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Some New Nyström Integrators

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Lyndon B. Johnson Space Center
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SHUTTLE PROGRAM

SOME NEW NYSTRÖM INTEGRATORS

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1. INTRODUCTION

Nyström integrators are self-starting integrators used to integrate the second order, vector, differential equations $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$ and $\ddot{\underline{x}} = \underline{f}(t,\underline{x},\dot{\underline{x}})$. Monuki, Reference 1, in a very extensive study has found the Nyström type integrators to be greatly superior to most Runge-Kutta type integrators in terms of speed and accuracy when integrating ballistic missile trajectories.

Nyström integrator parameters are defined by a set of nonlinear constraint equations. Frequently there are more parameters than there are equations. When this is the case, the investigator is free to choose some additional constraint equations. Additional equations are included for higher order (improved accuracy) integration when $\ddot{\underline{x}} = \underline{f}(t)$.

Two of the lower order integrators given here are not new, and are due to Nystrom, Reference 2. They are given so that the reader will have a single source for all orders of Nyström type integrators.

2. TWO FUNCTION INTEGRATION OF $\underline{\ddot{x}} = \underline{f}(t,\underline{x})$

This integrator evaluates $\underline{f}$ twice and hence the terminology "two function integration". It provides third order integration of $\underline{x}$ and $\underline{\dot{x}}$. If $\underline{\ddot{x}} = \underline{f}(t)$, the integration will still only be third order for $\underline{x}$ and $\underline{\dot{x}}$. The integration parameters are due to Nyström, Reference 2, and are the solution of six constraint equations in six unknowns, which are given in Reference 1.

$$\underline{k}_1 = \Delta T\underline{f}(t_n, \underline{x}_n)$$

$$\underline{k}_2 = \Delta T\underline{f}(t_n + \delta_2\Delta T, \underline{x}_n + \delta_2\Delta T\underline{\dot{x}}_n + a_1\Delta T\underline{k}_1)$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T\underline{\dot{x}}_n + \Delta T(\alpha_1\underline{k}_1 + \alpha_2\underline{k}_2) + 0(\Delta T^4)$$

$$\underline{\dot{x}}_{n+1} = \underline{\dot{x}}_n + \beta_1\underline{k}_1 + \beta_2\underline{k}_2 + 0(\Delta T^4)$$

Where $\Delta T = t_{n+1}-t_n$ and the integration coefficients are given by

$$\delta_2 = 2/3$$

$$a_1 = 2/9$$

$$\alpha_1 = 1/4$$

$$\alpha_2 = 1/4$$

$$\beta_1 = 1/4$$

$$\beta_2 = 3/4$$

3. THREE FUNCTION INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$

This integrator provides fourth order integration for both $\underline{x}$ and $\dot{\underline{x}}$. However, if $\ddot{\underline{x}} = \underline{f}(t)$, the integrator will be fifth order for both $\underline{x}$ and $\dot{\underline{x}}$. The integration parameters are due to Lear.

$$\underline{k}_1 = \Delta T\ \underline{f}(t_n,\underline{x}_n)$$

$$\underline{k}_2 = \Delta T\underline{f}(t_n + \delta_2\Delta T,\underline{x}_n + \delta_2\Delta T\dot{\underline{x}}_n + a_1\Delta T\underline{k}_1)$$

$$\underline{k}_3 = \Delta T\underline{f}[t_n + \delta_3\Delta T,\underline{x}_n + \delta_3\Delta T\dot{\underline{x}}_n + \Delta T(b_1\underline{k}_1 + b_2\underline{k}_2)]$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \Delta T\dot{\underline{x}}_n + \Delta T(\alpha_1\underline{k}_1 + \alpha_2\underline{k}_2 + \alpha_3\underline{k}_3)$$

$$\underline{x}_{n+1} = \underline{x}_n + \beta_1\underline{k}_1 + \beta_2\underline{k}_2 + \beta_3\underline{k}_3$$

