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Aeromechanical Stability of Helicopters with a Bearingless Main Rotor — Part I: Equations of Motion

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SYMBOLS

A	dimensionless flexbeam flapping stiffness, $\frac{EI_f}{I\Omega_0^2 \ell}$
a	rotor blade airfoil lift curve slope
$\mathbf{a}^{B*}$	acceleration of the body mass center, m/sec^2
$\mathbf{a}^{k*}$	acceleration of the kth blade mass center, m/sec^2
B	dimensionless flexbeam chordwise stiffness, $\frac{EI_c}{I\Omega_0^2 \ell}$
$[B]$	transformation matrix relating the blade-fixed axis system N_x^k, N_y^k, N_z^k to the flexbeam tip axis system n_x^k, n_y^k, n_z^k
b	number of blades (≥ 3)
C	dimensionless flexbeam torsion stiffness, $\frac{GJ}{I\Omega_0^2 \ell}$
$[C^A]$	damping matrix contribution from aerodynamics
$[C^D]$	damping matrix contribution from structural damping
c	rotor blade airfoil chord length, m
$\bar{c}$	$\frac{c}{L}$
c_i	steady component of c_i^k, m
c_i^k	components of the kth rotor blade mass center in the n_i^k direction, $i = 1,2,3$, from point J, equation (16), m
$\bar{c}_i$	steady component of c_i^k made dimensionless by L
c_X	damping coefficient for uncoupled X motion, N-sec/m
c_Y	damping coefficient for uncoupled Y motion, N-sec/m
c_ζ	damping coefficient for uncoupled ζ motion (blade lead-lag), N-m-sec
c_{Φ_x}	damping coefficient for uncoupled Φ_x motion, N-m-sec
c_{Φ_y}	damping coefficient for uncoupled Φ_y motion, N-m-sec
$c_{(\ )}$	$\cos(\ )$
c_{d_0}	rotor blade airfoil profile drag coefficient

D	dimensionless flexbeam axial stiffness, $\frac{EA\ell}{I\Omega_0^2}$; also, rotor blade drag per unit length, N/m
d	$\frac{c_{d_0}}{a}$
d_i^k	components of the kth rotor blade mass center in the n_i^k direction, $i = 1,2,3$, from point 0, equation (58), m
EA	flexbeam axial stiffness, N
EI_c	flexbeam chordwise bending stiffness, $N\text{-}m^2$
EI_f	flexbeam flap bending stiffness, $N\text{-}m^2$
e	hub radius, m
$\bar{e}$	$\frac{e}{\ell}$
e_1,e_2,e_3	space-fixed axis system, figure 1
$\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$	unit vectors parallel to e_1, e_2, e_3
[F]	transformation matrix relating the flexbeam axis system n_1^k,n_2^k,n_3^k to the rotating coordinate system N_1^k, N_2^k, N_3^k
F_r	generalized active force for the rth degree of freedom, $r = u_k, v_k, w_k, \zeta_k, \beta_k, \theta_k, X, Y, \Phi_x, \Phi_y$
F_r^*	generalized inertia force for the rth degree of freedom, $r = u_k, v_k, w_k, \zeta_k, \beta_k, \theta_k, X, Y, \Phi_x, \Phi_y$
$\mathscr{F}$	function to be minimized to produce rotor equilibrium solution
$\mathbf{F}_B$	force acting on the body, N
$\mathbf{F}_B^*$	inertia force acting on the body, N
$\mathbf{F}_k$	force acting on the kth blade, N
$\mathbf{F}_k^*$	inertia force acting on the kth blade, N
$\mathbf{F}_R$	force at the flexbeam root, made dimensionless by $\frac{I\Omega_0^2}{\ell}$
$\mathbf{F}_t$	force at the flexbeam tip, made dimensionless by $\frac{I\Omega_0^2}{\ell}$
$\bar{F}_u,\bar{F}_v,\bar{F}_w$	components of external force in the direction of u,v,w, respectively, acting on the flexbeam tip

$\bar{F}_\zeta,\bar{F}_\beta,\bar{F}_\theta$	components of external moments in the direction of ζ,β,θ rotations, respectively, acting on the flexbeam tip
$[G]$	$[B]\ [T]$
GJ	flexbeam torsional stiffness, $N\text{-}m^2$
$[G^I]$	gyroscopic matrix contribution from inertia forces
$[G^k]$	$[B]\ [T^k]$
g	rotor thrust per unit mass of total aircraft, m/sec^2
$\bar{g}$	$\dfrac{g}{\Omega_0{}^2 L}$
g_0	acceleration of gravity, m/sec^2
$\bar{g}_0$	$\dfrac{g_0}{\Omega_0{}^2 L}$
$[H]$	$[G]\ [F]$
$[H^k]$	$[G^k]\ [F]$
h	height of rotor-hub center above aircraft reference center, m
$\bar{h}$	$\dfrac{h}{L}$
$\dot{h}$	two-dimensional airfoil section plunge velocity, m/sec
I	rotor blade flapping inertia for point J; $I = I_2 + mL^2 x_b{}^2$
I_1,I_2,I_3	rotor blade mass moments of inertia for the blade mass center for axes $N_x{}^k$, $N_y{}^k$, $N_z{}^k$, respectively
$\bar{I}_1,\bar{I}_2,\bar{I}_3$	$\dfrac{I_i}{I}$, $i = 1, 2, 3$
I_x,I_y	body mass moment of inertia for the body mass center for axes N_A and N_B respectively, $kg\text{-}m^2$
$\bar{I}_x,\bar{I}_y$	$\dfrac{I_x}{I}$, $\dfrac{I_y}{I}$
$J_{33},\bar{J}_{13},\bar{J}_{23}$	geometric variables, equation (66)
$[K]$	flexbeam stiffness matrix
$[K^A]$	stiffness matrix contribution from aerodynamics

$[K^D]$	stiffness matrix contribution from structural damping
$[K^g]$	stiffness matrix contribution from gravity
$[K^I]$	stiffness matrix contribution from inertia
$[K^S]$	stiffness matrix contribution from structural loads
K_x, K_y, K_z	landing gear stiffness for each of four landing gear springs in the N_A, N_B, and N_C directions, respectively, N/m
$\bar{K}_x, \bar{K}_y, \bar{K}_z$	$\frac{K_x L^2}{I\Omega_0^2}, \frac{K_y L^2}{I\Omega_0^2}, \frac{K_z L^2}{I\Omega_0^2}$
L	blade length, m
L_C	two-dimensional airfoil circulatory lift per unit length, N/m
L_{NC}	two-dimensional airfoil noncirculatory lift per unit length, N/m
ℓ	flexbeam length, m
$\bar{\ell}$	$\frac{\ell}{L}$
ℓ_x, ℓ_y	longitudinal and lateral distances, respectively, from front to rear and from left to right landing gear, figure 7, m
ℓ_z	vertical distance of aircraft reference center above the landing gear, figure 7, m
$\bar{\ell}_x, \bar{\ell}_y, \bar{\ell}_z$	$\frac{\ell_x}{L}, \frac{\ell_y}{L}, \frac{\ell_z}{L}$
M	mass of aircraft; $M = bm + m_f$, kg; also two-dimensional airfoil pitching moment per unit length, N
$\bar{M}$	$\frac{ML^2}{I}$
$[M^A]$	mass matrix contribution from aerodynamics
$\mathbf{M}_B$	moments exerted on the body, N-m
$\mathbf{M}_B^*$	inertia moments exerted on the body, N-m
$[M^I]$	mass matrix contribution from inertia
$\mathbf{M}_k$	moments exerted on the kth blade, N-m

$\mathbf{M}_k^*$	inertia moments exerted on the kth blade, N-m
M_k	aerodynamic pitching moment per unit length on kth blade, N
M_m^k	$\int_0^L M_k \, d_x$, N-m
$\mathbf{M}_R$	moment at the flexbeam root, made dimensionless by $I\Omega_0^2$
$\mathbf{M}_t$	moment at the flexbeam tip, made dimensionless by $I\Omega_0^2$
m	mass of one rotor blade, kg
$\bar{m}$	$\frac{mL^2}{I}$
m_f	mass of the fuselage, kg
$\bar{m}_f$	$\frac{m_f L^2}{I}$
N_A, N_B, N_C	body-fixed coordinate system, figure 1
$\mathbf{N}_A, \mathbf{N}_B, \mathbf{N}_C$	unit vectors parallel to N_A, N_B, N_C
N_1^k, N_2^k, N_3^k	rotating coordinate system for kth blade, figure 8
$\mathbf{N}_1^k, \mathbf{N}_2^k, \mathbf{N}_3^k$	unit vectors parallel to N_1^k, N_2^k, N_3^k
N_x^k, N_y^k, N_z^k	coordinate system fixed in the kth blade, figure 8
$\mathbf{N}_x^k, \mathbf{N}_y^k, \mathbf{N}_z^k$	unit vectors parallel to N_x^k, N_y^k, N_z^k
n_1^k, n_2^k, n_3^k	coordinate system fixed in the kth flexbeam root, figure 8
$\mathbf{n}_1^k, \mathbf{n}_2^k, \mathbf{n}_3^k$	unit vectors parallel to n_1^k, n_2^k, n_3^k
n_x^k, n_y^k, n_z^k	coordinate system fixed in the kth flexbeam tip, figure 8
$\mathbf{n}_x^k, \mathbf{n}_y^k, \mathbf{n}_z^k$	unit vectors parallel to n_x^k, n_y^k, n_z^k
q_r	dummy symbol used to refer to the blade degrees of freedom
R	rotor radius; $R = e + \ell + L$, m
$\mathbf{r}$	position vector of point along flexbeam, made dimensionless by ℓ
$\mathbf{r}_t$	position vector of flexbeam tip, made dimensionless by ℓ
S	component of aerodynamic force per unit length along the chord-line in a two-dimensional airfoil section, N/m

S_k — component of aerodynamic force per unit length along the chordline (N_y^k axis) of the rotor blade, N/m

S_f^k — $\int_0^L S_k \, dx$, N

S_m^k — $\int_0^L x S_k \, dx$, N-m

s — arc length of deformed flexbeam elastic axis made dimensionless by ℓ

$s_{(\ )}$ — sin( )

T — component of aerodynamic force per unit length perpendicular to the chordline in a two-dimensional airfoil section N/m; also kinetic energy, kg-m^2/sec^2, also tension in flexbeam, N

$\mathscr{T}$ — rotor thrust, N

$[T]$ — steady part of $[T^k]$

T_k — component of aerodynamic force per unit length perpendicular to the chordline (along N_z^k axis) of the rotor blade, N/m

$[T^k]$ — transformation matrix relating the flexbeam tip axis system n_x^k, n_y^k, n_z^k to the flexbeam root axis system n_1^k, n_2^k, n_3^k

T_f^k — $\int_0^L T_k \, dx$, N

T_m^k — $\int_0^L x T_k \, dx$, N-m

U — total velocity of two-dimensional airfoil section, m/sec

U_1^k — components of rotor blade velocity with respect to air in n_1^k axis system, m/sec

U_P^k — component of rotor blade velocity with respect to air in N_z^k direction, m/sec

U_T^k — component of rotor blade velocity with respect to air in N_y^k direction, m/sec

u — steady component of u_k, m

$\bar{u}$ — $1 + \frac{u}{\ell} - \tilde{e}F_{11}$

u_b — component of flexbeam axial deflection due to bending alone, made dimensionless by ℓ

$\tilde{u}_c$ — $\frac{2}{b}\sum_{k=1}^{b} \tilde{u}_k \cos \psi_k$, m

u_k — axial deflection of the kth flexbeam tip, m

$\bar{u}_k$ — $1 + \frac{u_k}{\ell} + \bar{e}F_{11}$

$\tilde{u}_k$ — unsteady component of u_k, m

u_s — component of flexbeam axial deflection due to stretching, made dimensionless by ℓ

$\tilde{u}_s$ — $\frac{2}{b}\sum_{k=1}^{b} \tilde{u}_k \sin \psi_k$, m

u_t — assumed value for u in equilibrium deflection scheme, made dimensionless by ℓ

$\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ — coordinate system along the deformed flexbeam principal axes

V — free-stream velocity of two-dimensional airfoil section, m/sec

V_i^k — rotor blade mass center velocity components in n_i^k axis system, equation 16, m/sec

$\mathbf{V}^P$ — velocity of any point P

v — steady component of v_k, m

$\bar{v}$ — $\frac{v}{\ell} + \bar{e}F_{21}$

$\tilde{v}_c$ — $\frac{2}{b}\sum_{k=1}^{b} \tilde{v}_k \cos \psi_k$, m

v_i — induced inflow velocity, m/sec

v_k — chordwise deflection of kth flexbeam tip, m

$\bar{v}_k$ — $\frac{v_k}{\ell} + \bar{e}F_{21}$

$\tilde{v}_k$ — unsteady component of v_k, m

$\tilde{v}_s$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{v}_k \sin \psi_k$, m
v_t	assumed value for v in equilibrium deflection scheme, made dimensionless by ℓ
w	steady component of w_k, m
$\bar{w}$	$\frac{w}{\ell} + \bar{e}F_{31}$
$\tilde{w}_c$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{w}_k \cos \psi_k$, m
w_k	flapwise deflection of kth flexbeam tip, m
$\bar{w}_k$	$\frac{w_k}{\ell} + \bar{e}F_{31}$
$\tilde{w}_k$	unsteady component of w_k, m
$\tilde{w}_s$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{w}_k \sin \psi_k$, m
w_t	assumed value for w in equilibrium deflection-scheme, made dimensionless by ℓ
X	time integral of body velocity component in N_A direction, m
x	distance along N_x^k axis from point J, m
x_a	chordwise distance from N_x^k axis to aerodynamic center, made dimensionless by c, positive when aerodynamic center is ahead of N_x^k axis
x_b	axial distance from point J to blade mass center along N_x^k axis, made dimensionless by L
x_c	chordwise distance from N_x^k axis to blade mass center made dimensionless by c, positive when mass center is ahead of N_x^k axis
Y	time integral of body velocity component in N_B direction, m
z	vertical distance from aircraft reference center to body mass center, positive when body mass center is below reference center, m

$\bar{z}$	$\frac{z}{L}$
α	two-dimensional airfoil angle of attack, rad
β	steady component of β_k, rad
β_b	built-in coning angle of the blade with respect to the flexbeam, positive tip up, rad
$\tilde{\beta}_c$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\beta}_k \cos \psi_k$, rad
β_f	built-in coning angle of the flexbeam with respect to the hub, positive tip up, rad
β_k	elastic flap rotation of the kth flexbeam tip positive tip up, rad
$\tilde{\beta}_k$	unsteady component of β_k, rad
$\tilde{\beta}_s$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\beta}_k \sin \psi_k$, rad
β_t	assumed value for β in equilibrium deflection scheme, rad
γ	dimensionless airload parameter, $\frac{\rho acL^4}{I}$
ε	pitch angle of two-dimensional airfoil section, rad; also, perturbation of flexbeam tip loads; also, flexbeam middle surface strain, equation (73)
ζ	steady component of ζ_k, rad
ζ_b	built-in sweep angle of the blade with respect to the flexbeam, positive tip leading, rad
$\tilde{\zeta}_c$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\zeta}_k \cos \psi_k$, rad
ζ_f	built-in sweep angle of the flexbeam with respect to the hub, positive tip leading, rad
ζ_k	elastic lead angle of the kth flexbeam tip, positive tip leading, rad

$\tilde{\zeta}_k$	unsteady component of ζ_k, rad
$\tilde{\zeta}_s$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\zeta}_k \sin \psi_k$, rad
ζ_t	assumed value for ζ in equilibrium deflection scheme, rad
$\eta_1^{\,k}$	geometric parameter, equation (18)
η_ζ	isolated blade lead-lag structural damping ratio
$\eta_X,\eta_Y,\eta_{\Phi_X},\eta_{\Phi_Y}$	fuselage damping ratio for uncoupled motion in the X, Y, Φ_X, and Φ_Y direction, respectively
θ	steady component of θ_k, rad
θ_b	built-in pitch angle of the blade with respect to the flexbeam, positive leading edge up, rad
$\tilde{\theta}_c$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\theta}_k \cos \psi_k$, rad
θ_f	built-in pitch angle of the flexbeam with respect to the hub, positive leading edge up, rad
θ_k	elastic twist of the kth flexbeam tip, positive leading edge up, rad
$\tilde{\theta}_k$	unsteady component of θ_k, rad
$\tilde{\theta}_s$	$\frac{2}{b}\sum_{k=1}^{b} \tilde{\theta}_k \sin \psi_k$, rad
θ_t	assumed value for θ in equilibrium deflection scheme, rad
$\theta_{3/4}$	pitch angle at the blade 3/4-radius, rad
κ	curvature in the local flexbeam flap direction, made dimensionless by $\frac{1}{\ell}$
λ	curvature in the local flexbeam chordwise direction, made dimensionless by $\frac{1}{\ell}$
ρ	air density, kg/m^3

σ	rotor solidity $\frac{bc}{\pi R}$
τ	curvature in the local flexbeam torsional direction, made dimensionless by $\frac{1}{\ell}$
Φ_x	time integral of angular velocity component in N_A direction, rad
Φ_y	time integral of angular velocity component in N_B direction, rad
ϕ	inflow angle at the blade 3/4-radius from momentum theory, equation (22), rad
ψ_k	azimuth angle of kth blade
Ω	rotor angular velocity, rad/sec
Ω_0	nominal rotor angular velocity, rad/sec
$\bar{\Omega}$	$\frac{\Omega}{\Omega_0}$
$\boldsymbol{\omega}^B$	angular velocity of the body, rad/sec
$\boldsymbol{\omega}^k$	angular velocity of the kth blade, rad/sec
$\omega_i{}^k$	components of angular velocity in $N_x{}^k, N_y{}^k, N_z{}^k$ axis system, equation (18), rad/sec

AEROMECHANICAL STABILITY OF HELICOPTERS WITH A BEARINGLESS MAIN ROTOR — PART I. EQUATIONS OF MOTION

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SUMMARY

Equations of motion for a coupled rotor-body system are derived for the purpose of studying air and ground resonance characteristics of helicopters that have bearingless main rotors. For the fuselage, only four rigid body degrees of freedom are considered; longitudinal and lateral translations, pitch, and roll. The rotor is assumed to consist of three or more rigid blades. Each blade is joined to the hub by means of a flexible beam segment (flexbeam or strap). Pitch change is accomplished by twisting the flexbeam with the pitch-control system, the characteristics of which are variable. Thus, the analysis is capable of implicitly treating aeroelastic couplings generated by the flexbeam elastic deflections, the pitch-control system, and the angular offsets of the blade and flexbeam. The linearized equations are written in the nonrotating system retaining only the cyclic rotor modes; thus they comprise a system of homogeneous ordinary differential equations with constant coefficients. All contributions to the linearized perturbation equations from inertia, gravity, quasi-steady aerodynamics, and the flexbeam equilibrium deflections are retained exactly. Part II describes a computer program based on these equations of motion.

1. INTRODUCTION

The general problem of helicopter aeroelastic stability involves coupling between the motion of the individual blades through control system dynamics and the rotor wake, as well as coupling between the rotor and fuselage of the helicopter The complexity of the problem poses a challenge to the analyst, both in developing an analytical model and in understanding its physical behavior. An important part of analyzing the general rotor-body dynamic system involves the study of the dynamic behavior of rigid-body fuselage motions coupled with the rotor motion. The well-known phenomena of air and ground resonance occur in this way. The low-frequency rotor modes interact with the

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rigid-body fuselage motions to produce instabilities both in the air and on the ground (refs. 1-4).

Helicopter rotors without hinges to allow rigid-body flap and lead-lag motions of the blades are commonly called hingeless rotors. Hingeless rotors have distinct advantages over the more common hinged (articulated) rotors; these include fewer moving parts, light weight, and more control-power. In recent years there have been efforts to develop hingeless rotors without pitch bearings to further reduce mechanical complexity and weight. These rotors, called bearingless rotors, rely on a torsionally soft portion of the blade called the flexbeam (or strap) that is twisted by the pitch-control system to provide changes in pitch. There have been several totally different designs proposed within the helicopter industry (refs. 5-9). Of these designs, one has a snubber (ref. 6); some have cantilevered pitch-arms without snubbers (refs. 5, 7, 8); and one has a torsionally stiff torque tube that is very flexible in bending and acts much like a speedometer cable (ref. 9). Both hingeless and bearingless rotors have had some aeroelastic stability problems. In fact, most production hingeless rotors still must rely on an auxiliary lead-lag damper to suppress ground and air resonance instabilities One reason for this is believed to be the lack of suitable analytical capability.

The analytical treatment of hingeless rotor air and ground resonance has, for the most part, been limited to an equivalent-hinge, spring-restrained rigid blade model for the rotor blades (refs. 1-4). The need arises when applying these analytical models to somehow arrive at the proper orientations for the spring-restrained hinges so as to give the proper values for the aeroelastic couplings that arise due to blade elastic deflections, angular offsets such as precone or sweep, and the pitch-control geometry. Even with this apparent drawback, these relatively simple analytical models have been successful, to some degree, in assessing the air and ground resonance characteristics of some configurations. Bearingless rotors, however, have coupling parameters that change significantly as a function of the operating condition. For example, when the rotor is at high thrust the flexbeam may be highly twisted; whereas, at low thrust it may be untwisted. An alternative to a rigid-blade formulation is that of an elastic blade. Again, there is the need to do a separate analysis, here to obtain the free vibration modeshapes. This can be time consuming for a bearingless rotor because the coupled modeshapes will change as a function of operating condition. Also, these analyses are much more complicated than the rigid-blade approximations Bielawa has developed such an analysis for the bearingless rotor, but only for the hub-fixed case (ref. 10). Johnson has developed a rotorcraft aeroelastic stability program (ref. 11) that possibly could be adapted to treat bearingless rotors; however, this modification has not been made to date An analysis that possesses the simplicity of the rigid blade model but integrates the treatment of aeroelastic couplings would appear to be very useful. Such an analysis has been developed and is the subject of this report.

In this report, equations of motion are derived that are suitable for use in studying both air and ground resonance as well as the hub-fixed dynamic behavior of helicopters with a bearingless main rotor. The blade is modeled as rigid but connected to the hub with a structurally flexible appendage to simulate the flexbeam portion of a bearingless rotor blade. The analysis is restricted to rigid-body fuselage motions and all forces that would act along the flexbeam are neglected. Only rotor cyclic modes and body pitch, roll, and horizontal translations are considered. Each blade has six degrees of freedom in the rotating reference frame. This means a total of sixteen degrees of freedom in the analytical model. The analysis is tailored to treat the configurations mentioned in references 5-9 in an approximate way

In section 2, the physical model is described. In the text, the configuration described in reference 9 is analyzed; modifications necessary to treat other configurations are given in the appendixes. In section 3, Lagrange's form of D'Alembert's principle is introduced as a means of deriving the equations of motion. The coordinate systems used in the derivation are defined in section 4 and expressions for certain kinematical quantities are derived. The generalized active and inertia forces are derived in sections 5 and 6. The equations of motion are formed, linearized about equilibrium, and then written in the fixed system in section 7. Particular attention is given to the flexbeam structural representation in this analysis. In the appendixes are found the details of including blade and body structural damping and the modifications to the analysis that are required to treat the different pitch-control geometries.

An independent derivation of the equations of motion was performed by Dr. Donald L. Kunz of the Aeromechanics Laboratory in order to check the present derivation. His effort is gratefully acknowledged.

2. PHYSICAL MODEL

In this section the physical model used to represent the helicopter is described. Only those elements believed necessary to model air and ground resonance phenomena are retained. The aircraft dynamical system is composed of two parts: the fuselage and the rotor. The fuselage is assumed to be a rigid body. When in contact with the ground, the fuselage is suspended by a spring system to simulate the elastic restraint to fuselage motion in an actual helicopter imposed by the landing gear system. When the aircraft is airborne in hovering flight, it is unrestrained elastically. The rotor consists of three or more rigid blades attached to the hub by means of slender elastic beam segments. Both the fuselage and rotor are described in more detail below.

A schematic of the fuselage/hub is shown in figure 1. The hub, mast, and landing gear are all included in the mass and inertias of the fuselage. The total fuselage mass is m_f and the moments of inertia for the mass center are I_x and I_y, respectively, for the X and Y directions. The aircraft reference center, shown in figure 1, is a distance z above the body mass center and a distance h below the hub center. In the study of air resonance (in hover) and ground resonance, vertical translation and yaw rotation of the body uncouple from the other body motions and are, thus, not significant. The other four body degrees of freedom X, Y, Φ_x, and Φ_y are included. These quantities are time integrals of velocity and angular velocity components in the body fixed axis system, described in more detail in section 4. The landing gear provides stiffness restraining the body motion depending on the landing gear geometry shown in figure 2.

The rotor blades are attached to the hub with flexible beam segments called flexbeams and rotate at constant angular velocity Ω. A schematic of one rotor blade is shown in figure 3; here, the details of the pitch control system are not shown

The blade pitch angle may be changed by twisting the flexbeam with any of three types of pitch control mechanisms or by using built-in pitch angles θ_f and θ_b. In this report four cases are considered. Case I is the simplest case, with no pitch control system at all. This configuration corresponds to an experimental rotor set up to operate only at discrete pitch angles or to an operational rotor with a disabled control system The equations of motion are developed for this case throughout the text. The modifications for Cases II to IV are relatively simple and are given in appendixes A-C, respectively. Case II, shown in figure 4, has only a flexible cable-like torque tube torsionally stiff enough to twist the flexbeam. The cable is assumed to be flexible enough in bending to put only a pure twisting moment on the tip of the flexbeam. This configuration corresponds to that of reference 9 and also, approximately, to the pinned-pinned torque tube of reference 5. Case III shown in figure 5 corresponds to configurations with a pitch link and cantilevered pitch arm described in references 6, 8, and 9 Pitch change is accompanied by bending deflections and vice versa, giving sometimes large aeroelastic couplings Case IV, shown in figure 6, is identical to Case III except that a snubber, intended to reduce aeroelastic couplings, is added The snubber is modeled as an additional nonfunctioning pitch-link/pitch-arm assembly. The pitch link is connected to the rotating swashplate and the snubber link to a point stationary in the rotating reference frame. All the pitch mechanism and snubber mechanism flexibility should be lumped into the pitch link and snubber link, respectively. The mass and inertia of these components should be included with the blade. Parameters for each configuration are discussed in the appendixes.

3. DESCRIPTION OF THE DERIVATION

The equations of motion are derived through an application of Lagrange's form of D'Alembert's principle (ref. 12). This principle reduces to Lagrange's equations when all the degrees of freedom are generalized coordinates. This is the case only for the blade degrees of freedom in this analysis. The body degrees of freedom are based on components of velocity and angular velocity in the body fixed axis system (fig. 1) and thus are not generalized coordinates. They are "quasi-coordinates" and require special considerations. The analyst may choose to use the special form of Lagrange's equations for quasi-coordinates (ref 13), Lagrange's form of D'Alembert's principle (ref. 12) or Newton's laws.

Kane (ref. 12) has summarized the laws of motion in the form of a single equation

$$F_r + F_r{}^* = 0 \qquad r = 1, 2, \ldots, n \tag{1}$$

where F_r are the generalized forces associated with gravity, springs, contact forces, and aerodynamics. The $F_r{}^*$ are the generalized inertia forces for the n degrees of freedom. The generalized forces F_r are defined for the above physical model as

$$F_r = \mathbf{F}_B \cdot \frac{\partial \mathbf{V}^P}{\partial \dot{q}_r} + \sum_{k=1}^{b} \mathbf{F}_k \cdot \frac{\partial \mathbf{V}^{Q_k}}{\partial \dot{q}_r} + \mathbf{M}_B \cdot \frac{\partial \omega^B}{\partial \dot{q}_r} + \sum_{k=1}^{b} \mathbf{M}_k \cdot \frac{\partial \omega^k}{\partial \dot{q}_r} \qquad r = 1, 2, \ldots, n \tag{2}$$

The vectors $\mathbf{F}_B$ and $\mathbf{M}_B$ are the forces and moments acting on the body at a certain point P. The vectors $\mathbf{F}_k$ and $\mathbf{M}_k$ are the forces and moments acting on the kth blade at a certain point Q_k. The velocities $\mathbf{V}^P$ and $\mathbf{V}^{Q_k}$ are defined at the point P and Q_k in an inertial reference frame. The angular velocities ω^B and ω^k are written for the body and for the kth blade, respectively, also with respect to an inertial reference frame. The degrees of freedom q_r are u_k, v_k, w_k, ζ_k, β_k, θ_k for $k = 1, 2, \ldots, b$, and X, Y, Φ_x, Φ_y.

Similarly the generalized inertia force $F_r{}^*$ is defined as

$$F_r{}^* = \mathbf{F}_B{}^* \cdot \frac{\partial \mathbf{V}^{B^*}}{\partial \dot{q}_r} + \sum_{k=1}^{b} \mathbf{F}_k{}^* \cdot \frac{\partial \mathbf{V}^{k^*}}{\partial \dot{q}_r} + \mathbf{M}_B{}^* \cdot \frac{\partial \omega^B}{\partial \dot{q}_r} + \sum_{k=1}^{b} \mathbf{M}_k{}^* \cdot \frac{\partial \omega^k}{\partial \dot{q}_r} \tag{3}$$

where the inertia forces are

$$\begin{aligned} \mathbf{F}_B{}^* &= -m_f \mathbf{a}^{B^*} \\ \mathbf{F}_k{}^* &= -m\mathbf{a}^{k^*} \end{aligned} \tag{4}$$

The accelerations are $\mathbf{a}^{B^*}$ of the body mass center and $\mathbf{a}^{k^*}$ of the kth blade mass center. The inertia moments are given according to Euler's dynamical equations, written in detail below. The velocities $\mathbf{V}^{B^*}$ and $\mathbf{V}^{k^*}$ are for the body and blade mass centers, respectively.

In section 5, the generalized active forces due to aerodynamics, gravity, body springs, the flexbeam structure, and structural damping are derived. In section 6, the generalized inertia forces are derived. Before proceeding with the actual derivation, however, it is necessary to describe the coordinate systems used in the report and to develop expressions for certain velocities and angular velocities needed in the derivation. This is done in section 4 immediately following.

4. COORDINATE SYSTEMS

As an inertial reference frame, the space-fixed axes e_1, e_2, and e_3 shown in figure 1 are chosen, e_3 being positive down. The velocity of the aircraft reference center is defined to be

$$\mathbf{V}^C = \dot{X}\mathbf{N}_A + \dot{Y}\mathbf{N}_B \tag{5}$$

where the unit vectors $\mathbf{N}_A$ and $\mathbf{N}_B$ are parallel to the body-fixed axes N_A and N_B of the N_A, N_B, N_C axis system also shown in figure 1. The angular velocity of the body is defined to be

$$\boldsymbol{\omega}^B = \dot{\Phi}_x\mathbf{N}_A + \dot{\Phi}_y\mathbf{N}_B \tag{6}$$

Note that X, Y, Φ_x, and Φ_y are not geometric distances and angles, they are, instead, quasi-coordinates (ref. 13). In this report, the quantities X, Y, Φ_x, and Φ_y are assumed to be infinitesimal Thus,

$$\begin{Bmatrix} \mathbf{N}_A \\ \mathbf{N}_B \\ \mathbf{N}_C \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\Phi_y \\ 0 & 1 & \Phi_x \\ \Phi_y & -\Phi_x & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} \tag{7}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are unit vectors parallel to e_1, e_2, and e_3, respectively.

