta^{iy}\delta^{jy} - 3[1+\cos^2\chi(\mu/m)]\cos\chi\,\hat{r}^i\hat{r}^j \\
& + 4(2-\mu/m)\cos\chi\sin\chi\,\hat{r}^{(i}\delta^{j)y} + \tfrac{1}{3}(1-6\mu/m)\cos\chi[2\delta^{iy}\delta^{jy} \\
& - (1-3\sin^2\chi)\hat{r}^i\hat{r}^j - 4\sin\chi\,\hat{r}^{(i}\delta^{j)y}] + \tfrac{1}{3}(1-3\mu/m) \\
& \times [12n_y(\underset{\sim}{n}\cdot\hat{\underset{\sim}{\lambda}})\delta^{iy}\delta^{jy} + 12n_y{}^2\hat{\lambda}^{(i}\delta^{j)y} - 2n_y{}^2\cos\chi\,\hat{r}^i\hat{r}^j \\
& - 4n_y(\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})\cos\chi(8\hat{r}^{(i}\delta^{j)y} - 3\sin\chi\,\hat{r}^i\hat{r}^j) \\
& - (\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})^2\cos\chi(14\delta^{iy}\delta^{jy} - 3\hat{r}^i\hat{r}^j + 15\sin^2\chi\,\hat{r}^i\hat{r}^j \\
& - 30\sin\chi\,\hat{r}^{(i}\delta^{j)y})]\}\Big]_{TT} \ .
\end{aligned}
\tag{16}
$$

The variables in equation (16) are to be evaluated at "retarded time" t - R. The TT indicates the transverse-traceless part given by (Epstein and Wagoner 1975)

$$h^{ij}_{TT} = h_{k\ell}\, P^{ik}\, P^{j\ell} - \frac{1}{2} h_{k\ell}\, P^{k\ell}\, P^{ij} \ ,$$

$$P^{ik} = \delta^{ik} - n^i n^k \ . \tag{17}$$

Combining equations (15), (16) and (17) gives

$$
\begin{aligned}
h_+ \equiv h^{\theta\theta}_{TT} = -h^{\phi\phi}_{TT} = & \ \frac{1}{4}(1+\cos^2\theta)\cos 2\phi(h^{xx}-h^{yy}) \\
& + \frac{1}{2}(1+\cos^2\theta)\sin 2\phi\ h^{xy} - \frac{1}{4}\sin^2\theta(h^{xx}+h^{yy}) \ ,
\end{aligned}
\tag{18}
$$

$$h_\times \equiv h^{\phi\theta}_{TT} = h^{\theta\phi}_{TT} = \cos\theta[\cos 2\phi\ h^{xy} - \frac{1}{2}\sin 2\phi(h^{xx}-h^{yy})] \ . \tag{19}$$

When equations (18) and (19) are evaluated we obtain

$$
\begin{aligned}
h_+ = (2m_1m_2/Rb)\{N_+ & + \frac{1}{4}v_\infty(\delta m/m)\sin\theta(P_{1/2}N)_+ \\
& + \frac{1}{4}v_\infty{}^2(PN)_+\} \ ,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
h_\times = (2m_1m_2/Rb)\cos\theta\{N_\times & + \frac{1}{4}v_\infty(\delta m/m)\sin\theta(P_{1/2}N)_\times \\
& + \frac{1}{4}v_\infty{}^2(PN)_\times\} \ .
\end{aligned}
\tag{21}
$$

$N_+(N_\times)$ is the Newtonian-order contribution to $h_+(h_\times) \sim O(1)$,

$(P_{1/2}N)_+[(P_{1/2}N)_\times]$ is the post-1/2-order contribution to $h_+(h_\times) \sim O(v_\infty)$, and $(PN)_+[(PN)_\times]$ is the post-Newtonian-order contribution to $h_+(h_\times) \sim O(v_\infty^2)$. The N's, $(P_{1/2}N)$'s, and (PN)'s are given by

$$N_+ = -(1+\cos^2\theta)\{\tfrac{1}{2}\cos 2\phi(C_1+2C_3)+\sin 2\phi(S_1+S_3)\}-\tfrac{1}{2}\sin^2\theta C_1 , \quad (22)$$

$$N_\times = -2\cos 2\phi(S_1+S_3)+\sin 2\phi(C_1+2C_3) , \quad (23)$$

$$(P_{1/2}N)_+ = \tfrac{1}{2}(1+\cos^2\theta)\{-\cos 3\phi(6S_1+5S_3+12S_5) + \sin 3\phi(2C_1-C_3+12C_5)+\cos\phi(2S_1-S_3)-\sin\phi(2C_1-C_3)\} + \sin^2\theta\{-\cos\phi(2S_1+3S_3)+\sin\phi(2C_1+3C_3)\} , \quad (24)$$

$$(P_{1/2}N)_\times = \cos 3\phi(2C_1-C_3+12C_5)+\sin 3\phi(6S_1+5S_3+12S_5) - \cos\phi(2C_1-C_3)-\sin\phi(2S_1-S_3) , \quad (25)$$

$$(PN)_+ = \tfrac{1}{2}(1+\cos^2\theta)\{2\cos 2\phi(BC_1+3AC_3-2AC_5) - 4\sin 2\phi[(3+2\mu/m)S_1-AS_3+AS_5]\}+\sin^2\theta\{BC_1+\tfrac{1}{3}(23-3\mu/m)C_3\} + \tfrac{1}{2}A\sin^2\theta(1+\cos^2\theta)\{\tfrac{1}{2}\cos 4\phi(2C_1-C_3-16C_5+40C_7) + \sin 4\phi(4S_1+3S_3+2S_5+20S_7)-\cos 2\phi(2C_1+3C_3-2C_5) - 2\sin 2\phi(2S_1+S_3-S_5)+\tfrac{1}{6}(6C_1+5C_3)\} + \tfrac{1}{2}A\sin^4\theta\{\cos 2\phi(2C_1-C_3+10C_5) + 2\sin 2\phi(2S_1+2S_3+5S_5)-\tfrac{1}{3}(6C_1+5C_3)\} , \quad (26)$$

$$(PN)_\times = -4\cos 2\phi\{(3+2\mu/m)S_1-AS_3+AS_5\}-2\sin 2\phi(BC_1+3AC_3-2AC_5) + A\sin^2\theta\{\cos 4\phi(4S_1+3S_3+2S_5+20S_7) - \tfrac{1}{2}\sin 4\phi(2C_1-C_3-16C_5+40C_7)-2\cos 2\phi(2S_1+S_3-S_5) + \sin 2\phi(2C_1+3C_3-2C_5)-S_3\} , \quad (27)$$

where

$$A = 1-3\mu/m , \quad B = 5-4\mu/m , \quad (28)$$

$$C_n = \cos^n\chi = \{1+(t/\tau)^2\}^{-n/2} , \quad (29)$$

$$S_n = \sin\chi\cos^{n-1}\chi = (t/\tau)\{1+(t/\tau)^2\}^{-n/2} , \quad (30)$$

$$\tau = b/v_\infty . \quad (31)$$

The constant term (independent of time) in equation (16) has not been included in the expressions above since the quantities of physical interest involve the time-changing aspects of h^{ij}_{TT}.

We have chosen to express h^{ij}_{TT} as

$$\underset{\sim}{h}_{TT} = h_+(\hat{\underset{\sim}{\theta}}\otimes\hat{\underset{\sim}{\theta}} - \hat{\underset{\sim}{\phi}}\otimes\hat{\underset{\sim}{\phi}}) + h_\times(\hat{\underset{\sim}{\theta}}\otimes\hat{\underset{\sim}{\phi}} + \hat{\underset{\sim}{\phi}}\otimes\hat{\underset{\sim}{\theta}}) ; \quad (32)$$

one could have chosen to express $\underset{\sim}{h}_{TT}$ as a sum of tensor spherical harmonics (Wagoner 1976; we use the convention introduced by Thorne 1977)

$$h^{ij}_{TT} = \frac{1}{R}\sum_{LM}\{A^{E2}_{LM}T^{E2ij}_{LM} + A^{B2}_{LM}T^{B2ij}_{LM}\} . \quad (33)$$

The amplitudes A^{E2}_{LM} and A^{B2}_{LM} are related to h_+ and $h_\times$ by

$$A^{(E2,B2)}_{LM} = R\int h^{ij}_{TT}\, T^{(E2,B2)ij^*}_{LM}(\Omega)d\Omega$$

$$= 2R\int (h_+ T^{(E2,B2)\theta\theta^*}_{LM} + h_\times T^{(E2,B2)\theta\phi^*}_{LM})d\Omega \ . \qquad (34)$$

The "electric" and "magnetic" tensor spherical harmonics, $\underset{\sim}{T}^{E2}{}_{LM}$ and $\underset{\sim}{T}^{B2}{}_{LM}$, are given in the appendix for L = 2, 3, and 4. The Newtonian order terms (N_+ and $N_\times$) contribute only to $A^{E2}{}_{2M}$ (electric quadrupole). The post-1/2 order terms $[(P_{1/2}N)_+$ and $(P_{1/2}N)_\times]$ contribute to both $A^{B2}{}_{2M}$ and $A^{E2}{}_{3M}$. The post-Newtonian-order term $[(PN)_+$ and $(PN)_\times]$ contribute to $A^{E2}{}_{2M}$, $A^{B2}{}_{3M}$, and $A^{E2}{}_{4M}$.

The waveform h^{ij}_{TT} has the symmetry properties

$$h_{(^+_\times)}(\pi-\theta,\phi,t) = \pm h_{(^+_\times)}(\theta,\phi,t) \ ,$$

$$h^{ij}_{TT}(\pi-\theta,\pi-\phi,-t) = h^{ij}_{TT}(\theta,\phi,t) \ , \qquad (35)$$

so that the waveforms $h_\times(t)$ and $h_+(t)$ in one quadrant completely specify $\underset{\sim}{h}_{TT}$ over the whole sphere of observation. Figure 1 displays the waveforms h_+ and $h_\times$ (in units of $2m_1m_2/Rb$) in one quadrant for $m_2 \gg m_1$ (by universality ≡ waveforms in rest frame of m_2 for arbitrary mass ratio) and $v_\infty = 0$ and $v_\infty = 0.5$. The v_∞-dependence of the Newtonian order contribution $(2m_1m_2/Rb)$ is factored out, so h_+ and $h_\times$ for $v_\infty = 0$ are just the Newtonian contributions (N_+ and $\cos\theta N_\times$). The waveforms in Figure 1 are displayed at 45° intervals in the orbital plane, in a plane tilted 45° out of the orbital plane and intersecting the orbital plane along the y-axis, and in a plane perpendicular to the orbital plane and intersecting the orbital plane along the y-axis (the y-z plane). At different

observation directions the character of $h_\times$ (or h_+) is either step-like, pulse-like or a linear combination of the two. The shape of h_+ (or $h_\times$) in a given direction is somewhat arbitrary. An observer in the same direction using a different basis, $\underset{\sim}{\hat{\theta}}'$ and $\underset{\sim}{\hat{\phi}}'$, will see different components h_+' and $h_\times'$. The key point is that the waveform $\underset{\sim}{h}_{TT}(t)$ is a combination of a step-shape and a pulse-shape.