The integration coefficients are given by

$$\delta_2 = .6 - \sqrt{.06} = .35505\ 10257$$

$$\delta_3 = .6 + \sqrt{.06} = .84494\ 89743$$

$$a_1 = .21-.6\sqrt{.06} = .06303\ 06154$$

$$b_1 = (.15+4\sqrt{.06})/25 = .04519\ 18359$$

$$b_2 = (5.1+11\sqrt{.06})/25 = .31177\ 75487$$

$$\alpha_1 = 1/9 = .11111\ 11111$$

$$\alpha_2 = (7+20\sqrt{.06})/36 = .33052\ 72081$$

$$\alpha_3 = (7-20\sqrt{.06})/36 = .05836\ 16809$$

$$\beta_1 = 1/9 = .11111\ 11111$$

$$\beta_2 = (8+5\sqrt{.06})/18 = .51248\ 58262$$

$$\beta_3 = (8-5\sqrt{.06})/18 = .37640\ 30627$$

4. FOUR FUNCTION INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$

This integrator provides fifth order integration for $\underline{x}$ and $\dot{\underline{x}}$. However, if $\ddot{\underline{x}} = \underline{f}(t)$, the integrator will be seventh order for both $\underline{x}$ and $\dot{\underline{x}}$. The integration parameters are due to Lear.

$$\underline{k}_1 = \Delta T \underline{f}(t_n, \underline{x}_n)$$

$$\underline{k}_2 = \Delta T \underline{f}(t_n + \delta_2 \Delta T,\ \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1)$$

$$\underline{k}_3 = \Delta T \underline{f}[t_n + \delta_3 \Delta T,\ \underline{x}_n + \delta_3 \Delta T \dot{\underline{x}}_n + \Delta T(b_1 \underline{k}_1 + b_2 \underline{k}_2)]$$

$$\underline{k}_4 = \Delta T \underline{f}[t_n + \delta_4 \Delta T,\ \underline{x}_n + \delta_4 \Delta T \dot{\underline{x}}_n + \Delta T(c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3)]$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \alpha_3 \underline{k}_3 + \alpha_4 \underline{k}_4)$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4$$

The integration coefficients were obtained by numerically solving 17 non-linear equations in 17 unknowns. The solution is

$$\delta_2 = .21234\ 05385$$

$$\delta_3 = .59053\ 31358$$

$$\delta_4 = .91141\ 20406$$

$$a_1 = .022544\ 25214$$

$$b_1 = -.00114\ 39805$$

$$b_2 = .17550\ 86728$$

$$c_1 = .11715\ 41673$$

$$c_2 = .13937\ 54710$$

$$c_3 = .15880\ 63156$$

$$\alpha_1 = .06250\ 00001$$

$$\alpha_2 = .25901\ 73402$$

$$\alpha_3 = .15895\ 23623$$

$$\alpha_4 = .01953\ 02974$$

$$\beta_1 = .06250\ 00001$$

$$\beta_2 = .32884\ 43202$$

$$\beta_3 = .38819\ 34687$$

$$\beta_4 = .22046\ 22110$$

Note that probably both α_1 and β_1 are exactly .0625 = 1/16. Computer roundoff error probably prevented the correct solution. However, the above set of coefficients should be used as shown, since they satisfy many of the defining constraint equations exactly.

5. FIVE FUNCTION INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$

This integrator provides sixth order integration for $\underline{x}$ and $\dot{\underline{x}}$. There are 22 constraint equations in 24 unknowns, thus two more constraint equations were chosen so as to cause eighth order integration of $\ddot{\underline{x}} = \underline{f}(t)$. Unfortunately, the numerical solutions for the new constraint equations were unsatisfactory. Thus the integration constants shown here are only good to sixth order for integrating $\ddot{\underline{x}} = \underline{f}(t)$.