The rotating axes $N_1{}^k$, $N_2{}^k$, and $N_3{}^k$ associated with the kth rotor blade are also shown in figure 1. The angle $\psi_k = \Omega t + 2\pi(k - 1)/b$. The associated unit vectors are related by the following transformation

$$\begin{Bmatrix} \mathbf{N}_1{}^k \\ \mathbf{N}_2{}^k \\ \mathbf{N}_3{}^k \end{Bmatrix} = \begin{bmatrix} -c_{\psi_k} & s_{\psi_k} & 0 \\ s_{\psi_k} & c_{\psi_k} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \mathbf{N}_A \\ \mathbf{N}_B \\ \mathbf{N}_C \end{Bmatrix} \tag{8}$$

The kth rotor blade is attached to the hub (at hub radius = e) by means of an elastic beam segment. This flexbeam is built in at point 0 (cantilevered) along the $n_1{}^k$, $n_2{}^k$, $n_3{}^k$ axes, shown in figure 7. The associated unit vectors are related by the following transformation constructed from a sequence of angular rotations ζ_f, β_f, and θ_f

$$\begin{Bmatrix} \mathbf{n}_1{}^k \\ \mathbf{n}_2{}^k \\ \mathbf{n}_3{}^k \end{Bmatrix} = \begin{bmatrix} c_{\beta_f}c_{\zeta_f} & c_{\beta_f}s_{\zeta_f} & s_{\beta_f} \\ -s_{\theta_f}s_{\beta_f}c_{\zeta_f} - c_{\theta_f}s_{\zeta_f} & c_{\theta_f}c_{\zeta_f} - s_{\zeta_f}s_{\beta_f}s_{\theta_f} & c_{\beta_f}s_{\theta_f} \\ -c_{\theta_f}s_{\beta_f}s_{\zeta_f} + s_{\theta_f}s_{\zeta_f} & -s_{\theta_f}c_{\zeta_f} - s_{\zeta_f}s_{\beta_f}c_{\theta_f} & c_{\beta_f}c_{\theta_f} \end{bmatrix} \begin{Bmatrix} \mathbf{N}_1{}^k \\ \mathbf{N}_2{}^k \\ \mathbf{N}_3{}^k \end{Bmatrix} = [F]\begin{Bmatrix} \mathbf{N}_1{}^k \\ \mathbf{N}_2{}^k \\ \mathbf{N}_3{}^k \end{Bmatrix} \tag{9}$$

The flexbeam of the kth rotor blade is capable of all six beam deformations at the tip. The vector of translations is $\boldsymbol{\delta} = u_k \mathbf{n}_1{}^k + v_k \mathbf{n}_2{}^k + w_k \mathbf{n}_3{}^k$ There are also three angular rotations, ζ_k, β_k, and θ_k, so that the flexbeam tip is along the $n_x{}^k$, $n_y{}^k$, $n_z{}^k$ axes shown in figure 7. The associated unit vectors are related by the following transformation constructed from a sequence of angular rotations ζ_k, β_k, θ_k:

$$\begin{Bmatrix} \mathbf{n}_x{}^k \\ \mathbf{n}_y{}^k \\ \mathbf{n}_z{}^k \end{Bmatrix} = \begin{bmatrix} c_{\beta_k} c_{\zeta_k} & c_{\beta_k} s_{\zeta_k} & s_{\beta_k} \\ -s_{\theta_k} s_{\beta_k} c_{\zeta_k} - c_{\theta_k} s_{\zeta_k} & c_{\theta_k} c_{\zeta_k} - s_{\zeta_k} s_{\beta_k} s_{\theta_k} & c_{\beta_k} s_{\theta_k} \\ -c_{\theta_k} s_{\beta_k} s_{\zeta_k} + s_{\theta_k} s_{\zeta_k} & -s_{\theta_k} c_{\zeta_k} - s_{\zeta_k} s_{\beta_k} c_{\theta_k} & c_{\beta_k} c_{\theta_k} \end{bmatrix} \begin{Bmatrix} \mathbf{n}_1{}^k \\ \mathbf{n}_2{}^k \\ \mathbf{n}_3{}^k \end{Bmatrix} = [T^k] \begin{Bmatrix} \mathbf{n}_1{}^k \\ \mathbf{n}_2{}^k \\ \mathbf{n}_3{}^k \end{Bmatrix} \tag{10}$$

The kth blade is built in at point J along the axes $N_x{}^k$, $N_y{}^k$, $N_z{}^k$ shown in figure 7. The associated unit vectors are related by the following transformation constructed from a sequence of angular rotations ζ_b, β_b, and θ_b

$$\begin{Bmatrix} \mathbf{N}_x{}^k \\ \mathbf{N}_y{}^k \\ \mathbf{N}_z{}^k \end{Bmatrix} = \begin{bmatrix} c_{\beta_b} c_{\zeta_b} & c_{\beta_b} s_{\zeta_b} & s_{\beta_b} \\ -s_{\theta_b} s_{\beta_b} c_{\zeta_b} - c_{\theta_b} s_{\zeta_b} & c_{\theta_b} c_{\zeta_b} - s_{\zeta_b} s_{\beta_b} s_{\theta_b} & c_{\beta_b} s_{\theta_b} \\ -c_{\theta_b} s_{\beta_b} s_{\zeta_b} + s_{\theta_b} s_{\zeta_b} & -s_{\theta_b} c_{\zeta_b} - s_{\zeta_b} s_{\beta_b} c_{\theta_b} & c_{\beta_b} c_{\theta_b} \end{bmatrix} \begin{Bmatrix} \mathbf{n}_x{}^k \\ \mathbf{n}_y{}^k \\ \mathbf{n}_z{}^k \end{Bmatrix} = [B] \begin{Bmatrix} \mathbf{n}_x{}^k \\ \mathbf{n}_y{}^k \\ \mathbf{n}_z{}^k \end{Bmatrix} \tag{11}$$

The $N_x{}^k$, $N_y{}^k$, $N_z{}^k$ axes are the blade-fixed axes. Note that all the blades, flexbeams, and angular offsets are identical to each other. This is reflected in the absence of the subscript (or superscript) k in the [F] and [B] matrices. In the undeformed state $[T^k] = [I]$, the identity matrix. The following additional matrices are needed in the derivation below

$$\left.\begin{aligned} [G^k] &= [B][T^k] \\ [H^k] &= [G^k][F] \\ [T] &= [T^k]\Big|_{\zeta_k=\zeta,\beta_k=\beta,\theta_k=\theta} \qquad \text{for} \quad k=1,2,\ldots,b \\ [G] &= [B][T] \\ [H] &= [G][F] \end{aligned}\right\} \tag{12}$$

Extensive use will be made of these matrices in defining velocities and angular velocities needed in the derivation.

The velocities of several points in the system and the angular velocities of the body and kth blade are now developed. The velocity of the aircraft reference center is, by definition

$$\mathbf{V}^C = \dot{X}\mathbf{N}_A + \dot{Y}\mathbf{N}_B \tag{5}$$

The angular velocities of the body and kth blade are, respectively,

$$\boldsymbol{\omega}^B = \dot{\Phi}_x\mathbf{N}_A + \dot{\Phi}_y\mathbf{N}_B \tag{6}$$

$$\boldsymbol{\omega}^k = \dot{\Phi}_x\mathbf{N}_A + \dot{\Phi}_y\mathbf{N}_B + \Omega\mathbf{N}_3 + \dot{\zeta}_k\mathbf{n}_3{}^k - \dot{\beta}_k(c_{\zeta_k}\mathbf{n}_2{}^k - s_{\zeta_k}\mathbf{n}_1{}^k) + \dot{\theta}_k\mathbf{n}_x{}^k \tag{13}$$

The sequence of blade angular rotations, ζ_k, β_k, and θ_k, is shown in figure 8 from which ω^k is determined by inspection. Thus, the velocity of the body center of mass is

$$\mathbf{V}^{B^*} = (\dot{X} + z\dot{\Phi}_y)\mathbf{N}_A + (\dot{Y} - z\dot{\Phi}_x)\mathbf{N}_B \tag{14}$$

The position vector of the kth blade center of mass from the aircraft reference center is

$$\mathbf{r}^{k^*} = h\mathbf{N}_3 + e\mathbf{N}_1{}^k + (\ell + u_k)\mathbf{n}_1{}^k + v_k\mathbf{n}_2{}^k + w_k\mathbf{n}_3{}^k + Lx_b\mathbf{N}_x{}^k + cx_c\mathbf{N}_y{}^k \tag{15}$$

The velocity of k^* may be established by well-known laws of kinematics

$$\mathbf{V}^{k^*} = \sum_{i=1}^{3} V_i{}^k\mathbf{n}_i{}^k \tag{16}$$

where

$$\left.\begin{aligned}
V_1{}^k &= {}^kV_1{}^h + (w_k + c_3{}^k)\,{}^k\omega_2{}^h - (v_k + c_2{}^k)\,{}^k\omega_3{}^h + \dot{u}_k + \dot{c}_1{}^k \\
V_2{}^k &= {}^kV_2{}^h + (u_k + c_1{}^k)\,{}^k\omega_3{}^h - (w_k + c_3{}^k)\,{}^k\omega_1{}^h + \dot{v}_k + \dot{c}_2{}^k \\
V_3{}^k &= {}^kV_3{}^h + (v_k + c_2{}^k)\,{}^k\omega_1{}^h - (u_k + c_1{}^k)\,{}^k\omega_2{}^h + \dot{w}_k + \dot{c}_3{}^k \\
{}^kV_i{}^h &= -[(\dot{X} - h\dot{\Phi}_y)c_{\psi_k} - (\dot{Y} + h\dot{\Phi}_x)s_{\psi_k}]F_{i1} + [(\dot{X} - h\dot{\Phi}_y)s_{\psi_k} \\
&\quad + (\dot{Y} + h\dot{\Phi}_x)c_{\psi_k}]F_{i2} \qquad i = 1,2,3 \\
{}^k\omega_i{}^h &= -(\dot{\Phi}_x c_{\psi_k} - \dot{\Phi}_y s_{\psi_k})F_{i1} + (\dot{\Phi}_x s_{\psi_k} + \dot{\Phi}_y c_{\psi_k})F_{i2} + \Omega F_{i3} \qquad i = 1,2,3 \\
c_i{}^k &= Lx_b G_{1i}^k + cx_c G_{2i}^k \qquad i = 1,2,3 \\
\dot{c}_i{}^k &= \frac{\partial c_i{}^k}{\partial \zeta_k}\dot{\zeta}_k + \frac{\partial c_i{}^k}{\partial \beta_k}\dot{\beta}_k + \frac{\partial c_i{}^k}{\partial \theta_k}\dot{\theta}_k \qquad i = 1,2,3
\end{aligned}\right\} \tag{17}$$

It is necessary to express the angular velocity of the kth blade in the N_x^k, N_y^k, N_z^k axis system

$$\boldsymbol{\omega}^k = \omega_1{}^k\mathbf{N}_x{}^k + \omega_2{}^k\mathbf{N}_y{}^k + \omega_3{}^k\mathbf{N}_z{}^k \tag{18}$$

where

$$\left.\begin{aligned} \omega_i{}^k &= \sum_{j=1}^{N} G_{ij}^k\,{}^k\omega_j{}^h + \dot{\zeta}_k G_{i3}^k - \dot{\beta}_k \eta_i{}^k + \dot{\theta}_k B_{i1} \\ \eta_i{}^k &= c_{\theta_k} B_{i2} - s_{\theta_k} B_{i3} = c_{\zeta_k} G_{i2}^k - s_{\zeta_k} G_{i1}^k \end{aligned}\right\} \tag{19}$$

Also, the velocity of the flexbeam tip J_k is needed. This is available by inspection from equations (17) with $c_i{}^k = 0$

$$\mathbf{V}^J = \sum_{i=1}^{3} {}^kV_i{}^J \mathbf{n}_i{}^k \tag{20}$$

where

$$\left.\begin{aligned} {}^kV_1{}^J &= {}^kV_1{}^h + \dot{u}_k + w_k{}^k\omega_2{}^h - v_k{}^k\omega_3{}^h \\ {}^kV_2{}^J &= {}^kV_2{}^h + \dot{v}_k + u_k{}^k\omega_3{}^h - w_k{}^k\omega_1{}^h \\ {}^kV_3{}^J &= {}^kV_3{}^h + \dot{w}_k + v_k{}^k\omega_1{}^h - u_k{}^k\omega_2{}^h \end{aligned}\right\} \tag{21}$$

Finally, in developing the aerodynamics, the velocities of all points along the blade N_x^k axis are needed. The blade is moving in space and, in addition, there is an induced flow field assumed to remain along the e_3 axis. The magnitude of the induced inflow velocity is taken from momentum theory (as in ref. 14); for simplicity, it is assumed to vary linearly with the radius.

$$v_i = \Omega(e + \ell + u + x)\phi \qquad 0 \le x \le L \tag{22}$$

where

$$\left.\begin{aligned} u_k &= u + \tilde{u}_k(t) \\ \phi &= \frac{\pi\sigma}{6}\left[\left(1 + \frac{12|\theta_{3/4}|}{\pi\sigma}\right)^{1/2} - 1\right]\operatorname{sgn}(\theta_{3/4}) \\ \theta_{3/4} &= H_{23} \end{aligned}\right\} \tag{23}$$

When the induced inflow is superposed on the blade velocity the result is

$$\mathbf{V}^k = \sum_{i=1}^{3} U_i^{\,k} \mathbf{n}_i^{\,k} \tag{24}$$

where

$$\left.\begin{aligned}
U_1^{\,k} &= {}^kU_1^{\,h} + \dot{u}_k + xG_{11}^k + (\ell\bar{w}_k + xG_{13}^k)\,{}^k\omega_2^{\,h} - (\ell\bar{v}_k + xG_{12}^k)\,{}^k\omega_3^{\,h} \\
U_2^{\,k} &= {}^kU_2^{\,h} + \dot{v}_k + xG_{12}^k + (\ell\bar{u}_k + xG_{11}^k)\,{}^k\omega_3^{\,h} - (\ell\bar{w}_k + xG_{13}^k)\,{}^k\omega_1^{\,h} \\
U_3^{\,k} &= {}^kU_3^{\,h} + \dot{w}_k + xG_{13}^k + (\ell\bar{v}_k + xG_{12}^k)\,{}^k\omega_1^{\,h} - (\ell\bar{u}_k + xG_{11}^k)\,{}^k\omega_2^{\,h} \\
{}^kU_i^{\,h} &= -\left[(\dot{X} - h\dot{\Phi}_y + v_i\Phi_y)c_{\psi_k} - (\dot{Y} + h\dot{\Phi}_x - v_i\Phi_x)s_{\psi_k}\right]F_{i1} \\
&\quad + \left[(\dot{X} - h\dot{\Phi}_y + v_i\Phi_y)s_{\psi_k} + (\dot{Y} + h\dot{\Phi}_x - v_i\Phi_x)c_{\psi_k}\right]F_{i2} \\
&\quad + v_iF_{i3} \qquad i = 1,2,3
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
\ell\bar{u}_k &= \ell + u_k + eF_{11} \\
\ell\bar{v}_k &= v_k + eF_{21} \\
\ell\bar{w}_k &= w_k + eF_{31}
\end{aligned}\right\} \tag{26}$$

5 DERIVATION OF GENERALIZED ACTIVE FORCES

In this section, the generalized forces due to aerodynamics, gravity, body springs, the flexbeam structure, and structural damping are derived Some details are omitted for the sake of brevity.

5.1 Aerodynamic Loads

The aerodynamic loads are derived from a quasi-steady version of Greenberg's equations (ref. 15) for lift and pitching moment. Also included is a quasi-steady profile drag contribution The circulatory lift per unit length is given by

$$L_C = \frac{\rho aVc}{2}\left[\dot{h} + V\varepsilon + \frac{c}{2}\,(1 - 2x_a)\dot{\varepsilon}\right] \tag{27}$$

The noncirculatory lift per unit length is

$$L_{NC} = \frac{\rho a c^2}{8}\left[\ddot{h} + \dot{V}\varepsilon + V\dot{\varepsilon} + \frac{c}{4}(1 - 4x_a)\ddot{\varepsilon}\right] \tag{28}$$

The pitching moment per unit length is

$$M = -\frac{\rho a c^3}{32}\left[2V\dot{\varepsilon}(1 - 2x_a) + \dot{V}\varepsilon(1 - 4x_a) + \ddot{h}(1 - 4x_a) + \left(c\,\frac{3}{8} - 2x_a + 4x_a^2\right)\ddot{\varepsilon}\right] + \frac{\rho a V c^2 x_a}{2}\left[\dot{h} + V\varepsilon + \frac{c}{2}(1 - 2x_a)\dot{\varepsilon}\right] \tag{29}$$

The air density is ρ and the free-stream velocity is V. The airfoil section is pitched at the angle ε with respect to the free-stream as shown in figure 9. In accordance with small disturbance airfoil theory, we may set

$$\left.\begin{aligned} U &= \sqrt{U_T^2 + U_P^2} \cong V \\ U_P &\cong -\dot{h} - V\varepsilon \end{aligned}\right\} \tag{30}$$

The blade airfoil velocity components are U_T and U_P, expressed in the N_y^k and N_z^k direction, respectively. Substitution of equations (30) into (27) to (29) yields

$$\left.\begin{aligned} L_C &= \frac{\rho a c U}{2}\left[-U_P + \frac{c}{2}(1 - 2x_a)\dot{\varepsilon}\right] \\ L_{NC} &= \frac{\rho a c^2}{8}\left[-\dot{U}_P + \frac{c}{4}(1 - 4x_a)\ddot{\varepsilon}\right] \end{aligned}\right\} \tag{31}$$

$$M = -\frac{\rho a c^2 x_a U U_P}{2} - \frac{\rho a c^3}{32}\left[U\dot{\varepsilon}(1 - 8x_a + 16x_a^2) - \dot{U}_P(1 - 4x_a) + \frac{3c}{8}\left(1 - \frac{16}{3}x_a + \frac{32}{3}x_a^2\right)\ddot{\varepsilon}\right] \tag{32}$$

Next, the total aerodynamic force per unit length on the blade airfoil section is considered The noncirculatory lift is taken to act in the direction normal to the chord line as shown in figure 10. The circulatory lift is taken normal to the resultant velocity U. An aerodynamic profile drag force per unit length D, acting parallel to the velocity resultant, is included, based on a constant profile drag coefficient c_{d_0}.

$$D = \frac{\rho c_{d_0} c U^2}{2} \tag{33}$$

The force components and directions as shown in figure 10 give the following expressions for S and T

$$\left.\begin{aligned} S &= L_C \sin \alpha - D \cos \alpha \\ T &= L_C \cos \alpha + L_{NC} + D \sin \alpha \end{aligned}\right\} \tag{34}$$

where, from figure 9

$$\left.\begin{aligned} \cos \alpha &= \frac{U_T}{U} \\ \sin \alpha &= -\frac{U_P}{U} \end{aligned}\right\} \tag{35}$$

Substitution of equations (31), (33), and (35) into equations (34) yields

$$\left.\begin{aligned} S &= \frac{\rho ac}{2}\left[U_P{}^2 - \frac{c}{2}(1 - 2x_a)U_P\dot{\varepsilon} - \frac{c_{d_0}}{a} UU_T\right] \\ T &= \frac{\rho ac}{2}\left[-U_PU_T + \frac{c}{2}(1 - 2x_a)U_T\dot{\varepsilon} - \frac{c}{4}\dot{U}_P + \left(\frac{c}{4}\right)^2(1 - 4x_a)\varepsilon - \frac{c_{d_0}}{a} UU_P\right] \end{aligned}\right\} \tag{36}$$

Since c_{d_0}/a is small with respect to unity and because the magnitude of the aerodynamic pitching moment is small, it is permissible to set $U = U_T$ in S, T, and M. The aerodynamic force per unit length acting along the blade is

$$\left(\frac{d\mathbf{F}_k}{dx}\right)_{aero} = S_k\mathbf{N}_y{}^k + T_k\mathbf{N}_z{}^k \tag{37}$$

The aerodynamic pitching moment per unit length is

$$\left(\frac{d\mathbf{M}_k}{dx}\right)_{aero} = M_k\mathbf{N}_x{}^k \tag{38}$$

Now, for the kth blade

$$\left.\begin{aligned} S_k &= \frac{\rho ac}{2}\left[U_P{}^{k^2} - \frac{c}{2}(1 - 2x_a)U_P{}^k\omega_1{}^k - dU_T{}^{k^2}\right] \\ T_k &= \frac{\rho ac}{2}\left[-U_P{}^kU_T{}^k - \frac{c}{4}\dot{U}_P{}^k + \frac{c}{2}(1 - 2x_a)U_T{}^k\omega_1{}^k + \left(\frac{c}{4}\right)^2(1 - 4x_a)\dot{\omega}_1{}^k\right] \\ M_k &= -\frac{\rho ac}{2}\left\{x_acU_P{}^kU_T{}^k + \left(\frac{c}{4}\right)^2\left[U_T{}^k(1 - 8x_a + 16x_a{}^2)\omega_1{}^k - (1 - 4x_a)\dot{U}_P{}^k \right.\right. \\ &\qquad \left.\left. + \frac{3c}{8}\left(1 - \frac{16}{3}x_a + \frac{32}{3}x_a{}^2\right)\dot{\omega}_1{}^k\right]\right\} \end{aligned}\right\} \tag{39}$$

where $d = c_{d_0}/a$ has been neglected with respect to unity, and $\dot{\varepsilon}$ has been replaced with the component of angular velocity of the blade in the $N_x{}^k$ direction $\omega_1{}^k$. The velocity components $U_P{}^k$ and $U_T{}^k$ are given by

$$\left.\begin{aligned} U_P{}^k &= \sum_{i=1}^{3} U_i{}^k G_{3i}^k \\ U_T{}^k &= \sum_{i=1}^{3} U_i{}^k G_{2i}^k \end{aligned}\right\} \tag{40}$$

where the velocity components $U_i{}^k$ are defined in equations (25). The aerodynamic force and moment resultants at the point J are given by

$$\left.\begin{aligned} \left(\mathbf{F}_k\right)_{aero} &= S_f{}^k \mathbf{N}_y{}^k + T_f{}^k \mathbf{N}_z{}^k \\ \left(\mathbf{M}_k\right)_{aero} &= M_m{}^k \mathbf{N}_x{}^k - T_m{}^k \mathbf{N}_y{}^k + S_m{}^k \mathbf{N}_z{}^k \end{aligned}\right\} \tag{41}$$

where

$$\left.\begin{aligned} S_f{}^k &= \int_0^L S_k \, dx \qquad S_m{}^k = \int_0^L x S_k \, dx \\ T_f{}^k &= \int_0^L T_k \, dx \qquad T_m{}^k = \int_0^L x T_k \, dx \\ M_m{}^k &= \int_0^L M_k \, dx \end{aligned}\right\} \tag{42}$$

The generalized forces due to aerodynamics now follow immediately from equations (2) and (4)

$$\left.\begin{aligned} \left(F_{u_k}\right)_{aero} &= \left(\mathbf{F}_k\right)_{aero} \cdot \mathbf{n}_1{}^k \\ \left(F_{v_k}\right)_{aero} &= \left(\mathbf{F}_k\right)_{aero} \cdot \mathbf{n}_2{}^k \\ \left(F_{w_k}\right)_{aero} &= \left(\mathbf{F}_k\right)_{aero} \cdot \mathbf{n}_3{}^k \\ \left(F_{\zeta_k}\right)_{aero} &= \left(\mathbf{M}_k\right)_{aero} \cdot \mathbf{n}_3{}^k \\ \left(F_{\beta_k}\right)_{aero} &= \left(\mathbf{M}_k\right)_{aero} \cdot \left(c_{\zeta_k} \mathbf{n}_2{}^k - s_{\zeta_k} \mathbf{n}_1{}^k\right) \\ \left(F_{\theta_k}\right)_{aero} &= \left(\mathbf{M}_k\right)_{aero} \cdot \mathbf{n}_x{}^k \end{aligned}\right\} k = 1, 2, \ldots, b \tag{43}$$

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OF POOR QUALITY

$$\left(F_x\right)_{aero} = \sum_{k=1}^{b} \left(\mathbf{F}_k\right)_{aero} \cdot \mathbf{N}_A$$

$$\left(F_y\right)_{aero} = \sum_{k=1}^{b} \left(\mathbf{F}_k\right)_{aero} \cdot \mathbf{N}_B$$

$$\begin{aligned}\left(F_{\Phi_x}\right)_{aero} = \sum_{k=1}^{b} \Bigg\langle \left(\mathbf{F}_k\right)_{aero} \cdot \Bigg\{ & h \sum_{i=1}^{3} \left(s_{\psi_k} F_{i1} + c_{\psi_k} F_{i2}\right) \mathbf{n}_i^k \\ & + \left[\ell\bar{v}_k\left(c_{\psi_k} F_{31} - s_{\psi_k} F_{32}\right) - \ell\bar{w}_k\left(c_{\psi_k} F_{21} - s_{\psi_k} F_{22}\right)\right] \mathbf{n}_1^k \\ & + \left[\ell\bar{w}_k\left(c_{\psi_k} F_{11} - s_{\psi_k} F_{12}\right) - \ell\bar{u}_k\left(c_{\psi_k} F_{31} - s_{\psi_k} F_{32}\right)\right] \mathbf{n}_2^k \\ & + \left[\ell\bar{u}_k\left(c_{\psi_k} F_{21} - s_{\psi_k} F_{22}\right) - \ell\bar{v}_k\left(c_{\psi_k} F_{11} - s_{\psi_k} F_{12}\right)\right] \mathbf{n}_3^k \Bigg\} \\ & + \left(\mathbf{M}_k\right)_{aero} \cdot \mathbf{N}_A \Bigg\rangle\end{aligned}$$

$$\begin{aligned}\left(F_{\Phi_y}\right)_{aero} = \sum_{k=1}^{b} \Bigg\langle \left(\mathbf{F}_k\right)_{aero} \cdot \Bigg\{ & h \sum_{i=1}^{3} \left(c_{\psi_k} F_{i1} - s_{\psi_k} F_{i2}\right) \mathbf{n}_i^k \\ & + \left[\ell\bar{w}_k\left(s_{\psi_k} F_{21} + c_{\psi_k} F_{22}\right) - \ell\bar{v}_k\left(s_{\psi_k} F_{31} + c_{\psi_k} F_{32}\right)\right] \mathbf{n}_1^k \\ & + \left[\ell\bar{u}_k\left(s_{\psi_k} F_{31} + c_{\psi_k} F_{32}\right) - \ell\bar{w}_k\left(s_{\psi_k} F_{11} + c_{\psi_k} F_{12}\right)\right] \mathbf{n}_2^k \\ & + \left[\ell\bar{v}_k\left(s_{\psi_k} F_{11} + c_{\psi_k} F_{12}\right) - \ell\bar{u}_k\left(s_{\psi_k} F_{21} + c_{\psi_k} F_{22}\right)\right] \mathbf{n}_3^k \Bigg\} \\ & + \left(\mathbf{M}_k\right)_{aero} \cdot \mathbf{N}_B \Bigg\rangle\end{aligned}$$

(43) concluded

Equations (43) define the contribution of aerodynamics to the generalized forces.

5.2 Gravity

In this section the gravitational contribution to the generalized forces is expressed. For a system of rigid bodies, gravity applies a force acting through the mass center of each component. For the body

$$(\mathbf{F}_B)_g = m_f g\ \mathbf{e}_3 \tag{44}$$

and similarly, for the kth blade

$$(\mathbf{F}_k)_g = mg\ \mathbf{e}_3 \tag{45}$$

The subscript g refers to gravity. The symbol g is the rotor thrust per unit mass of the aircraft ($\mathscr{T}$/M) if the aircraft is airborne. When the aircraft is in ground contact, the analysis must be slightly modified. This is done at the end of section 5.3.

The generalized forces due to gravity follow immediately from equation (2):

$$\left.\begin{aligned}
&\left(F_{u_k}\right)_g = mg\ \mathbf{e}_3 \cdot \mathbf{n}_1{}^k \qquad\qquad \left(F_{\zeta_k}\right)_g = mg\ \mathbf{e}_3 \cdot \sum_{i=1}^{3} \frac{\partial c_i{}^k}{\partial \zeta_k}\,\mathbf{n}_i{}^k \\
&\left(F_{v_k}\right)_g = mg\ \mathbf{e}_3 \cdot \mathbf{n}_2{}^k \\
&\left(F_{w_k}\right)_g = mg\ \mathbf{e}_3 \cdot \mathbf{n}_3{}^k \qquad\qquad \left(F_{\beta_k}\right)_g = mg\ \mathbf{e}_3 \cdot \sum_{i=1}^{3} \frac{\partial c_i{}^k}{\partial \beta_k}\,\mathbf{n}_i{}^k \\
&(F_x)_g = Mg\ \mathbf{e}_3 \cdot \mathbf{N}_A \\
&\qquad\qquad\qquad\qquad\qquad\quad \left(F_{\theta_k}\right)_g = mg\ \mathbf{e}_3 \cdot \sum_{i=1}^{3} \frac{\partial c_i{}^k}{\partial \theta_k}\,\mathbf{n}_i{}^k \\
&(F_y)_g = Mg\ \mathbf{e}_3 \cdot \mathbf{N}_B \\
&\left(F_{\Phi_x}\right)_g = (mgbh - m_f gz)\mathbf{e}_3 \cdot \mathbf{N}_B - mg\ \mathbf{e}_3 \cdot \sum_{k=1}^{b} \Bigg\{ \bigg[\left(\ell\bar{w}_k + c_3{}^k\right)\left(c_{\psi_k}F_{21} - s_{\psi_k}F_{22}\right) \\
&\qquad - \left(\ell\bar{v}_k + c_2{}^k\right)\left(c_{\psi_k}F_{31} - s_{\psi_k}F_{32}\right)\bigg]\mathbf{n}_1{}^k + \bigg[\left(\ell\bar{u}_k + c_1{}^k\right)\left(c_{\psi_k}F_{31} - s_{\psi_k}F_{32}\right) \\
&\qquad - \left(\ell\bar{v}_k + c_3{}^k\right)\left(c_{\psi_k}F_{11} - s_{\psi_k}F_{12}\right)\bigg]\mathbf{n}_2{}^k + \bigg[\left(\ell\bar{v}_k + c_2{}^k\right)\left(c_{\psi_k}F_{11} - s_{\psi_k}F_{12}\right) \\
&\qquad - \left(\ell\bar{u}_k + c_1{}^k\right)\left(c_{\psi_k}F_{21} - s_{\psi_k}F_{22}\right)\bigg]\mathbf{n}_3{}^k\Bigg\} \\
&\left(F_{\Phi_y}\right)_g = -(mgbh - m_f gz)\mathbf{e}_3 \cdot \mathbf{N}_A + mg\ \mathbf{e}_3 \cdot \sum_{k=1}^{b} \Bigg\{ \bigg[\left(w_k + c_3{}^k\right)\left(s_{\psi_k}F_{21} + c_{\psi_k}F_{22}\right) \\
&\qquad - \left(v_k + c_2{}^k\right)\left(s_{\psi_k}F_{31} + c_{\psi_k}F_{32}\right)\bigg]\mathbf{n}_1 + \bigg[\left(u_k + c_1{}^k\right)\left(s_{\psi_k}F_{31} + c_{\psi_k}F_{32}\right) \\
&\qquad - \left(w_k + c_3{}^k\right)\left(s_{\psi_k}F_{11} + c_{\psi_k}F_{12}\right)\bigg]\mathbf{n}_2 + \bigg[\left(v_k + c_2{}^k\right)\left(s_{\psi_k}F_{11} + c_{\psi_k}F_{12}\right) \\
&\qquad - \left(u_k + c_1{}^k\right)\left(s_{\psi_k}F_{21} + c_{\psi_k}F_{22}\right)\bigg]\mathbf{n}_3\Bigg\}
\end{aligned}\right\} \tag{46}$$

Equations (46) define the generalized forces due to gravity when the aircraft is airborne. If the aircraft is hovering at thrust = weight, g becomes the acceleration of gravity g. When the aircraft is on the ground, the body forces F_x and F_y need slight modification. This modification is closely connected with the body spring forces treated in the next section.