The Newtonian and post-Newtonian contributions to h^{ij}_{TT} depend on sines and cosines of even multiples (0, 2, and 4) of ϕ, while the post-1/2 contribution to h^{ij}_{TT} depends on sines and cosines of odd multiples (1 and 3) of ϕ. In the forward ($\theta = \pi/2$, $\phi = \pi/2$) and backward ($\theta = \pi/2$, $\phi = -\pi/2$) directions the Newtonian and post-Newtonian contributions are the same, but the post-1/2 contribution changes sign. The post-1/2 contribution is proportional to $\delta m/m$, so if $m_1 \neq m_2$ there is a front-back asymmetry, i.e., beaming; if $m_1 = m_2$ there is no beaming (more correctly there is beaming in both directions due to the post-Newtonian term). The beaming is in the direction of motion of the lighter body. In Figure 1 where $\delta m/m = -1$ beaming in the forward direction ($\phi > 0$) is apparent; the amplitude of h_+ (or $h_\times$) is greater for ϕ (> 0) and θ than the corresponding amplitude for $-\phi$ (< 0) and θ.

III. LUMINOSITY $dL/d\Omega$

The energy emitted in gravitational waves per unit time per solid angle is given by

$$\frac{dL}{d\Omega} = \frac{R^2}{32\pi} \left\langle \frac{\partial h^{ij}_{TT}}{\partial t} \frac{\partial h^{ij}_{TT}}{\partial t} \right\rangle , \tag{36}$$

where the brackets indicate an average over several wavelengths. As noted earlier, in the bremsstrahlung process the energy is released in a burst of length $\sim \tau$ ($= b/v_\infty$), so this averaging process is not possible. Therefore the formal significance of $dL/d\Omega$ is not clear. We will present the formula for $dL/d\Omega$ with that caveat.

We can write $\partial h^{ij}_{TT}/\partial t$ in the same form h^{ij}_{TT} was written in equations (20) and (21)

$$\partial h_+/\partial t = (2m_1m_2/R)(v_\infty/b^2)\{\dot{N}_+ + \frac{1}{4} v_\infty \frac{\delta m}{m} \sin\theta (P_{1/2}\dot{N})_+ + \frac{1}{4} v_\infty{}^2 (P\dot{N})_+\} , \tag{37}$$

$$\partial h_\times/\partial t = (2m_1m_2/R)(v_\infty/b^2)\cos\theta\{\dot{N}_\beta + \frac{1}{4} v_\infty \frac{\delta m}{m} \sin\theta (P_{1/2}\dot{N})_\times + \frac{1}{4} v_\infty{}^2 (P\dot{N})_\times\} , \tag{38}$$

where dot ($\dot{}$) means $\partial/\partial(t/\tau)$. The Newtonian, post-1/2, and post-Newtonian terms are $\dot{N}$, $(P_{1/2}\dot{N})$, and $(P\dot{N})$ respectively, given by

$$\dot{N}_+ = (1+\cos^2\theta)\{\frac{1}{2}\cos2\phi(S_3+6S_5)+\sin2\phi(C_3-3C_5)\}+\frac{1}{2}\sin^2\theta S_3 , \tag{39}$$

$$\dot{N}_\times = 2\cos2\phi(C_3-3C_5)-\sin2\phi(S_3+6S_5) , \tag{40}$$

$$\begin{aligned}(P_{1/2}\dot{N})_+ = \frac{1}{2}(1+\cos^2\theta)\{&\cos3\phi(4C_3+33C_5-60C_7)-\sin3\phi(2S_3-3S_5+60S_7) \\ &+ \cos\phi(4C_3-3C_5)+\sin\phi(2S_3-3S_5)\} \\ &+ \sin^2\theta\{\cos\phi(4C_3-9C_5)-\sin\phi(2S_3+9S_5)\} ,\end{aligned} \tag{41}$$

$$\begin{aligned}(P_{1/2}\dot{N})_\times = &-\cos3\phi(2S_3-3S_5+60S_7)-\sin3\phi(4C_3+33C_5-60C_7) \\ &+ \cos\phi(2S_3-3S_5)-\sin\phi(4C_3-3C_5) ,\end{aligned} \tag{42}$$

$$\begin{aligned}(P\dot{N})_+ = \frac{1}{2}(1+\cos^2\theta)\{&-2\cos2\phi(BS_3+9AS_5-10AS_7) \\ &- 4\sin2\phi(BC_3-7AC_5+5AC_7)\}-\sin^2\theta\left[BS_3+(23-3\mu/m)S_5\right] \\ &+\frac{1}{2}A\sin^2\theta(1+\cos^2\theta)\{-\frac{1}{2}\cos4\phi(2S_3-3S_5-80S_7+280S_9) \\ &- \sin4\phi(2C_3- C_5+110C_7-140C_9)+\cos2\phi(2S_3+9S_5-10S_7) \\ &- 2\sin2\phi(7C_5-5C_7)-\frac{1}{2}(2S_3+5S_5)\} \\ &+\frac{1}{2}A\sin^4\theta\{-\cos2\phi(2S_3-3S_5+50S_7)-2\sin2\phi(2C_3+14C_5-25C_7) \\ &+ (2S_3+5S_5)\} ,\end{aligned} \tag{43}$$

$$\begin{aligned}(P\dot{N})_\times = &-4\cos2\phi(BC_3-7AC_5+5AC_7)+2\sin2\phi(BS_3+9AS_5-10AS_7) \\ &+ A\sin^2\theta\{-\cos4\phi(2C_3-C_5+110C_7-140C_9) \\ &+\frac{1}{2}\sin4\phi(2S_3-3S_5-80S_7+280S_9)-2\cos2\phi(7C_5-5C_7) \\ &- \sin2\phi(2S_3+9S_5-10S_7)+(2C_3-3C_5)\} .\end{aligned} \tag{44}$$

Combining equation (36) for $dL/d\Omega$ and equations (37) and (38) for $\partial h^{ij}_{TT}/\partial t$ we obtain

$$\frac{dL}{d\Omega} = \frac{1}{4\pi} \frac{m_1^2 m_2^2}{b^4} v_\infty^2 \Big[(\dot{N}_+^2 + \cos^2\theta \dot{N}_\times^2) + \frac{1}{2} v_\infty \frac{\delta m}{m} \sin\theta [\dot{N}_+ (P_{1/2}^{\cdot} N)_+ + \cos^2\theta \dot{N}_\times (P_{1/2}^{\cdot} N)_\times] + \frac{1}{2} v_\infty^2 \{ \frac{1}{8} \left(\frac{\delta m}{m}\right)^2 \sin^2\theta [(P_{1/2}^{\cdot} N)_+^2 + \cos^2\theta (P_{1/2}^{\cdot} N)_\times^2] + [\dot{N}_+ (\dot{P} N)_+ + \cos^2\theta \dot{N}_\times (\dot{P} N)_\times] \} \Big] . \quad (45)$$

The "$\dot{N}^2$-terms" are the Newtonian-order contribution; the "$\dot{N}\cdot(P_{1/2}^{\cdot}N)$-terms" are the post-1/2-order contribution, and the "$(P_{1/2}^{\cdot}N)^2$-terms" and the "$\dot{N}\cdot(\dot{P}N)$-terms" comprise the post-Newtonian-order contribution to $dL/d\Omega$.

Much algebraic strength is required to plug expressions (39)-(44) into equation (45) for $dL/d\Omega$; however, it is a simple task to evaluate equation (45) for $dL/d\Omega$ numerically. To illustrate the result, we show in Figure 4 the time evolution of $dL/d\Omega$ in the orbital plane for $m_2 = m_1$ (by universality $\equiv$ the pattern seen in the center-of-velocity frame) and $m_2 \gg m_1$ ($\equiv$ the pattern seen in the rest frame of m_2) and $v_\infty = 0.5$. In Fig. 4(a) ($m_1 = m_2$), as time evolves, the quadrupole pattern present at $t = 0$ rotates counterclockwise (as $\underset{\sim}{r}$ does) and the lobes along the direction of motion of m_1 and m_2 become dominant so that the pattern becomes almost dipole-like. In Fig. 4(b) ($m_2 \gg m_1$) the lobes in the direction of m_1's motion ($\phi = 90$) are enhanced and are pushed forward (beaming) at $t = 0$; as time evolves this pattern also rotates counterclockwise and the lobe along m_1's direction of motion becomes dominant. Note, $dL/d\Omega$ has the symmetry property in the orbital plane that $dL/d\Omega\ (\pi-\phi, t) = dL/d\Omega\ (\phi, -t)$; so $dL/d\Omega$ for $t < 0$ can be obtained by reflecting $dL/d\Omega$ for $-t > 0$ across the line $\phi = 90$. The results for $m_2 \gg m_1$ were actually computed using the "trick" described in §X. Both luminosity patterns are accurate through $O(v_\infty^3)$.

IV. POWER

The energy radiated per unit time (power) in gravitational waves is given by

$$P(t) = \int dL/d\Omega \, d\Omega \, . \tag{46}$$

Once again the same caveat regarding the physical meaning of this quantity applies. When equation (45) is integrated over solid angle we find

$$P(t) = (4\pi)^{-1}(m_1{}^2 m_2{}^2 v_\infty{}^2/b^4)\{A(t) + \tfrac{1}{2}v_\infty{}^2[\tfrac{1}{8}(\delta m/m)^2 B(t) + C(t)]\} \, , \tag{47}$$

where $A(t)$ is the Newtonian contribution, and $B(t)$ and $C(t)$ are the post-Newtonian contributions to $P(t)$. Expression B contains the "$(P_{1/2}\dot{N})^2$-terms" and C contains the "$\dot{N}\cdot(P\dot{N})$-terms". There is no post-1/2 contribution to the power. Expressions A, B, and C are given by

$$A = (32\pi/15)\cos^4\chi(1+11\cos^2\chi) \, , \tag{48}$$

$$B = (512\pi/105)\cos^4\chi(1+11\cos^2\chi + \frac{297}{4}\cos^4\chi) \, , \tag{49}$$

$$C = -(64\pi/105)\cos^4\chi \, \{(33-22\mu/m) + (298-278\mu/m)\cos^2\chi - (441-168\mu/m)\cos^4\chi\} \, . \tag{50}$$

Evaluating equation (47) we find

$$P(t) = (8/15)(m_1{}^2 m_2{}^2 v_\infty{}^2/b^4)\cos^4\chi\{(1+11\cos^2\chi) + \frac{1}{7}v_\infty{}^2[-(32-18\mu/m) - (287-234\mu/m)\cos^2\chi + \frac{1}{4}(2061-1860\mu/m)\cos^4\chi]\} \, . \tag{51}$$

In Figure 5 we have plotted our results [good through $O(v_\infty{}^2)$] and the results of Peters (1970) for $m_2 \gg m_1$ and $v_\infty = .1, .2, .35,$ and .5. Kovács and Thorne (1977b) did not compute $P(t)$, but had they done so they would presumably have agreed with Peters (1970) to high accuracy. The energy carried off in gravitational waves is released in a burst of width $\sim \tau$ ($= b/v_\infty$). Neither Peters nor Kovács and Thorne were able to supply us with tabular data, so we plotted our results on figures supplied by Peters. To within the accuracy of our ability to represent our data on his figures ($\sim$ a few percent) our computations and his are identical up to $v_\infty = .35$; at $v_\infty = .5$ our curve falls below his by $\sim$ 10 percent. Further comparisons between the Peters-Kovacs and Thorne works and ours are discussed in §X.