$$\underline{k}_1 = \Delta T \underline{f}(t_n, \underline{x}_n)$$
$$\underline{k}_2 = \Delta T \underline{f}(t_n + \delta_2 \Delta T,\ \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta t \underline{k}_1)$$
$$\underline{k}_3 = \Delta T \underline{f}[t_n + \delta_3 \Delta T,\ \underline{x}_n + \delta_3 \Delta T \dot{\underline{x}}_n + \Delta T(b_1 \underline{k}_1 + b_2 \underline{k}_2)]$$
$$\underline{k}_4 = \Delta T \underline{f}[t_n + \delta_4 \Delta T,\ \underline{x}_n + \delta_4 \Delta T \dot{\underline{x}}_n + \Delta T(c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3)]$$
$$\underline{k}_5 = \Delta T \underline{f}[t_n + \delta_5 \Delta T,\ \underline{x}_n + \delta_5 \Delta T \dot{\underline{x}}_n + \Delta T(d_1 \underline{k}_1 + d_2 \underline{k}_2 + d_3 \underline{k}_3 + d_4 \underline{k}_4)]$$
$$\underline{x}_{n+1} = \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \alpha_3 \underline{k}_3 + \alpha_4 \underline{k}_4 + \alpha_5 \underline{k}_5)$$
$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4 + \beta_5 \underline{k}_5$$

The integration coefficients, due to Lear, are

$\delta_2 = \frac{1}{2}$ $\quad \delta_3 = \frac{1}{3}$ $\quad \delta_4 = \frac{2}{3}$ $\quad \delta_5 = 1$

$a_1 = \frac{1}{8}$ $\quad b_1 = \frac{1}{18}$ $\quad b_2 = 0$

$c_1 = \frac{1}{9}$ $\quad c_2 = 0$ $\quad c_3 = \frac{1}{9}$

$d_1 = 0$ $\quad d_2 = -\frac{8}{11}$ $\quad d_3 = \frac{9}{11}$ $\quad d_4 = \frac{9}{22}$

$\alpha_1 = \frac{11}{120}$ $\quad \alpha_2 = -\frac{4}{15}$ $\quad \alpha_3 = \frac{9}{20}$ $\quad \alpha_4 = \frac{9}{40}$ $\quad \alpha_5 = 0$

$\beta_1 = \frac{11}{120}$ $\quad \beta_2 = -\frac{8}{15}$ $\quad \beta_3 = \beta_4 = \frac{27}{40}$ $\quad \beta_5 = \frac{11}{120}$

Monuki's integration coefficients (Reference 1) are

$\delta_2 = .3$ $\delta_3 = .6$ $\delta_4 = 2/3$ $\delta_5 = 1$

$a_1 = .045$ $b_1 = .18$ $b_2 = 0$

$c_1 = .13671\ 69639$

$c_2 = .08047\ 55373$

$c_3 = .00502\ 97211$

$d_1 = .00740\ 74074$

$d_2 = .49023\ 56902$

$d_3 = -.57037\ 03704$

$d_4 = .57272\ 72727$

$\alpha_1 = .08796\ 29630$

$\alpha_2 = .33670\ 03367$

$\alpha_3 = -.23148\ 14815$

$\alpha_4 = .30681\ 81818$

$\alpha_5 = 0$

$\beta_1 = .08796\ 29630$

$\beta_2 = .48100\ 04810$

$\beta_3 = -.57870\ 37037$

$\beta_4 = .92045\ 45455$

$\beta_5 = .08928\ 57143$

6. SIX FUNCTION INTEGRATION OF $\underline{\dot{x}} = \underline{f}(t,\underline{x})$

This integrator provides seventh order integration for $\underline{x}$ and $\underline{\dot{x}}$. There were 33 constraint equations in 32 unknowns, impossible to solve at first glance. However, Monuki, Reference 1, using a brilliant set of assumptions, was able to show how to obtain a solution.