5.3 Body Springs

When the aircraft is in ground contact, a system of springs is introduced into the mathematical model to account for restraint of fuselage motion due to the landing gear flexibility. These springs, shown in figure 2, are symmetric about the aircraft reference center both in the longitudinal and lateral directions In deriving the generalized forces, the analysis will be carried out in only one plane, the e_1, e_3 plane and the results extrapolated to the total system.

Consider the two-dimensional system of figure 11. The velocities of certain points are needed in order to derive the generalized forces. The reference center velocity is

$$\mathbf{V}^C = \dot{X}\mathbf{N}_A + \dot{Z}\mathbf{N}_C \tag{47}$$

and the body angular velocity is

$$\omega^B = \dot{\Phi}_y\mathbf{N}_B \tag{48}$$

The velocity of points A_1 and A_2 is, respectively,

$$\left.\begin{aligned}\mathbf{V}^{A_1} &= (\dot{X} + \ell_z\dot{\Phi}_y)\mathbf{N}_A + \left(\dot{Z} + \frac{\ell_x}{2}\,\dot{\Phi}_y\right)\mathbf{N}_C \\ \mathbf{V}^{A_2} &= (\dot{X} + \ell_z\dot{\Phi}_y)\mathbf{N}_A + \left(\dot{Z} - \frac{\ell_x}{2}\,\dot{\Phi}_y\right)\mathbf{N}_C\end{aligned}\right. \tag{49}$$

The force applied by the springs at each of the attachment points is

$$\left.\begin{aligned}\mathbf{F}_{A_1} &= -2K_x(e_1 + \ell_z\Phi_y)\mathbf{e}_1 - 2K_z\left(e_3 + \frac{\ell_x}{2}\,\Phi_y\right)\mathbf{e}_3 \\ \mathbf{F}_{A_2} &= -2K_x(e_1 + \ell_z\Phi_y)\mathbf{e}_1 - 2K_z\left(e_3 - \frac{\ell_x}{2}\,\Phi_y\right)\mathbf{e}_3\end{aligned}\right\} \tag{50}$$

where e_1 and e_3 are the space-fixed deflections of the points A_1 and A_2 along the e_1 and e_2 axes. The vertical degree of freedom is retained only to aid in modifying the gravity terms in F_x and F_y for the ground contact case. When the aircraft is on the ground, there is a steady vertical deflection of the ground springs that varies with rotor thrust and gross weight. Thus, for first order in X and Φ_y the thrust $\mathscr{T}$ and weight Mg_0 must be included for this case. Each of these forces may be assumed to act at the reference center for the purpose of calculating spring forces.

$$\mathbf{F}_{B^*} = -\mathscr{T}\mathbf{N}_C + Mg_0\,\mathbf{e}_3 \tag{51}$$

From equation (2)

$$\begin{aligned} F_x &= -4K_x(e_1 + \ell_z\Phi_y) + 4K_z\,e_3\Phi_y - Mg_0\Phi_y \\ F_z &= -4K_z\,e_3 - \mathscr{T} + Mg_0 \\ F_{\Phi_y} &= -(4K_x\ell_z^2 + K_z\ell_x^2)\Phi_y - 4K_x\ell_z\,e_1 \end{aligned} \tag{52}$$

For the equilibrium of the coupled rotor-body system (all degrees of freedom = constant), $F_z = 0$. Therefore,

$$4K_z e_3 = Mg_0 - \mathscr{T} \tag{53}$$

Also, the deflections along e_1 and e_3 are identical to X and Z for infinitesimal Φ_y. Thus, for X and Φ_y degrees of freedom

$$\left.\begin{aligned} (F_x)_{spr} &= -4K_x(X + \ell_z\Phi_y) + (Mg_0 - \mathscr{T})\Phi_y - Mg_0\Phi_y \\ &= -4K_x(X + \ell_z\Phi_y) - \mathscr{T}\Phi_y \\ \left(F_{\Phi_y}\right)_{spr} &= -(4K_x\ell_z^2 + K_z\ell_x^2)\Phi_y - 4K_x\ell_z X \end{aligned}\right\} \tag{54}$$

Similarly, for Y and Φ_x degrees of freedom

$$\begin{aligned} (F_y)_{spr} &= -4K_y(Y - \ell_z\Phi_x) + \mathscr{T}\Phi_x \\ \left(F_{\Phi_x}\right)_{spr} &= -(4K_y\ell_z^2 + K_z\ell_y^2)\Phi_x + 4K_y\ell_z Y \end{aligned} \tag{55}$$

Equations (54) and (55) express the generalized forces due to body springs. The gravitational terms in F_x and F_y are replaced by rotor thrust terms when the aircraft is in ground contact. This is equivalent to replacing g in F_x and F_y with g^* where

$$g^* \equiv \frac{\mathscr{T}}{M} \tag{56}$$

when the aircraft is on the ground and replacing g in all other terms with g_0.

5.4 Flexbeam Structure

The structural loads exerted by the flexbeam are represented symbolically by the flexbeam stiffness matrix [K]. The matrix [K] is determined numerically by perturbing the equilibrium solution. This operation is described in section 7.2 below. With the stiffness matrix formulation, the generalized forces are

$$\left(F_{q_i^k}\right)_{flex} = -\sum_{j=1}^{6} K_{ij} q_j^k \qquad q_j^k = u_k, v_k, w_k, \zeta_k, \beta_k, \theta_k, \quad k = 1,2,\ldots,b \tag{57}$$

The fact that all the b flexbeams are identical is reflected in the lack of dependence of the [K] matrix on k.

When $\bar{\Omega} = 0$ and the flexbeam is undeformed, the matrix [K] is given by

$$[K] = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_c}{\ell^3} & 0 & \frac{-6EI_c}{\ell^2} & 0 & 0 \\ & & \frac{12EI_f}{\ell^3} & 0 & \frac{-6EI_f}{\ell^2} & 0 \\ & & & \frac{4EI_c}{\ell} & 0 & 0 \\ & & & & \frac{4EI_f}{\ell} & 0 \\ & & & & & \frac{GJ}{\ell} \end{bmatrix} \tag{58}$$

where EA, EI_c, EI_f, and GJ are the flexbeam axial stiffness, chordwise bending stiffness, flapwise bending stiffness, and torsional stiffness, respectively. When the rotor is at some general operating condition, the matrix [K] fully couples the blade equations structurally.

5.5 Structural Damping

In the coupled rotor-body system only the blade lead-lag motion and some body modes have small enough damping from aerodynamics to be significantly affected by a small amount of structural damping. The non-zero generalized forces due to structural damping are

$$\left.\begin{aligned}
\left(F_{\zeta_k}\right)_{damp} &= -c_\zeta \dot{\zeta}_k \\
\left(F_x\right)_{damp} &= -c_X \dot{X} \\
\left(F_y\right)_{damp} &= -c_Y \dot{Y} \\
\left(F_{\Phi_x}\right)_{damp} &= -c_{\Phi_x} \dot{\Phi}_x \\
\left(F_{\Phi_y}\right)_{damp} &= -c_{\Phi_y} \dot{\Phi}_y
\end{aligned}\right\} \tag{59}$$

The explicit form of c_ζ, c_X, c_Y, c_{Φ_x}, c_{Φ_y} is derived in appendix D in terms of damping ratios for both blade and body motion.

This concludes the development of the generalized active forces. In the next section the analysis continues with the generalized inertia forces.

6. DERIVATION OF GENERALIZED INERTIA FORCES

In this section, the generalized inertia forces due to body motion and blade motion are derived. As in the last section some details are omitted for the sake of brevity.

6.1 Motion of the kth Blade

Generalized inertia forces that are associated with generalized coordinates, as the blade degrees of freedom are, may be derived directly from the kinetic energy. For the rotor system the kinetic energy may be expressed as

$$T = \frac{m}{2} \sum_{k=1}^{b} \mathbf{v}^{k^*} \cdot \mathbf{v}^{k^*} + \frac{1}{2} \sum_{i=1}^{3} \sum_{k=1}^{b} I_i {\omega_i^k}^2 \tag{60}$$

Once this quantity has been expressed in the system degrees of freedom, the generalized forces for the blade degrees of freedom are given by

$$F^*_{q_r{}^k} = \frac{\partial T}{\partial q_r{}^k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r{}^k}\right) \qquad \begin{aligned} q_r{}^k &= u_k, v_k, w_k, \zeta_k, \beta_k, \theta_k \\ k &= 1, 2, \ldots, b \end{aligned} \tag{61}$$

This operation is straightforward and the details are not given here.

6.2 Motion of the Body

Unfortunately, the generalized inertia forces associated with body motion cannot be derived from equations (61). This is because the degrees of freedom describing body motion are not generalized coordinates. They are, instead, quasi-coordinates and require special considerations. The portions of body generalized inertia forces that are linear in blade variables are already known because of symmetry considerations and equations (61) Hence, only terms linear in X, Y, Φ_x, and Φ_y and their time derivatives need to be retained in this section.

The accelerations of B^* and k^* used in equation (4) to define the inertia forces are needed. These may be derived from equations (5) to (17) by standard laws of kinematics

$$\left.\begin{aligned}
\mathbf{a}^{B^*} &= (X + z\Phi_y)\mathbf{N}_A + (Y - z\Phi_x)\mathbf{N}_B + \ldots \\
\mathbf{a}^{k^*} &= \left[X - \left(h + \sum_{i=1}^{3} d_i^{\,k}F_{i3}\right)\Phi_y\right]\mathbf{N}_A + \left[Y + \left(h + \sum_{i=1}^{3} d_i^{\,k}F_{i3}\right)\Phi_x\right]\mathbf{N}_B \\
&\quad + \left[\left(\Phi_x - 2\Omega\dot{\Phi}_y\right) \sum_{i=1}^{3} d_i^{\,k}\left(F_{i1}s_{\psi_k} + F_{i2}c_{\psi_k}\right)\right. \\
&\quad \left. + \left(\Phi_y + 2\Omega\dot{\Phi}_x\right) \sum_{i=1}^{3} d_i^{\,k}\left(F_{i1}c_{\psi_k} - F_{i2}s_{\psi_k}\right)\right]\mathbf{N}_C + \ldots
\end{aligned}\right\} \quad (62)$$

where

$$\left.\begin{aligned}
d_1^{\,k} &= \ell\bar{u}_k + c_1^{\,k} \\
d_2^{\,k} &= \ell\bar{v}_k + c_2^{\,k} \\
d_3^{\,k} &= \ell\bar{w}_k + c_3^{\,k}
\end{aligned}\right\} \quad (63)$$

Here, the dots refer to blade terms and nonlinear terms in X, Y, Φ_x, and Φ_y and their time derivatives. Also, in equation (3) the inertial moments must be written. These may be expressed directly from Euler's dynamical relations and the angular velocities in equations (6)

$$\left.\begin{aligned}
\mathbf{M}_B^{\,*} &= -I_x\Phi_x\mathbf{N}_A - I_y\Phi_y\mathbf{N}_B + \ldots \\
\mathbf{M}_k^{\,*} &= [\omega_2^{\,k}\omega_3^{\,k}(I_2 - I_3) - \dot{\omega}_1^{\,k}I_1]\mathbf{N}_x^{\,k} \\
&\quad + [\omega_3^{\,k}\omega_1^{\,k}(I_3 - I_1) - \dot{\omega}_2^{\,k}I_2]\mathbf{N}_y^{\,k} \\
&\quad + [\omega_1^{\,k}\omega_2^{\,k}(I_1 - I_2) - \dot{\omega}_3^{\,k}I_3]\mathbf{N}_z^{\,k}
\end{aligned}\right\} \quad (64)$$

Now the body terms of the generalized inertia forces for X, Y, Φ_x, and Φ_y may be written from equations (3), (4), (62), and (64)

$$\left.\begin{aligned}
F_x^* &= \mathbf{N}_A \cdot \left(\mathbf{F}_B^* + \sum_{k=1}^{b} \mathbf{F}_k^*\right) + \; . \; . \\
F_y^* &= \mathbf{N}_B \cdot \left(\mathbf{F}_B^* + \sum_{k=1}^{b} \mathbf{F}_k^*\right) + \; . \; . \\
F_{\Phi_x}^* &= -z\mathbf{N}_B \cdot \mathbf{F}_B^* + \mathbf{N}_A \cdot \left(\mathbf{M}_B^* + \sum_{k=1}^{b} \mathbf{M}_k^*\right) \\
&\quad + \sum_{k=1}^{b} \left[\left(h + \sum_{i=1}^{3} d_i{}^k F_{i3}\right)\mathbf{N}_B + \sum_{i=1}^{3} d_i{}^k\left(F_{i1} s_{\psi_k} + F_{i2} c_{\psi_k}\right)\mathbf{N}_C\right] \cdot \mathbf{F}_k^* \\
F_{\Phi_y}^* &= z\mathbf{N}_A \cdot \mathbf{F}_B^* + \mathbf{N}_B \cdot \left(\mathbf{M}_B^* + \sum_{k=1}^{b} \mathbf{M}_k^*\right) \\
&\quad + \sum_{k=1}^{b} \left[-\left(h + \sum_{i=1}^{3} d_i{}^k F_{i3}\right)\mathbf{N}_A + \sum_{i=1}^{3} d_i{}^k\left(F_{i1} c_{\psi_k} - F_{i2} s_{\psi_k}\right)\mathbf{N}_C\right] \cdot \mathbf{F}_k^*
\end{aligned}\right\} \quad (65)$$

Equations (65) are the body terms of the generalized inertia forces for the body degrees of freedom. In the next section, the equations of motion are written from equation (1).

7. FORMATION OF THE EQUATIONS OF MOTION

The equations of motion are to be written from equation (1) using all the components of the generalized forces in sections 5 and 6. The ultimate purpose of this analysis is to provide a means of assessing the linear stability of small motions about equilibrium. In this section the equilibrium solution is discussed separately before the final linearized perturbation equations are written. Because from the outset the blade and flexbeam properties have been assumed identical for all blades, the equilibrium values of u_k, v_k, w_k, ζ_k, β_k, θ_k are identical constants for all k. Thus, the blade degrees of freedom have steady and oscillatory components

$$\left.\begin{aligned}
u_k &= u + \tilde{u}_k(t) \; ; & \zeta_k &= \zeta + \tilde{\zeta}_k(t) \\
v_k &= v + \tilde{v}_k(t) \; ; & \beta_k &= \beta + \tilde{\beta}_k(t) \\
w_k &= w + \tilde{w}_k(t) \; ; & \theta_k &= \theta + \tilde{\theta}_k(t)
\end{aligned}\right\} \quad (66)$$

Recall that the equilibrium values of X, Y, Φ_x, and Φ_y are zero so that

$$\left.\begin{aligned} X &= \tilde{X}(t) \\ Y &= \tilde{Y}(t) \\ \Phi_x &= \tilde{\Phi}_x(t) \\ \Phi_y &= \tilde{\Phi}_y(t) \end{aligned}\right\} \tag{67}$$

The tilde refers to an infinitesimal perturbation motion. Collection of terms from sections 5 and 6 yields a set of relations like the following

$$\left.\begin{aligned} F_{u_k} + F^*_{u_k} &= F_u - \tilde{F}_{u_k}(t) = 0 \\ F_{v_k} + F^*_{v_k} &= F_v - \tilde{F}_{v_k}(t) = 0 \\ F_{w_k} + F^*_{w_k} &= F_w - \tilde{F}_{w_k}(t) = 0 \\ F_{\zeta_k} + F^*_{\zeta_k} &= F_\zeta - \tilde{F}_{\zeta_k}(t) = 0 \\ F_{\beta_k} + F^*_{\beta_k} &= F_\beta - \tilde{F}_{\beta_k}(t) = 0 \\ F_{\theta_k} + F^*_{\theta_k} &= F_\theta - \tilde{F}_{\theta_k}(t) = 0 \\ F_x + F_x{}^* &= -\frac{b}{2}\tilde{F}_x(t) = 0 \\ F_y + F_y{}^* &= -\frac{b}{2}\tilde{F}_y(t) = 0 \\ F_{\Phi_x} + F^*_{\Phi_x} &= -\frac{b}{2}\tilde{F}_{\Phi_x}(t) = 0 \\ F_{\Phi_y} + F^*_{\Phi_y} &= -\frac{b}{2}\tilde{F}_{\Phi_y}(t) = 0 \end{aligned}\right\} \tag{68}$$

Here all perturbation quantities have been linearized in $\tilde{u}_k$, $\tilde{v}_k$, $\tilde{w}_k$, $\tilde{\zeta}_k$, $\tilde{\beta}_k$, $\tilde{\theta}_k$, $\tilde{X}$, $\tilde{Y}$, $\tilde{\Phi}_x$, and $\tilde{\Phi}_y$ and their first two time derivatives Below, the rotor equilibrium solution is obtained by performing an iterative static structural analysis for the flexbeam. The flexbeam structural stiffness matrix is next obtained from a numerical perturbation of the flexbeam equilibrium position. The perturbation equations, although linearized in all the perturbation degrees of freedom, contain terms with periodic coefficients (s_{ψ_k}, c_{ψ_k}). In order to most efficiently solve the system of linear, ordinary differential equations, the system is transformed to fixed (nonrotating) coordinates by the so-called multiblade coordinate transformation (ref. 16).

7.1 Equilibrium Solution for the Flexbeam

The equilibrium generalized forces F_u, F_v, F_w, F_ζ, F_β, and F_θ are, physically, the components from aerodynamics, gravity, blade inertial loads, and flexbeam structural loads, of force and moments acting at the tip of the flexbeam. Explicit expressions for these generalized forces are known except that expressions for the flexbeam structural loads are unknown. This means that all loads external to the flexbeam are known explicitly in terms of the deflections of the tip. When the details of the generalized forces are carried out from sections 5 and 6, equations (43), (46), and (60), the flexbeam tip external force $\mathbf{F}_t$ and moment $\mathbf{M}_t$ are given by

$$\left.\begin{aligned}
\mathbf{F}_t &= \bar{F}_u \mathbf{n}_1 + \bar{F}_v \mathbf{n}_2 + \bar{F}_w \mathbf{n}_3 \\
\mathbf{M}_t &= \sum_{i=1}^{3} M_i \mathbf{n}_i \\
M_1 &= -\frac{c_\zeta s_\beta}{c_\beta} \bar{F}_\zeta + s_\zeta \bar{F}_\beta + \frac{c_\zeta}{c_\beta} \bar{F}_\theta \\
M_2 &= -\frac{s_\zeta s_\beta}{c_\beta} \bar{F}_\zeta - c_\zeta \bar{F}_\beta + \frac{s_\zeta}{c_\beta} \bar{F}_\theta \\
M_3 &= \bar{F}_\zeta
\end{aligned}\right\} \tag{69}$$

The bars over $\bar{F}_u$, $\bar{F}_v$, $\bar{F}_w$, $\bar{F}_\zeta$, $\bar{F}_\beta$, $\bar{F}_\theta$ indicate that these are external forces and moments, excluding the structural part. These quantities are listed below in nondimensional form. Forces are nondimensionalized with respect to $I\Omega_0^2/\ell$ and moments by $I\Omega_0^2$.

$$\left.\begin{aligned}
\bar{F}_u &= \bar{m}\bar{\Omega}^2\bar{\ell}[\bar{\ell}\bar{u} + \bar{c}_1 - F_{13}(J_{33} - \bar{h})] - \bar{m}\bar{g}\bar{\ell}F_{13} + \bar{\ell}(S_{f_0}G_{21} + T_{f_0}G_{31}) \\
\bar{F}_v &= \bar{m}\bar{\Omega}^2\bar{\ell}[\bar{\ell}\bar{v} + \bar{c}_2 - F_{23}(J_{33} - \bar{h})] - \bar{m}\bar{g}\bar{\ell}F_{23} + \bar{\ell}(S_{f_0}G_{22} + T_{f_0}G_{32}) \\
\bar{F}_w &= \bar{m}\bar{\Omega}^2\bar{\ell}[\bar{\ell}\bar{w} + \bar{c}_3 - F_{33}(J_{33} - \bar{h})] - \bar{m}\bar{g}\bar{\ell}F_{33} + \bar{\ell}(S_{f_0}G_{22} + T_{f_0}G_{32}) \\
\bar{F}_\zeta &= \bar{\Omega}^2 \sum_{i=1}^{3} \bar{I}_i H_{i3} \frac{\partial H_{i3}}{\partial \zeta} + \bar{m}\bar{\Omega}^2 \left[(\bar{\ell}\bar{u} + \bar{c}_1) \frac{\partial \bar{c}_1}{\partial \zeta} + (\bar{\ell}\bar{v} + \bar{c}_2) \frac{\partial \bar{c}_2}{\partial \zeta} \right. \\
&\quad \left. + (\bar{\ell}\bar{w} + \bar{c}_3) \frac{\partial \bar{c}_3}{\partial \zeta} - \frac{\partial J_{33}}{\partial \zeta} (J_{33} - \bar{h}) \right] - \bar{m}\bar{g} \frac{\partial J_{33}}{\partial \zeta} + G_{13}M_{m_0} - G_{23}T_{m_0} + G_{33}S_{m_0}
\end{aligned}\right\} \tag{70}$$

$$\bar{F}_\beta = \bar{\Omega}^2 \sum_{i=1}^{3} \bar{I}_i H_{i3} \frac{\partial H_{i3}}{\partial \beta} + \bar{m}\bar{\Omega}^2 \left[(\bar{\ell}\bar{u} + \bar{c}_1) \frac{\partial \bar{c}_1}{\partial \beta} + (\bar{\ell}\bar{v} + \bar{c}_2) \frac{\partial \bar{c}_2}{\partial \beta} \right.$$

$$\left. + (\bar{\ell}\bar{w} + \bar{c}_3) \frac{\partial \bar{c}_3}{\partial \beta} - \frac{\partial J_{33}}{\partial \beta} (J_{33} - \bar{h}) \right] - \bar{m}\bar{g} \frac{\partial J_{33}}{\partial \beta} - \eta_1 M_{m_0} + \eta_2 T_{m_0} - \eta_3 S_{m_0}$$

$$\bar{F}_\theta = \bar{\Omega}^2 \sum_{i=1}^{3} \bar{I}_i H_{i3} \frac{\partial H_{i3}}{\partial \theta} + \bar{m}\bar{\Omega}^2 \left[(\bar{\ell}\bar{u} + \bar{c}_1) \frac{\partial \bar{c}_1}{\partial \theta} + (\bar{\ell}\bar{v} + \bar{c}_2) \frac{\partial \bar{c}_2}{\partial \theta} \right.$$

$$\left. + (\bar{\ell}\bar{w} + \bar{c}_3) \frac{\partial \bar{c}_3}{\partial \theta} - \frac{\partial J_{33}}{\partial \theta} (J_{33} - \bar{h}) \right] - \bar{m}\bar{g} \frac{\partial J_{33}}{\partial \theta} + B_{11} M_{m_0} - B_{21} T_{m_0} + B_{31} S_{m_0}$$

(70) concluded

where

$$\bar{m} = \frac{mL^2}{I}$$

$$\bar{\Omega} = \frac{\Omega}{\Omega_0}$$

$$\bar{\ell} = \frac{\ell}{L}$$

$$\bar{c}_i = \frac{c_i}{L}$$

$$\bar{h} = \frac{h}{L}$$

$$J_{33} = \bar{h} + (\bar{\ell}\bar{u} + \bar{c}_1)F_{13} + (\bar{\ell}\bar{v} + \bar{c}_2)F_{23} + (\bar{\ell}\bar{w} + \bar{c}_3)F_{33}$$

$$\bar{g} = \frac{g}{\Omega_0{}^2 L}$$

$$\bar{I}_i = \frac{I_i}{I}$$

$$S_{f_0} = \frac{\gamma}{6} \left[V_{P_0}^2 - dV_{T_0}^2 + 3\left(U_{P_0} V_{P_0} - dU_{T_0} V_{T_0} + U_{P_0}^2 - dU_{T_0}^2\right) - \frac{3\bar{c}}{4} (1 - 2x_a)\omega_{1_0}\left(V_{P_0} + 2U_{P_0}\right)\right]$$

(71)

$$
\left.
\begin{aligned}
T_{f_0} &= -\frac{\gamma}{6}\Bigg[V_{P_0}V_{T_0} + \frac{3}{2}\left(U_{P_0}V_{T_0} + V_{P_0}U_{T_0}\right) + 3U_{P_0}U_{T_0} \\
&\qquad - \frac{3\bar{c}}{4}\,(1 - 2x_a)\omega_{1_0}\left(V_{T_0} + 2U_{T_0}\right)\Bigg] \\
M_{m_0} &= -\frac{\gamma\bar{c}x_a}{6}\left[V_{P_0}V_{T_0} + \frac{3}{2}\left(V_{T_0}U_{P_0} + V_{P_0}U_{T_0}\right) + 3U_{P_0}U_{T_0}\right] \\
&\qquad - \frac{\gamma\bar{c}^2}{64}\,\omega_{1_0}\left(V_{T_0} + 2U_{T_0}\right)(1 - 8x_a + 16x_a{}^2) \\
T_{m_0} &= -\frac{\gamma}{8}\Bigg[V_{P_0}V_{T_0} + \frac{4}{3}\left(U_{P_0}V_{T_0} + V_{P_0}U_{T_0}\right) + 2U_{P_0}U_{T_0} \\
&\qquad - \frac{2\bar{c}}{3}\,(1 - 2x_a)\omega_{1_0}\left(V_{T_0} + \frac{3U_{T_0}}{2}\right)\Bigg] \\
S_{m_0} &= \frac{\gamma}{8}\Bigg[V_{P_0}^2 - dV_{T_0}^2 + \frac{8}{3}\left(U_{P_0}V_{P_0} - dU_{T_0}V_{T_0}\right) + 2\left(U_{P_0}^2 - dU_{T_0}^2\right) \\
&\qquad - \frac{2\bar{c}}{3}\,(1 - 2x_a)\omega_{1_0}\left(V_{P_0} + \frac{3U_{P_0}}{2}\right)\Bigg] \\
\gamma &= \frac{\rho acL^4}{I} \\
\bar{c} &= \frac{c}{L} \\
\omega_{1_0} &= \bar{\Omega}H_{13} \\
U_{P_0} &= \bar{\Omega}(-\bar{J}_{23}H_{31} + \bar{J}_{13}H_{32} + \bar{\ell}\bar{u}\phi H_{33}) \\
U_{T_0} &= \bar{\Omega}(-\bar{J}_{23}H_{21} + \bar{J}_{13}H_{22} + \bar{\ell}\bar{u}\phi H_{23}) \\
V_{P_0} &= \bar{\Omega}(\phi H_{33} - H_{23}) \\
V_{T_0} &= \bar{\Omega}(\phi H_{23} + H_{33}) \\
\bar{J}_{13} &= \bar{\ell}(\bar{u}F_{11} + \bar{v}F_{21} + \bar{w}F_{31}) \\
\bar{J}_{23} &= \bar{\ell}(\bar{u}F_{12} + \bar{v}F_{22} + \bar{w}F_{32})
\end{aligned}
\right\}
\begin{array}{l}(71)\\ \text{con-}\\ \text{cluded}\end{array}
$$

Each of the force and moment components acting at the tip of the flexbeam are known functions of u, v, w, ζ, β, and θ which are unknown. In order to solve for u, v, w, ζ, β, and θ we assume a set of values u_t, v_t, w_t, ζ_t,

β_t, and θ_t. We can then calculate the forces and moments $\mathbf{F}_t$ and $\mathbf{M}_t$ based on equations (69) and (70). The force $\mathbf{F}_R$ and moment $\mathbf{M}_R$ at the flexbeam root are then determined from statics

$$\left.\begin{aligned}
&\mathbf{F}_R = \mathbf{F}_t \\
&\mathbf{M}_R = \mathbf{M}_t + \mathbf{r}_t \times \mathbf{F}_t \\
&\mathbf{r}_t = (1 + u_t)\mathbf{n}_1 + v_t\mathbf{n}_2 + w_t\mathbf{n}_3 \\
&\mathbf{M}_R = \mathbf{M}_t + (v_t\bar{F}_w - w_t\bar{F}_v)\mathbf{n}_1 + [w_t\bar{F}_u - (1+u_t)\bar{F}_w]\mathbf{n}_2 + [(1+u_t)\bar{F}_v - v_t\bar{F}_u]\mathbf{n}_3
\end{aligned}\right\} \tag{72}$$

At any station s along the deformed flexbeam length, the force $\mathbf{F}$ and moment $\mathbf{M}$ are

$$\left.\begin{aligned}
&\mathbf{F} = \mathbf{F}_R \\
&\mathbf{M} = \mathbf{M}_R - \mathbf{r} \times \mathbf{F}_R \\
&\mathbf{r} = [s + u_b(s)]\mathbf{n}_1 + v(s)\mathbf{n}_2 + w(s)\mathbf{n}_3
\end{aligned}\right\} \tag{73}$$

where $u_b(s)$ is the geometric component of $u(s)$ due to bending only. The bending moments in the principal axis system for the flexbeam cross section $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are

$$\left.\begin{aligned}
\mathbf{M} \cdot \mathbf{u}_2 &= A\kappa \\
\mathbf{M} \cdot \mathbf{u}_3 &= B\lambda
\end{aligned}\right\} \tag{74}$$

where A and B are flap and chord flexbeam bending stiffnesses made dimensionless by $I\Omega_0{}^2\ell$. The curvatures are κ and λ, made dimensionless by $1/\ell$. The torsion moment is, according to reference 17,

$$\mathbf{M} \cdot \mathbf{u}_1 = \left[C + \frac{E\tau^2}{2} + \left(\frac{A + B}{D}\right)T\right]\tau \tag{75}$$

where, from purely geometric considerations, it can be shown that curvatures and slopes are related by the following:

$$\left.\begin{aligned}
\frac{d\beta}{ds} &= -\kappa c_\theta + \lambda s_\theta \\
\frac{d\zeta}{ds} &= \frac{\kappa s_\theta + \lambda c_\theta}{c_\beta} \\
\frac{d\theta}{ds} &= \tau - \frac{d\zeta}{ds} s_\beta
\end{aligned}\right\} \tag{76}$$

and where

$$\left.\begin{aligned} E &\cong \frac{4(A^2 + B^2)}{5D} \quad \text{(for a rectangular cross section)} \\ T &= \mathbf{F} \cdot \mathbf{u}_1 \end{aligned}\right\} \tag{77}$$

The torsion stiffness C is made dimensionless by $I\Omega_0{}^2\ell$ and the axial stiffness D is made dimensionless by $I\Omega_0{}^2/\ell$. The longitudinal strain is given by

$$\varepsilon = \frac{du}{ds}\sqrt{1 - \left(\frac{dv}{ds}\right)^2 - \left(\frac{dw}{ds}\right)^2} + \frac{1}{2}\left(\frac{dv}{ds}\right)^2 + \frac{1}{2}\left(\frac{dw}{ds}\right)^2 - \frac{1}{2}\left(\frac{du}{ds}\right)^2 \tag{78}$$

which is equal to, according to reference 17,

$$\varepsilon = \frac{T}{D} - \left(\frac{A + B}{D}\right)\frac{\tau^2}{2} \tag{79}$$

We define $u(s)$ as being

$$u(s) = u_b(s) + u_s(s) \tag{80}$$

where u_s is due to axial stretching and u_b is a purely geometric deflection due to bending only. We then set the strain due to bending equal to zero and obtain

$$\varepsilon = \frac{du_s}{ds} - \frac{1}{2}\left(\frac{du_s}{ds}\right)^2 \tag{81}$$

In combination with equation (79), we have

$$\frac{du_s}{ds} = 1 - \sqrt{1 - \frac{2T}{D} + \left(\frac{A + B}{D}\right)\tau^2} \tag{82}$$

Also, from purely geometric considerations (see the appendix of ref. 17)

$$\left.\begin{aligned} \frac{du_b}{ds} &= c_\beta c_\zeta - 1 \\ \frac{dv}{ds} &= c_\beta s_\zeta \\ \frac{dw}{ds} &= s_\beta \end{aligned}\right\} \tag{83}$$

Now, with initial conditions $u_s = u_b = v = w = \zeta = \beta = \theta$ for $s = 0$ equations (76), (82), and (83) may be integrated numerically, using equations (73) to (75) and (77), from $s = 0$ to $s = \ell + u_s(s)$. This will yield a set of values for u, v, w, ζ, β, and θ at the tip of the flexbeam which will be the same as the set of u_t, v_t, w_t, ζ_t, β_t, and θ_t assumed if and only if the trail set is the correct set. Since the correct u_t, v_t, w_t, ζ_t, β_t, and θ_t are unknown, the following function may be minimized

$$\min_{\substack{u_t,v_t,w_t,\\ \zeta_t,\beta_t,\theta_t}} \quad \mathscr{F} = (u-u_t)^2 + (v-v_t)^2 + (w-w_t)^2 + (\zeta-\zeta_t)^2 + (\beta-\beta_t)^2 + (\theta-\theta_t)^2 \tag{84}$$

using the modified Levenberg-Marquardt algorithm (ref. 18). The minimum $\mathscr{F} = 0$ is the exact solution for the rotor equilibrium deflections.