V. ANTENNA PATTERN $d(\delta E)/d\Omega$

A useful and interesting quantity is $d(\delta E)/d\Omega$, the angular distribution of total radiated gravitational wave energy. It is easily obtained (in symbols) by integrating $dL/d\Omega$ (given by equation [45]) over time,

$$d(\delta E)/d\Omega = \int_{-\infty}^{\infty} dL/d\Omega \; dt \; . \tag{52}$$

The structure of $d(\delta E)/d\Omega$ is the same as the structure of $dL/d\Omega$,

$$d(\delta E)/d\Omega = (4\pi)^{-1}(m_1{}^2 m_2{}^2 v_\infty/b^3)\{R_1 + \tfrac{1}{2}v_\infty(\delta m/m)R_2 + \tfrac{1}{2}v_\infty{}^2[\tfrac{1}{8}(\delta m/m)^2 R_3 + R_4]\} \; . \tag{53}$$

R_1 is the "$\int \dot{N}^2 dt$" contribution; R_2 is the "$\int \dot{N}\cdot(P_{1/2}\dot{N})dt$" contribution (post-1/2 order); R_3 is the "$\int (P_{1/2}\dot{N})^2 dt$" contribution (post-Newtonian order), and R_4 is the "$\int \dot{N}\cdot(P\dot{N})dt$" contribution (post-Newtonian order). Expressions R_1, R_2, R_3, and R_4 are given by

$$R_1 = (\pi/128)\{4\cos^2\theta(73+50\cos^2 2\phi)+(1+\cos^2\theta)^2(73+50\sin^2 2\phi) + 4\sin^4\theta+32\sin^2\theta(1+\cos^2\theta)\cos 2\phi\} \; , \tag{54}$$

$$\begin{aligned}R_2 = (\pi/256)\sin\theta\{&4\cos^2\theta(-1137\sin 3\phi\cos 2\phi+93\cos 2\phi\sin\phi \\ &+787\cos 3\phi\sin 2\phi-7\sin 2\phi\cos\phi)+(1+\cos^2\theta)^2(1137\cos 3\phi\sin 2\phi \\ &-93\sin 2\phi\cos\phi-787\sin 3\phi\cos 2\phi+7\cos 2\phi\sin\phi) \\ &+\sin^2\theta(1+\cos^2\theta)(594\sin 2\phi\cos\phi-420\cos 2\phi\sin\phi-154\sin 3\phi \\ &+4\sin\phi)+\sin^4\theta(-104\sin\phi)\} \; ,\end{aligned} \tag{55}$$

$$\begin{aligned}R_3 = (\pi/128)\sin^2\theta\{&\cos^2\theta(9416+2450\sin^2 3\phi+110\sin^2\phi-1254\sin 3\phi\sin\phi \\ &+34\cos 3\phi\cos\phi)+\tfrac{1}{4}(1+\cos^2\theta)^2(9416+2450\cos^2 3\phi+110\cos^2\phi \\ &-1254\cos 3\phi\cos\phi+34\sin 3\phi\sin\phi)+\sin^2\theta(1+\cos^2\theta)(-25-182\cos^2\phi \\ &+2823\cos 3\phi\cos\phi+2515\sin 3\phi\sin\phi)+\sin^4\theta(723+34\sin^2\phi)\} ,\end{aligned} \tag{56}$$

$$\begin{aligned}R_4 = (\pi/256)[&2\cos^2\theta[(1620+1476\mu/m)-(2424+472\mu/m)\sin^2 2\phi] \\ &+2A\cos^2\theta\sin^2\theta(210+420\cos^2 2\phi-2187\cos 4\phi\cos 2\phi-1662\sin 4\phi\sin 2\phi \\ &+102\cos 2\phi)+(1+\cos^2\theta)^2[(810+738\mu/m)-(1212+236\mu/m)\cos^2 2\phi] \\ &+\sin^2\theta(1+\cos^2\theta)[(-1296+500\mu/m)\cos 2\phi]+\sin^4\theta(-264+88\mu/m) \\ &+A(1+\cos^2\theta)^2\sin^2\theta(105+210\sin^2 2\phi-79.5\cos 2\phi-831\cos 4\phi\cos 2\phi \\ &-1093.5\sin 4\phi\sin 2\phi)+A\sin^4\theta(1+\cos^2\theta)(-675-210\sin^2 2\phi \\ &+186\cos 2\phi-147\cos 4\phi)+A\sin^6\theta(36-129\cos 2\phi)\} \; .\end{aligned} \tag{57}$$

Formula (53) for $d(\delta E)/d\Omega$ is accurate through post-Newtonian order (errors of $O[v_\infty{}^3]$). In §X we show how this accuracy can be increased to post-3/2 order (errors of $O[v_\infty{}^4]$) with the use of a "trick". Therefore, if one wishes to compute $d(\delta E)/d\Omega$ the procedure in §X should be followed to obtain the best accuracy.

Figure 2 shows the antenna patterns for $v_\infty = 0$ (the v_∞-dependence has been factored out; this pattern represents the Newtonian contribution), $v_\infty = 0.5$ and $m_2 = m_1$, and $v_\infty = 0.5$ and $m_2 \gg m_1$ (the pattern for $m_2 \gg m_1$ was computed using the procedure in § X). For comparison the antenna pattern for electromagnetic bremsstrahlung, $v_\infty = 0.5$ and $m_2 \gg m_1$ (e.g. an electron passing a heavy nucleus), is also shown in Figure 2. The difference between dipole and quadrupole radiation is very apparent. A comparison of the radiation pattern in the orbital plane to the corresponding results of Peters (his only available results are in the orbital plane, 1970) is presented in §X.

VI. RADIATED ENERGY SPECTRUM $d(\delta E)/d\sigma$

The total energy radiated per solid angle can be written as

$$\frac{d(\delta E)}{d\Omega} = \int_{-\infty}^{\infty} \frac{dL}{d\Omega}\, dt = \frac{R^2}{32\pi} \int_{-\infty}^{\infty} \left\langle \frac{\partial h^{ij}_{TT}}{\partial t} \frac{\partial h^{ij}_{TT}}{\partial t} \right\rangle dt \quad . \qquad (58)$$

If we write

$$\partial h_+(t)/\partial t = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathcal{H}_+(\omega)\, e^{-i\omega t}\, d\omega \quad , \qquad (59)$$

$$\partial h_\times(t)/\partial t = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathcal{H}_\times(\omega)\, e^{-i\omega t}\, d\omega \quad , \qquad (60)$$

then equation (58) can be written as

$$d(\delta E)/d\Omega = (R^2/16\pi^2) \int_0^{\infty} \{|\mathcal{H}_+(\omega)|^2 + |\mathcal{H}_\times(\omega)|^2\}\, d\omega \ , \qquad (61)$$

by the use of Parseval's theorem. Define σ (dimensionless frequency) by $\sigma = \omega\tau$ and integrate equation (61) over solid angle. Then $d(\delta E)/d\sigma$ can be written as

$$d(\delta E)/d\sigma = (4\pi^2)^{-1}(m_1^{\ 2} m_2^{\ 2} v_\infty/b^3)\{E_1 + \tfrac{1}{2} v_\infty^{\ 2}[\tfrac{1}{8}(\delta m/m)^2 E_2 + E_3]\} . \qquad (62)$$

E_1 is the Newtonian contribution to the spectrum; E_2 and E_3 are the post-Newtonian contributions give by

$$E_1 = (8\pi/15)[12(\tilde{C}_3 - 3\tilde{C}_5)^2 + 3(\tilde{S}_3 + 6\tilde{S}_5)^2 + \tilde{S}_3^{\ 2}] \ , \qquad (63)$$

$$\tfrac{1}{8}(\delta m/m)^2 E_2 + E_3 = (64\pi/105)\{[-(97-48\mu/m)\tilde{C}_3^{\ 2} + (423-588\mu/m)\tilde{C}_3\tilde{C}_5$$
$$- (105-390\mu/m)\tilde{C}_3\tilde{C}_7 - (58.5+18\mu/m)\tilde{C}_5^{\ 2} - (528.75-2205\mu/m)\tilde{C}_5\tilde{C}_7$$
$$+ (562.5-2250\mu/m)\tilde{C}_7^{\ 2}] + [-(32-18\mu/m)\tilde{S}_3^{\ 2} - (222-204\mu/m)\tilde{S}_3\tilde{S}_5$$

(cont'd)

$$+ (32.5-195\mu/m)\tilde{S}_3\tilde{S}_7 - (182.25-522\mu/m)\tilde{S}_5^{\,2} + (33.75-45\mu/m)\tilde{S}_5\tilde{S}_7$$

$$+ (562.6-2250\mu/m)\tilde{S}_7^{\,2}]\} \quad . \tag{64}$$

The $\tilde{S}_n$ and $\tilde{C}_n$ are related to the S_n and C_n [eqs. [29], [30]] by

$$\tilde{C}_n = 2\int_0^\infty C_n \cos(\sigma t/\tau)\, d(t/\tau)$$

$$= 2\pi^{1/2}(\sigma/2)^{\frac{1}{2}(n-1)}[\Gamma(n/2)]^{-1} K_{\frac{1}{2}(n-1)}(\sigma) \quad , \tag{65}$$

$$\tilde{S}_n = 2\int_0^\infty S_n \sin(\sigma t/\tau)\, d(t/\tau)$$

$$= 2\pi^{1/2}(\sigma/2)^{\frac{1}{2}(n-1)}[\Gamma(n/2)]^{-1} K_{\frac{1}{2}(n-3)}(\sigma) \quad , \tag{66}$$

where $K_n(\sigma)$ is the modified Bessel function of the second kind, of order n, and $\Gamma(n)$ is the gamma function.

The energy radiated and its spectrum can also be described by its multipole distribution,

$$\delta E = \frac{1}{32\pi} \sum_{L,M} \int_{-\infty}^{\infty} \{ |\frac{\partial}{\partial t} A^{E2}{}_{LM}|^2 + |\frac{\partial}{\partial t} A^{B2}{}_{LM}|^2 \}\, dt \quad , \tag{67}$$

or

$$\delta E = \frac{1}{32\pi^2} \sum_{L,M} \int_0^\infty \{ |\mathcal{A}^{E2}{}_{LM}(\omega)|^2 + |\mathcal{A}^{B2}{}_{LM}(\omega)|^2 \}\, d\omega \quad , \tag{68}$$

where $\mathcal{A}^{(E2,B2)}_{LM}(\omega)$ is given by

$$\mathcal{A}^{(E2,B2)}_{LM}(\omega) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} A^{(E2,B2)}_{LM}(t)\, e^{i\omega t}\, dt \quad . \tag{69}$$

The energy spectrum consists of a L = 2, "electric" component, from the Newtonian and the Newtonian-post-Newtonian cross terms, and a L = 3 "Electric" and a L = 2, "magnetic" component, both from the post-1/2

squared terms. These contributions are,

$$d(\delta E)/d\sigma = \frac{1}{4\pi^2} \frac{m_1^{\,2} m_2^{\,2}}{b^3} v_\infty \{ \begin{pmatrix} E2 \\ L=2 \end{pmatrix} + \frac{v_\infty^{\,2}}{16} \left(\frac{\delta m}{m}\right)^2 \left[\begin{pmatrix} B2 \\ L=2 \end{pmatrix} + \begin{pmatrix} E2 \\ L=3 \end{pmatrix} \right] \}, \tag{70}$$

where $\binom{(E2,B2)}{L}$ represent the various multipole contributions,

$$\begin{pmatrix} E2 \\ L=2 \end{pmatrix} = E_1 - \frac{32\pi}{105} v_\infty^{\,2} \{ [(109-96\mu/m)\tilde{C}_3^{\,2} - (411-540\mu/m)\tilde{C}_3\tilde{C}_5$$

$$+ 30A\tilde{C}_3\tilde{C}_7 + 252A\tilde{C}_5^{\,2} - 90A\tilde{C}_5\tilde{C}_7] + [(33-22\mu/m)\tilde{S}_3^{\,2} + (222-204\mu/m)\tilde{S}_3\tilde{S}_5$$

$$- 15A\tilde{S}_3\tilde{S}_7 + 207A\tilde{S}_5^{\,2} - 90A\tilde{S}_5\tilde{S}_7]\} \quad , \tag{71}$$

$$\begin{pmatrix} B2 \\ L=2 \end{pmatrix} = \frac{512\pi}{45} \{ (2\tilde{C}_3 - 3\tilde{C}_5)^2 + (3\tilde{S}_5)^2 \} \quad , \tag{72}$$

$$\begin{pmatrix} E2 \\ L=3 \end{pmatrix} = \frac{128\pi}{315} \{ (32\tilde{C}_3^{\,2} + 480\tilde{C}_3\tilde{C}_5 + 2070\tilde{C}_5^{\,2} - 900\tilde{C}_3\tilde{C}_7 - 7425\tilde{C}_5\tilde{C}_7$$

$$+ 6750\tilde{C}_7^{\,2}) + (12\tilde{S}_3^{\,2} + 450\tilde{S}_3\tilde{S}_7 + 45\tilde{S}_5^{\,2} - 675\tilde{S}_5\tilde{S}_7 + 6750\tilde{S}_7^{\,2})\} \quad . \tag{73}$$

Figure 6 shows the energy spectrum $d(\delta E)/d\sigma$ for $v_\infty = 0$ (the Newtonian v_∞-dependence is factored out, so this curve corresponds to the Newtonian contribution only), and $v_\infty = 0.5$ for $m_1 = m_2$ and $m_2 \gg m_1$. The primary effect of the post-Newtonian contributions is to flatten the spectrum near $\sigma = 0$. Figure 7 exhibits the multipole structure of the energy spectrum. The "electric" multipoles dominate the "magnetic" multipoles, and even at $v_\infty = 0.5$ the L = 2 "electric" contribution is the dominant one.