$$\underline{k}_1 = \Delta T\underline{f}(t_n, \underline{x}_n)$$

$$\underline{k}_2 = \Delta T\underline{f}(t_n + \delta_2\Delta T, \underline{x}_n + \delta_2\Delta T\underline{\dot{x}}_n + a_1\Delta T\underline{k}_1)$$

$$\underline{k}_3 = \Delta T\underline{f}[t_n + \delta_3\Delta T, \underline{x}_n + \delta_3\Delta T\underline{\dot{x}}_n + \Delta T(b_1\underline{k}_1 + b_2\underline{k}_2)]$$

$$\underline{k}_4 = \Delta T\underline{f}[t_n + \delta_4\Delta T, \underline{x}_n + \delta_4\Delta T\underline{\dot{x}}_n + \Delta T(c_1\underline{k}_1 + c_2\underline{k}_2 + c_3\underline{k}_3)]$$

$$\underline{k}_5 = \Delta T\underline{f}[t_n + \delta_5\Delta T, \underline{x}_n + \delta_5\Delta T\underline{\dot{x}}_n + \Delta T(d_1\underline{k}_1 + d_2\underline{k}_2 + d_3\underline{k}_3 + d_4\underline{k}_4)]$$

$$\underline{k}_6 = \Delta T\underline{f}[t_n + \delta_6\Delta T, \underline{x}_n + \delta_6\Delta T\underline{\dot{x}}_n + \Delta T(e_1\underline{k}_1 + e_2\underline{k}_2 + e_3\underline{k}_3 + e_4\underline{k}_4 + e_5\underline{k}_5)]$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T\underline{\dot{x}}_n + \Delta T(\alpha_1\underline{k}_1 + \alpha_2\underline{k}_2 + \alpha_3\underline{k}_3 + \alpha_4\underline{k}_4 + \alpha_5\underline{k}_5 + \alpha_6\underline{k}_6)$$

$$\underline{\dot{x}}_{n+1} = \underline{\dot{x}}_n + \beta_1\underline{k}_1 + \beta_2\underline{k}_2 + \beta_3\underline{k}_3 + \beta_4\underline{k}_4 + \beta_5\underline{k}_5 + \beta_6\underline{k}_6$$

The integration coefficients, due to Monuki, are

$$\delta_2 = .10654\ 17886$$

$$\delta_3 = .21308\ 35772$$

$$\delta_4 = .59267\ 23008$$

$$\delta_5 = .916$$

$$\delta_6 = .972$$

$$a_1 = .56755\ 76359\cdot 10^{-2}$$

$$b_1 = .07567\ 43515\cdot 10^{-1}$$

$$b_2 = .15134\ 87029\cdot 10^{-1}$$

$c_1 = .14003\ 61674$

$c_2 = -.25447\ 80570$

$c_3 = .29007\ 21177$

$d_1 = -1.02164\ 36141$

$d_2 = 2.65397\ 01073$

$d_3 = -1.48615\ 90950$

$d_4 = .27336\ 06017$

$e_1 = -20.40832\ 94915$

$e_2 = 50.31431\ 81086$

$e_3 = -32.30441\ 78724$

$e_4 = 2.94949\ 60939$

$e_5 = -.07867\ 48385$

$\alpha_1 = .06271\ 70177$

$\alpha_2 = 0$

$\alpha_3 = .25968\ 74616$

$\alpha_4 = .15875\ 55586$

$\alpha_5 = .01912\ 37845$

$\alpha_6 = -.00028\ 38224$

$\beta_1 = .06271\ 70177$

$\beta_2 = 0$

$\beta_3 = .33000\ 64074$

$\beta_4 = .38974\ 89881$

$\beta_5 = .22766\ 41014$

$\beta_6 = -.01013\ 65146$

7. TWO FUNCTION INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x},\dot{\underline{x}})$

This integrator provides third order integration for $\underline{x}$ and second order for $\dot{\underline{x}}$. However, if $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$ or if $\ddot{\underline{x}} = \underline{f}(t)$ then the integration will be third order for both $\underline{x}$ and $\dot{\underline{x}}$. The integration parameters are due to Nyström, Reference 2, and are the solution of seven constraint equations in seven unknowns.