7.2 Flexbeam Stiffness Coefficients

The external equilibrium generalized forces at the flexbeam tip $\bar{F}_u$, $\bar{F}_v$, $\bar{F}_w$, $\bar{F}_\zeta$, $\bar{F}_\beta$, $\bar{F}_\theta = Q_i$, $i = 1, 2, \ldots, 6$ may be used to obtain the flexibility coefficients for the flexbeam structure. The Q_i may be perturbed in succession, starting with Q_1, by a small number ε This will yield six different sets of the deflections u, v, w, ζ, β, $\theta = x_i$, $i = 1, 2, \ldots, 6$ only slightly different from the equilibrium values. Flexibility influence coefficients are simply

$$F_{ij} = \frac{x_i(Q_j + \varepsilon) - x_i(Q_j)}{\varepsilon} \tag{85}$$

The smaller value of ε that is chosen, the closer the coefficients will be to a pure linear perturbation. If the deflections are calculated to N significant digits, ε should be chosen at $10^{-N/2}$ to minimize both nonlinearities and truncation errors, simultaneously.

The stiffness matrix for the flexbeam is simply

$$[K] = [F^{-1}] \tag{86}$$

When $\bar{\Omega} = 0$, the structural stiffness matrix is given by equation (58). The numerical scheme described above gives this stiffness matrix to five significant figures when $\bar{\Omega} = 0$ and exhibits a fully coupled system structurally, for general flight conditions.

7.3 Transformation to Fixed System Coordinates

Although the perturbation equations are linearized in $\tilde{u}_k$, $\tilde{u}_k$, $\tilde{w}_k$, $\tilde{\zeta}_k$, $\tilde{\beta}_k$, $\tilde{\theta}_k$, $\tilde{X}$, $\tilde{Y}$, $\tilde{\Phi}_x$, $\tilde{\Phi}_y$ and their first two time derivatives, the equations have periodic coefficients in the form of $\sin\psi_k$ and $\cos\psi_k$. These terms may be eliminated with the multiblade coordinate transformation (ref. 16) when $b \geq 3$. It is necessary to retain only the rotor cyclic modes since the collective modes couple only with vertical translation and yaw rotation, which are also decoupled from the retained degrees of freedom in hover. The differential collective modes and the warping modes (for $b \geq 5$) are uncoupled from all other degrees of freedom in hover. This reduces the 6b rotor blade degrees of freedom to only 12 (two cyclic modes per blade degree of freedom); thus a total of 16 degrees of freedom remain for the coupled rotor-body analysis. The 12 rotor degrees of freedom, in the order that the matrices below are written, are

$$\left.\begin{aligned}
\tilde{u}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{u}_k \cos\psi_k \\
\tilde{u}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{u}_k \sin\psi_k \\
\tilde{v}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{v}_k \cos\psi_k \\
\tilde{v}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{v}_k \sin\psi_k \\
\tilde{w}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{w}_k \cos\psi_k \\
\tilde{w}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{w}_k \sin\psi_k \\
\tilde{\zeta}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\zeta}_k \cos\psi_k \\
\tilde{\zeta}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\zeta}_k \sin\psi_k \\
\tilde{\beta}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\beta}_k \cos\psi_k \\
\tilde{\beta}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\beta}_k \sin\psi_k \\
\tilde{\theta}_c &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\theta}_k \cos\psi_k \\
\tilde{\theta}_s &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\theta}_k \sin\psi_k
\end{aligned}\right\} \tag{87}$$

The use of the variables in equations (87) will eliminate all terms with periodic coefficients in the body equations. The transformation is carried out on each kth blade equation in the following manner:

$$\left.\begin{aligned}
\tilde{F}_{u_c} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{F}_{u_k} \cos \psi_k = 0 \\
\tilde{F}_{u_s} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{F}_{u_k} \sin \psi_k = 0 \\
&\vdots \\
\tilde{F}_{\theta_c} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{F}_{\theta_k} \cos \psi_k = 0 \\
\tilde{F}_{\theta_s} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{F}_{\theta_k} \sin \psi_k = 0
\end{aligned}\right\} \tag{88}$$

where $\tilde{F}_{u_c}$, $\tilde{F}_{u_s}$, . . ., $\tilde{F}_{\theta_c}$, $\tilde{F}_{\theta_s}$, $\tilde{F}_X$, $\tilde{F}_Y$, $\tilde{F}_{\Phi_x}$, and $\tilde{F}_{\Phi_y}$ become linear equations with constant coefficients in $\tilde{u}_c$, $\tilde{u}_s$, $\tilde{v}_c$, $\tilde{v}_s$, $\tilde{w}_c$, $\tilde{w}_s$, $\tilde{\zeta}_c$, $\tilde{\zeta}_s$, $\tilde{\beta}_c$, $\tilde{\beta}_s$, $\tilde{X}$, $\tilde{Y}$, $\tilde{\Phi}_x$, $\tilde{\Phi}_y$ and their first two time derivatives upon substitution of equations (87). The details of these operations are omitted and only the final results are given in the next section.

7.4 Linearized Perturbation Equations of Motion

The linearized perturbation equations of motion for infinitesimal motions about equilibrium are expressed in matrix form as

$$[M^I + M^A]\{\ddot{X}\} + [G^I + C^A + C^D]\{\dot{X}\} + [K^I + K^S + K^g + K^A + K^D]\{X\} = 0 \tag{89}$$

where

$[M^I] \equiv$ mass matrix due to inertial forces, symmetric

$[M^A] \equiv$ mass matrix due to aerodynamic forces

$[G^I] \equiv$ gyroscopic matrix due to inertial forces, antisymmetric

$[C^A] \equiv$ damping matrix due to aerodynamic forces

$[C^D] \equiv$ damping matrix due to structural damping, diagonal

$[K^I] \equiv$ stiffness matrix due to inertial forces, symmetric

$[K^S] \equiv$ stiffness matrix due to springs and structural forces, symmetric

$[K^g] \equiv$ stiffness matrix due to gravitational forces, symmetric except as specified

$[K^D] \equiv$ stiffness matrix due to structural damping, antisymmetric

Thus, $M^I_{1,1}$ is the $\ddot{u}_c$ inertial term in the $\ddot{u}_c$ equation The elements of the above matrices are given below. Only nonzero elements are specified, for matrices with symmetry or antisymmetry, only the terms on or above the diagonal are given.

Modifications to the above analysis for different pitch-control geometries are described in appendices A to C.

$$M^I_{1,1} = \bar{m}\bar{\ell}^2$$

$$M^I_{1,7} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_1}{\partial \zeta}$$

$$M^I_{1,9} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_1}{\partial \beta}$$

$$M^I_{1,11} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_1}{\partial \theta}$$

$$M^I_{1,13} = -\bar{m}\bar{\ell}F_{11}$$

$$M^I_{1,14} = \bar{m}\bar{\ell}F_{12}$$

$$M^I_{1,15} = \bar{m}\bar{\ell}J_{12}$$

$$M^I_{1,16} = \bar{m}\bar{\ell}J_{11}$$

$$M^I_{2,2} = M^I_{1,1}$$

$$M^I_{2,8} = M^I_{1,7}$$

$$M^I_{2,10} = M^I_{1,9}$$

$$M^I_{2,12} = M^I_{1,11}$$

$$M^I_{2,13} = M^I_{1,14}$$

$$M^I_{2,14} = -M^I_{1,13}$$

$$M^I_{2,15} = M^I_{1,16}$$

$$M^I_{2,16} = -M^I_{1,15}$$

$$M^I_{3,3} = \bar{m}\bar{\ell}^2$$

$$M^I_{3,7} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_2}{\partial \zeta}$$

$$M^I_{3,9} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_2}{\partial \beta}$$

$$M^I_{3,11} = \bar{m}\bar{\ell}\,\frac{\partial \bar{c}_2}{\partial \theta}$$

$$M^I_{3,13} = -\bar{m}\bar{\ell}F_{21}$$

$$M^I_{3,14} = \bar{m}\bar{\ell}F_{22}$$

$$M^{I}_{3,15} = \bar{m}\bar{\ell}J_{22}$$

$$M^{I}_{3,16} = \bar{m}\bar{\ell}J_{21}$$

$$M^{I}_{4,4} = M^{I}_{3,3}$$

$$M^{I}_{4,8} = M^{I}_{3,7}$$

$$M^{I}_{4,10} = M^{I}_{3,9}$$

$$M^{I}_{4,12} = M^{I}_{3,11}$$

$$M^{I}_{4,13} = M^{I}_{3,14}$$

$$M^{I}_{4,14} = -M^{I}_{3,13}$$

$$M^{I}_{4,15} = M^{I}_{3,16}$$

$$M^{I}_{4,16} = -M^{I}_{3,15}$$

$$M^{I}_{5,5} = \bar{m}\bar{\ell}^{2}$$

$$M^{I}_{5,7} = \bar{m}\bar{\ell}\,\frac{\partial\bar{c}_3}{\partial\zeta}$$

$$M^{I}_{5,9} = \bar{m}\bar{\ell}\,\frac{\partial\bar{c}_3}{\partial\beta}$$

$$M^{I}_{5,11} = \bar{m}\bar{\ell}\,\frac{\partial\bar{c}_3}{\partial\theta}$$

$$M^{I}_{5,13} = -\bar{m}\bar{\ell}F_{31}$$

$$M^{I}_{5,14} = \bar{m}\bar{\ell}F_{32}$$

$$M^{I}_{5,15} = \bar{m}\bar{\ell}J_{32}$$

$$M^{I}_{5,16} = \bar{m}\bar{\ell}J_{31}$$

$$M^{I}_{6,6} = M^{I}_{5,5}$$

$$M^{I}_{6,8} = M^{I}_{5,7}$$

$$M^{I}_{6,10} = M^{I}_{5,9}$$

$$M^{I}_{6,12} = M^{I}_{5,11}$$

$$M^{I}_{6,13} = M^{I}_{5,14}$$

$$M^{I}_{6,14} = -M^{I}_{5,13}$$

$$M^{I}_{6,15} = M^{I}_{5,16}$$

$$M^{I}_{6,16} = -M^{I}_{5,15}$$

$$M^{I}_{7,7} = \bar{m}\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\zeta}\right)^{2} + \sum_{i=1}^{3}\bar{I}_i\left[1 - \sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\zeta}\right)^{2}\right]$$

$$M^{I}_{7,9} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}\,\frac{\partial\bar{c}_i}{\partial\beta} - \sum_{i=1}^{3}\bar{I}_i\sum_{j=1}^{3}\frac{\partial G_{ij}}{\partial\zeta}\,\frac{\partial G_{ij}}{\partial\beta}$$

$$M^I_{7,11} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}\frac{\partial\bar{c}_i}{\partial\theta} + \sum_{i=1}^{3}\bar{I}_i\left(T_{13} - \sum_{j=1}^{3}\frac{\partial G_{ij}}{\partial\zeta}\frac{\partial G_{ij}}{\partial\theta}\right)$$

$$M^I_{7,13} = -\bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}F_{i1}$$

$$M^I_{7,14} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}F_{i2}$$

$$M^I_{7,15} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}J_{i2} - \sum_{i=1}^{3}\bar{I}_i G_{i3} H_{i1}$$

$$M^I_{7,16} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}J_{i1} + \sum_{i=1}^{3}\bar{I}_i G_{i3} H_{i2}$$

$$M^I_{8,8} = M^I_{7,7}$$

$$M^I_{8,10} = M^I_{7,9}$$

$$M^I_{8,12} = M^I_{7,11}$$

$$M^I_{8,13} = M^I_{7,14}$$

$$M^I_{8,14} = -M^I_{7,13}$$

$$M^I_{8,15} = M^I_{7,16}$$

$$M^I_{8,16} = -M^I_{7,15}$$

$$M^I_{9,9} = \bar{m}\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\beta}\right)^2 + \sum_{i=1}^{3}\bar{I}_i\left[1 - \sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\beta}\right)^2\right]$$

$$M^I_{9,11} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}\,\frac{\partial\bar{c}_i}{\partial\theta} - \sum_{i=1}^{3}\bar{I}_i\sum_{j=1}^{3}\frac{\partial G_{ij}}{\partial\beta}\,\frac{\partial G_{ij}}{\partial\theta}$$

$$M^I_{9,13} = -\bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}\,F_{i1}$$

$$M^I_{9,14} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}\,F_{i2}$$

$$M^I_{9,15} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}\,J_{i2} + \sum_{i=1}^{3}\bar{I}_i\eta_i H_{i1}$$

$$M^I_{9,16} = \bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}\,J_{i1} - \sum_{i=1}^{3}\bar{I}_i\eta_i H_{i2}$$

$$M^I_{10,10} = M^I_{9,9}$$

$$M^I_{10,12} = M^I_{9,11}$$

$$M^I_{10,13} = M^I_{9,14}$$

$$M^I_{10,14} = -M^I_{9,13}$$

$$M^I_{10,15} = M^I_{9,16}$$

$$M^I_{10,16} = -M^I_{9,15}$$

$$M^I_{11,11} = \bar{m}\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\theta}\right)^2 + \sum_{i=1}^{3}\bar{I}_i\left[1 - \sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\theta}\right)^2\right]$$

$$M^I_{11,13} = -\bar{m}\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\theta}\,F_{i1}$$

$$M^{I}_{11,14} = \bar{m} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} F_{i2}$$

$$M^{I}_{11,15} = \bar{m} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} J_{i2} - \sum_{i=1}^{3} \bar{I}_i B_{i1} H_{i1}$$

$$M^{I}_{11,16} = \bar{m} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} J_{i1} + \sum_{i=1}^{3} \bar{I}_i B_{i1} H_{i2}$$

$$M^{I}_{12,12} = M^{I}_{11,11}$$

$$M^{I}_{12,13} = M^{I}_{11,14}$$

$$M^{I}_{12,14} = -M^{I}_{11,13}$$

$$M^{I}_{12,15} = M^{I}_{11,16}$$

$$M^{I}_{12,16} = -M^{I}_{11,15}$$

$$M^{I}_{13,13} = \frac{2\bar{M}}{b}$$

$$M^{I}_{13,16} = 2\bar{m}_f \frac{\bar{z}}{b} - 2\bar{m}J_{33}$$

$$M^{I}_{14,14} = \frac{2\bar{M}}{b}$$

$$M^{I}_{14,15} = -M^{I}_{13,16}$$

$$M^{I}_{15,15} = \frac{2(\bar{I}_x + \bar{m}_f \bar{z}^2)}{b} + \bar{m}\Big[2J_{33}^2 - (J_{33} - \bar{h})^2 + (\bar{\ell}\bar{u} + \bar{c}_1)^2 + (\bar{\ell}\bar{v} + \bar{c}_2)^2$$

$$+ (\bar{\ell}\bar{w} + \bar{c}_3)^2\Big] + \sum_{i=1}^{3} \bar{I}_i (1 - H_{i3}^2)$$

$$M^{I}_{16,16} = \frac{2\bar{I}_y}{b} + M^{I}_{15,15} - \frac{2\bar{I}_x}{b}$$

$$G^{I}_{1,2} = 2\bar{\Omega}M^{I}_{1,1}$$

$$G^{I}_{1,3} = -2\bar{\Omega}\bar{m}\bar{\ell}^2 F_{33}$$

$$G^{I}_{1,5} = 2\bar{\Omega}\bar{m}\bar{\ell}^2 F_{23}$$

$$G^{I}_{1,7} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_3}{\partial\zeta} F_{23} - \frac{\partial\bar{c}_2}{\partial\zeta} F_{33}\right)$$

$$G^{I}_{1,8} = 2\bar{\Omega}M^{I}_{1,7}$$

$$G^{I}_{1,9} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_3}{\partial\beta} F_{23} - \frac{\partial\bar{c}_2}{\partial\beta} F_{33}\right)$$

$$G^{I}_{1,10} = 2\bar{\Omega}M^{I}_{1,9}$$

$$G^{I}_{1,11} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_3}{\partial\theta} F_{23} - \frac{\partial\bar{c}_2}{\partial\theta} F_{33}\right)$$

$$G^{I}_{1,12} = 2\bar{\Omega}M^{I}_{1,11}$$

$$G^{I}_{1,15} = -2\bar{\Omega}\bar{m}\bar{\ell}F_{13}J_{13}$$

$$G^{I}_{1,16} = 2\bar{\Omega}\bar{m}\bar{\ell}F_{13}J_{23}$$

$$G^{I}_{2,4} = G^{I}_{1,3}$$

$$G^{I}_{2,6} = G^{I}_{1,5}$$

$$G^{I}_{2,7} = -G^{I}_{1,8}$$

$$G^{I}_{2,8} = G^{I}_{1,7}$$

$$G^{I}_{2,9} = -G^{I}_{1,10}$$

$$G^{I}_{2,10} = G^{I}_{1,9}$$

$$G^{I}_{2,11} = -G^{I}_{1,12}$$

$$G^{I}_{2,12} = G^{I}_{1,11}$$

$$G^{I}_{2,15} = G^{I}_{1,16}$$

$$G^{I}_{2,16} = -G^{I}_{1,15}$$

$$G^{I}_{3,4} = 2\bar{\Omega}M^{I}_{3,3}$$

$$G^{I}_{3,5} = -2\bar{\Omega}\bar{m}\bar{\ell}^2 F_{13}$$

$$G^{I}_{3,7} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_1}{\partial\zeta} F_{33} - \frac{\partial\bar{c}_3}{\partial\zeta} F_{13}\right)$$

$$G^{I}_{3,8} = 2\bar{\Omega}M^{I}_{3,7}$$

$$G^{I}_{3,9} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_1}{\partial\beta} F_{33} - \frac{\partial\bar{c}_3}{\partial\beta} F_{13}\right)$$

$$G^{I}_{3,10} = 2\bar{\Omega}M^{I}_{3,9}$$

$$G^{I}_{3,11} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_1}{\partial\theta} F_{33} - \frac{\partial\bar{c}_3}{\partial\theta} F_{13}\right)$$

$$G^{I}_{3,12} = 2\bar{\Omega}M^{I}_{3,11}$$

$$G^{I}_{3,15} = -2\bar{\Omega}\bar{m}\bar{\ell}F_{23}J_{13}$$

$$G^{I}_{3,16} = 2\bar{\Omega}\bar{m}\bar{\ell}F_{23}J_{23}$$

$$G^{I}_{4,6} = G^{I}_{3,5}$$

$$G^{I}_{4,7} = -G^{I}_{3,8}$$

$$G^I_{4,8} = G^I_{3,7}$$

$$G^I_{4,9} = -G^I_{3,10}$$

$$G^I_{4,10} = G^I_{3,9}$$

$$G^I_{4,11} = -G^I_{3,12}$$

$$G^I_{4,12} = G^I_{3,11}$$

$$G^I_{4,15} = G^I_{3,16}$$

$$G^I_{4,16} = -G^I_{3,15}$$

$$G^I_{5,6} = 2\bar{\Omega}M^I_{5,5}$$

$$G^I_{5,7} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_2}{\partial\zeta} F_{13} - \frac{\partial\bar{c}_1}{\partial\zeta} F_{23}\right)$$

$$G^I_{5,8} = 2\bar{\Omega}M^I_{5,7}$$

$$G^I_{5,9} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_2}{\partial\beta} F_{13} - \frac{\partial\bar{c}_1}{\partial\beta} F_{23}\right)$$

$$G^I_{5,10} = 2\bar{\Omega}M^I_{5,9}$$

$$G^I_{5,11} = 2\bar{\Omega}\bar{m}\bar{\ell}\left(\frac{\partial\bar{c}_2}{\partial\theta} F_{13} - \frac{\partial\bar{c}_1}{\partial\theta} F_{23}\right)$$

$$G^I_{5,12} = 2\bar{\Omega}M^I_{5,11}$$

$$G^I_{5,15} = -2\bar{\Omega}\bar{m}\bar{\ell}F_{33}J_{13}$$

$$G^I_{5,16} = 2\bar{\Omega}\bar{m}\bar{\ell}F_{33}J_{23}$$

$$G^I_{6,7} = -G^I_{5,8}$$

$$G^I_{6,8} = G^I_{5,7}$$

$$G^I_{6,9} = -G^I_{5,10}$$

$$G^I_{6,10} = G^I_{5,9}$$

$$G^I_{6,11} = -G^I_{5,12}$$

$$G^I_{6,12} = G^I_{5,11}$$

$$G^I_{6,15} = G^I_{5,16}$$

$$G^I_{6,16} = -G^I_{5,15}$$

$$G^I_{7,8} = 2\bar{\Omega}M^I_{7,7}$$

$$G^I_{7,9} = 2\bar{\Omega}\left[\sum_{i=1}^{3} \bar{I}_i H_{i3} \frac{\partial G_{i3}}{\partial\beta} - \bar{m}\frac{\partial\bar{c}_3}{\partial\beta}\sum_{i=1}^{3} \bar{c}_i F_{i3} - (F_{13}c_\zeta + F_{23}s_\zeta)\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{7,10} = 2\bar{\Omega}M^I_{7,9}$$

$$G^I_{7,11} = 2\bar{\Omega}\left[\sum_{i=1}^{3} \bar{I}_i H_{i3} \frac{\partial G_{i3}}{\partial \theta} - \bar{m}\frac{\partial \bar{c}_3}{\partial \theta}\sum_{i=1}^{3} \bar{c}_i F_{i3} + (F_{13}T_{12} - F_{23}T_{11})\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{7,12} = 2\bar{\Omega}M^I_{7,11}$$

$$G^I_{7,15} = 2\bar{\Omega}\left[-\bar{m}\frac{\partial J_{33}}{\partial \zeta} J_{13} + \sum_{i=1}^{3}\bar{I}_i H_{i1}\frac{\partial H_{i3}}{\partial \zeta} + F_{32}\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{7,16} = 2\bar{\Omega}\left[m\frac{\partial J_{33}}{\partial \zeta} J_{23} - \sum_{i=1}^{3}\bar{I}_i H_{i2}\frac{\partial H_{i3}}{\partial \zeta} + F_{31}\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{8,9} = -G^I_{7,10}$$

$$G^I_{8,10} = G^I_{7,9}$$

$$G^I_{8,11} = -G^I_{7,12}$$

$$G^I_{8,12} = G^I_{7,11}$$

$$G^I_{8,15} = G^I_{7,16}$$

$$G^I_{8,16} = -G^I_{7,15}$$

$$G^I_{9,10} = 2\bar{\Omega}M^I_{9,9}$$

$$G^I_{9,11} = 2\bar{\Omega}\left\{\bar{m}\left(c_\zeta\frac{\partial \bar{c}_2}{\partial \theta} - s_\zeta\frac{\partial \bar{c}_1}{\partial \theta}\right)\sum_{i=1}^{3}\bar{c}_i F_{i3} - \sum_{i=1}^{3}\bar{I}_i H_{i3}\left(c_\zeta\frac{\partial G_{i2}}{\partial \theta} - s_\zeta\frac{\partial G_{i1}}{\partial \theta}\right)\right.$$

$$\left. + \left[s_\beta(F_{13}c_\zeta + F_{23}s_\zeta) - c_\beta F_{33}\right]\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right\}$$

$$G^I_{9,12} = 2\bar{\Omega}M^I_{9,11}$$

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$$G^I_{9,15} = 2\bar{\Omega}\left[-\bar{m}\,\frac{\partial J_{33}}{\partial\beta}\,J_{13} + \sum_{i=1}^{3}\bar{I}_i H_{i1}\,\frac{\partial H_{i3}}{\partial\beta} + (F_{12}s_\zeta - F_{22}c_\zeta)\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{9,16} = 2\bar{\Omega}\left[\bar{m}\,\frac{\partial J_{33}}{\partial\beta}\,J_{23} - \sum_{i=1}^{3}\bar{I}_i H_{i2}\,\frac{\partial H_{i3}}{\partial\beta} + (F_{11}s_\zeta - F_{21}c_\zeta)\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{10,11} = -G^I_{9,12}$$

$$G^I_{10,12} = G^I_{9,11}$$

$$G^I_{10,15} = G^I_{9,16}$$

$$G^I_{10,16} = -G^I_{9,15}$$

$$G^I_{11,12} = 2\bar{\Omega}M^I_{11,11}$$

$$G^I_{11,15} = 2\bar{\Omega}\left[-\bar{m}\,\frac{\partial J_{33}}{\partial\theta}\,J_{13} + \sum_{i=1}^{3}\bar{I}_i H_{i1}\,\frac{\partial H_{i3}}{\partial\theta} + (T_{11}F_{12} + T_{12}F_{22} + T_{13}F_{32})\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{11,16} = 2\bar{\Omega}\left[\bar{m}\,\frac{\partial J_{33}}{\partial\theta}\,J_{23} - \sum_{i=1}^{3}\bar{I}_i H_{i2}\,\frac{\partial H_{i3}}{\partial\theta} + (T_{11}F_{11} + T_{12}F_{21} + T_{13}F_{31})\sum_{i=1}^{3}\frac{\bar{I}_i}{2}\right]$$

$$G^I_{12,15} = G^I_{11,16}$$

$$G^I_{12,16} = -G^I_{11,15}$$

$$G^I_{15,16} = 2\bar{\Omega}\left\{\bar{m}[(J_{33} - \bar{h})^2 - (\bar{\ell}\bar{u} + \bar{c}_1)^2 - (\bar{\ell}\bar{v} + \bar{c}_2)^2 - (\bar{\ell}\bar{w} + \bar{c}_3)^2] - \sum_{i=1}^{3}\bar{I}_i H^2_{i3}\right\}$$

$$K^I_{1,1} = -\bar{m}\bar{\ell}^2\bar{\Omega}^2(2 - F^2_{13})$$

$$K^I_{1,3} = \bar{m}\bar{\ell}^2\bar{\Omega}^2F_{13}F_{23}$$

$$K^I_{1,4} = \bar{\Omega} G^I_{1,3}$$

$$K^I_{1,5} = \bar{m}\bar{\ell}^2\bar{\Omega}^2 F_{13} F_{33}$$

$$K^I_{1,6} = \bar{\Omega} G^I_{1,5}$$

$$K^I_{1,7} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \zeta} F_{13} - 2 \frac{\partial \bar{c}_1}{\partial \zeta} \right)$$

$$K^I_{1,8} = \bar{\Omega} G^I_{1,7}$$

$$K^I_{1,9} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \beta} F_{13} - 2 \frac{\partial \bar{c}_1}{\partial \beta} \right)$$

$$K^I_{1,10} = \bar{\Omega} G^I_{1,9}$$

$$K^I_{1,11} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \theta} F_{13} - 2 \frac{\partial \bar{c}_1}{\partial \theta} \right)$$