VII. TOTAL ENERGY RADIATED δE

The total energy released in gravitational radiation can be computed by integrating the power over time. The result (also given in WW, eqs. [13] and [14]; note the correction of the original result) is

$$\delta E = \frac{37\pi}{15} \frac{m_1^2 m_2^2}{b^3} v_\infty \{1 + \frac{1}{2} v_\infty^2 g(\mu/m)\} , \quad (74)$$

where

$$g(\mu/m) = 404/259 + \frac{3}{4}(\delta m/m)^2$$

$$= (1/1036)\{2393-3108\mu/m\} , \quad (75)$$

and

$$1.56 < g(\mu/m) < 2.31 . \quad (76)$$

Figure 8 shows δE plotted as a function of v_∞ for $m_2 \gg m_1$. Three curves are shown: the Newtonian result, the Newtonian + post-Newtonian result (equation 39), and the Peters-KT result. For small v_∞ all three curves coalesce; at $v_\infty = 0.25$ the Newtonian result begins to deviate significantly ($\sim$ 10%) from the Peters-KT curve, and finally at $v_\infty = 0.5$ the post-Newtonian result of this paper begins to deviate significantly ($\sim$ 15%). Neither Kovacs and Thorne nor Peters were able to supply us with tabular data. Peters kindly supplied us with a figure (from Peters 1970) and we plotted our results on this figure. Note: the energy scale was labeled too small by a factor of 10 when it appeared in Peters (1970).

VIII. THE ZERO FREQUENCY LIMIT

Using a quantum-mechanical technique, Smarr (1977) has evaluated the zero-frequency limit (ZFL) of the gravitational bremsstrahlung spectrum for $m_2 \gg m_1$ and arbitrary velocities. In our post-Newtonian formalism, this limit may be evaluated by noting that the quantities that determine the ZFL are $\mathcal{H}_+(0)$ and $\mathcal{H}_\times(0)$, (cf. eqs. [59] and [60]). These quantities are simply the differences in the corresponding h's between $t = -\infty$ and $t = +\infty$, that is,

$$\mathcal{H}_{+(\times)}(0) = h_{+(\times)}(\infty) - h_{+(\times)}(-\infty) . \quad (77)$$

Combining equations (20)-(31), (61) and (77) we obtain

$$\frac{d[\delta E(0)]}{d\sigma d\Omega} = \frac{4}{\pi^2} \frac{m_1^2 m_2^2}{b^3} v_\infty (1-\sin^2\theta\sin^2\phi)\{\mathcal{R}_1 - v_\infty \frac{\delta m}{m} \sin\theta \mathcal{R}_2$$

$$+ v_\infty^2 [\frac{1}{4} \left(\frac{\delta m}{m}\right)^2 \sin^2\theta \mathcal{R}_3 + \mathcal{R}_4]\} , \quad (78)$$

where

$$\mathcal{R}_1 = \cos^2\theta + \sin^2\theta\sin^2\phi , \quad \mathcal{R}_2 = \sin\phi(2-3\sin^2\theta\cos^2\phi) ,$$

$$\mathcal{R}_3 = 1+3\sin^2\phi-9\sin^2\theta\sin^2\phi\cos^2\phi ,$$

$$\mathcal{R}_4 = (3+2\mu/m)(\cos^2\theta+\sin^2\theta\sin^2\phi)$$

$$- 2(1-3\mu/m)\sin^2\theta\sin^2\phi(1-2\sin^2\theta\cos^2\phi) . \quad (79)$$

Integration over solid angle yields

$$d[\delta E(0)]/d\sigma = (32/5\pi)(m_1^2 m_2^2 v_\infty/b^3)(1 + \frac{25}{7} v_\infty^2) . \quad (80)$$

For the case $m_2 \gg m_1$, these results are in complete agreement with the post-Newtonian limit of Smarr's equations (2.12) and (2.13) (after reconciling our different coordinate systems). However, Smarr goes on to use equation (2.13) to estimate the total bremsstrahlung energy by defining a cut-off frequency $\sigma_c = 3.8(1-v_\infty^2)^{-1/2}$, and writing

$$\delta E = \sigma_c \, d[\delta E(0)]/d\sigma \ . \qquad (81)$$

The factor 3.8 was chosen to fit the low-velocity, Newtonian results. But for ultrarelativistic velocities, Smarr and Peters-KT disagree on the velocity dependence of δE. KT interpret the disagreement as being due to Smarr's incorrect use of an angle-independent cut-off frequency. Because of beaming effects, different frequencies dominate the spectrum in different directions; our frequency spectra shown in Figure 7 illustrate this dependence of σ_c on angle (or multipole) even at low velocities. Furthermore, our equations (74) and (80) may be combined to yield an effective average cut-off frequency, given by

$$\langle\sigma_c\rangle = \delta E(d[\delta E(0)]/d\sigma)^{-1} = 3.8(1 - \frac{5007}{2072} v_\infty^2) \ . \qquad (82)$$

This average cut-off frequency, $\langle\sigma_c\rangle$, does not agree with the post-Newtonian limit of Smarr's σ_c nor does it have an obvious interpretation as the post-Newtonian limit of a relativistic expression of a simple form envisioned by Smarr.

IX. UNIVERSALITY OF GRAVITATIONAL BREMSSTRAHLUNG

Using the post-linear formalism, Kovács and Thorne (1977) discovered that the general gravitational bremsstrahlung waveform could be written as

$$h^{ij}_{TT} = (2/R)(m_1 m_2/b)\, \tilde{h}^{ij}_{TT} \ , \qquad (83)$$

where $\tilde{h}^{ij}_{TT}$ is a function of frame-invariant quantities associated with the straight-line unperturbed orbits (such as particle four-velocities, proper times measured along particle world lines, etc.), but is not a function of the masses m_1 and m_2. Thus, $\tilde{h}^{ij}_{TT}$ is a "universal" function of orbital parameters. On the other hand, the post-Newtonian form of h^{ij}_{TT} given in equation (16) contains explicit mass dependence (terms involving $\delta m/m$ and μ/m). This is simply because we chose to locate the origin of our coordinates at the center-of-mass of the system. Our post-Newtonian waveform does in fact obey the universality predicted by KT. To show this, we have rederived the post-Newtonian equations of motion and the gravitational waveform, using a more general coordinate origin, in which the center of mass of the system moves according to

$$\underset{\sim}{X}_{CM}(t) = \frac{1}{2}(\delta m/m + \alpha)\, \underset{\sim}{r}_0(t) \qquad (84)$$

where $\underset{\sim}{r}_0(t) = b\underset{\sim}{e}_x + v_\infty t\, \underset{\sim}{e}_y$. Since the use of a moving center of mass will affect only post-Newtonian terms, we can use Newtonian variables to describe it. The origin of this coordinate system lies along the line joining the unperturbed positions of the two bodies. Following KT, it is useful to define the midpoint or the center-of-velocity of the orbit by

$$\underset{\sim}{X}_{CV} = \frac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2) \ , \quad \underset{\sim}{V}_{CV} = \frac{1}{2}(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) \ , \tag{85}$$

where $\underset{\sim}{r}_1(\underset{\sim}{r}_2)$ and $\underset{\sim}{v}_1(\underset{\sim}{v}_2)$ are the position and velocity of $m_1(m_2)$. Then in our new coordinate system, the center of velocity moves according to

$$\underset{\sim}{V}_{CV} = \frac{1}{2}\alpha \, v_\infty \, \underset{\sim}{e}_y \ . \tag{86}$$

If $\alpha = 1$ (-1), the coordinates are centered on and at rest with respect to particle 2 (1); if $\alpha = 0$ the coordinates are centered on the center-of-velocity; if $\alpha = -\delta m/m$, the coordinates are centered on the center of mass. To keep the resulting formulas simple we have not considered an arbitrary velocity $\underset{\sim}{V}_0$ for the center of mass or an arbitrary constant shift $\underset{\sim}{X}_0$ in its location. Universality can be verified even in this more general case. The use of a moving center of mass (eq. [84]) produces modifications in the post-Newtonian corrections to the equations of motion (WW eq. [38]) and additional contributions to the waveform given in WW, eq. (97). After dropping the constant terms from h^{ij}_{TT}, and making use of the relation $v_\infty{}^2/e = m/b$, we obtain the final result

$$\begin{aligned} h^{ij}_{TT}(t) = (2m_1m_2/Rb) \Big[& (4\hat{\lambda}^{(i}\delta^{j)y} - 2\cos\chi \hat{r}^i\hat{r}^j) \\ & - \alpha v_\infty [3\cos\chi(\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})(2\hat{r}^{(i}\delta^{j)y} - \sin\chi\hat{r}^i\hat{r}^j) - 2\delta^{iy}\delta^{jy}(\underset{\sim}{n}\cdot\hat{\underset{\sim}{\lambda}}) \\ & - n_y(4\hat{\lambda}^{(i}\delta^{j)y} - \cos\chi\hat{r}^i\hat{r}^j)] \\ & + v_\infty{}^2\{\tfrac{1}{2}[15 - \sin^2\chi - \alpha^2(3 - \sin^2\chi)]\hat{\lambda}^{(i}\delta^{j)y} - (15+\alpha^2)\cos\chi\delta^{iy}\delta^{jy} \\ & - \tfrac{3}{4}[4 + (1-\alpha^2)\cos^2\chi]\cos\chi\hat{r}^i\hat{r}^j + (7+\alpha^2)\cos\chi\sin\chi\hat{r}^{(i}\delta^{j)y} \\ & - \tfrac{1}{6}(1-3\alpha^2)\cos\chi[2\delta^{iy}\delta^{jy} - (1-3\sin^2\chi)\hat{r}^i\hat{r}^j - 4\sin\chi\hat{r}^{(i}\delta^{j)y}] \end{aligned}$$

(cont'd)

$$\begin{aligned} & + \tfrac{1}{12}(1+3\alpha^2)[12n_y(n\cdot\hat{\lambda})\delta^{iy}\delta^{jy} + 12n_y{}^2\hat{\lambda}^{(i}\delta^{j)y} - 2n_y{}^2\cos\chi\hat{r}^i\hat{r}^j \\ & - 4n_y(\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})\cos\chi(8\hat{r}^{(i}\delta^{j)y} - 3\sin\chi\hat{r}^i\hat{r}^j) \\ & - (\underset{\sim}{n}\cdot\hat{\underset{\sim}{r}})^2\cos\chi(14\delta^{iy}\delta^{jy} - 3\hat{r}^i\hat{r}^j + 15\sin^2\chi\hat{r}^i\hat{r}^j \\ & - 30\sin\chi\hat{r}^{(i}\delta^{j)y})]\} \Big]_{TT} \ . \end{aligned} \tag{87}$$

The only mass dependence in equation (87) is in the overall m_1m_2 factor and the waveform depends on the velocity of the center of velocity in the chosen frame (α), hence our result does satisfy the universality property found by KT. The waveform given in §II equation (16) is merely equation (87) evaluated in the center-of-mass frame, where $\alpha = -\delta m/m$ and $\alpha^2 = 1-4\mu/m$.