$$\underline{k}_1 = \Delta T \underline{f}(t_n, \underline{x}_n, \dot{\underline{x}}_n)$$

$$\underline{k}_2 = \Delta T \underline{f}(t_n + \delta_2 \Delta T, \underline{x}_n + \delta_2 \Delta T \dot{\underline{x}}_n + a_1 \Delta T \underline{k}_1, \dot{\underline{x}}_n + \dot{a}_1 \underline{k}_1)$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T \dot{\underline{x}}_n + \Delta T(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2)$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2$$

The integration coefficients are given by

$$\delta_2 = 2/3$$

$$a_1 = 2/9$$

$$\alpha_1 = 1/4$$

$$\alpha_2 = 1/4$$

$$\dot{a}_1 = 2/3$$

$$\beta_1 = 1/4$$

$$\beta_2 = 3/4$$

8. THREE FUNCTION INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x},\dot{\underline{x}})$

This integrator provides fourth order integration for $\underline{x}$ and third order for $\dot{\underline{x}}$. If $\dot{\underline{x}} = \underline{f}(t,\underline{x})$ the integrator will be fourth order for both $\underline{x}$ and $\dot{\underline{x}}$. If $\ddot{\underline{x}} = \underline{f}(t)$ the integrator will be fifth order for both $\underline{x}$ and $\dot{\underline{x}}$. The integration constants are due to Lear.

$$\underline{k}_1 = \Delta T\underline{f}(t_n,\underline{x}_n,\dot{\underline{x}}_n)$$
$$\underline{k}_2 = \Delta T\underline{f}(t_n + \delta_2\Delta T,\ \underline{x}_n + \delta_2\Delta T\dot{\underline{x}}_n + a_1\Delta T\underline{k}_1,\ \dot{\underline{x}}_n + \dot{a}_1\underline{k}_1)$$
$$\underline{k}_3 = \Delta T\underline{f}[t_n + \delta_3\Delta T,\ \underline{x}_n + \delta_3\Delta T\dot{\underline{x}}_n + \Delta T(b_1\underline{k}_1 + b_2\underline{k}_2), \dot{\underline{x}}_n + \dot{b}_1\underline{k}_1 + \dot{b}_2\underline{k}_2]$$
$$\underline{x}_{n+1} = \underline{x}_n + \Delta T\dot{\underline{x}}_n + \Delta T(\alpha_1\underline{k}_1 + \alpha_2\underline{k}_2 + \alpha_3\underline{k}_3)$$
$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_n + \beta_1\underline{k}_1 + \beta_2\underline{k}_2 + \beta_3\underline{k}_3$$

The integration coefficients are given by

$$\delta_2 = .6-\sqrt{.06} = .35505\ 10257$$
$$\delta_3 = .6+\sqrt{.06} = .84494\ 89743$$
$$a_1 = .21-.6\sqrt{.06} = .06303\ 06154$$
$$b_1 = (.15+4\sqrt{.06})/25 = .04519\ 18359$$
$$b_2 = (5.1+11\sqrt{.06})/25 = .31177\ 75487$$
$$\alpha_1 = 1/9 = .11111\ 11111$$
$$\alpha_2 = (7+20\sqrt{.06})/36 = .33052\ 72081$$
$$\alpha_3 = (7-20\sqrt{.06})/36 = .05836\ 16809$$
$$\dot{a}_1 = .6-\sqrt{.06} = .35505\ 10257$$
$$\dot{b}_1 = -(5.4+19\sqrt{.06})/25 = -.40216\ 12204$$
$$\dot{b}_2 = (20.4+44\sqrt{.06})/25 = 1.24711\ 01947$$
$$\beta_1 = 1/9 = .11111\ 11111$$
$$\beta_2 = (8+5\sqrt{.06})/18 = .51248\ 58262$$
$$\beta_3 = (8-5\sqrt{.06})/18 = .37640\ 30627$$

9. FOUR FUNCTION INTEGRATION OF $\underline{x} = \underline{f}(t,\underline{x},\underline{\dot{x}})$

This integrator provides fourth order integration for both $\underline{x}$ and $\underline{\dot{x}}$. If $\underline{\ddot{x}} = \underline{f}(t,\underline{x})$ the integration will be fifth order for both $\underline{x}$ and $\underline{\dot{x}}$. If $\underline{\ddot{x}} = \underline{f}(t)$ the integrator will be sixth order for both $\underline{x}$ and $\underline{\dot{x}}$. The integration constants are due to Lear.