$$K^I_{1,12} = \bar{\Omega} G^I_{1,11}$$

$$K^I_{2,2} = K^I_{1,1}$$

$$K^I_{2,3} = -K^I_{1,4}$$

$$K^I_{2,4} = K^I_{1,3}$$

$$K^I_{2,5} = -K^I_{1,6}$$

$$K^I_{2,6} = K^I_{1,5}$$

$$K^I_{2,7} = -K^I_{1,8}$$

$$K^I_{2,8} = K^I_{1,7}$$

$$K^I_{2,9} = -K^I_{1,10}$$

$$K^I_{2,10} = K^I_{1,9}$$

$$K^I_{2,11} = -K^I_{1,12}$$

$$K^I_{2,12} = K^I_{1,11}$$

$$K^I_{3,3} = -\bar{m}\bar{\ell}\bar{\Omega}^2 (2 - F^2_{23})$$

$$K^I_{3,5} = \bar{m}\bar{\ell}^2\bar{\Omega}^2 F_{23} F_{33}$$

$$K^I_{3,6} = \bar{\Omega} G^I_{3,5}$$

$$K^I_{3,7} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \zeta} F_{23} - 2 \frac{\partial \bar{c}_2}{\partial \zeta} \right)$$

$$K^I_{3,8} = \bar{\Omega} G^I_{3,7}$$

$$K^I_{3,9} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \beta} F_{23} - 2 \frac{\partial \bar{c}_2}{\partial \beta} \right)$$

$$K^I_{3,10} = \bar{\Omega} G^I_{3,9}$$

$$K^I_{3,11} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \theta} F_{23} - \frac{\partial \bar{c}_2}{\partial \theta} \right)$$

$$K^I_{3,12} = \bar{\Omega} G^I_{3,11}$$

$$K^I_{4,4} = K^I_{3,3}$$

$$K^I_{4,5} = -K^I_{3,6}$$

$$K^I_{4,6} = K^I_{3,5}$$

$$K^I_{4,7} = -K^I_{3,8}$$

$$K^I_{4,8} = K^I_{3,7}$$

$$K^I_{4,9} = -K^I_{3,10}$$

$$K^I_{4,10} = K^I_{3,9}$$

$$K^I_{4,11} = -K^I_{3,12}$$

$$K^I_{4,12} = K^I_{3,11}$$

$$K^I_{5,5} = -\bar{m}\bar{\ell}^2\bar{\Omega}^2(2 - F^2_{33})$$

$$K^I_{5,7} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \zeta} F_{33} - 2 \frac{\partial \bar{c}_3}{\partial \zeta}\right)$$

$$K^I_{5,8} = \bar{\Omega}G^I_{5,7}$$

$$K^I_{5,9} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \beta} F_{33} - 2 \frac{\partial \bar{c}_3}{\partial \beta}\right)$$

$$K^I_{5,10} = \bar{\Omega}G^I_{5,9}$$

$$K^I_{5,11} = \bar{m}\bar{\ell}\bar{\Omega}^2 \left(\frac{\partial J_{33}}{\partial \theta} F_{33} - 2 \frac{\partial \bar{c}_3}{\partial \theta}\right)$$

$$K^I_{5,12} = \bar{\Omega}G^I_{5,11}$$

$$K^I_{6,6} = K^I_{5,5}$$

$$K^I_{6,7} = -K^I_{5,8}$$

$$K^I_{6,8} = K^I_{5,7}$$

$$K^I_{6,9} = -K^I_{5,10}$$

$$K^I_{6,10} = K^I_{5,9}$$

$$K^I_{6,11} = -K^I_{5,12}$$

$$K^I_{6,12} = K^I_{5,11}$$

$$K^I_{7,7} = \bar{\Omega}^2\Bigg\{\bar{m}\Bigg[(J_{33} - \bar{h})\sum_{i=1}^{3}\frac{\partial^2\bar{c}_i}{\partial\zeta^2}F_{i3} + \left(\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}F_{i3}\right)^2 - (\ell\bar{u} + \bar{c}_1)\frac{\partial^2\bar{c}_1}{\partial\zeta^2}$$

$$- (\ell\bar{v} + \bar{c}_2)\frac{\partial^2\bar{c}_2}{\partial\zeta^2} - (\ell\bar{w} + \bar{c}_3)\frac{\partial^2\bar{c}_3}{\partial\zeta^2} - 2\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\zeta}\right)^2\Bigg] + \sum_{i=1}^{3}\bar{I}_i\Bigg[\sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\zeta}\right)^2$$

$$- 1 - \left(\frac{\partial H_{i3}}{\partial\zeta}\right)^2 - H_{i3}\frac{\partial^2 H_{i3}}{\partial\zeta^2}\Bigg]\Bigg\}$$

$$K^I_{7,9} = \bar{\Omega}^2\Bigg\{\bar{m}\Bigg[(J_{33} - \bar{h})\sum_{i=1}^{3}\frac{\partial^2\bar{c}_i}{\partial\zeta\partial\beta}F_{i3} + \left(\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}F_{i3}\right)\left(\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\beta}F_{i3}\right)$$

$$- (\ell\bar{u} + \bar{c}_1)\frac{\partial^2\bar{c}_1}{\partial\zeta\partial\beta} - (\ell\bar{v} + \bar{c}_2)\frac{\partial^2\bar{c}_2}{\partial\zeta\partial\beta} - (\ell\bar{w} + \bar{c}_3)\frac{\partial^2\bar{c}_3}{\partial\zeta\partial\beta} - 2\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\zeta}\right)\left(\frac{\partial\bar{c}_i}{\partial\beta}\right)\Bigg]$$

$$+ \sum_{i=1}^{3}\bar{I}_i\Bigg[\sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\zeta}\right)\left(\frac{\partial G_{ij}}{\partial\beta}\right) - \left(\frac{\partial H_{i3}}{\partial\zeta}\right)\left(\frac{\partial H_{i3}}{\partial\beta}\right) - H_{i3}\frac{\partial^2 H_{i3}}{\partial\zeta\partial\beta}\Bigg]\Bigg\}$$

$$K^I_{7,10} = \bar{\Omega}G^I_{7,9}$$

$$K^I_{7,11} = \bar{\Omega}^2\Bigg\{\bar{m}\Bigg[(J_{33} - \bar{h})\sum_{i=1}^{3}\frac{\partial^2\bar{c}_i}{\partial\zeta\partial\theta}F_{i3} + \left(\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\zeta}F_{i3}\right)\left(\sum_{i=1}^{3}\frac{\partial\bar{c}_i}{\partial\theta}F_{i3}\right)$$

$$- (\ell\bar{u} + \bar{c}_1)\frac{\partial^2\bar{c}_1}{\partial\zeta\partial\theta} - (\ell\bar{v} + \bar{c}_2)\frac{\partial^2\bar{c}_2}{\partial\zeta\partial\theta} - (\ell\bar{w} + \bar{c}_3)\frac{\partial^2\bar{c}_3}{\partial\zeta\partial\theta} - 2\sum_{i=1}^{3}\left(\frac{\partial\bar{c}_i}{\partial\zeta}\right)\left(\frac{\partial\bar{c}_i}{\partial\theta}\right)\Bigg]$$

$$+ \sum_{i=1}^{3}\bar{I}_i\Bigg[\sum_{j=1}^{3}\left(\frac{\partial G_{ij}}{\partial\zeta}\right)\left(\frac{\partial G_{ij}}{\partial\theta}\right) - \left(\frac{\partial H_{i3}}{\partial\zeta}\right)\left(\frac{\partial H_{i3}}{\partial\theta}\right) - H_{i3}\frac{\partial^2 H_{i3}}{\partial\zeta\partial\theta}\Bigg]\Bigg\}$$

$$K^I_{7,12} = \bar{\Omega} G^I_{7,11}$$

$$K^I_{8,8} = K^I_{7,7}$$

$$K^I_{8,9} = -K^I_{7,10}$$

$$K^I_{8,10} = K^I_{7,9}$$

$$K^I_{8,11} = -K^I_{7,12}$$

$$K^I_{8,12} = K^I_{7,11}$$

$$K^I_{9,9} = \bar{\Omega}^2 \left\{ \bar{m} \left[(J_{33} - \bar{h}) \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \beta^2} F_{i3} + \left(\sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \beta} F_{i3} \right)^2 - (\ell \bar{u} + \bar{c}_1) \frac{\partial^2 \bar{c}_1}{\partial \beta^2} \right. \right.$$

$$\left. - (\ell \bar{v} + \bar{c}_2) \frac{\partial^2 \bar{c}_2}{\partial \beta^2} - (\ell \bar{w} + \bar{c}_3) \frac{\partial^2 \bar{c}_3}{\partial \beta^2} - 2 \sum_{i=1}^{3} \left(\frac{\partial \bar{c}_i}{\partial \beta} \right)^2 \right] + \sum_{i=1}^{3} \bar{I}_i \left[\sum_{j=1}^{3} \left(\frac{\partial G_{ij}}{\partial \beta} \right)^2 \right.$$

$$\left. \left. - 1 - \left(\frac{\partial H_{i3}}{\partial \beta} \right)^2 - H_{i3} \frac{\partial^2 H_{i3}}{\partial \beta^2} \right] \right\}$$

$$K^I_{9,11} = \bar{\Omega}^2 \left\{ \bar{m} \left[(J_{33} - \bar{h}) \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \beta \partial \theta} F_{i3} + \left(\sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \beta} F_{i3} \right) \left(\sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} F_{i3} \right) \right. \right.$$

$$\left. - (\ell \bar{u} + \bar{c}_1) \frac{\partial^2 \bar{c}_1}{\partial \beta \partial \theta} - (\ell \bar{v} + \bar{c}_2) \frac{\partial^2 \bar{c}_2}{\partial \beta \partial \theta} - (\ell \bar{w} + \bar{c}_3) \frac{\partial^2 \bar{c}_3}{\partial \beta \partial \theta} - 2 \sum_{i=1}^{3} \left(\frac{\partial \bar{c}_i}{\partial \beta} \right) \left(\frac{\partial \bar{c}_i}{\partial \theta} \right) \right]$$

$$\left. + \sum_{i=1}^{3} \bar{I}_i \left[\sum_{j=1}^{3} \left(\frac{\partial G_{ij}}{\partial \beta} \right) \left(\frac{\partial G_{ij}}{\partial \theta} \right) - \left(\frac{\partial H_{i3}}{\partial \beta} \right) \left(\frac{\partial H_{i3}}{\partial \theta} \right) - H_{i3} \frac{\partial^2 H_{i3}}{\partial \beta \partial \theta} \right] \right\}$$

$$K^I_{9,12} = \bar{\Omega} G^I_{9,11}$$

$$K^I_{10,10} = K^I_{9,9}$$

$$K^I_{10,11} = -K^I_{9,12}$$

$$K^I_{10,12} = K^I_{9,11}$$

$$K^I_{11,11} = \bar{\Omega}^2 \Bigg\{ \bar{m} \Bigg[(J_{33} - \bar{h}) \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \theta^2} F_{i3} + \left(\sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} F_{i3} \right)^2 - (\bar{\ell}\bar{u} + \bar{c}_1) \frac{\partial^2 \bar{c}_1}{\partial \theta^2}$$

$$- (\bar{\ell}\bar{v} + \bar{c}_2) \frac{\partial^2 \bar{c}_2}{\partial \theta^2} - (\bar{\ell}\bar{w} + \bar{c}_3) \frac{\partial^2 \bar{c}_3}{\partial \theta^2} - 2 \sum_{i=1}^{3} \left(\frac{\partial \bar{c}_i}{\partial \theta} \right)^2 \Bigg] + \sum_{i=1}^{3} \bar{I}_i \Bigg[\sum_{j=1}^{3} \left(\frac{\partial G_{ij}}{\partial \theta} \right)^2$$

$$- 1 - \left(\frac{\partial H_{i3}}{\partial \theta} \right)^2 - H_{i3} \frac{\partial^2 H_{i3}}{\partial \theta^2} \Bigg] \Bigg\}$$

$$K^I_{12,12} = K^I_{11,11}$$

$$K^g_{1,15} = -\bar{m}\bar{g}\bar{\ell}F_{12}$$

$$K^g_{1,16} = -\bar{m}\bar{g}\bar{\ell}F_{11}$$

$$K^g_{2,15} = K^g_{1,16}$$

$$K^g_{2,16} = -K^g_{1,15}$$

$$K^g_{3,15} = -\bar{m}\bar{g}\bar{\ell}F_{22}$$

$$K^g_{3,16} = -\bar{m}\bar{g}\bar{\ell}F_{21}$$

$$K^g_{4,15} = K^g_{3,16}$$

$$K^g_{4,16} = -K^g_{3,15}$$

$$K^g_{5,15} = -\bar{m}\bar{g}\bar{\ell}F_{32}$$

$$K^g_{5,16} = -\bar{m}\bar{g}\bar{\ell}F_{33}$$

$$K^g_{6,15} = K^g_{5,16}$$

$$K^g_{6,16} = -K^g_{5,15}$$

$$K^g_{7,7} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \zeta^2} F_{i3}$$

$$K^g_{7,9} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \zeta \partial \beta} F_{i3}$$

$$K^g_{7,11} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \zeta \partial \theta} F_{i3}$$

$$K^g_{7,15} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \zeta} F_{i2}$$

$$K^g_{7,16} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \zeta} F_{i1}$$

$$K^g_{8,8} = K^g_{7,7}$$

$$K^g_{8,10} = K^g_{7,9}$$

$$K^g_{8,12} = K^g_{7,11}$$

$$K^g_{8,15} = K^g_{7,16}$$

$$K^g_{8,16} = -K^g_{7,15}$$

$$K^g_{9,9} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \beta^2} F_{i3}$$

$$K^g_{9,11} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \beta \partial \theta} F_{i3}$$

$$K^g_{9,15} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \beta} F_{i2}$$

$$K^g_{9,16} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \beta} F_{i1}$$

$$K^g_{10,10} = K^g_{9,9}$$

$$K^g_{10,12} = K^g_{9,11}$$

$$K^g_{10,15} = K^g_{9,16}$$

$$K^g_{10,16} = -K^g_{9,15}$$

$$K^g_{11,11} = \bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial^2 \bar{c}_i}{\partial \theta^2} F_{i3}$$

$$K^g_{11,15} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} F_{i2}$$

$$K^g_{11,16} = -\bar{m}\bar{g} \sum_{i=1}^{3} \frac{\partial \bar{c}_i}{\partial \theta} F_{i1}$$

$$K^g_{12,12} = K^g_{11,11}$$

$$K^g_{12,15} = K^g_{11,16}$$

$$K^g_{12,16} = -K^g_{11,15}$$

$$K^g_{13,16} = \frac{2\bar{M}\bar{g}}{b} \quad ; \quad K^g_{16,13} = 0$$

$$K^g_{14,15} = -K^g_{13,16} \quad ; \quad K^g_{16,14} = 0$$

$$K^g_{15,15} = -2\bar{g}\left(\bar{m}J_{33} - \frac{\bar{m}_f\bar{z}}{b}\right)$$

$$K^g_{16,16} = K^g_{15,15}$$

Recall from section 5 2 that when the aircraft is airborne the parameter $\bar{g}$ is given by $N\bar{g}_0$ where

$$\bar{g}_0 = \frac{g_0}{\Omega_0{}^2 L}$$

and N is the load factor = thrust/weight. When the aircraft is in "1-G" hovering flight or in simulated hovering flight on a gimballed model test stand, $N = 1$ and $\bar{g} = \bar{g}_0$. When the aircraft is in ground contact $\bar{g}$ is given by $\bar{g}_0$ in all elements of $[K^g]$ except

$$K^g_{13,16} = -K^g_{14,15} = \frac{2}{b} N\bar{M}\bar{g}_0$$

where $N \leq 1$.

All elements of $[C^D]$ and $[K^D]$ are zero except

$$C^D_{7,7} = C^D_{8,8} = c_\zeta$$

$$C^D_{13,13} = \frac{2}{b} c_X \quad , \quad C^D_{14,14} = \frac{2}{b} c_Y$$

$$C^D_{15,15} = \frac{2}{b} c_{\Phi_x} \quad ; \quad C^D_{16,16} = \frac{2}{b} c_{\Phi_y}$$

The above structural damping terms are derived in detail in appendix D.

The structural terms are

$$K^S_{2i-1,2j-1} = K^S_{2i,2j} = K_{ij} \quad , \quad i,j = 1,\ 2,\ \ldots,\ 6$$

The following elements of $[K^S]$ are zero except when the aircraft is in contact with the ground:

$$K^S_{13,13} = \frac{8}{b}\bar{K}_x \quad , \quad K^S_{14,14} = \frac{8}{b}\bar{K}_y$$

$$K^S_{13,16} = \frac{8}{b}\,\bar{K}_x\bar{\ell}_z\;, \qquad K^S_{14,15} = -\frac{8}{b}\,\bar{K}_y\bar{\ell}_z$$

$$K^S_{15,15} = \frac{2}{b}\,(4\bar{K}_y\bar{\ell}_z{}^2 + \bar{K}_z\bar{\ell}_y{}^2)$$

$$K^S_{16,16} = \frac{2}{b}\,(4\bar{K}_x\bar{\ell}_z{}^2 + \bar{K}_z\bar{\ell}_x{}^2)$$

$$M^A_{1,1} = M^A_{2,2} = -G_{31}\bar{\ell}^2\,\frac{\partial P_1}{\partial \dot{U}_P}$$

$$M^A_{1,3} = M^A_{2,4} = -G_{32}\bar{\ell}^2\,\frac{\partial P_1}{\partial \dot{U}_P}$$

$$M^A_{1,5} = M^A_{2,6} = -G_{33}\bar{\ell}^2\,\frac{\partial P_1}{\partial \dot{U}_P}$$

$$M^A_{1,7} = M^A_{2,8} = \left(G_{23}\,\frac{\partial P_1}{\partial \dot{V}_P} - G_{13}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\bar{\ell}$$

$$M^A_{1,9} = M^A_{2,10} = \left(-\eta_2\,\frac{\partial P_1}{\partial \dot{V}_P} + \eta_1\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\bar{\ell}$$

$$M^A_{1,11} = M^A_{2,12} = \left(B_{21}\,\frac{\partial P_1}{\partial \dot{V}_P} - B_{11}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\bar{\ell}$$

$$M^A_{1,13} = H_{31}\,\frac{\partial P_1}{\partial \dot{U}_P}\;\bar{\ell}$$

$$M^A_{1,14} = -H_{32}\,\frac{\partial P_1}{\partial \dot{U}_P}\;\bar{\ell}$$

$$M^A_{1,15} = \left(-J^H_{32}\,\frac{\partial P_1}{\partial \dot{U}_P} - H_{21}\,\frac{\partial P_1}{\partial \dot{V}_P} + H_{11}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\bar{\ell}$$

$$M^A_{1,16} = \left(-J^H_{31}\,\frac{\partial P_1}{\partial \dot{U}_P} + H_{22}\,\frac{\partial P_1}{\partial \dot{V}_P} - H_{12}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\bar{\ell}$$

$$M^A_{2,13} = M^A_{1,14}$$

$$M^A_{2,14} = -M^A_{1,13}$$

$$M^A_{2,15} = M^A_{1,16}$$

$$M^A_{2,16} = -M^A_{1,15}$$

Rows 3 and 4 of M^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_2.

Rows 5 and 6 of M^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_3.

Rows 7 and 8 of M^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\zeta/\bar{\ell}$.

Rows 9 and 10 of M^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\beta/\bar{\ell}$.

Rows 11 and 12 of M^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\theta/\bar{\ell}$.

$$M^A_{13,1} = G_{31} \frac{\partial P_x}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,2} = -G_{31} \frac{\partial P_y}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,3} = G_{32} \frac{\partial P_x}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,4} = -G_{32} \frac{\partial P_y}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,5} = G_{33} \frac{\partial P_x}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,6} = -G_{33} \frac{\partial P_y}{\partial \dot{U}_P} \bar{\ell}$$

$$M^A_{13,7} = -G_{23} \frac{\partial P_x}{\partial \dot{V}_P} + G_{13} \frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$M^A_{13,8} = G_{23} \frac{\partial P_y}{\partial \dot{V}_P} - G_{13} \frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$M^A_{13,9} = \eta_2 \frac{\partial P_x}{\partial \dot{V}_P} - \eta_1 \frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$M^A_{13,10} = -\eta_2 \frac{\partial P_y}{\partial \dot{V}_P} + \eta_1 \frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$M^A_{13,11} = -B_{21} \frac{\partial P_x}{\partial \dot{V}_P} + B_{11} \frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$M^A_{13,12} = B_{21} \frac{\partial P_y}{\partial \dot{V}_P} - B_{11} \frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$M^A_{13,13} = -H_{31} \frac{\partial P_x}{\partial \dot{U}_P} - H_{32} \frac{\partial P_y}{\partial \dot{U}_P}$$

$$M^A_{13,14} = H_{32} \frac{\partial P_x}{\partial \dot{U}_P} - H_{31} \frac{\partial P_y}{\partial \dot{U}_P}$$

$$M^A_{13,15} = J^H_{32} \frac{\partial P_x}{\partial \dot{U}_P} + H_{21} \frac{\partial P_x}{\partial \dot{V}_P} - H_{11} \frac{\partial P_x}{\partial \dot{\omega}_1} - J^H_{31} \frac{\partial P_y}{\partial \dot{U}_P} + H_{22} \frac{\partial P_y}{\partial \dot{V}_P} - H_{12} \frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$M^A_{13,16} = J^H_{31} \frac{\partial P_x}{\partial \dot{U}_P} - H_{22} \frac{\partial P_x}{\partial \dot{V}_P} + H_{12} \frac{\partial P_x}{\partial \dot{\omega}_1} + J^H_{32} \frac{\partial P_y}{\partial \dot{U}_P} + H_{21} \frac{\partial P_y}{\partial \dot{V}_P} - H_{11} \frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$M^A_{14,1} = M^A_{13,2}$$

$$M^A_{14,2} = -M^A_{13,1}$$

$$M^A_{14,3} = M^A_{13,4}$$

$$M^A_{14,4} = -M^A_{13,3}$$

$$M^A_{14,5} = M^A_{13,6}$$

$$M^A_{14,6} = -M^A_{13,5}$$

$$M^A_{14,7} = M^A_{13,8}$$

$$M^A_{14,8} = -M^A_{13,7}$$

$$M^A_{14,9} = M^A_{13,10}$$

$$M^A_{14,10} = -M^A_{13,9}$$

$$M^A_{14,11} = M^A_{13,12}$$

$$M^A_{14,12} = -M^A_{13,11}$$

$$M^A_{14,13} = -M^A_{13,14}$$

$$M^A_{14,14} = M^A_{13,13}$$

$$M^A_{14,15} = -M^A_{13,16}$$

$$M^A_{14,16} = M^A_{13,15}$$

Rows 15 and 16 of M^A are identical to rows 13 and 14, respectively, except that P_x and P_y are replaced by Q_x and Q_y, respectively.

$$C^A_{1,1} = -\bar{\ell}^2\left(G_{31}\frac{\partial P_1}{\partial U_P} + G_{21}\frac{\partial P_1}{\partial U_T}\right) + \bar{\Omega}\bar{\ell}^2\frac{\partial P_1}{\partial \dot{U}_P}(F_{12}H_{31} - F_{11}H_{32})$$

$$C^A_{1,2} = 2\bar{\Omega}M^A_{1,1}$$

$$C^A_{1,3} = -\bar{\ell}^2\left(G_{32}\frac{\partial P_1}{\partial U_P} + G_{22}\frac{\partial P_1}{\partial U_T}\right) + \bar{\Omega}\bar{\ell}^2\frac{\partial P_1}{\partial \dot{U}_P}(F_{22}H_{31} - F_{21}H_{32})$$

$$C^A_{1,4} = 2\bar{\Omega}M^A_{1,3}$$

$$C^A_{1,5} = -\bar{\ell}^2\left(G_{33}\frac{\partial P_1}{\partial U_P} + G_{23}\frac{\partial P_1}{\partial U_T}\right) + \bar{\Omega}\bar{\ell}^2\frac{\partial P_1}{\partial \dot{U}_P}(F_{32}H_{31} - F_{31}H_{32})$$

$$C^A_{1,6} = 2\bar{\Omega}M^A_{1,5}$$

$$C^A_{1,7} = \bar{\ell}\left(G_{23}\frac{\partial P_1}{\partial V_P} - G_{33}\frac{\partial P_1}{\partial V_T} - G_{13}\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$C^A_{1,8} = 2\bar{\Omega}M^A_{1,7}$$

$$C^A_{1,9} = \bar{\ell}\left(-\eta_2\frac{\partial P_1}{\partial V_P} + \eta_3\frac{\partial P_1}{\partial V_T} + \eta_1\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$C^A_{1,10} = 2\bar{\Omega}M^A_{1,9}$$

$$C^A_{1,11} = \bar{\ell}\left(B_{21}\,\frac{\partial P_1}{\partial V_P} - B_{31}\,\frac{\partial P_1}{\partial V_T} - B_{11}\,\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \theta}\,\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \theta}\,\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\,\frac{\partial H_{13}}{\partial \theta}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$C^A_{1,12} = 2\bar{\Omega}M^A_{1,11}$$

$$C^A_{1,13} = \bar{\ell}\left(H_{31}\,\frac{\partial P_1}{\partial U_P} + H_{21}\,\frac{\partial P_1}{\partial U_T} - \bar{\Omega}H_{32}\,\frac{\partial P_1}{\partial \dot{U}_P}\right)$$

$$C^A_{1,14} = \bar{\ell}\left(-H_{32}\,\frac{\partial P_1}{\partial U_P} - H_{22}\,\frac{\partial P_1}{\partial U_T} - \bar{\Omega}H_{31}\,\frac{\partial P_1}{\partial \dot{U}_P}\right)$$

$$C^A_{1,15} = \bar{\ell}\left[-J^H_{32}\,\frac{\partial P_1}{\partial U_P} - H_{21}\,\frac{\partial P_1}{\partial V_P} - J^H_{22}\,\frac{\partial P_1}{\partial U_T} + H_{31}\,\frac{\partial P_1}{\partial V_T} + H_{11}\,\frac{\partial P_1}{\partial \omega_1}\right.$$
$$\left. + (\bar{\ell}\bar{u}\phi H_{32} - J^H_{31})\bar{\Omega}\,\frac{\partial P_1}{\partial \dot{U}_P} + (H_{22} + \phi H_{32})\bar{\Omega}\,\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}H_{12}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right]$$

$$C^A_{1,16} = \bar{\ell}\left[-J^H_{31}\,\frac{\partial P_1}{\partial U_P} + H_{22}\,\frac{\partial P_1}{\partial V_P} - J^H_{21}\,\frac{\partial P_1}{\partial U_T} - H_{32}\,\frac{\partial P_1}{\partial V_T} - H_{12}\,\frac{\partial P_1}{\partial \omega_1}\right.$$
$$\left. + (\bar{\ell}\bar{u}\phi H_{31} + J^H_{32})\bar{\Omega}\,\frac{\partial P_1}{\partial \dot{U}_P} + (H_{21} + \phi H_{31})\bar{\Omega}\,\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}H_{11}\,\frac{\partial P_1}{\partial \dot{\omega}_1}\right]$$

$$C^A_{2,1} = -C^A_{1,2}$$

$$C^A_{2,2} = C^A_{1,1}$$

$$C^A_{2,3} = -C^A_{1,4}$$

$$C^A_{2,4} = C^A_{1,3}$$

$$C^A_{2,5} = -C^A_{1,6}$$

$$C^A_{2,6} = C^A_{1,5}$$

$$C^A_{2,7} = -C^A_{1,8}$$

$$C^A_{2,8} = C^A_{1,7}$$

$$C^A_{2,9} = -C^A_{1,10}$$

$$C^A_{2,10} = C^A_{1,9}$$

$$C^A_{2,11} = -C^A_{1,12}$$

$$C^A_{2,12} = C^A_{1,11}$$

$$C^A_{2,13} = C^A_{1,14}$$

$$C^A_{2,14} = -C^A_{1,13}$$

$$C^A_{2,15} = C^A_{1,16}$$

$$C^A_{2,16} = -C^A_{1,15}$$

Rows 3 and 4 of C^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_2.

Rows 5 and 6 of C^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_3.

Rows 7 and 8 of C^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\zeta/\bar{\ell}$.

Rows 9 and 10 of C^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\beta/\bar{\ell}$.