Although equation (87) allows one to determine h^{ij}_{TT} in any frame, the two-body system does select three "preferred" frames, the rest-frame of one body ($\alpha = \pm 1$), the center-of-velocity frame ($\alpha = 0$) and the center-of-mass frame ($\alpha = -\delta m/m$) in which the gravitational bremsstrahlung has special characteristics. These characteristics are summarized in Table 1. One conclusion that can be drawn from the property of universality is that Peters' formulae, while derived assuming $m_2 \gg m_1$ (test-body perturbations of Schwarzschild), are actually valid for arbitrary mass ratios, with the observer in the rest frame of one body. The numerical comparison that KT made with Peters employed KT's waveforms evaluated in the rest-frame of one body. Our numerical comparison with KT also used rest-frame formulae and universality makes our agreement with their results completely general.

X. BOOST

The universality discussed in the previous section provides a means of improving the accuracy of the post-Newtonian formalism, from errors of $O(v_\infty^3)$ to errors of $O(v_\infty^4)$. Because the waveform $\tilde{h}^{ij}_{TT}$ depends only on the velocity of the coordinate system relative to the bodies, its form in different frames may be obtained by Lorentz transformations (because we work to first order in m/R), plus a possible additional gauge transformation to preserve the TT gauge. In the center-of-velocity frame ($\alpha = 0$), h^{ij}_{TT} has complete front-back symmetry, thus contains no terms of odd order in v_∞, and so the post-Newtonian method makes errors of $O(v_\infty^4)$ instead of $O(v_\infty^3)$. Therefore, an exact Lorentz boost to a frame of arbitrary α will yield h^{ij}_{TT} (modulo gauge transformations) correct through $O(v_\infty^3)$, for arbitrary mass ratios. We have tested this improved accuracy numerically by a comparison with the Peters-KT results. However, because of the possible gauge complications involving h^{ij}_{TT}, we have performed a boost only on the luminosity $dL/d\Omega$ and the antenna pattern $d(\delta E)/d\Omega$ (both are gauge invariant quantities). Consider the bremsstrahlung process in the center-of-velocity frame and perform a boost along the orbital direction with velocity $\underset{\sim}{u} = \beta v_\infty \underset{\sim}{e}_y$, then

$$\delta E' = \gamma\delta E(1-\underset{\sim}{u}\cdot\underset{\sim}{n}), \quad d\Omega' = \gamma^{-2}(1-\underset{\sim}{u}\cdot\underset{\sim}{n})^{-2}\, d\Omega ,$$

$$dt' = \gamma^{-1}(1-\underset{\sim}{u}\cdot\underset{\sim}{n})^{-1}\, dt \quad , \tag{88}$$

where $\underset{\sim}{n}$ is the direction to the observer, and $\gamma = (1-u^2)^{-1/2}$. In the new frame, the relative velocity at infinite separation v_∞' has the form

$$v_\infty' = [(v_1-\beta v_\infty)(1-\beta v_\infty v_1)^{-1}+(v_2+\beta v_\infty)(1+\beta v_\infty v_2)^{-1}]_\infty$$

$$= v_\infty \gamma^{-2}(1-\tfrac{1}{4}\beta^2 v_\infty^{\,4})^{-1} \ . \tag{89}$$

To the necessary order $O(v_\infty^4)$, we may write in place of equation (89)

$$v_\infty = v_\infty'(1+\beta^2 v_\infty'^{\,2}) \ . \tag{90}$$

Then we find that

$$dL'/d\Omega'|_{v_\infty'} = \gamma^4(1-\beta v_\infty n_y)^4\, dL/d\Omega|_{v_\infty} \ , \tag{91}$$

$$d(\delta E')/d\Omega'|_{v_\infty'} = \gamma^3(1-\beta v_\infty n_y)^3\, d(\delta E)/d\Omega|_{v_\infty} \ , \tag{92}$$

where v_∞ and v_∞' are related by equation (90) and $dL/d\Omega$ and $d(\delta E)/d\Omega$ are to be evaluated in the center-of-velocity frame. Since $dL/d\Omega$ and $d(\delta E)/d\Omega$ in the center-of-velocity frame (obtained from equations [45] and [53] with $m_1 = m_2$) are accurate through $O(v_\infty^3)$, the "doppler-shift expressions" (equations [91] and [92]) generate the correct $O(v_\infty^3)$ terms in the boosted patterns $dL'/d\Omega'$ and $d(\delta E')/d\Omega'$. By boosting to the rest frame of particle 2, $\beta = -\frac{1}{2}$, we can compare the boosted antenna patterns with those presented graphically by Peters (1970), and with the standard rest-frame or "$m_2 \gg m_1$" post-Newtonian patterns evaluated from equations (53) with $\delta m/m = -1$ and $\mu/m = 0$. That comparison for patterns in the orbital plane, and for velocities up to $v_\infty = 0.5$, is shown in Figure 9, and demonstrates the improved agreement with the exact results of Peters-Kovács and Thorne. The luminosity shown in Figure 4(b) was also calculated using this method.

Finally, one further verification of the universality of gravitational bremsstrahlung may be made by integrating equation (92) over solid angle, to obtain (dropping primes)

$$\delta E = \left(\frac{37\pi}{15}\right) \frac{m_1^2 m_2^2}{b^3} v_\infty [1 + \frac{1}{2} v_\infty^2 g(\beta)] , \quad (93)$$

where

$$g(\beta) = \frac{404}{259} + 3\beta^2 . \quad (94)$$

The quantity δE is the total energy radiated as seen in a frame moving at velocity $\underset{\sim}{u} = \beta v_\infty \underset{\sim}{e}_y$ with respect to the center-of-velocity frame. In the center-of-mass frame, $\beta = \frac{1}{2}\delta m/m$, $\beta^2 = \frac{1}{4}(1-4\mu/m)$ and we recover equations (74) and (75).

XI. CONCLUDING REMARKS

We have presented a detailed analysis of gravitational bremsstrahlung at the post-Newtonian level, and have compared our results with those obtained using more accurate, although more complicated, techniques. In making these comparisons, we have found as a rule that the post-Newtonian method alone is most reliable when dealing with equal masses in the center-of-mass frame (or equivalently when working with arbitrary masses in the center-of-velocity frame) and least reliable when dealing with test masses ($m_2 \gg m_1$) in the center-of-mass frame (or with arbitrary masses in either rest frame). (In this sense, the numerical comparisons that we discussed in §II represent a worst case for errors.) There are essentially three reasons for this (i) for equal masses or in the center-of-velocity frame the waveforms are automatically more accurate, with errors of $O(v_\infty^4)$ instead of $O(v_\infty^3)$, (ii) the velocities of the individual particles in the center-of-velocity frame are each one half their relative velocity v_∞, so for a given v_∞ the system is not as "relativistic" as might have been expected, and (iii) the absence of strong beaming in the center-of-velocity frame is well suited to the post-Newtonian method which contains only a limited number of multipoles ($L = 4$), and so has difficulty "mocking up" strongly beamed radiation. For this reason, the property of universality is crucial to obtaining the best accuracy, by permitting exact Lorentz boosts of center-of-velocity formulae, and yielding results automatically accurate through $O(v_\infty^3)$. We have attempted to exploit this feature wherever possible.

ACKNOWLEDGMENTS

We wish to thank S. J. Kovács, P. C. Peters, K. S. Thorne, and R. V. Wagoner for helpful comments.

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APPENDIX

The "electric" and "magnetic" tensor spherical harmonics are given by (Wagoner 1976)

$$\underset{\sim}{T}^{(m)}_{LM} = A_{LM}(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})-iB_{LM}(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}) , \tag{A1}$$

$$\underset{\sim}{T}^{(e)}_{LM} = B_{LM}(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})-iA_{LM}(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta}) . \tag{A2}$$

The functions A_{LM} and B_{LM} are

$$A_{LM} = 2C_L\{ \frac{\partial^2}{\partial\theta^2} + \frac{L(L+1)}{2} \} Y_{LM} , \tag{A3}$$

$$B_{LM} = - 2C_L \frac{M}{\sin\theta} \{ \frac{\partial}{\partial\theta} - \cot\theta \} Y_{LM} , \tag{A4}$$

where $C = [2L(L+1)(L-1)(L+2)]^{-1/2}$ and Y_{LM} are scalar spherical harmonics (Jackson 1962). Recently Thorne (1977) has proposed a new generalized notation for tensor spherical harmonics. We shall adopt his notation. Thorne's tensor spherical harmonics, $\underset{\sim}{T}^{E2}_{LM}$ and $\underset{\sim}{T}^{B2}_{LM}$, are related to Wagoner's by

$$\underset{\sim}{T}^{E2}_{LM} = \underset{\sim}{T}^{(m)}_{LM} , \quad \underset{\sim}{T}^{B2}_{LM} = -i\, \underset{\sim}{T}^{(e)}_{LM} . \tag{A5}$$

From the identity, $Y_{LM} = (-1)^M Y^*_{LM}$, it follows that

$$\underset{\sim}{T}^{(E2,B2)}_{L-M} = (-1)^M \underset{\sim}{T}^{(E2,B2)*}_{LM} , \tag{A6}$$

so that it is sufficient to give the tensor spherical harmonics only for $M \geq 0$. For $L = 2$ they are

$$\underset{\sim}{T}^{E2}_{22} = (5/128\pi)^{1/2}\{(1+\cos^2\theta)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})+2i\cos\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{2i\phi} , \tag{A7}$$

$$\underset{\sim}{T}^{E2}_{21} = (5/32\pi)^{1/2}\{\sin\theta\cos\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})+i\sin\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi} . \tag{A8}$$

$$\underset{\sim}{T}^{E2}_{20} = (15/64\pi)^{1/2}\{\sin^2\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})\}\ , \tag{A9}$$

$$\underset{\sim}{T}^{B2}_{22} = (5/128\pi)^{1/2}\{2i\cos\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})-(1+\cos^2\theta)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{2i\phi}\ , \tag{A10}$$

$$\underset{\sim}{T}^{B2}_{21} = (5/32\pi)^{1/2}\{i\sin\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})-\sin\theta\cos\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi}\ , \tag{A11}$$

$$\underset{\sim}{T}^{B2}_{20} = -(15/64\pi)^{1/2}\sin^2\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\ . \tag{A12}$$