$$\underline{k}_1 = \Delta T\underline{f}(t_n, \underline{x}_n, \underline{\dot{x}}_n)$$

$$\underline{k}_2 = \Delta T\underline{f}(t_n + \delta_2\Delta T,\ \underline{x}_n + \delta_2\Delta T\underline{\dot{x}}_n + a_1\Delta T\underline{k}_1,\ \underline{\dot{x}}_n + \dot{a}_1\underline{k}_1)$$

$$\underline{k}_3 = \Delta T\underline{f}[t_n + \delta_3\Delta T,\ \underline{x}_n + \delta_3\Delta T\underline{\dot{x}}_n + \Delta T(b_1\underline{k}_1 + b_2\underline{k}_2),\ \underline{\dot{x}}_n + \dot{b}_1\underline{k}_1 + \dot{b}_2\underline{k}_2]$$

$$\underline{k}_4 = \Delta T\underline{f}[t_n + \delta_4\Delta T,\ \underline{x}_n + \delta_4\Delta T\underline{\dot{x}}_n + \Delta T(c_1\underline{k}_1 + c_2\underline{k}_2 + c_3\underline{k}_3),\ \underline{\dot{x}}_n$$
$$+ \dot{c}_1\underline{k}_1 + \dot{c}_2\underline{k}_2 + \dot{c}_3\underline{k}_3]$$

$$\underline{x}_{n+1} = \underline{x}_n + \Delta T\underline{\dot{x}}_n + \Delta T(\alpha_1\underline{k}_1 + \alpha_2\underline{k}_2 + \alpha_3\underline{k}_3 + \alpha_4\underline{k}_4)$$

$$\underline{\dot{x}}_{n+1} = \underline{\dot{x}}_n + \beta_1\underline{k}_1 + \beta_2\underline{k}_2 + \beta_3\underline{k}_3 + \beta_4\underline{k}_4$$

The integration coefficients are given by

$$\delta_2 = (5-\sqrt{5})/10 = .27639\ 32023$$

$$\delta_3 = (5+\sqrt{5})/10 = .72360\ 67977$$

$$\delta_4 = 1.$$

$$a_1 = (3-\sqrt{5})/20 = .038196\ 60115$$

$$b_1 = 0$$

$$b_2 = (3+\sqrt{5})/20 = .26180\ 33989$$

$$c_1 = (-1+\sqrt{5})/4 = .30901\ 69943$$

$$c_2 = 0$$

$$c_3 = (3-\sqrt{5})/4 = .19098\ 30058$$

$$\alpha_1 = 1/12 = .08333\ 33333$$

$$\alpha_2 = (5+\sqrt{5})/24 = .30150\ 28324$$

$$\alpha_3 = (5-\sqrt{5})/24 = .11516\ 38343$$

$$\alpha_4 = 0$$

$$\dot{a}_1 = (5-\sqrt{5})/10 = .27639\ 32023$$

$$\dot{b}_1 = -(5+3\sqrt{5})/20 = -.58541\ 01966$$

$$\dot{b}_2 = (3+\sqrt{5})/4 = 1.30901\ 69944$$

$$\dot{c}_1 = -(1-5\sqrt{5})/4 = 2.54508\ 49719$$

$$\dot{c}_2 = -(5+3\sqrt{5})/4 = -2.92705\ 09831$$

$$\dot{c}_3 = (5-\sqrt{5})/2 = 1.38196\ 60113$$

$$\beta_1 = 1/12 = .08333\ 33333$$

$$\beta_2 = 5/12 = .41666\ 66667$$

$$\beta_3 = 5/12 = .41666\ 66667$$

$$\beta_4 = 1/12 = .08333\ 33333$$

10. INTEGRATION OF $\ddot{\underline{x}} = \underline{f}(t,\underline{x},\dot{\underline{x}}) + \underline{\varepsilon}(t,\underline{x},\dot{\underline{x}})$