Rows 11 and 12 of C^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\theta/\bar{\ell}$.

$$C^A_{13,1} = \bar{\ell}\left[G_{31}\frac{\partial P_x}{\partial U_P} + G_{21}\frac{\partial P_x}{\partial U_T} - \bar{\Omega}\frac{\partial P_x}{\partial \dot{U}_P}(F_{12}H_{31} - F_{11}H_{32}) + 2\bar{\Omega}G_{31}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$C^A_{13,2} = \bar{\ell}\left[2\bar{\Omega}G_{31}\frac{\partial P_x}{\partial \dot{U}_P} - G_{31}\frac{\partial P_y}{\partial U_P} - G_{21}\frac{\partial P_y}{\partial U_T} + \bar{\Omega}\frac{\partial P_y}{\partial \dot{U}_P}(F_{12}H_{31} - F_{11}H_{32})\right]$$

$$C^A_{13,3} = \bar{\ell}\left[G_{32}\frac{\partial P_x}{\partial U_P} + G_{22}\frac{\partial P_x}{\partial U_T} - \bar{\Omega}\frac{\partial P_x}{\partial \dot{U}_P}(F_{22}H_{31} - F_{21}H_{32}) + 2\Omega G_{32}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$C^A_{13,4} = \bar{\ell}\left[2\bar{\Omega}G_{32}\frac{\partial P_x}{\partial \dot{U}_P} - G_{32}\frac{\partial P_y}{\partial U_P} - G_{22}\frac{\partial P_y}{\partial U_T} + \bar{\Omega}\frac{\partial P_y}{\partial \dot{U}_P}(F_{22}H_{31} - F_{21}H_{32})\right]$$

$$C^A_{13,5} = \bar{\ell}\left[G_{33}\frac{\partial P_x}{\partial U_P} + G_{23}\frac{\partial P_x}{\partial U_T} - \bar{\Omega}\frac{\partial P_x}{\partial \dot{U}_P}(F_{32}H_{31} - F_{31}H_{32}) + 2\bar{\Omega}G_{33}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$C^{A}_{13,6} = \bar{\ell}\left[2\bar{\Omega}G_{33}\frac{\partial P_x}{\partial \dot{U}_P} - G_{33}\frac{\partial P_y}{\partial U_P} - G_{23}\frac{\partial P_y}{\partial U_T} + \bar{\Omega}\frac{\partial P_y}{\partial \dot{U}_P}(F_{32}H_{31} - F_{31}H_{32})\right]$$

$$C^{A}_{13,7} = -G_{23}\frac{\partial P_x}{\partial V_P} + G_{33}\frac{\partial P_x}{\partial V_T} + G_{13}\frac{\partial P_x}{\partial \omega_1} + \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial \dot{V}_P} + \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$- 2\bar{\Omega}\left(G_{23}\frac{\partial P_y}{\partial \dot{V}_P} - G_{13}\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$C^{A}_{13,8} = -2\bar{\Omega}\left(G_{23}\frac{\partial P_x}{\partial \dot{V}_P} - G_{13}\frac{\partial P_x}{\partial \dot{\omega}_1}\right) + G_{23}\frac{\partial P_y}{\partial V_P} - G_{33}\frac{\partial P_y}{\partial V_T} - G_{13}\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{U}_P}$$

$$- \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$C^{A}_{13,9} = \eta_2\frac{\partial P_x}{\partial V_P} - \eta_3\frac{\partial P_x}{\partial V_T} - \eta_1\frac{\partial P_x}{\partial \omega_1} + \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial \dot{U}_P} + \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$- 2\bar{\Omega}\left(-\eta_2\frac{\partial P_y}{\partial \dot{V}_P} + \eta_1\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$C^{A}_{13,10} = -2\bar{\Omega}\left(-\eta_2\frac{\partial P_x}{\partial \dot{V}_P} + \eta_1\frac{\partial P_x}{\partial \dot{\omega}_1}\right) - \eta_2\frac{\partial P_y}{\partial V_P} + \eta_3\frac{\partial P_y}{\partial V_T} + \eta_1\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial \dot{U}_P}$$

$$- \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$C^{A}_{13,11} = -B_{21}\frac{\partial P_x}{\partial V_P} + B_{31}\frac{\partial P_x}{\partial V_T} + B_{11}\frac{\partial P_x}{\partial \omega_1} + \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial \dot{U}_P} + \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$- 2\bar{\Omega}\left(B_{21}\frac{\partial P_y}{\partial \dot{V}_P} - B_{11}\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$C^{A}_{13,12} = -2\bar{\Omega}\left(B_{21}\frac{\partial P_x}{\partial \dot{V}_P} - B_{11}\frac{\partial P_x}{\partial \dot{\omega}_1}\right) + B_{21}\frac{\partial P_y}{\partial V_P} - B_{31}\frac{\partial P_y}{\partial V_T} - B_{11}\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial \dot{U}_P}$$

$$- \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$C^{A}_{13,13} = -H_{31}\frac{\partial P_x}{\partial U_P} - H_{21}\frac{\partial P_x}{\partial U_T} + \bar{\Omega}\left(H_{32}\frac{\partial P_x}{\partial \dot{U}_P} - H_{31}\frac{\partial P_y}{\partial \dot{U}_P}\right) - H_{32}\frac{\partial P_y}{\partial U_P} - H_{22}\frac{\partial P_x}{\partial U_T}$$

$$C^{A}_{13,14} = H_{32}\frac{\partial P_x}{\partial U_P} + H_{22}\frac{\partial P_x}{\partial U_T} + \bar{\Omega}\left(H_{31}\frac{\partial P_x}{\partial \dot{U}_P} + H_{32}\frac{\partial P_y}{\partial \dot{U}_P}\right) - H_{31}\frac{\partial P_y}{\partial U_P} - H_{21}\frac{\partial P_y}{\partial U_T}$$

$$\begin{aligned} C^{A}_{13,15} = {} & J^{H}_{32}\frac{\partial P_x}{\partial U_P} + H_{21}\frac{\partial P_x}{\partial V_P} + J^{H}_{22}\frac{\partial P_x}{\partial U_T} - H_{31}\frac{\partial P_x}{\partial V_T} - H_{11}\frac{\partial P_x}{\partial \omega_1} - (\bar{\ell}\bar{u}\phi H_{32} - J^{H}_{31})\bar{\Omega}\frac{\partial P_x}{\partial \dot{U}_P} \\ & - (H_{22} + \phi H_{32})\bar{\Omega}\frac{\partial P_x}{\partial \dot{V}_P} + \bar{\Omega}H_{12}\frac{\partial P_x}{\partial \dot{\omega}_1} - J^{H}_{31}\frac{\partial P_y}{\partial U_P} + H_{22}\frac{\partial P_y}{\partial V_P} - J^{H}_{21}\frac{\partial P_y}{\partial U_T} \\ & - H_{32}\frac{\partial P_y}{\partial V_T} - H_{12}\frac{\partial P_y}{\partial \omega_1} + (\bar{\ell}\bar{u}\phi H_{31} + J^{H}_{32})\bar{\Omega}\frac{\partial P_y}{\partial \dot{U}_P} + (H_{21} + \phi H_{31})\bar{\Omega}\frac{\partial P_y}{\partial \dot{V}_P} \\ & - \bar{\Omega}H_{11}\frac{\partial P_y}{\partial \dot{\omega}_1} \end{aligned}$$

$$\begin{aligned} C^{A}_{13,16} = {} & J^{H}_{31}\frac{\partial P_x}{\partial U_P} - H_{22}\frac{\partial P_x}{\partial V_P} + J^{H}_{21}\frac{\partial P_x}{\partial U_T} + H_{32}\frac{\partial P_x}{\partial V_T} + H_{12}\frac{\partial P_x}{\partial \omega_1} - (\bar{\ell}\bar{u}\phi H_{31} + J^{H}_{32})\bar{\Omega}\frac{\partial P_x}{\partial \dot{U}_P} \\ & - (H_{21} + \phi H_{31})\bar{\Omega}\frac{\partial P_x}{\partial \dot{V}_P} + \bar{\Omega}H_{11}\frac{\partial P_x}{\partial \dot{\omega}_1} + J^{H}_{32}\frac{\partial P_y}{\partial U_P} + H_{21}\frac{\partial P_y}{\partial V_P} + J^{H}_{22}\frac{\partial P_y}{\partial U_T} \\ & - H_{31}\frac{\partial P_y}{\partial V_T} - H_{11}\frac{\partial P_y}{\partial \omega_1} - (\bar{\ell}\bar{u}\phi H_{32} - J^{H}_{31})\bar{\Omega}\frac{\partial P_y}{\partial \dot{U}_P} - (H_{22} + \phi H_{32})\bar{\Omega}\frac{\partial P_y}{\partial \dot{V}_P} \\ & + \bar{\Omega}H_{12}\frac{\partial P_y}{\partial \dot{\omega}_1} \end{aligned}$$

$$C^{A}_{14,1} = C^{A}_{13,2}$$

$$C^{A}_{14,2} = -C^{A}_{13,1}$$

$$C^{A}_{14,3} = C^{A}_{13,4}$$

$$C^{A}_{14,4} = -C^{A}_{13,3}$$

$C^A_{14,5} = C^A_{13,6}$

$C^A_{14,6} = -C^A_{13,5}$

$C^A_{14,7} = C^A_{13,8}$

$C^A_{14,8} = -C^A_{13,7}$

$C^A_{14,9} = C^A_{13,10}$

$C^A_{14,10} = -C^A_{13,9}$

$C^A_{14,11} = C^A_{13,12}$

$C^A_{14,12} = -C^A_{13,11}$

$C^A_{14,13} = -C^A_{13,14}$

$C^A_{14,14} = C^A_{13,13}$

$C^A_{14,15} = -C^A_{13,16}$

$C^A_{14,16} = C^A_{13,15}$

Rows 15 and 16 of C^A are identical to rows 13 and 14, respectively, except that P_x and P_y are replaced by Q_x and Q_y.

$$K^A_{1,1} = \bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}(F_{12}H_{31} - F_{11}H_{32}) + \frac{\partial P_1}{\partial U_T}(F_{12}H_{21} - F_{11}H_{22}) + \bar{\Omega}G_{31}\frac{\partial P_1}{\partial \dot{U}_P}\right]$$

$$K^A_{1,2} = -\bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}G_{31} + \frac{\partial P_1}{\partial U_T}G_{21} - \frac{\partial P_1}{\partial \dot{U}_P}(F_{12}H_{31} - F_{11}H_{32})\bar{\Omega}\right]$$

$$K^A_{1,3} = \bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}(F_{22}H_{31} - F_{21}H_{32}) + \frac{\partial P_1}{\partial U_T}(F_{22}H_{21} - F_{21}H_{22}) + \bar{\Omega}G_{32}\frac{\partial P_1}{\partial \dot{U}_P}\right]$$

$$K^A_{1,4} = -\bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}G_{32} + \frac{\partial P_1}{\partial U_T}G_{22} - \frac{\partial P_1}{\partial \dot{U}_P}(F_{22}H_{31} - F_{21}H_{32})\bar{\Omega}\right]$$

$$K^A_{1,5} = \bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}(F_{32}H_{31} - F_{31}H_{32}) + \frac{\partial P_1}{\partial U_T}(F_{32}H_{21} - F_{31}H_{22}) + \bar{\Omega}G_{33}\frac{\partial P_1}{\partial \dot{U}_P}\right]$$

$$K^A_{1,6} = -\bar{\Omega}\bar{\ell}^2 \left[\frac{\partial P_1}{\partial U_P}G_{33} + \frac{\partial P_1}{\partial U_T}G_{23} - \frac{\partial P_1}{\partial \dot{U}_P}(F_{32}H_{31} - F_{31}H_{32})\bar{\Omega}\right]$$

$$K^A_{1,7} = -\bar{\ell}P_1\zeta - \left[\frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \zeta}\frac{\partial P_1}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \zeta}\frac{\partial P_1}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_1}{\partial \omega_1}\right.$$
$$\left. + \bar{\Omega}^2\left(G_{23}\frac{\partial P_1}{\partial \dot{V}_P} - G_{13}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\right]\bar{\ell}$$

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$$K^A_{1,8} = \bar{\Omega}\bar{\ell}\left(G_{23}\frac{\partial P_1}{\partial V_P} - G_{33}\frac{\partial P_1}{\partial V_T} - G_{13}\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$K^A_{1,9} = -\bar{\ell}P_1{}^{\beta} - \left[\frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \beta}\frac{\partial P_1}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \beta}\frac{\partial P_1}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_1}{\partial \omega_1}\right.$$
$$\left. - \bar{\Omega}^2\left(\eta_2\frac{\partial P_1}{\partial \dot{V}_P} - \eta_1\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\right]\bar{\ell}$$

$$K^A_{1,10} = \bar{\Omega}\bar{\ell}\left(-\eta_2\frac{\partial P_1}{\partial V_P} + \eta_3\frac{\partial P_1}{\partial V_T} + \eta_1\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$K^A_{1,11} = -\bar{\ell}P_1{}^{\theta} - \left[\frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_1}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_1}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \theta}\frac{\partial P_1}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \theta}\frac{\partial P_1}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_1}{\partial \omega_1}\right.$$
$$\left. + \bar{\Omega}^2\left(B_{21}\frac{\partial P_1}{\partial \dot{V}_P} - B_{11}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)\right]\bar{\ell}$$

$$K^A_{1,12} = \bar{\Omega}\bar{\ell}\left(B_{21}\frac{\partial P_1}{\partial V_P} - B_{31}\frac{\partial P_1}{\partial V_T} - B_{11}\frac{\partial P_1}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_1}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_1}{\partial \dot{V}_P} - \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_1}{\partial \dot{\omega}_1}\right)$$

$$K^A_{1,15} = \bar{\Omega}\bar{\ell}\phi\left[\left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial U_P} + \frac{\partial P_1}{\partial V_P}\right)H_{32} + \left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial U_T} + \frac{\partial P_1}{\partial V_T}\right)H_{22} + \left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial \dot{U}_P} + \frac{\partial P_1}{\partial \dot{V}_P}\right)\bar{\Omega}H_{31}\right]$$

$$K^A_{1,16} = \bar{\Omega}\bar{\ell}\phi\left[\left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial U_P} + \frac{\partial P_1}{\partial V_P}\right)H_{31} + \left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial U_T} + \frac{\partial P_1}{\partial V_T}\right)H_{21} - \left(\bar{\ell}\bar{u}\frac{\partial P_1}{\partial \dot{U}_P} + \frac{\partial P_1}{\partial \dot{V}_P}\right)\bar{\Omega}H_{32}\right]$$

$$K^A_{2,1} = -K^A_{1,2}$$

$$K^A_{2,2} = K^A_{1,1}$$

$$K^A_{2,3} = -K^A_{1,4}$$

$$K^A_{2,4} = K^A_{1,3}$$

$$K^A_{2,5} = -K^A_{1,6}$$

$$K^A_{2,6} = K^A_{1,5} \qquad K^A_{2,11} = -K^A_{1,12}$$

$$K^A_{2,7} = -K^A_{1,8} \qquad K^A_{2,12} = K^A_{1,11}$$

$$K^A_{2,8} = K^A_{1,7} \qquad K^A_{2,15} = K^A_{1,16}$$

$$K^A_{2,9} = -K^A_{1,10} \qquad K^A_{2,16} = -K^A_{1,15}$$

$$K^A_{2,10} = K^A_{1,9}$$

Rows 3 and 4 of K^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_2.

Rows 5 and 6 of K^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by P_3.

Rows 7 and 8 of K^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\zeta/\bar{\ell}$.

Rows 9 and 10 of K^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\beta/\bar{\ell}$.

Rows 11 and 12 of K^A are identical to rows 1 and 2, respectively, except that P_1 is replaced by $Q_\theta/\bar{\ell}$.

$$K^A_{13,1} = -\bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}(F_{12}H_{31} - F_{11}H_{32}) + \frac{\partial P_x}{\partial U_T}(F_{12}H_{21} - F_{11}H_{22}) + \bar{\Omega}G_{31}\frac{\partial P_x}{\partial \dot{U}_P}\right.$$
$$\left. - \frac{\partial P_y}{\partial U_P}G_{31} - \frac{\partial P_y}{\partial U_T}G_{21} + \bar{\Omega}(F_{12}H_{31} - F_{11}H_{32})\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$K^A_{13,2} = \bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}G_{31} + \frac{\partial P_x}{\partial U_T}G_{21} - \bar{\Omega}(F_{12}H_{31} - F_{11}H_{32})\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial P_y}{\partial U_P}(F_{12}H_{31} - F_{11}H_{32})\right.$$
$$\left. + \frac{\partial P_y}{\partial U_T}(F_{12}H_{21} - F_{11}H_{22}) + \bar{\Omega}G_{31}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

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$$K^A_{13,3} = -\bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}(F_{22}H_{31} - F_{21}H_{32}) + \frac{\partial P_x}{\partial U_T}(F_{22}H_{21} - F_{21}H_{22}) + \bar{\Omega}G_{32}\frac{\partial P_x}{\partial \dot{U}_P}\right.$$

$$\left. - \frac{\partial P_y}{\partial U_P}G_{32} - \frac{\partial P_y}{\partial U_T}G_{22} + \bar{\Omega}(F_{22}H_{31} - F_{21}H_{32})\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$K^A_{13,4} = \bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}G_{32} + \frac{\partial P_x}{\partial U_T}G_{22} - \bar{\Omega}(F_{22}H_{31} - F_{21}H_{32})\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial P_y}{\partial U_P}(F_{22}H_{31} - F_{21}H_{32})\right.$$

$$\left. + \frac{\partial P_y}{\partial U_T}(F_{22}H_{21} - F_{21}H_{22}) + \bar{\Omega}G_{32}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$K^A_{13,5} = -\bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}(F_{32}H_{31} - F_{31}H_{32}) + \frac{\partial P_x}{\partial U_T}(F_{32}H_{21} - F_{31}H_{22}) + \bar{\Omega}G_{33}\frac{\partial P_x}{\partial \dot{U}_P}\right.$$

$$\left. - \frac{\partial P_y}{\partial U_P}G_{33} - \frac{\partial P_y}{\partial U_T}G_{23} + \bar{\Omega}(F_{32}H_{31} - F_{31}H_{32})\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$K^A_{13,6} = \bar{\Omega}\bar{\ell}\left[\frac{\partial P_x}{\partial U_P}G_{33} + \frac{\partial P_x}{\partial U_T}G_{23} - \bar{\Omega}(F_{32}H_{31} - F_{31}H_{32})\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial P_y}{\partial U_P}(F_{32}H_{31} - F_{31}H_{32})\right.$$

$$\left. + \frac{\partial P_y}{\partial U_T}(F_{32}H_{21} - F_{31}H_{22}) + \bar{\Omega}G_{33}\frac{\partial P_y}{\partial \dot{U}_P}\right]$$

$$K^A_{13,7} = P_x{}^{\zeta} + \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \zeta}\frac{\partial P_x}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \zeta}\frac{\partial P_x}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_x}{\partial \omega_1}$$

$$+ \bar{\Omega}^2\left(G_{23}\frac{\partial P_x}{\partial \dot{V}_P} - G_{13}\frac{\partial P_x}{\partial \dot{\omega}_1}\right) - \bar{\Omega}\left(G_{23}\frac{\partial P_y}{\partial V_P} - G_{33}\frac{\partial P_y}{\partial V_T} - G_{13}\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{U}_P}\right.$$

$$\left. - \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$K^A_{13,8} = -\bar{\Omega}\left(G_{23}\frac{\partial P_x}{\partial V_P} - G_{33}\frac{\partial P_x}{\partial V_T} - G_{13}\frac{\partial P_x}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_x}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \dot{\omega}_1} - P_y{}^{\zeta}$$

$$- \frac{\partial U_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial U_P} - \frac{\partial V_{P_0}}{\partial \zeta}\frac{\partial P_y}{\partial V_P} - \frac{\partial U_{T_0}}{\partial \zeta}\frac{\partial P_y}{\partial U_T} - \frac{\partial V_{T_0}}{\partial \zeta}\frac{\partial P_y}{\partial V_T} - \bar{\Omega}\frac{\partial H_{13}}{\partial \zeta}\frac{\partial P_y}{\partial \omega_1}$$

$$- \bar{\Omega}^2\left(G_{23}\frac{\partial P_y}{\partial \dot{V}_P} - G_{13}\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$K^A_{13,9} = P_x{}^{\beta} + \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \beta}\frac{\partial P_x}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \beta}\frac{\partial P_x}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_x}{\partial \omega_1}$$

$$- \bar{\Omega}^2\left(\eta_2\frac{\partial P_x}{\partial \dot{V}_P} - \eta_1\frac{\partial P_x}{\partial \dot{\omega}_1}\right) - \bar{\Omega}\left(-\eta_2\frac{\partial P_y}{\partial V_P} + \eta_3\frac{\partial P_y}{\partial V_T} + \eta_1\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial \dot{U}_P}\right.$$

$$\left. - \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$K^A_{13,10} = -\bar{\Omega}\left(-\eta_2\frac{\partial P_x}{\partial V_P} + \eta_3\frac{\partial P_x}{\partial V_T} + \eta_1\frac{\partial P_x}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_x}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$- P_y{}^{\beta} - \frac{\partial U_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial U_P} - \frac{\partial V_{P_0}}{\partial \beta}\frac{\partial P_y}{\partial V_P} - \frac{\partial U_{T_0}}{\partial \beta}\frac{\partial P_y}{\partial U_T} - \frac{\partial V_{T_0}}{\partial \beta}\frac{\partial P_y}{\partial V_T} - \bar{\Omega}\frac{\partial H_{13}}{\partial \beta}\frac{\partial P_y}{\partial \omega_1}$$

$$+ \bar{\Omega}^2\left(\eta_2\frac{\partial P_y}{\partial \dot{V}_P} - \eta_1\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$K^A_{13,11} = P_x{}^{\theta} + \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial U_P} + \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial V_P} + \frac{\partial U_{T_0}}{\partial \theta}\frac{\partial P_x}{\partial U_T} + \frac{\partial V_{T_0}}{\partial \theta}\frac{\partial P_x}{\partial V_T} + \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_x}{\partial \omega_1}$$

$$+ \bar{\Omega}^2\left(B_{21}\frac{\partial P_x}{\partial \dot{V}_P} - B_{11}\frac{\partial P_x}{\partial \dot{\omega}_1}\right) - \bar{\Omega}\left(B_{21}\frac{\partial P_y}{\partial V_P} - B_{31}\frac{\partial P_y}{\partial V_T} - B_{11}\frac{\partial P_y}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial \dot{U}_P}\right.$$

$$\left. - \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_y}{\partial \dot{\omega}_1}$$

$$K^A_{13,12} = -\bar{\Omega}\left(B_{21}\frac{\partial P_x}{\partial V_P} - B_{31}\frac{\partial P_x}{\partial V_T} - B_{11}\frac{\partial P_x}{\partial \omega_1} - \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial \dot{U}_P} - \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_x}{\partial \dot{V}_P}\right) + \bar{\Omega}^2\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_x}{\partial \dot{\omega}_1}$$

$$- P_{y}{}^{\theta} - \frac{\partial U_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial U_P} - \frac{\partial V_{P_0}}{\partial \theta}\frac{\partial P_y}{\partial V_P} - \frac{\partial U_{T_0}}{\partial \theta}\frac{\partial P_y}{\partial U_T} - \frac{\partial V_{T_0}}{\partial \theta}\frac{\partial P_y}{\partial V_T}$$

$$- \bar{\Omega}\frac{\partial H_{13}}{\partial \theta}\frac{\partial P_y}{\partial \omega_1} - \bar{\Omega}^2\left(B_{21}\frac{\partial P_y}{\partial \dot{V}_P} - B_{11}\frac{\partial P_y}{\partial \dot{\omega}_1}\right)$$

$$K^A_{13,15} = -\bar{\Omega}\phi\left[\left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial U_P} + \frac{\partial P_x}{\partial V_P}\right)H_{32} + \left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial U_T} + \frac{\partial P_x}{\partial V_T}\right)H_{22} + \bar{\Omega}\left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial P_x}{\partial \dot{V}_P}\right)H_{31}\right.$$

$$\left. - \left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial U_P} + \frac{\partial P_y}{\partial V_P}\right)H_{31} - \left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial U_T} + \frac{\partial P_y}{\partial V_T}\right)H_{21} + \bar{\Omega}\left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial \dot{U}_P} + \frac{\partial P_y}{\partial \dot{V}_P}\right)H_{32}\right]$$

$$K^A_{13,16} = -\bar{\Omega}\phi\left[\left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial U_P} + \frac{\partial P_x}{\partial V_P}\right)H_{31} + \left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial U_T} + \frac{\partial P_x}{\partial V_T}\right)H_{21} - \bar{\Omega}\left(\bar{\ell}\bar{u}\frac{\partial P_x}{\partial \dot{U}_P} + \frac{\partial P_x}{\partial \dot{V}_P}\right)H_{32}\right.$$

$$\left. + \left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial U_P} + \frac{\partial P_y}{\partial V_P}\right)H_{32} + \left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial U_T} + \frac{\partial P_y}{\partial V_T}\right)H_{22} + \bar{\Omega}\left(\bar{\ell}\bar{u}\frac{\partial P_y}{\partial \dot{U}_P} + \frac{\partial P_y}{\partial \dot{V}_P}\right)H_{31}\right]$$

$$K^A_{14,1} = K^A_{13,2}$$

$$K^A_{14,2} = -K^A_{13,1}$$

$$K^A_{14,3} = K^A_{13,4}$$

$$K^A_{14,4} = -K^A_{13,3}$$

$$K^A_{14,5} = K^A_{13,6}$$

$$K^A_{14,6} = -K^A_{13,5}$$

$$K^A_{14,7} = K^A_{13,8}$$

$$K^A_{14,8} = -K^A_{13,7}$$

$$K^A_{14,9} = K^A_{13,10} \qquad K^A_{14,12} = -K^A_{13,11}$$

$$K^A_{14,10} = -K^A_{13,9} \qquad K^A_{14,15} = -K^A_{13,16}$$

$$K^A_{14,11} = K^A_{13,12} \qquad K^A_{14,16} = K^A_{13,15}$$

Rows 15 and 16 of K^A are identical to rows 13 and 14, respectively, except that P_x and P_y are replaced by Q_x and Q_y, respectively, and the following terms are added to the elements indicated:

$$K^A_{15,1} = P_z{}^0F_{12} - P_y{}^0F_{13} + \ldots \qquad K^A_{16,1} = K^A_{15,2}$$

$$K^A_{15,2} = P_z{}^0F_{11} - P_x{}^0F_{13} + \ldots \qquad K^A_{16,2} = -K^A_{15,1}$$

$$K^A_{15,3} = P_z{}^0F_{22} - P_y{}^0F_{23} + \ldots \qquad K^A_{16,3} = K^A_{15,4}$$

$$K^A_{15,4} = P_z{}^0F_{21} - P_x{}^0F_{23} + \ldots \qquad K^A_{16,4} = -K^A_{15,3}$$

$$K^A_{15,5} = P_z{}^0F_{32} - P_y{}^0F_{33} + \ldots \qquad K^A_{16,5} = K^A_{15,6}$$

$$K^A_{15,6} = P_z{}^0F_{31} - P_x{}^0F_{33} + \ldots \qquad K^A_{16,6} = -K^A_{15,5}$$

The quantities appearing in the elements of the above matrices not defined elsewhere in the report are defined as follows:

$$J_{12} = \bar{h}F_{12} + (\bar{\ell}\bar{v} + \bar{c}_2)F_{31} - (\bar{\ell}\bar{w} + \bar{c}_3)F_{21}$$

$$J_{11} = \bar{h}F_{11} - (\bar{\ell}\bar{v} + \bar{c}_2)F_{32} + (\bar{\ell}\bar{w} + \bar{c}_3)F_{22}$$

$$J_{22} = \bar{h}F_{22} + (\bar{\ell}\bar{w} + \bar{c}_3)F_{11} - (\bar{\ell}\bar{u} + \bar{c}_1)F_{31}$$

$$J_{21} = \bar{h}F_{21} - (\bar{\ell}\bar{w} + \bar{c}_3)F_{12} + (\bar{\ell}\bar{u} + \bar{c}_1)F_{32}$$

$$J_{32} = \bar{h}F_{32} + (\bar{\ell}\bar{u} + \bar{c}_1)F_{21} - (\bar{\ell}\bar{v} + \bar{c}_2)F_{11}$$

$$J_{31} = \bar{h}F_{31} - (\bar{\ell}\bar{u} + \bar{c}_1)F_{22} + (\bar{\ell}\bar{v} + \bar{c}_2)F_{12}$$

$$\bar{M} = b\bar{m} + \bar{m}_f$$

$$\bar{m}_f = \frac{m_f L^2}{I}$$

$$\bar{z} = \frac{z}{L}$$

$$\bar{I}_x = \frac{I_x}{I}$$

$$\bar{I}_y = \frac{I_y}{I}$$

$$J_{13} = (\bar{\ell}\bar{u} + \bar{c}_1)F_{11} + (\bar{\ell}\bar{v} + \bar{c}_2)F_{21} + (\bar{\ell}\bar{w} + \bar{c}_3)F_{31}$$

$$J_{23} = (\bar{\ell}\bar{u} + \bar{c}_1)F_{12} + (\bar{\ell}\bar{v} + \bar{c}_2)F_{22} + (\bar{\ell}\bar{w} + \bar{c}_3)F_{32}$$

$$P_i = G_{2i}S_f + G_{3i}T_f$$

$$Q_\zeta = G_{13}M_m - G_{23}T_m + G_{33}S_m$$

$$Q_\beta = -\eta_1 M_m + \eta_2 T_m - \eta_3 S_m$$

$$Q_\theta = B_{11}M_m - B_{21}T_m + B_{31}S_m$$

$$P_x = H_{21}S_f + H_{31}T_f$$

$$P_y = H_{22}S_f + H_{32}T_f$$

$$P_z = H_{23}S_f + H_{33}T_f$$

$$Q_x = H_{11}M_m - H_{21}T_m + H_{31}S_m - \bar{J}_{33}P_y + \bar{J}_{23}P_z$$

$$Q_y = H_{12}M_m - H_{22}T_m + H_{32}S_m + \bar{J}_{33}P_x - \bar{J}_{13}P_z$$

$$\bar{J}_{33} = \bar{h} + \bar{\ell}(\bar{u}F_{13} + \bar{v}F_{23} + \bar{w}F_{33})$$

$$S_f = \frac{\gamma}{2}\left[V_{P_0} + 2U_{P_0} - \frac{\bar{c}}{2}\left(1 - 2x_a\right)\omega_{1_0}\right]U_P + \frac{\gamma}{3}\left[V_{P_0} + \frac{3}{2}U_{P_0}\right.$$

$$\left. - \frac{3\bar{c}}{8}\left(1 - 2x_a\right)\omega_{1_0}\right]V_P - \frac{\gamma d}{2}\left(V_{T_0} + 2U_{T_0}\right)U_T$$

$$- \frac{\gamma d}{3}\left(V_{T_0} + \frac{3}{2}U_{T_0}\right)V_T - \frac{\gamma\bar{c}}{8}\left(1 - 2x_a\right)\left(V_{P_0} + 2U_{P_0}\right)\omega_1$$

$$T_f = -\frac{\gamma}{4}\left(V_{T_0} + 2U_{T_0}\right)U_P - \frac{\gamma}{6}\left(V_{T_0} + \frac{3}{2}U_{T_0}\right)V_P - \frac{\gamma}{4}\left[V_{P_0} + 2U_{P_0} - \bar{c}(1 - 2x_a)\omega_{1_0}\right]U_T$$

$$- \frac{\gamma}{6}\left[V_{P_0} + \frac{3}{2}U_{P_0} - \frac{3\bar{c}}{4}(1 - 2x_a)\omega_{1_0}\right]V_T + \frac{\gamma\bar{c}}{8}(1 - 2x_a)\left(V_{T_0} + 2U_{T_0}\right)\omega_1 - \frac{\gamma\bar{c}}{8}\dot{U}_P$$

$$- \frac{\gamma\bar{c}}{16}\dot{V}_P + \frac{\gamma\bar{c}^2}{32}(1 - 4x_a)\dot{\omega}_1$$

$$S_m = \frac{\gamma}{3}\left[V_{P_0} + \frac{3}{2}U_{P_0} - \frac{3\bar{c}}{8}(1 - 2x_a)\omega_{1_0}\right]U_P + \frac{\gamma}{4}\left[V_{P_0} + \frac{4}{3}U_{P_0} - \frac{\bar{c}}{3}(1 - 2x_a)\omega_{1_0}\right]V_P$$

$$- \frac{\gamma d}{3}\left(V_{T_0} + \frac{3}{2}U_{T_0}\right)U_T - \frac{\gamma d}{4}\left(V_{T_0} + \frac{4}{3}U_{T_0}\right)V_T - \frac{\gamma\bar{c}}{12}(1 - 2x_a)\left(V_{P_0} + \frac{3U_{P_0}}{2}\right)\omega_1$$

$$T_m = -\frac{\gamma}{6}\left(V_{T_0} + \frac{3}{2}U_{T_0}\right)U_P - \frac{\gamma}{8}\left(V_{T_0} + \frac{4}{3}U_{T_0}\right)V_P - \frac{\gamma}{6}\left[V_{P_0} + \frac{3}{2}U_{P_0}\right.$$

$$\left. - \frac{3\bar{c}}{4}(1 - 2x_a)\omega_{1_0}\right]U_T - \frac{\gamma}{8}\left[V_{P_0} + \frac{4}{3}U_{P_0} - \frac{2\bar{c}}{3}(1 - 2x_a)\omega_{1_0}\right]V_T$$