For L = 3 they are

$$\underset{\sim}{T}^{E2}_{33} = -(21/256\pi)^{1/2}\sin\theta\{(1+\cos^2\theta)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + 2i\cos\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{3i\phi}\ , \tag{A13}$$

$$\underset{\sim}{T}^{E2}_{32} = (7/128\pi)^{1/2}\{\cos\theta(3\cos^2\theta-1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + 2i(2\cos^2\theta-1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{2i\phi}\ , \tag{A14}$$

$$\underset{\sim}{T}^{E2}_{31} = (35/256\pi)^{1/2}\sin\theta\{(3\cos^2\theta-1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + 2i\cos\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi}\ , \tag{A15}$$

$$\underset{\sim}{T}^{E2}_{30} = (105/64\pi)^{1/2}\cos\theta\sin^2\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})\ , \tag{A16}$$

$$\underset{\sim}{T}^{B2}_{33} = -(21/256\pi)^{1/2}\sin\theta\{2i\cos\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - (1+\cos^2\theta)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{3i\phi}\ , \tag{A17}$$

$$\underset{\sim}{T}^{B2}_{32} = (7/128\pi)^{1/2}\{2i(2\cos^2\theta-1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - \cos\theta(3\cos^2\theta-1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{2i\phi}\ , \tag{A18}$$

$$\underset{\sim}{T}^{B2}_{31} = (35/256\pi)^{1/2}\sin\theta\{2i\cos\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - (3\cos^2\theta-1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi}\ , \tag{A19}$$

$$\underset{\sim}{T}^{B2}_{30} = -(105/64\pi)^{1/2}\cos\theta\sin^2\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\ . \tag{A20}$$

Finally, for L = 4 they are

$$\underset{\sim}{T}^{E2}_{44} = (63/512\pi)^{1/2}\sin^2\theta\{(1+\cos^2\theta)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + 2i\cos\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{4i\phi}\ , \tag{A21}$$

$$\underset{\sim}{T}^{E2}_{43} = -(63/256\pi)^{1/2}\sin\theta\{2\cos^3\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + i(3\cos^2\theta-1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{3i\phi}\ , \tag{A22}$$

$$\underset{\sim}{T}^{E2}_{42} = (9/128\pi)^{1/2}\{(7\cos^4\theta-6\cos^2\theta+1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + i\cos\theta(7\cos^2\theta-5)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{2i\phi}\ , \tag{A23}$$

$$\underset{\sim}{T}^{E2}_{41} = (9/256\pi)^{1/2}\sin\theta\{2\cos\theta(7\cos^2\theta-4)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) + i(7\cos^2\theta-1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi}\ , \tag{A24}$$

$$\underset{\sim}{T}^{E2}_{40} = -(45/256\pi)^{1/2}(7\cos^4\theta-8\cos^2\theta+1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})\ , \tag{A25}$$

$$\underset{\sim}{T}^{B2}_{44} = (63/512\pi)^{1/2}\sin^2\theta\{2i\cos\theta(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - (1+\cos^2\theta)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{4i\phi}\ , \tag{A26}$$

$$\underset{\sim}{T}^{B2}_{43} = -(63/256\pi)^{1/2}\sin\theta\{i(3\cos^2\theta-1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - 2\cos^3\theta(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{3i\phi}\ , \tag{A27}$$

$$\underset{\sim}{T}^{B2}_{42} = (9/128\pi)^{1/2}\{i\cos\theta(7\cos^2\theta-5)(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}) - (7\cos^4\theta-6\cos^2\theta+1)(\hat{\theta}\hat{\phi}-\hat{\phi}\hat{\theta})\}e^{2i\phi}\ , \tag{A28}$$

$$\underset{\sim}{T}^{B2}_{41} = (9/256\pi)^{1/2}\sin\theta\{i(7\cos^2\theta-1)(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi}) - 2\cos\theta(7\cos^2\theta-4)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\}e^{i\phi}\ , \tag{A29}$$

$$\underset{\sim}{T}^{B2}_{40} = (45/256\pi)^{1/2}(7\cos^4\theta-8\cos^2\theta+1)(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\ . \tag{A30}$$

TABLE I

CHARACTERISTICS OF GRAVITATIONAL BREMSSTRAHLUNG IN SPECIAL FRAMES

Value of α	Frame	Characteristics of Post-Newtonian Bremsstrahlung	Other Methods
+1 (-1)	Rest Frame of $m_2(m_1)$	Results universal (i.e. $\tilde{h}$ independent of m_1 and m_2). Front-back asymmetry (beaming). Identical to results in CM frame if $m_2 \gg (\ll) m_1$.	Peters (1970) Kovács and Thorne (1977b) Smarr (1977)
0	Center-of-Velocity Frame	Results universal (i.e. $\tilde{h}$ independent of m_1 and m_2). No front-back asymmetry. No $O(v_\infty)$, $O(v_\infty^3)$ terms - results correct to $O(v_\infty^4)$. Same as results in CM frame if $m_2 = m_1$.	Kovács and Thorne (1977b)
$-\delta m/m$	Center-of-Mass Frame	Waveform $\tilde{h}$ depends on m_1 and m_2. If $m_1 = m_2$, $\alpha = 0$. If $m_2 \gg (\ll) m_1$ $\alpha = +1$ (-1).	

FIGURE CAPTIONS

Fig. 1. The waveforms $h_+(t)$ and $h_\times(t)$ displayed in three observation planes (the orbital plane, a plane tilted 45° out of the orbital plane, and a plane normal to the orbital plane) at 45° intervals for $m_2 \gg m_1$. The graphs labeled $\alpha = 45°$, $\psi = \pm 45°$ are in the plane tilted 45° and are halfway between $\phi = 0$ and $\phi = \pm 90$. The orbit is shown for reference. Each scale mark represents $(2m_1m_2/Rb)$ on the h-axis and 5τ on the t-axis. The solid surves are for $v_\infty = 0.5$ and the broken curves $v_\infty \approx 0$. By universality, these are also the waveforms seen in the rest frame of m_2 for arbitrary mass ratio.

Fig. 2. Antenna patterns, $d(\delta E)/d\Omega$ (unnormalized); the smallest lobes lie in the orbital plane and the arrow represents the direction of m_1's velocity. In (a) the Newtonian pattern is pictured ($v_\infty \approx 0$). In (b) the antenna pattern for $v_\infty = 0.5$ seen in the CM frame for $m_1 = m_2$ ($\equiv$ pattern seen in the center-of-velocity frame for arbitrary mass ratio) is shown. Note the front-back symmetry and broadening of the lobes relative to (a). Pictured in (c) is the pattern for $v_\infty = 0.5$ seen in the CM frame for $m_2 \gg m_1$ ($\equiv$ pattern seen in the rest frame of m_2 for arbitrary mass ratio). Note the strong beaming in the direction of m_1's travel. For comparison, the antenna pattern from electromagnetic bremsstrahlung is shown in (d) for $m_2 \gg m_1$ and $v_\infty = 0.5$ in the rest frame of m_2 (m_2 is indicated by the dot).

Fig. 3. The coordinate system used throughout this paper. The xy-plane is the orbital plane, $\underset{\sim}{n}$ is the observation direction, and $\hat{\underset{\sim}{\theta}}$ and $\hat{\underset{\sim}{\phi}}$ are unit basis vectors.

Fig. 4. The luminosity, $(dL/d\Omega)/[m_1^2 m_2^2 v_\infty^2/b^4]$, shown at $t = 0.0$ (solid), $0.25\,\tau$ (broken), and $0.50\,\tau$ (dotted) for $v_\infty = 0.5$. The patterns for $t = -0.25\,\tau$ and $t = -0.50\,\tau$ can be obtained by reflecting the patterns for $t = 0.25\,\tau$ and $t = 0.50\,\tau$ across the vertical axis. In (a) the luminosity is shown in the CM frame with $m_1 = m_2$ ($\equiv$ luminosity seen in the center-of-velocity frame for arbitrary mass ratio). In (b) the luminosity is shown in the CM frame with $m_2 \gg m_1$ ($\equiv$ luminosity seen in the rest frame of m_2 for arbitrary mass ratio). The curves drawn in (b) were obtained using the "trick" described in §X.

Fig. 5. The power in units of $m_1^2 m_2^2/b^4$ plotted as a function of t/τ in the CM frame for $m_2 \gg m_1$ ($\equiv$ power seen in the rest frame of m_2 for arbitrary mass ratio) and $v_\infty = 0.1$, 0.2, 0.35, and 0.5. The curves are the post-Newtonian results; the symbols represent Peters' results. Up to $v_\infty = 0.35$ the differences are too small to show graphically. We have plotted our results on a figure supplied by Peters.

Fig. 6. The energy spectrum, $d(\delta E)/d\sigma$, plotted in units of $m_1^2 m_2^2 v_\infty/b^3$ as a function of dimensionless frequency σ ($\equiv \omega\tau$). The solid curve represents the Newtonian result ($v_\infty \approx 0$). The broken and dotted curves represent our post-Newtonian results for $v_\infty = 0.5$ and for $m_2 \gg m_1$ and $m_2 = m_1$ respectively. The post-Newtonian contributions tend to flatten the spectrum near $\sigma = 0$.

Fig. 7. The multipole structure of the energy spectrum for $v_\infty = 0.5$ and $m_2 \gg m_1$. Each multipole contribution is plotted in units of $m_1^2 m_2^2 v_\infty/b^3$ as a function of σ ($\equiv \omega\tau$). Note that each multipole has a different frequency dependence.

Fig. 8. The total energy radiated in units of $m_1^2 m_2^2/b^3$ as a function of v for $m_2 \gg m_1$. The Newtonian results begin to deviate from Peters-Kovacs and Thorne at $v_\infty = 0.25$ and our post-Newtonian results (equation [74]) begin to deviate at $v_\infty = 0.5$. The Newtonian and post-Newtonian results were plotted on a figure supplied to us by Peters.

Fig. 9. The antenna pattern, $[d(\delta E)/d\Omega]/[m_1^2 m_2^2 v_\infty/b^3]$, in the orbital plane with $m_2 \gg m_1$ and $v_\infty = 0.2$ (a), $v_\infty = 0.35$ (b), and $v_\infty = 0.5$ (c). The dotted curves represent Peters' results. The broken cruves are the post-Newtonian results from equation (53) and are accurate through $O(v_\infty^2)$. The solid curves are the results obtained by the boost described in §X and are accurate through $O(v_\infty^3)$. The boost significantly improves the accuracy.