In many problems there may be a very small perturbing acceleration, $\underline{\varepsilon}(t,\underline{x},\dot{\underline{x}})$, adding to a dominate acceleration, $\underline{f}(t,\underline{x},\dot{\underline{x}})$. For example, $\underline{\varepsilon}(t)$ could be a small venting acceleration on an orbiting vehicle. $\underline{\varepsilon}(\underline{x})$ could be due to higher order harmonics in the earth's gravity field. $\underline{\varepsilon}(\dot{\underline{x}})$ could be due to small drag or solar pressure forces acting on a space vehicle.

When $\underline{\varepsilon}$ is time consuming to evaluate, it may be desirable to not evaluate $\underline{\varepsilon}$ as often as $\underline{f}$ is evaluated in the integration step. Or, we may desire to use an integrator for $\underline{x} = \underline{f}(t,\underline{x})$ when $\underline{\varepsilon}$ has velocity terms in it. In these cases we may try the following scheme.

Integrate $\ddot{\underline{x}} = \underline{f}(t,\underline{x},\dot{\underline{x}})$ or $\ddot{\underline{x}} = \underline{f}(t,\underline{x})$ normally. Then to the solution add

$$\underline{x}_{n+1} = \underline{x}_{n+1} + \frac{\Delta T^2}{2} \underline{\varepsilon}(t_{n+.5}, \underline{x}_{n+.5}, \dot{\underline{x}}_{n+.5})$$

$$\dot{\underline{x}}_{n+1} = \dot{\underline{x}}_{n+1} + \frac{\Delta T}{1} \underline{\varepsilon}(t_{n+.5}, \underline{x}_{n+.5}, \dot{\underline{x}}_{n+.5})$$

where, we see, $\underline{\varepsilon}$ is evaluated at the midpoint of the integration step. The equations for the midpoint are

$$t_{n+.5} = t_n + \Delta T/2$$

$$\underline{x}_{n+.5} = \underline{x}_n + \frac{1}{2} (\underline{x}_{n+1} - \underline{x}_n) - \frac{\Delta T}{8} (\dot{\underline{x}}_{n+1} - \dot{\underline{x}}_n) + 0(\Delta T^4)$$

$$\dot{\underline{x}}_{n+.5} = \frac{3}{2\Delta T} (\underline{x}_{n+1} - \underline{x}_n) - \frac{1}{4} (\dot{\underline{x}}_{n+1} - \dot{\underline{x}}_n) - \frac{1}{2} \dot{\underline{x}}_n + 0(\Delta T^4)$$

$$= (\dot{\underline{x}}_{n+1} + \dot{\underline{x}}_n)/2 \text{ for } \Delta T = 0 \text{ or } \Delta T \text{ small}$$

Note that the equation for $\dot{\underline{x}}_{n+.5}$ is afflicted with roundoff error for small ΔT. A more stable form is

$$\underline{x}_{n+.5} = \underline{x}_n + \frac{\Delta T}{2} [\dot{\underline{x}}_n + (\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \cdots) - .25(\beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \cdots)]$$

$$\dot{\underline{x}}_{n+.5} = \dot{\underline{x}}_n + 1.5(\alpha_1 \underline{k}_1 + \alpha_2 \underline{k}_2 + \cdots) -.25(\beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \cdots)$$

where the α_i, β_i and $\underline{k}_i$ are those used in the Nyström integrator.

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11. REFERENCES

1. Albert T. Monuki, "Improved Integration Method Development", 1977 IR&D Study, Volume II, "Improved Integrator". This excellent lengthy document is currently available in rough-draft by contacting

   Albert T. Monuki
   TRW DSSG
   One Space Park
   Redondo Beach, Ca. 90278

2. E.J. Nyström, "Uber die Numerische Integration von Differential Gleichungen", Acta Soc. Sci. Fenn. 50 (1925), No. 13, 1-55.

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