$$+ \frac{\gamma\bar{c}}{12}(1 - 2x_a)\left(V_{T_0} + \frac{3U_{T_0}}{2}\right)\omega_1 - \frac{\gamma\bar{c}}{16}\dot{U}_P - \frac{\gamma\bar{c}}{24}\dot{V}_P + \frac{\gamma\bar{c}^2}{64}(1 - 4x_a)\dot{\omega}_1$$

$$M_m = -\frac{\gamma\bar{c}x_a}{4}\left(V_{T_0} + 2U_{T_0}\right)U_P - \frac{\gamma\bar{c}x_a}{6}\left(V_{T_0} + \frac{3U_{T_0}}{2}\right)V_P - \frac{\gamma\bar{c}}{4}\left[x_a\left(V_{P_0} + 2U_{P_0}\right)\right.$$

$$\left. + \frac{\bar{c}}{8}\omega_{1_0}(1 - 8x_a + 16x_a^2)\right]U_T - \frac{\gamma\bar{c}}{6}\left[x_a\left(V_{P_0} + \frac{3U_{P_0}}{2}\right) + \frac{3\bar{c}}{32}\omega_{1_0}(1 - 8x_a + 16x_a^2)\right]V_T$$

$$- \frac{\gamma\bar{c}^2}{64}\left(V_{T_0} + 2U_{T_0}\right)(1 - 8x_a + 16x_a^2)\omega_1 + \frac{\gamma\bar{c}^2}{32}(1 - 4x_a)\dot{U}_P + \frac{\gamma\bar{c}^2}{64}(1 - 4x_a)\dot{V}_P$$

$$- \frac{3\gamma\bar{c}^3}{256}\left(1 - \frac{16x_a}{3} + \frac{32x_a^2}{3}\right)\dot{\omega}_1$$

$$\bar{u} = 1 + \frac{u}{\ell} + \bar{e}F_{11}$$

$$\bar{v} = \frac{v}{\ell} + \bar{e}F_{21}$$

$$\bar{w} = \frac{w}{\ell} + \bar{e}F_{31}$$

$$\bar{J}^{H}_{32} = \bar{J}_{33}H_{32} - \bar{J}_{23}H_{33}$$

$$\bar{J}^{H}_{31} = \bar{J}_{33}H_{31} - \bar{J}_{13}H_{33}$$

$$\bar{J}^{H}_{22} = \bar{J}_{33}H_{22} - \bar{J}_{23}H_{23}$$

$$\bar{J}^{H}_{21} = \bar{J}_{33}H_{21} - \bar{J}_{13}H_{23}$$

$$P_{1}{}^{\zeta} = S_{f_0} \frac{\partial G_{21}}{\partial \zeta} + T_{f_0} \frac{\partial G_{31}}{\partial \zeta}$$

$$P_{1}{}^{\beta} = S_{f_0} \frac{\partial G_{21}}{\partial \beta} + T_{f_0} \frac{\partial G_{31}}{\partial \beta}$$

$$P_{1}{}^{\theta} = S_{f_0} \frac{\partial G_{21}}{\partial \theta} + T_{f_0} \frac{\partial G_{31}}{\partial \theta}$$

$$Q_{\zeta}{}^{\zeta} = M_{m_0} \frac{\partial G_{13}}{\partial \zeta} - T_{m_0} \frac{\partial G_{23}}{\partial \zeta} + S_{m_0} \frac{\partial G_{33}}{\partial \zeta}$$

$$Q_{\zeta}{}^{\beta} = M_{m_0} \frac{\partial G_{13}}{\partial \beta} - T_{m_0} \frac{\partial G_{23}}{\partial \beta} + S_{m_0} \frac{\partial G_{33}}{\partial \beta}$$

$$Q_{\zeta}{}^{\theta} = M_{m_0} \frac{\partial G_{13}}{\partial \theta} - T_{m_0} \frac{\partial G_{23}}{\partial \theta} + S_{m_0} \frac{\partial G_{33}}{\partial \theta}$$

$$Q_{\beta}{}^{\zeta} = 0$$

$$Q_{\beta}{}^{\beta} = 0$$

$$Q_{\beta}{}^{\theta} = M_{m_0}(s_\theta B_{12} + c_\theta B_{13}) - T_{m_0}(s_\theta B_{22} + c_\theta B_{23}) + S_{m_0}(s_\theta B_{32} + c_\theta B_{33})$$

$$Q_{\theta}{}^{\zeta} = 0$$

$$Q_{\theta}{}^{\beta} = 0$$

$$Q_{\theta}{}^{\theta} = 0$$

$$P_x{}^{\zeta} = S_{f_0} \frac{\partial H_{21}}{\partial \zeta} + T_{f_0} \frac{\partial H_{31}}{\partial \zeta}$$

$$P_x{}^{\beta} = S_{f_0} \frac{\partial H_{21}}{\partial \beta} + T_{f_0} \frac{\partial H_{31}}{\partial \beta}$$

$$P_x{}^{\theta} = S_{f_0} \frac{\partial H_{21}}{\partial \theta} + T_{f_0} \frac{\partial H_{31}}{\partial \theta}$$

$$P_y{}^{\zeta} = S_{f_0} \frac{\partial H_{22}}{\partial \zeta} + T_{f_0} \frac{\partial H_{32}}{\partial \zeta}$$

$$P_y{}^{\beta} = S_{f_0} \frac{\partial H_{22}}{\partial \beta} + T_{f_0} \frac{\partial H_{32}}{\partial \beta}$$

$$P_y{}^{\theta} = S_{f_0} \frac{\partial H_{22}}{\partial \theta} + T_{f_0} \frac{\partial H_{32}}{\partial \theta}$$

$$P_z{}^{\zeta} = S_{f_0} \frac{\partial H_{23}}{\partial \zeta} + T_{f_0} \frac{\partial H_{33}}{\partial \zeta}$$

$$P_z{}^{\beta} = S_{f_0} \frac{\partial H_{23}}{\partial \beta} + T_{f_0} \frac{\partial H_{33}}{\partial \beta}$$

$$P_z{}^{\theta} = S_{f_0} \frac{\partial H_{23}}{\partial \theta} + T_{f_0} \frac{\partial H_{33}}{\partial \theta}$$

$$Q_x{}^{\zeta} = M_{m_0} \frac{\partial H_{11}}{\partial \zeta} - T_{m_0} \frac{\partial H_{21}}{\partial \zeta} + S_{m_0} \frac{\partial H_{31}}{\partial \zeta} + \bar{J}_{23} P_z{}^{\zeta} - \bar{J}_{33} P_y{}^{\zeta}$$

$$Q_x{}^{\beta} = M_{m_0} \frac{\partial H_{11}}{\partial \beta} - T_{m_0} \frac{\partial H_{21}}{\partial \beta} + S_{m_0} \frac{\partial H_{31}}{\partial \beta} + \bar{J}_{23} P_z{}^{\beta} - \bar{J}_{33} P_y{}^{\beta}$$

$$Q_x{}^{\theta} = M_{m_0} \frac{\partial H_{11}}{\partial \theta} - T_{m_0} \frac{\partial H_{21}}{\partial \theta} + S_{m_0} \frac{\partial H_{31}}{\partial \theta} + \bar{J}_{23} P_z{}^{\theta} - \bar{J}_{33} P_y{}^{\theta}$$

$$Q_y{}^{\zeta} = M_{m_0} \frac{\partial H_{12}}{\partial \zeta} - T_{m_0} \frac{\partial H_{22}}{\partial \zeta} + S_{m_0} \frac{\partial H_{32}}{\partial \zeta} - \bar{J}_{13} P_z{}^{\zeta} + \bar{J}_{33} P_x{}^{\zeta}$$

$$Q_y{}^{\beta} = M_{m_0} \frac{\partial H_{12}}{\partial \beta} - T_{m_0} \frac{\partial H_{22}}{\partial \beta} + S_{m_0} \frac{\partial H_{32}}{\partial \beta} - \bar{J}_{13} P_z{}^{\beta} + \bar{J}_{33} P_x{}^{\beta}$$

$$Q_y{}^{\theta} = M_{m_0} \frac{\partial H_{12}}{\partial \theta} - T_{m_0} \frac{\partial H_{22}}{\partial \theta} + S_{m_0} \frac{\partial H_{32}}{\partial \theta} - \bar{J}_{13} P_z{}^{\theta} + \bar{J}_{33} P_x{}^{\theta}$$

$$P_x^0 = S_{f_0}H_{21} + T_{f_0}H_{31}$$

$$P_y^0 = S_{f_0}H_{22} + T_{f_0}H_{32}$$

$$P_z^0 = S_{f_0}H_{23} + T_{f_0}H_{33}$$

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APPENDIX A

FLEXIBLE TORQUE TUBE

The effect of a pitch-control system similar to that of the Boeing BMR (ref. 9) can be easily incorporated into the analysis. This device has the effect of placing an additional twisting moment at the tip of the flexbeam. This moment is proportional to $\theta - \theta_0$ where θ_0 is the control input (see fig. 4). We assume that the torque tube is sufficiently flexible in bending so that no bending moments are applied by the control system at the flexbeam tip. With torsion stiffness K_θ, the additional moment in the equilibrium generalized force expression is

$$\bar{F}_\theta = \bar{K}_\theta(\theta - \theta_0) + \ldots \tag{A1}$$

where $\bar{K}_\theta = K_\theta/I\Omega_0{}^2$. By changing θ_0, any pitch angle that is desired can be obtained from the iterative process described in the text. The stiffness matrix is also modified so that

$$K^S_{11,11} = K^S_{12,12} = K_{66} + \bar{K}_\theta \tag{A2}$$

Equations (A1) and (A2) show the modifications necessary to account for a flexible torque tube.

APPENDIX B

CANTILEVER PITCH ARM WITH FLEXIBLE PITCH LINK

A pitch-control system with geometry similar to those of references 7 and 8 may be easily added to the analysis described in the text. In figure 5, a schematic of this configuration is shown, and in figure 12 a more detailed schematic of the pitch-link geometry is shown. Here the blade and flexbeam are shown with $\zeta_f = \beta_f = \theta_f = u = v = w = \zeta = \beta = \theta = \zeta_b = \beta_b = \theta_b = 0$. The parameters u_1, v_1, w_1 define the position of the swashplate-end of the pitch link S with respect to point 0 in the n_1^k, n_2^k, n_3^k axis system (there are no k's, indicating that the geometry is the same for all k). The parameters x_1, y_1, and z_1 define the pitch-arm/pitch-link junction J with respect to point 0 in the n_x^k, n_y^k, n_z^k axis system. Geometric constraints may be formulated by writing the position vector of P in two different ways One way is by the flexbeam tip through the pitch arm:

$$\mathbf{r}^{P/0} = (1 + u_k)\mathbf{n}_1{}^k + v_k\mathbf{n}_2{}^k + w_k\mathbf{n}_3{}^k + x_1\mathbf{n}_x{}^k + y_1\mathbf{n}_y{}^k + z_1\mathbf{n}_z{}^k \tag{B1}$$

Another way is by the point S and through the pitch link

$$\mathbf{r}^{P/0} = u_1\mathbf{n}_1{}^k + v_1\mathbf{n}_2{}^k + w_1\mathbf{n}_3{}^k + (p_0 + p_k)(\sin\mu_k \sin\delta_k\mathbf{n}_1{}^k + \cos\mu_k \sin\delta_k\mathbf{n}_2{}^k + \cos\delta_k\mathbf{n}_3{}^k) \tag{B2}$$

Here, all lengths are made dimensionless by ℓ and the pitch-link length is p_0 (undeformed) + p_k (deformation) The angles μ_k and δ_k define the orientation of the pitch link, shown in figure 13. These equations yield geometric constraints as follows:

$$\left.\begin{aligned} f_x{}^k &= 1 + u_k + x_1T_{11}^k + y_1T_{21}^k + z_1T_{31}^k - u_1 - (p_0+p_k)\sin\mu_k \sin\delta_k = 0 \\ f_y{}^k &= v_k + x_1T_{12}^k + y_1T_{22}^k + z_1T_{32}^k - v_1 - (p_0 + p_k)\cos\mu_k \sin\delta_k = 0 \\ f_z{}^k &= w_k + x_1T_{13}^k + y_1T_{23}^k + z_1T_{33}^k - w_1 - (p_0 + p_k)\cos\delta_k = 0 \end{aligned}\right\} \tag{B3}$$

where $f_x{}^k$ is obtained by taking the dot product of (B1) and (B2) with $\mathbf{n}_1{}^k$; $f_y{}^k$ is obtained by taking the dot product of (B1) and (B2) with $\mathbf{n}_2{}^k$, $f_z{}^k$ is obtained by taking the dot product of (B1) and (B2) with $\mathbf{n}_3{}^k$

These geometric constraints form relationships between the geometric variables and enable us to eliminate superfluous variables from the analysis. Meirovitch (ref. 13) has shown that the generalized forces due to the constraints are

$$\alpha_{q_r}{}^k = \lambda_x{}^k \frac{\partial f_x{}^k}{\partial q_r{}^k} + \lambda_y{}^k \frac{\partial f_y{}^k}{\partial q_r{}^k} + \lambda_z{}^k \frac{\partial f_z{}^k}{\partial q_r{}^k} \qquad q_r{}^k = u_k, v_k, w_k, \zeta_k, \beta_k, \theta_k, \mu_k, \delta_k, p_k \tag{B4}$$

The $\lambda_x{}^k$, $\lambda_y{}^k$, $\lambda_z{}^k$ are Lagrange multipliers and the $\alpha_{q_r}{}^k$ are the generalized forces. Thus

$$\begin{aligned}
\alpha_{u_k} &= \lambda_x{}^k \\
\alpha_{v_k} &= \lambda_y{}^k \\
\alpha_{w_k} &= \lambda_z{}^k \\
\alpha_{\zeta_k} &= x_1\left(\lambda_y{}^k T_{11}^k - \lambda_x{}^k T_{12}^k\right) + y_1\left(\lambda_y{}^k T_{21}^k - \lambda_x{}^k T_{22}^k\right) + z_1\left(\lambda_y{}^k T_{31}^k - \lambda_x{}^k T_{32}^k\right) \\
\alpha_{\beta_k} &= -\left(\lambda_x{}^k c_{\zeta_k} + \lambda_y{}^k s_{\zeta_k}\right)\left(x_1 T_{13}^k + y_1 T_{23}^k + z_1 T_{33}^k\right) + \lambda_z{}^k\left[x_1 c_{\beta_k} - s_{\beta_k}\left(y_1 s_{\theta_k} + z_1 c_{\theta_k}\right)\right] \\
\alpha_{\theta_k} &= \lambda_x{}^k\left(y_1 T_{31}^k - z_1 T_{21}^k\right) + \lambda_y{}^k\left(y_1 T_{32}^k - z_1 T_{22}^k\right) + \lambda_z{}^k\left(y_1 T_{33}^k - z_1 T_{23}^k\right) \\
\alpha_{\mu_k} &= (p_0 + p_k)\sin\delta_k\left(\lambda_y{}^k \sin\mu_k - \lambda_x{}^k \cos\mu_k\right) = 0 \\
\alpha_{\delta_k} &= (p_0 + p_k)\left[-\cos\delta_k\left(\lambda_x{}^k \sin\mu_k + \lambda_y{}^k \cos\mu_k\right) + \lambda_z{}^k \sin\delta_k\right] = 0 \\
\alpha_{p_k} &= -\lambda_x{}^k \sin\mu_k \sin\delta_k - \lambda_y{}^k \cos\mu_k \sin\delta_k - \lambda_z{}^k \cos\delta_k = \bar{K}_p p_k \\
\bar{K}_p &\equiv \frac{K_p \ell^2}{I\Omega_0{}^2}
\end{aligned} \tag{B5}$$

Here, $\alpha_{\delta_k} = \alpha_{\mu_k} = \alpha_{p_k} - \bar{K}_p p_k = 0$ because the mass and inertia of the link are neglected. From $\alpha_{\mu_k} = \alpha_{\delta_k} = \alpha_{p_k} - \bar{K}_p p_k = 0$, the Lagrange multipliers, which are the unknown reactions at the point P, are

$$\left.\begin{aligned}
\lambda_x{}^k &= -\bar{K}_p p_k \sin\mu_k \sin\delta_k \\
\lambda_y{}^k &= -\bar{K}_p p_k \cos\mu_k \sin\delta_k \\
\lambda_z{}^k &= -\bar{K}_p p_k \cos\delta_k
\end{aligned}\right\} \tag{B6}$$

The relations (B6) and (B3) are six relations between the twelve variables u_k, v_k, w_k, ζ_k, β_k, θ_k, μ_k, δ_k, p_k, $\lambda_x{}^k$, $\lambda_y{}^k$, and $\lambda_z{}^k$. Thus, the blade system retains its six degrees of freedom. The scheme for calculating the equilibrium deflections needs to be modified slightly. Given u_{1^-}, v_{1^-}, w_{1^-}, x_{1^-}, y_{1^-}, z_{1^-}, and p_{0^-} and an estimate of u, $\bar{v}$, w, ζ, $\bar{\beta}$, and θ we can calculate μ, δ, p, equilibrium components of μ_k, δ_k, p_k, respectively, from (B3) and then $\lambda_x{}^k$, $\lambda_y{}^k$, and $\lambda_z{}^k$ from (B6). Now the total generalized forces including α_u, α_v, etc., from (B5), are known. Integration along the flexbeam will yield slopes and deflections to be included in the minimization scheme, equation (84). The iteration may include a variation in w_1, to achieve the desired pitch angle or thrust as an extra independent variable in the minimization scheme.

The perturbation equations are obtained by expressing equations (B3)-(B6) in perturbation from and eliminating $\tilde{\lambda}_{x_c}$, $\tilde{\lambda}_{x_s}$, $\tilde{\lambda}_{y_c}$, $\tilde{\lambda}_{y_s}$, $\tilde{\lambda}_{z_c}$, $\tilde{\lambda}_{z_s}$, $\tilde{u}_c$, $\tilde{u}_s$, $\tilde{v}_c$, $\tilde{v}_s$, $\tilde{w}_c$, and $\tilde{w}_s$. This operation yields

$$[R^T MR]\{X\} + [R^T CR]\{\dot{X}\} + [R^T KR + D]\{X\} = 0 \tag{B7}$$

where

$$\left.\begin{aligned}
[M] &= [M^I + M^A] \\
[C] &= [G^I + C^A + C^D] \\
[K] &= [K^I + K^S + K^g + K^A + K^D] \\
\{X\} &= \lfloor \mu_c, \mu_s, \delta_c, \delta_s, p_c, p_s, \zeta_c, \zeta_s, \beta_c, \beta_s, \theta_c, \theta_s, X, Y, \Phi_x, \Phi_y \rfloor^T \\
[R] &= \begin{bmatrix} R_2 & R_1 & 0 \\ 0 & I_6 & 0 \\ 0 & 0 & I_4 \end{bmatrix} \\
[I_4] &= 4\times 4 \text{ identity matrix} \\
[I_6] &= 6\times 6 \text{ identity matrix}
\end{aligned}\right\} \tag{B8}$$

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$$[R_i] = \begin{bmatrix} R^i_{11} & 0 & R^i_{12} & 0 & R^i_{13} & 0 \\ 0 & R^i_{11} & 0 & R^i_{12} & 0 & R^i_{13} \\ R^i_{21} & 0 & R^i_{22} & 0 & R^i_{23} & 0 \\ 0 & R^i_{21} & 0 & R^i_{22} & 0 & R^i_{23} \\ R^i_{31} & 0 & R^i_{32} & 0 & R^i_{33} & 0 \\ 0 & R^i_{31} & 0 & R^i_{32} & 0 & R^i_{33} \end{bmatrix} \qquad i = 1,\ 2$$

$$R^1_{11} = x_1T_{12} + y_1T_{22} + z_1T_{32}$$

$$R^1_{12} = c_\zeta(x_1T_{13} + y_1T_{23} + z_1T_{33})$$

$$R^1_{13} = z_1T_{21} - y_1T_{31}$$

$$R^1_{21} = -(x_1T_{11} + y_1T_{21} + z_1T_{31})$$

$$R^1_{22} = s_\zeta(x_1T_{13} + y_1T_{23} + z_1T_{33})$$

$$R^1_{23} = z_1T_{22} - y_1T_{32}$$

$$R^1_{31} = 0$$

$$R^1_{32} = -x_1c_\beta + s_\beta(y_1s_\theta + z_1c_\theta)$$

$$R^1_{33} = z_1T_{23} - y_1T_{33}$$

$$R^2_{11} = (p_0 + p)\cos\mu\sin\delta$$

$$R^2_{12} = (p_0 + p)\sin\mu\cos\delta$$

$$R^2_{13} = \sin\mu\sin\delta$$

$$R^2_{21} = -(p_0 + p)\sin\mu\sin\delta$$

$$R^2_{22} = (p_0 + p)\cos\mu\cos\delta$$

$$R^2_{23} = \cos\mu\sin\delta$$

(B8) continued

$$R^2_{31} = 0$$

$$R^2_{32} = -(p_0 + p)\sin \delta$$

$$R^2_{33} = \cos \delta$$

$$[D] = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

$$[D_1] = \begin{bmatrix} D^1_{11} & 0 & D^1_{12} & 0 & D^1_{13} & 0 \\ & D^1_{11} & 0 & D^1_{12} & 0 & D^1_{13} \\ & & D^1_{22} & 0 & D^1_{23} & 0 \\ \text{Symmetric} & & & D^1_{22} & 0 & D^1_{23} \\ & & & & D^1_{33} & 0 \\ & & & & & D^1_{33} \end{bmatrix}$$

$$D^1_{11} = (p_0 + p)(\bar{K}_p p + \lambda_z \cos \delta)$$

$$D^1_{12} = 0$$

$$D^1_{13} = 0$$

$$D^1_{22} = (p_0 + p)\bar{K}_p p$$

$$D^1_{23} = 0$$

$$D^1_{33} = \bar{K}_p$$

$$D^2_{11} = -\lambda_x R^1_{21} + \lambda_y R^1_{11}$$

$$D^2_{12} = -\lambda_x R^1_{22} + \lambda_y R^1_{12}$$

$$D^2_{13} = -\lambda_x R^1_{23} + \lambda_y R^1_{13}$$

$$D^2_{22} = -(\lambda_x c_\zeta + \lambda_y s_\zeta)R^1_{32} + \lambda_z (c_\zeta R^1_{12} + s_\zeta R^1_{22})$$

(B8) continued

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$$\left.\begin{aligned}
D^2_{23} &= -(\lambda_x c_\zeta + \lambda_y s_\zeta)R^1_{33} + \lambda_z(c_\zeta R^1_{13} + s_\zeta R^1_{23}) \\
D^2_{33} &= \lambda_x(y_1 T_{21} + z_1 T_{31}) + \lambda_y(y_1 T_{22} + z_1 T_{32}) + \lambda_z(y_1 T_{23} + z_1 T_{33}) \\
\lambda_x{}^k &= \lambda_x + \tilde{\lambda}_x{}^k \\
\lambda_y{}^k &= \lambda_y + \tilde{\lambda}_y{}^k \\
\lambda_z{}^k &= \lambda_z + \tilde{\lambda}_z{}^k \\
\tilde{\lambda}_{x_c} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\lambda}_x{}^k \cos \psi_k \\
\tilde{\lambda}_{x_s} &= \frac{2}{b}\sum_{k=1}^{b} \tilde{\lambda}_x{}^k \sin \psi_k \\
&\vdots
\end{aligned}\right\} \quad \text{(B8) concluded}$$

APPENDIX C

SNUBBER AND PITCH ARM WITH FLEXIBLE PITCH LINK AND SNUBBER CONNECTION

For the case of a pitch arm and snubber as in the Sikorsky UTTAS tail rotor (ref. 6), the geometry may be treated as two flexible pitch links with two pitch arms and two different sets of geometric variables: u_1, v_1, w_1, x_1, y_1, z_1, and p_{0_1}; and u_2, v_2, w_2, x_2, y_2, z_2, and p_{0_2}. The two dimensionless flexibilities are $\bar{K}_1$ and $\bar{K}_2$ and the deflections, analogous to appendix B, are $\mu_1{}^k$, $\delta_1{}^k$, $p_1{}^k$ and $\mu_2{}^k$, $\delta_2{}^k$, $p_2{}^k$. A typical configuration of this type is shown in figure 6.

The mathematical development of the generalized forces and the iterative equilibrium solution is identical with that of appendix B except that there is a respective contribution from both pitch arms.

$$\left.\begin{aligned}
f^k_{x_1} &= 1 + u_k - u_1 + x_1 T^k_{11} + y_1 T^k_{21} + z_1 T^k_{31} - (p_{0_1} + p_1{}^k)\sin \mu_1{}^k \sin \delta_1{}^k = 0 \\
f^k_{x_2} &= 1 + u_k - u_2 + x_2 T^k_{11} + y_2 T^k_{21} + z_2 T^k_{31} - (p_{0_2} + p_2{}^k)\sin \mu_2{}^k \sin \delta_2{}^k = 0 \\
&\quad \text{etc} \\
\alpha_{u_k} &= \lambda^k_{x1} + \lambda^k_{x2} \\
&\quad \text{etc} \\
\alpha_{\zeta_k} &= x_1(\lambda^k_{y_1} T^k_{11} - \lambda^k_{x_1} T^k_{12}) + y_1(\lambda^k_{y_1} T^k_{21} - \lambda^k_{x_1} T^k_{22}) + z_1(\lambda^k_{y_1} T^k_{31} - \lambda^k_{x_1} T^k_{32}) \\
&\quad + x_2(\lambda^k_{y_2} T^k_{11} - \lambda^k_{x_2} T^k_{12}) + y_2(\lambda^k_{y_2} T^k_{21} - \lambda^k_{x_2} T^k_{22}) + z_2(\lambda^k_{y_2} T^k_{31} - \lambda^k_{x_2} T^k_{32}) \\
&\quad \text{etc} \\
\lambda^k_{x_i} &= -\sin \mu_i{}^k \sin \delta_i{}^k \bar{K}_i p_i{}^k \qquad i = 1,\ 2 \\
&\quad \text{etc.}
\end{aligned}\right\} \qquad \text{(C1)}$$

Therefore, the equilibrium scheme is altered in a way similar to that of appendix B except that both pitch arms contribute. One pitch link will, of course, change the pitch angle where the other one remains fixed as a snubber.

The perturbation equations are given by

$$[M]\{x\} + [C]\{\dot{x}\} + [K + Q]\{x\} = 0 \qquad \text{(C2)}$$

where

$$\{X\} = \lfloor \tilde{u}_c, \tilde{u}_s, \tilde{v}_c, \tilde{v}_s, \tilde{w}_c, \tilde{w}_s, \tilde{\zeta}_c, \tilde{\zeta}_s, \tilde{\beta}_c, \tilde{\beta}_s, \tilde{\theta}_c, \tilde{\theta}_s, \tilde{X}, \tilde{Y}, \tilde{\phi}_x, \tilde{\phi}_y \rfloor^T$$

$$\left.\begin{aligned}
[Q] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_1 + Q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} I & I & 0 \\ S_1{}^T & S_2{}^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1^{-1} & 0 & 0 \\ 0 & E_2^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & S_1 & 0 \\ I & S_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
[Q_i] &= [D_2]_i \qquad i = 1, 2
\end{aligned}\right\} \tag{C3}$$

where the subscript i on the matrix $[D_2]$ refers to subscripting all the parameters in $[D_2]$ sub i (x_i, y_i, z_i, λ_{x_i}, λ_{y_i}, λ_{z_i}, etc.) and similarly for the following

$$\left.\begin{aligned}
[S_i] &= -[R_1]_i & i &= 1, 2 \\
[E_i] &= [A_i{}^T \; C_i^{-1} \; A_i] & i &= 1, 2 \\
[A_i] &= [R_2]_i & i &= 1, 2 \\
[C_i] &= [D_1]_i & i &= 1, 2
\end{aligned}\right\} \tag{C4}$$

See appendix B for the definitions of the matrices on the right-hand side of equation (C4).

APPENDIX D

STRUCTURAL DAMPING COEFFICIENTS

In section 5.5, the blade and body structural damping coefficients are used in the analysis. We first consider the rotor blade for $\bar{\Omega} = 0$ in vacuo It is desirable to specify a structural damping ratio isolated blade chordwise motion. The single blade motion is given by

$$\begin{bmatrix} \bar{m}\bar{\ell}^2 & \bar{m}\bar{\ell}x_b \\ \bar{m}\bar{\ell}x_b & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{v}} \\ \ddot{\tilde{\zeta}} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_\zeta \end{bmatrix} \begin{Bmatrix} \dot{\tilde{v}} \\ \dot{\tilde{\zeta}} \end{Bmatrix} + \begin{bmatrix} 12B & -6B \\ -6B & 4B \end{bmatrix} \begin{Bmatrix} \tilde{v} \\ \tilde{\zeta} \end{Bmatrix} = 0 \qquad \text{(D1)}$$

where c_ζ is unknown. We wish to choose c_ζ so that the motion is critically damped and then multiply by η_ζ, a dimensionless number given by the structural damping ratio. For critically damped motion we require that the eigenvalues of (D1) be a pair of complex conjugates $-\sigma \pm i\omega$ and a pair of negative real numbers $-a$, $-a$. The characteristic equation for such roots is

$$s^4 + 2(a+\sigma)s^3 + [(a+\sigma)^2 + 2a\sigma + \omega^2]s^2 + 2[a\sigma(a+\sigma) + a\omega^2]s + a^2(\sigma^2+\omega^2) = 0 \qquad \text{(D2)}$$

The characteristic equation for equation (D1) is

$$s^4 + p_3 c_\zeta s^3 + p_2 s^2 + p_1 c_\zeta s + p_0 = 0 \qquad \text{(D3)}$$

where

$$\left.\begin{aligned} p_0 &= p_1 B \\ p_1 &= \frac{12B}{\bar{m}\bar{\ell}^2(1 - \bar{m}x_b{}^2)} \\ p_2 &= p_1\left(1 + \bar{m}\bar{\ell}x_b + \frac{\bar{m}\bar{\ell}^2}{3}\right) \\ p_3 &= \frac{1}{1 - \bar{m}x_b{}^2} \end{aligned}\right\} \qquad \text{(D4)}$$

When the coefficients of equations (D2) and (D3) are equated, it can be shown that

$$c_\zeta = \frac{3a^4 + p_2 a^2 - p_0}{2a^3 p_3} \qquad \text{(D5)}$$

where a^2 is the single real, positive root of

$$a^6 + a^4\left(\frac{3p_1}{p_3} - p_2\right) + a^2\left(\frac{p_1 p_2}{p_3} - 3p_0\right) - \frac{p_1 p_0}{3} = 0 \qquad \text{(D6)}$$

It is difficult to achieve an accurate numerical solution, for small $\bar{\ell}$, to equations (D5) and (D6). An asymptotic expansion, however, presents no problem numerically and yields a simple formula accurate to second order in $\bar{\ell}$:

$$c_\zeta = \sqrt{B}\left[2 + \bar{m}\bar{\ell}x_b + \frac{\bar{m}\bar{\ell}^2}{4}\left(1 - \frac{4}{3}\,\bar{m}x_b{}^2\right) + 0(\bar{\ell}^3)\right] \qquad \text{(D7)}$$

which gives accuracy within 1% for $\bar{\ell} \le 0.27$.