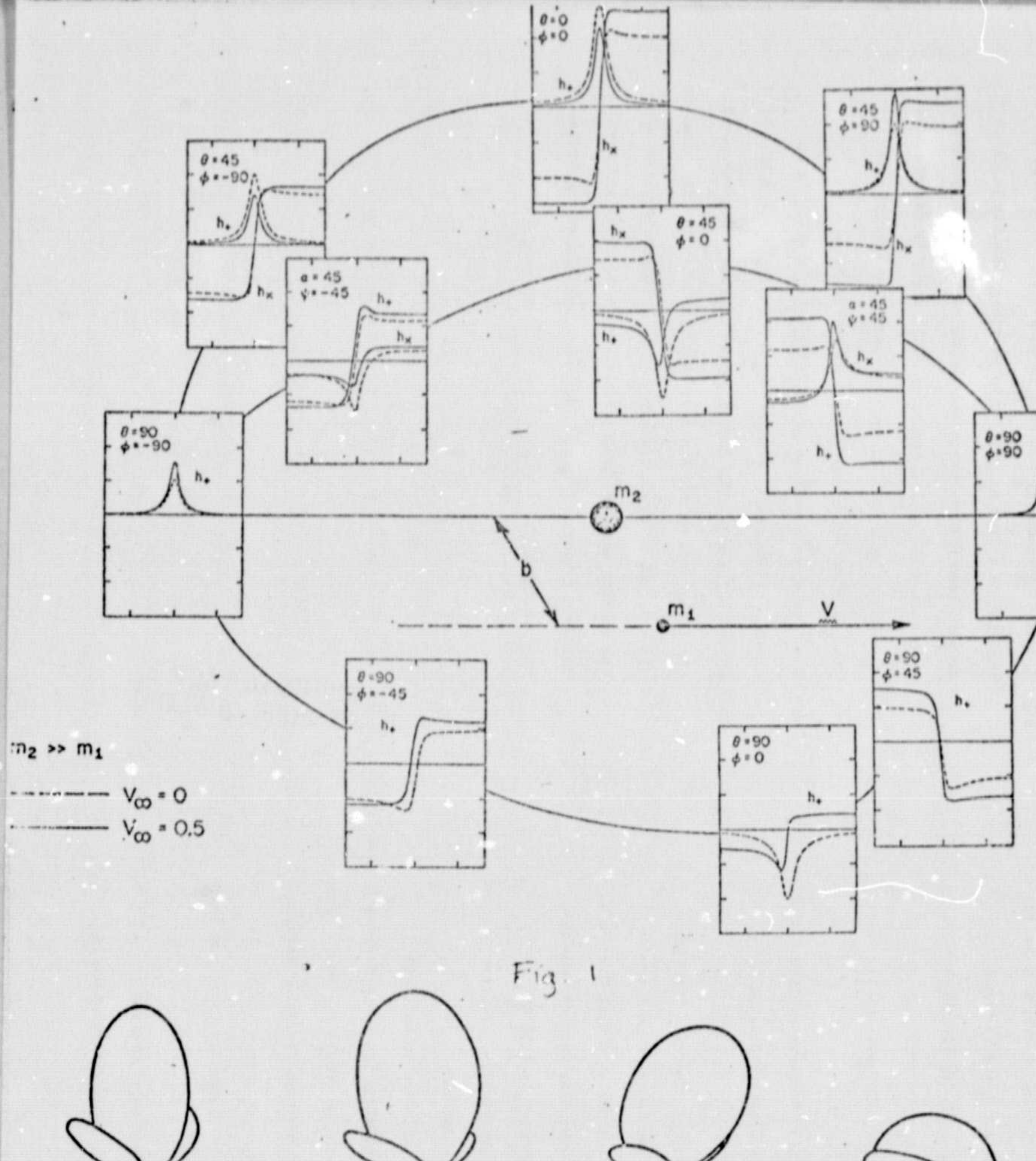

Fig. 1

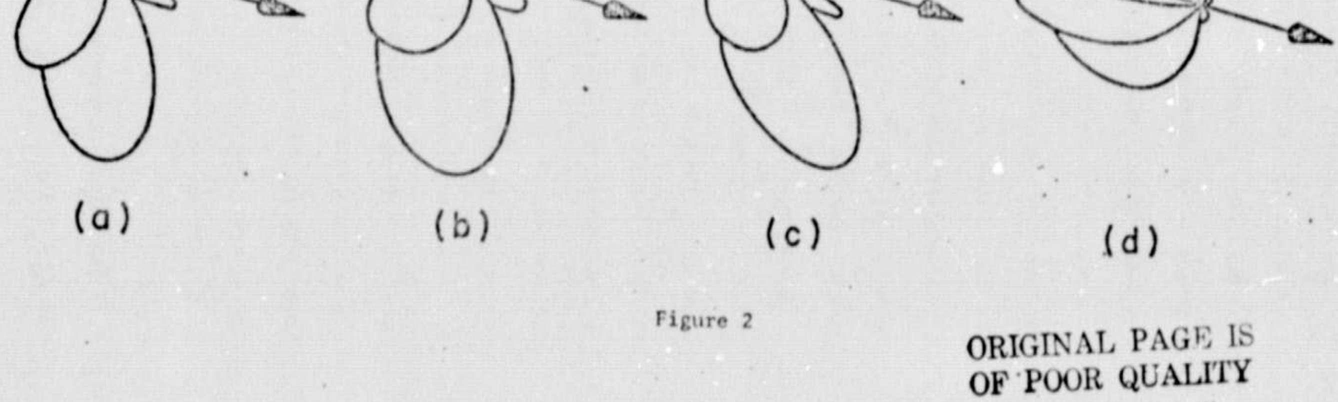

Figure 2

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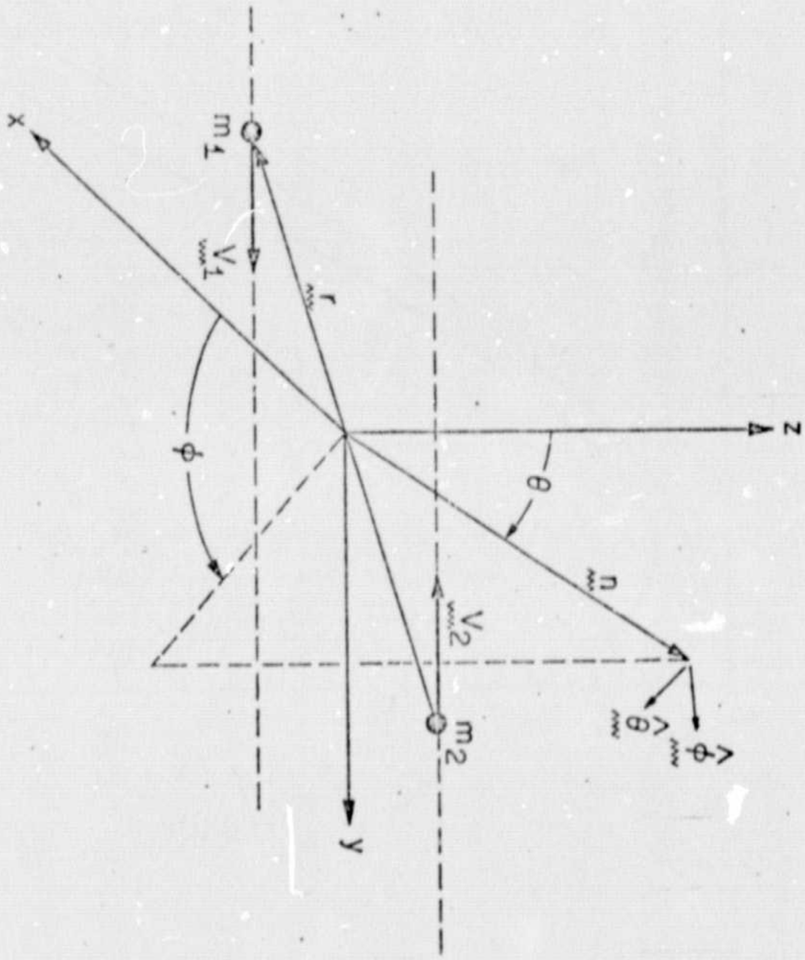
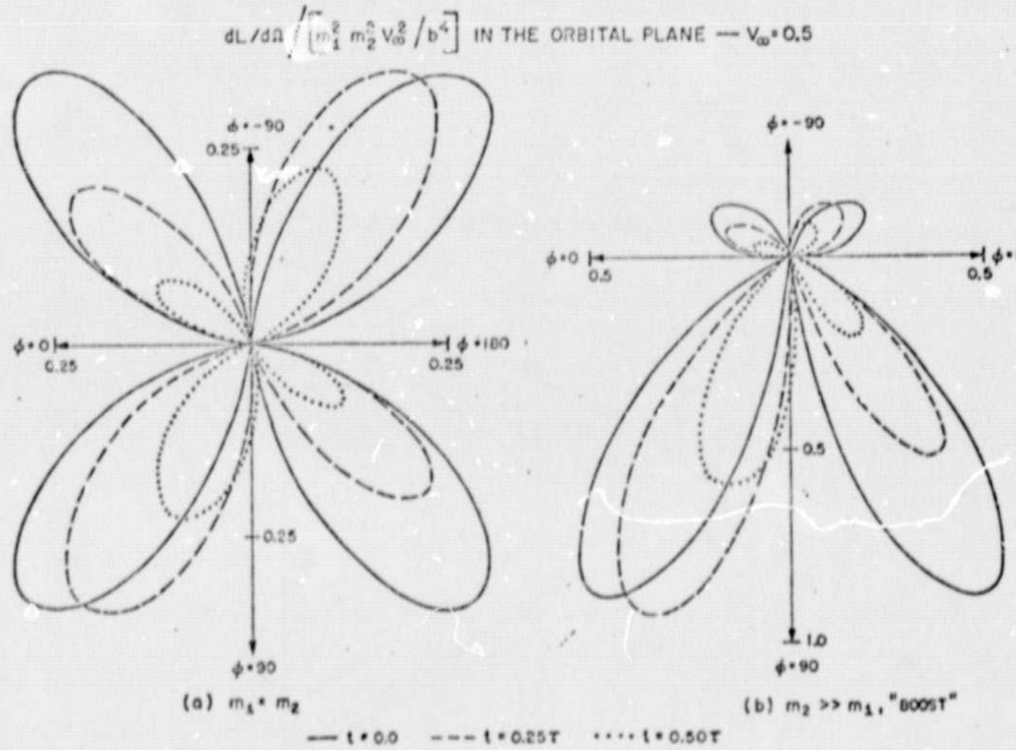


Figure 3



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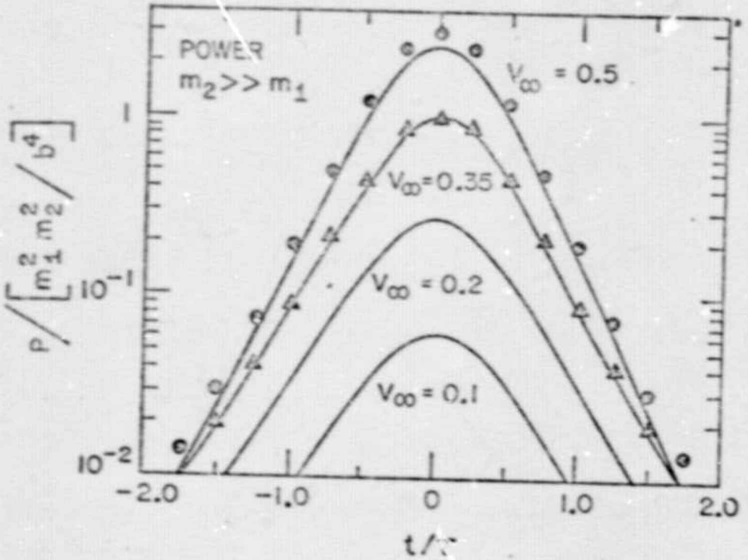

Figure 5

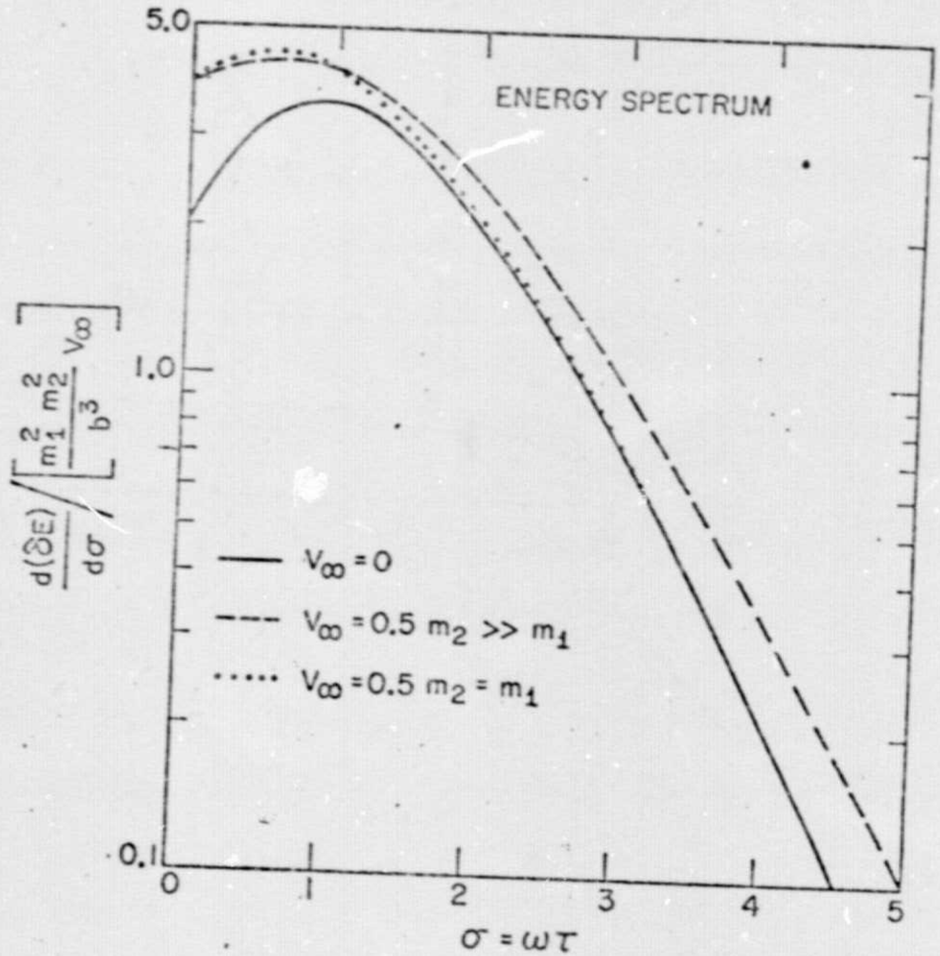

Figure 6

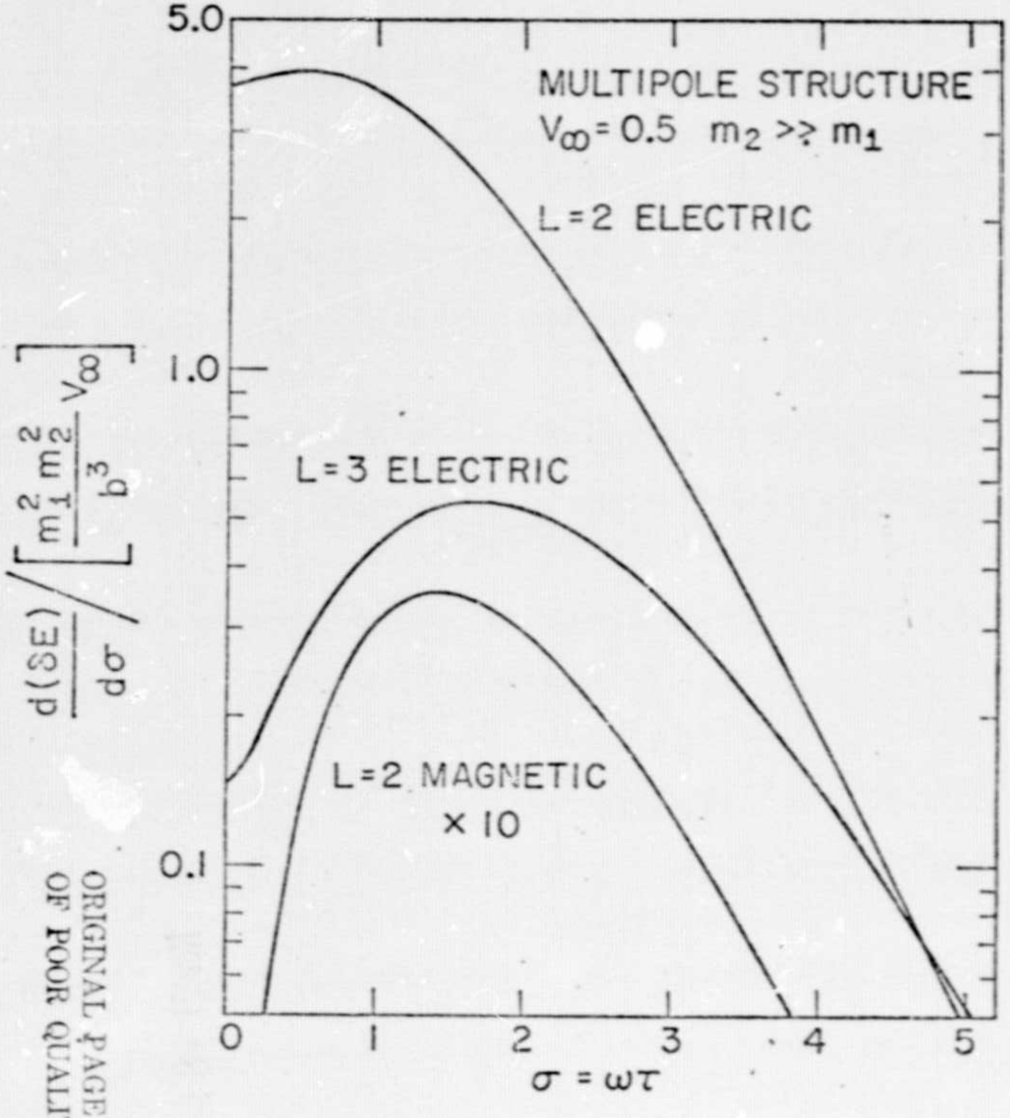

Figure 7

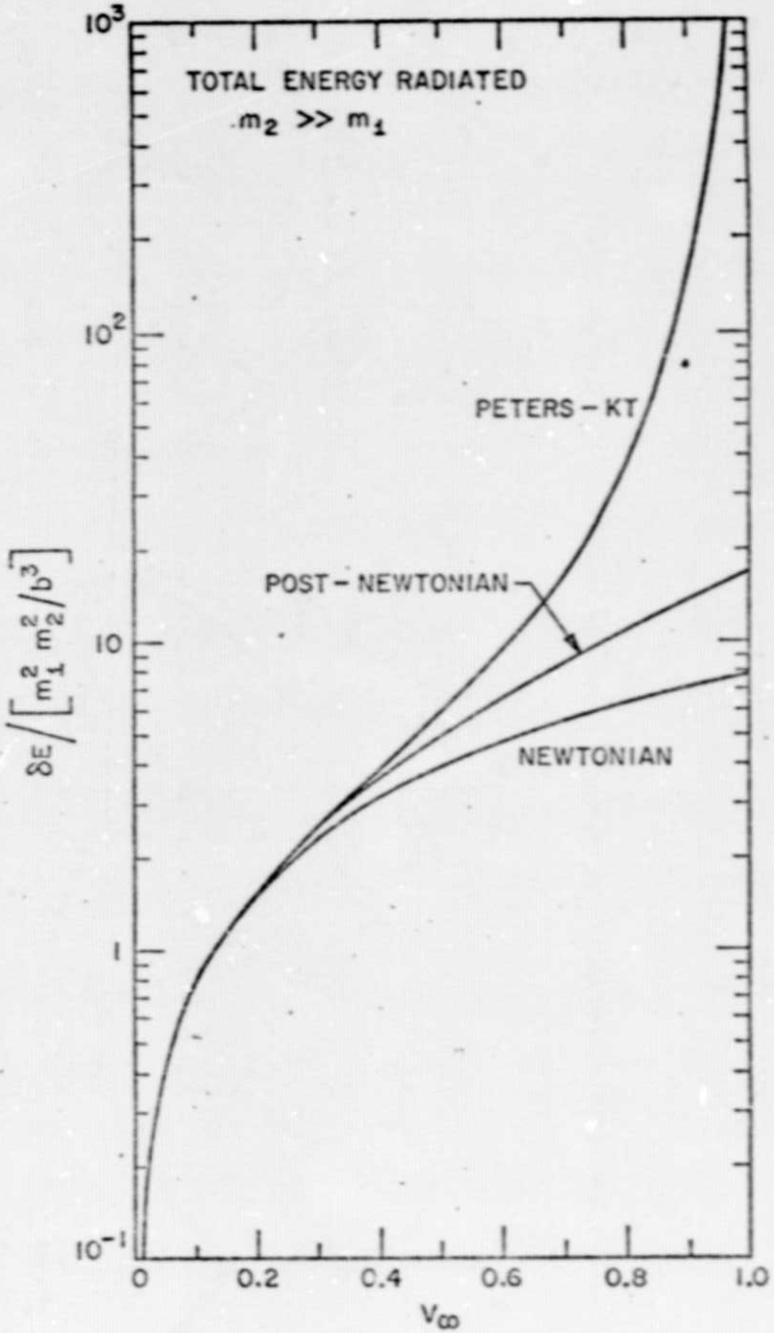

Figure 8

RADIATION PATTERN $d(\delta E)/d\Omega \Big/ \left[m_1^2 m_2^2 V_\infty / b^3\right]$, IN THE ORBITAL PLANE — $m_2 >> m_1$

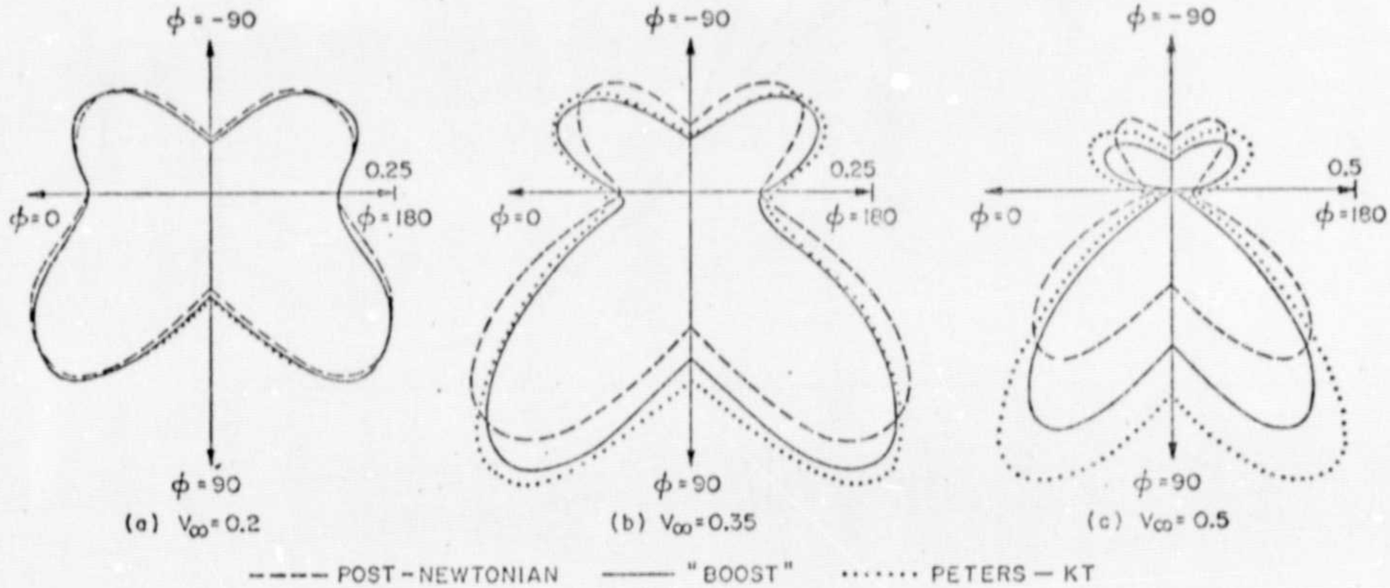

Figure 9