The structural damping value is thus

$$c_\zeta = \eta_\zeta\sqrt{B}\left[2 + \bar{m}\bar{\ell}x_b + \frac{\bar{m}\bar{\ell}^2}{4}\left(1 - \frac{4\bar{m}x_b{}^2}{3}\right)\right]$$

where typically $0.005 \le \eta_\zeta \le 0.03$

For the body motion, we choose structural damping coefficients as if X, Y, Φ_x, and Φ_y motions were uncoupled:

$$c_X = 2\eta_X\sqrt{K^S_{13,13}M^I_{13,13}}$$

$$c_Y = 2\eta_Y\sqrt{K^S_{14,14}M^I_{14,14}}$$

$$c_{\Phi_x} = 2\eta_{\Phi_x}\sqrt{K^S_{15,15}M^I_{15,15}}$$

$$c_{\Phi_y} = 2\eta_{\Phi_y}\sqrt{K^S_{16,16}M^I_{16,16}}$$

Here η_X, η_Y, η_{Φ_x}, and η_{Φ_y} are the body structural damping coefficients, typically $0.005 \le \eta_X, \eta_Y, \eta_{\Phi_x}, \eta_{\Phi_y} \le 0.05$

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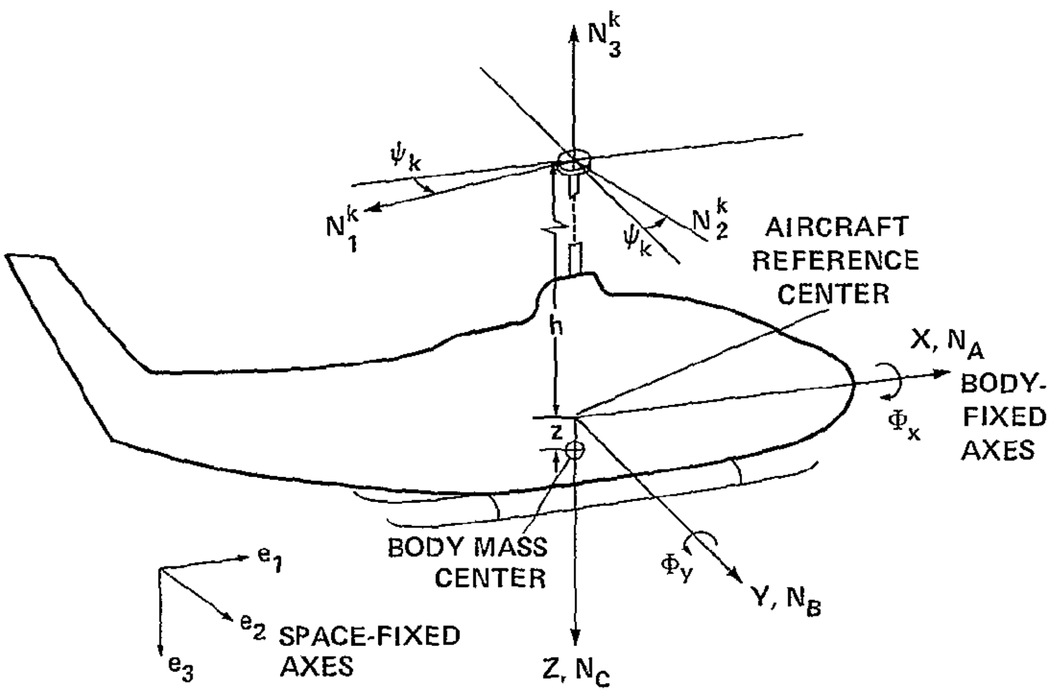

Figure 1 - Rotorcraft fuselage model.

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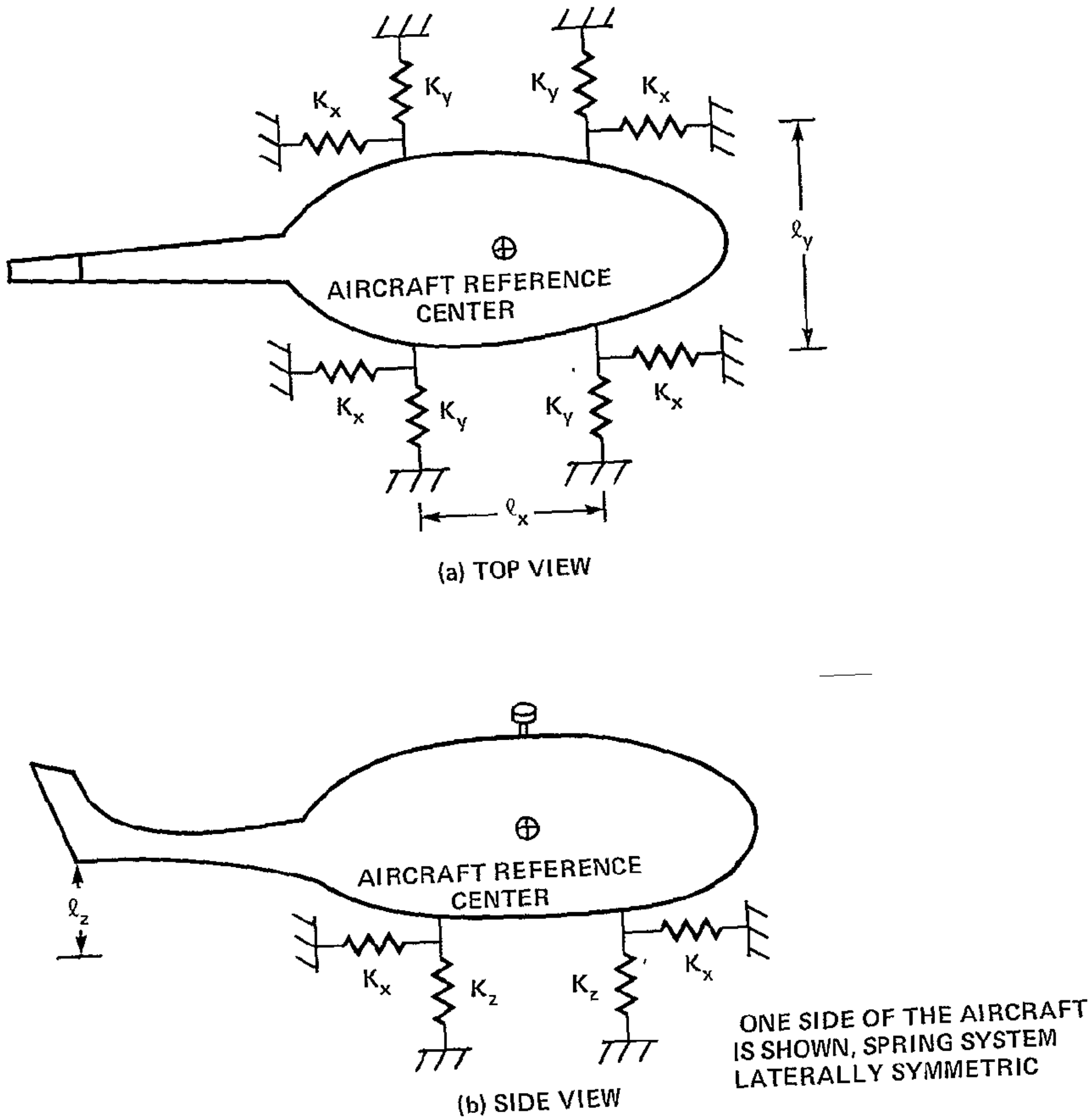

Figure 2 - Landing gear geometry.

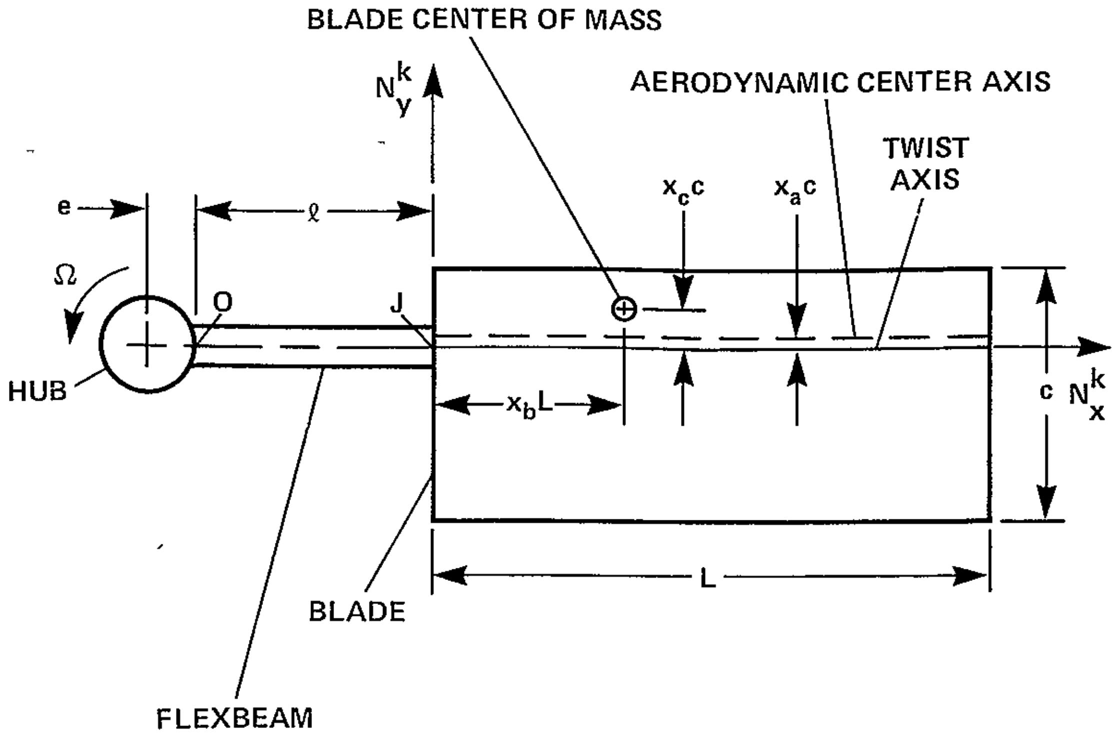

Figure 3.- Rotor blade configuration.

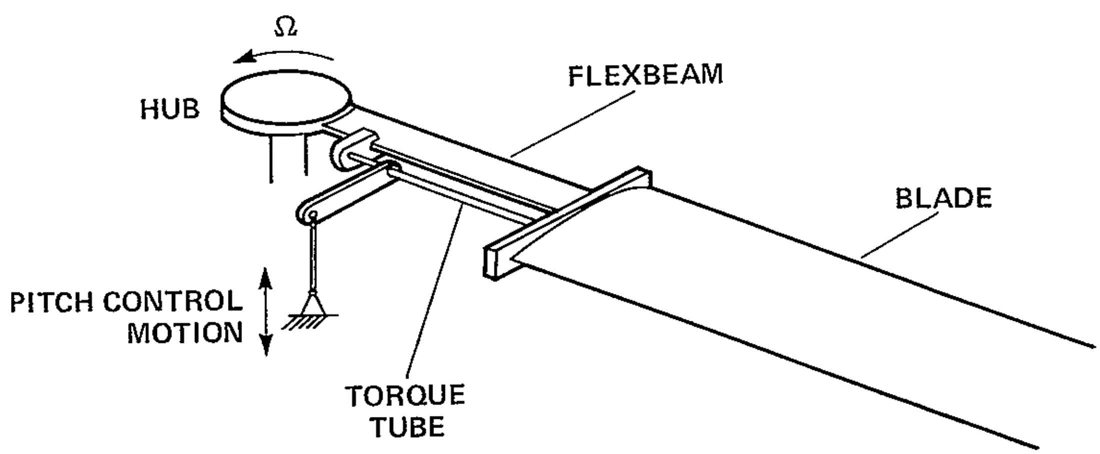

Figure 4.- Flexible torque tube, Case II.

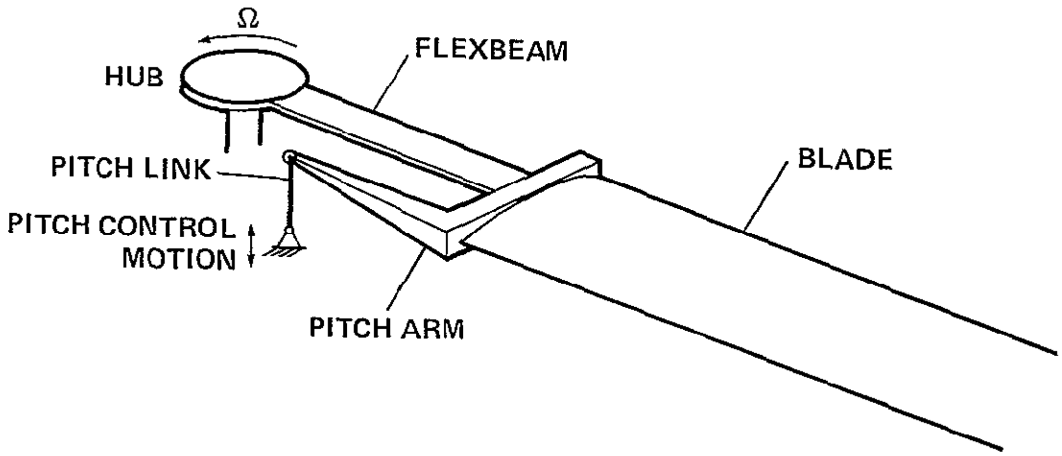

Figure 5.- Cantilever pitch arm, Case III.

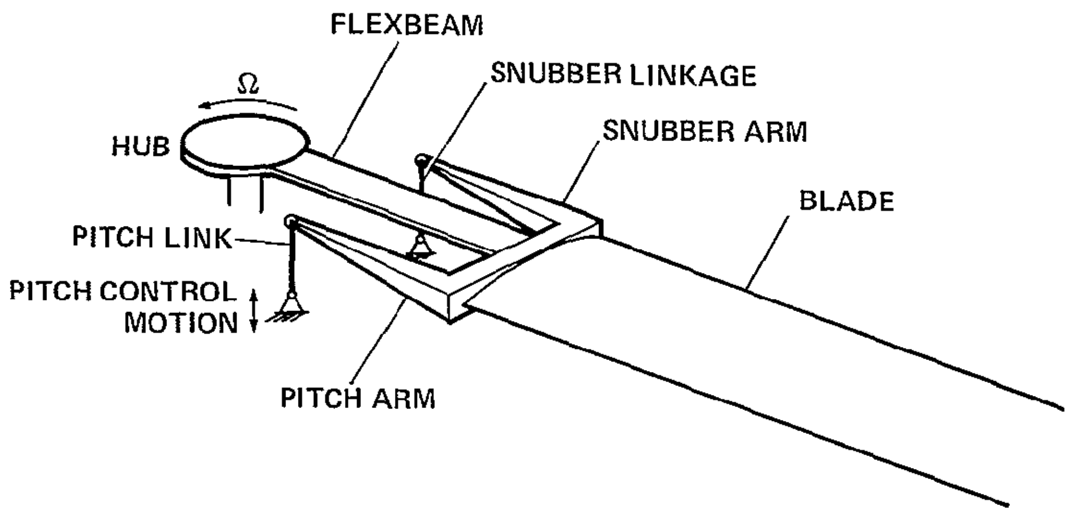

Figure 6.- Cantilever pitch arm with snubber, Case IV.

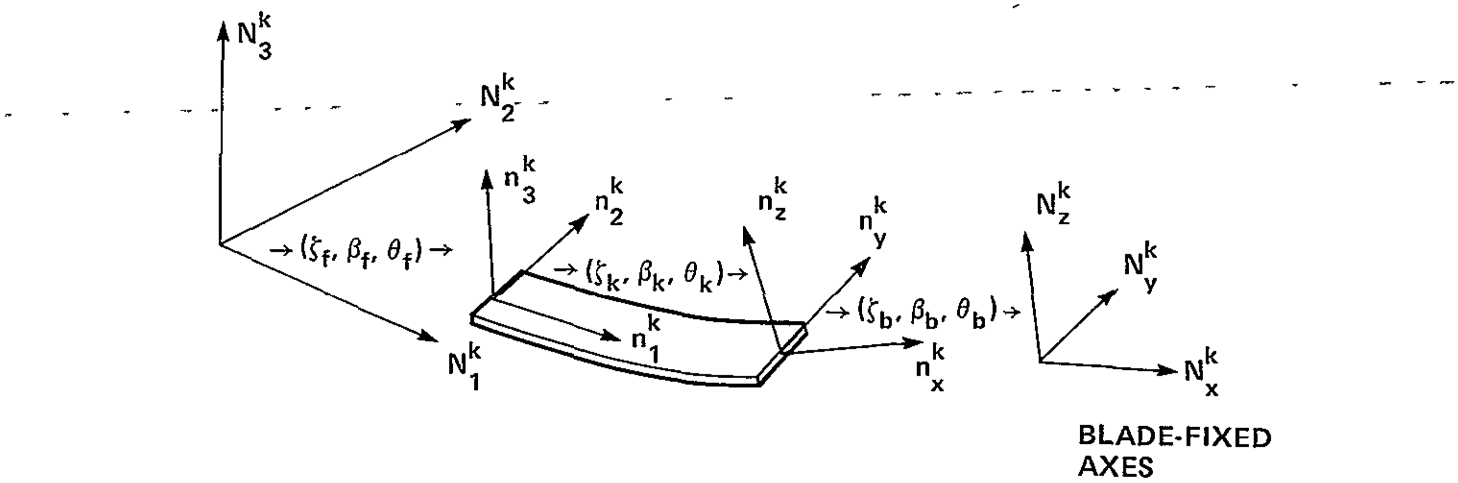

Figure 7.- Rotating rotor blade coordinate systems.

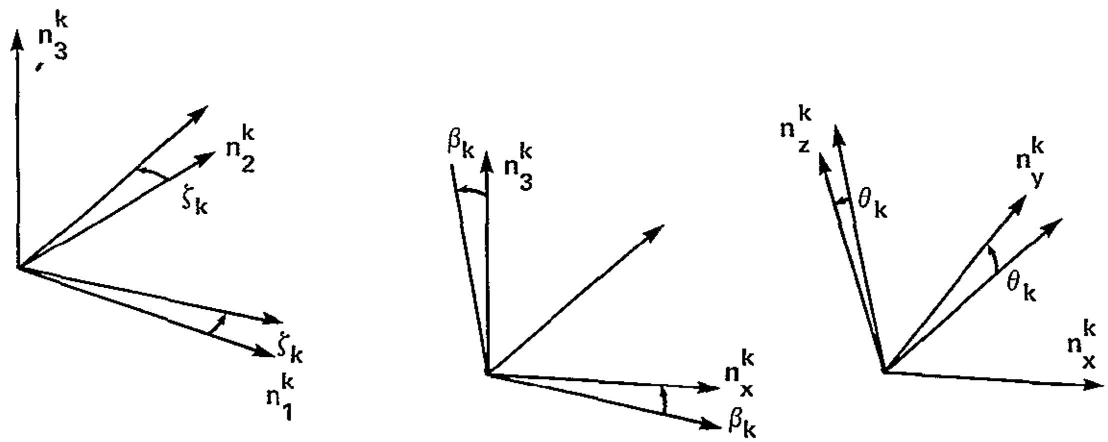

Figure 8.- Sequence of angles describing blade orientation.

C-2

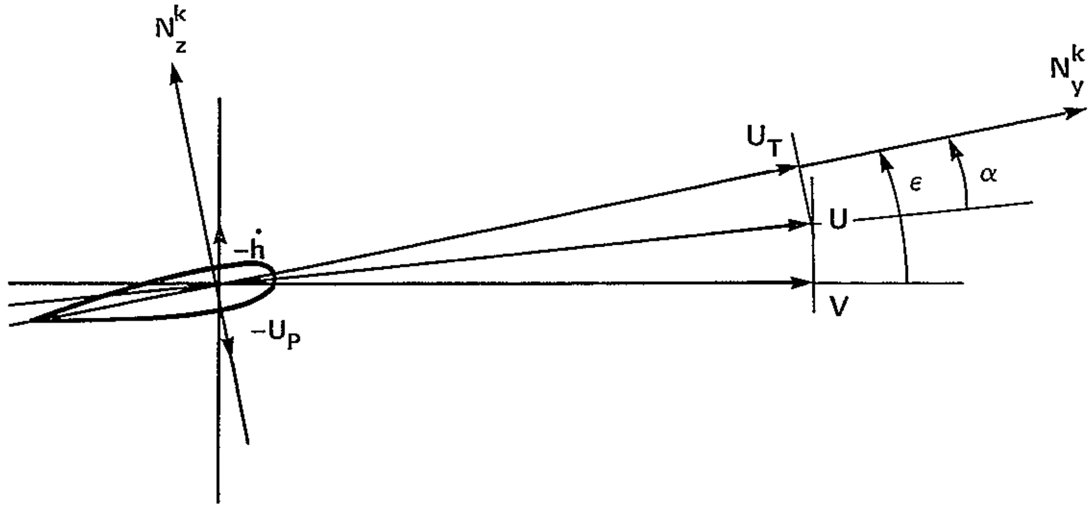

Figure 9.- Rotor blade airfoil section in general unsteady motion.

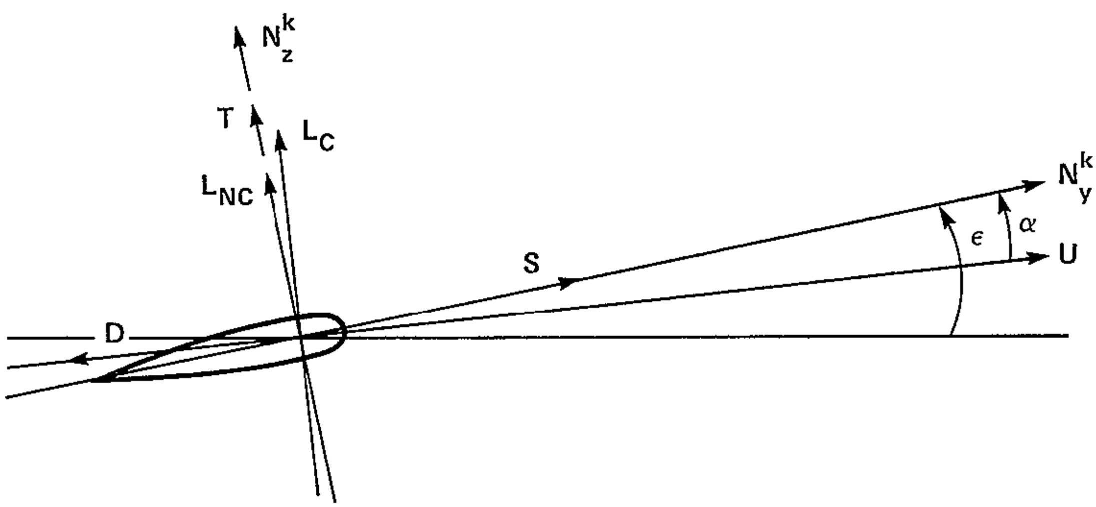

Figure 10.- Orientation of components of aerodynamic loading.

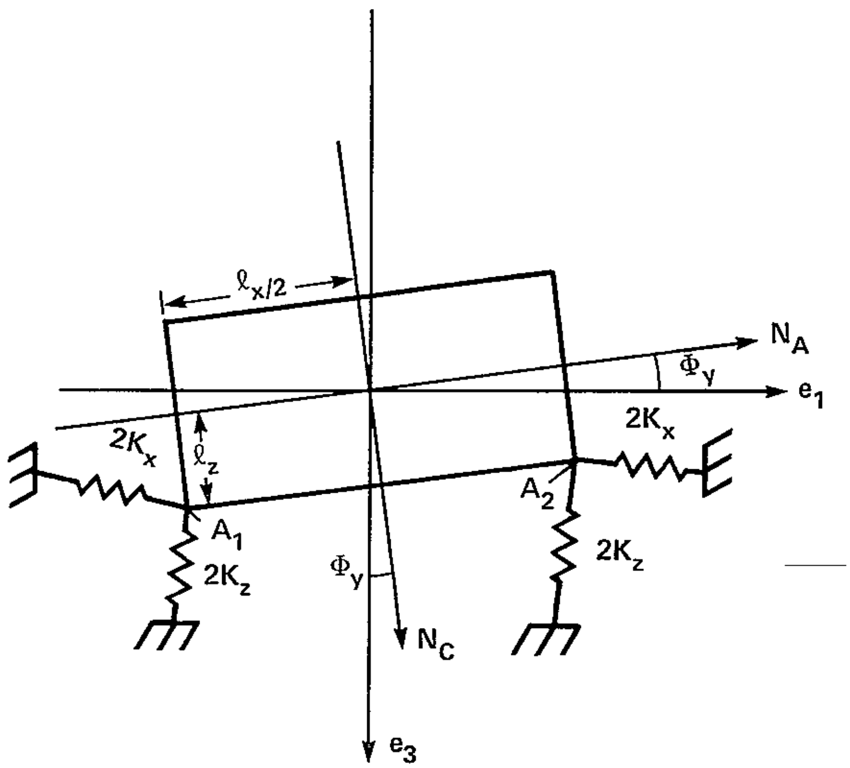

Figure 11.- Longitudinal plane of body-spring configuration.

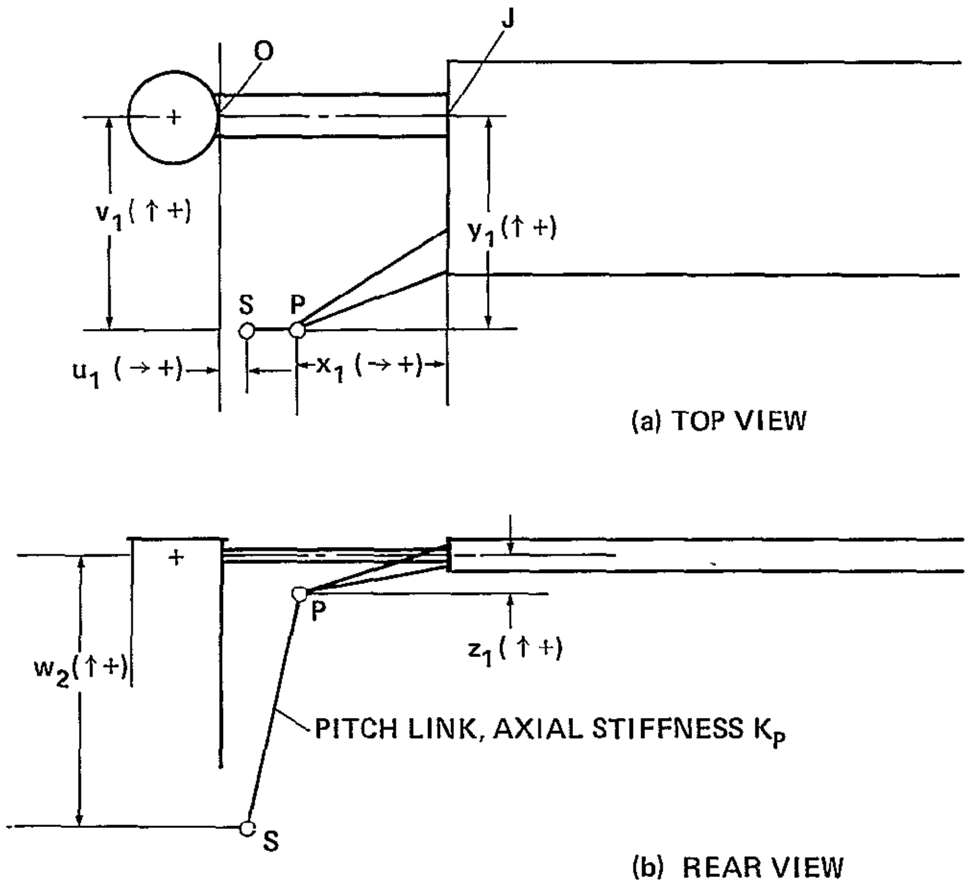

Figure 12.- Detailed schematic pitch link configuration.

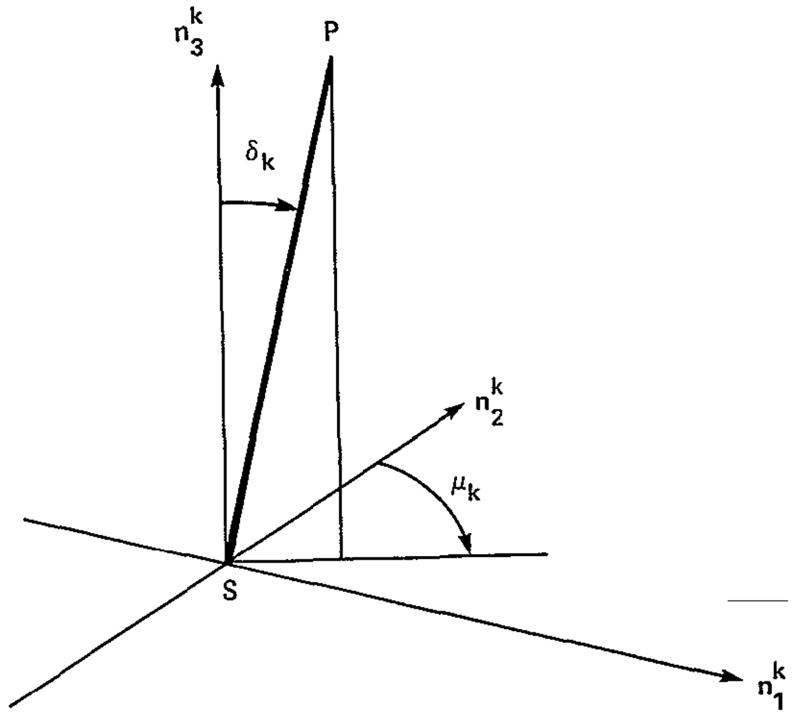

Figure 13.- Angles μ_k and δ_k describing pitch link orientation.

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16 Abstract

Equations of motion for a coupled rotor-body system are derived for the purpose of studying air and ground resonance characteristics of helicopters that have bearingless main rotors. For the fuselage, only four rigid body degrees of freedom are considered; longitudinal and lateral translations, pitch, and roll. The rotor is assumed to consist of three or more rigid blades. Each blade is joined to the hub by means of a flexible beam segment (flexbeam or strap). Pitch change is accomplished by twisting the flexbeam with the pitch-control system, the characteristics of which are variable. Thus, the analysis is capable of implicitly treating aeroelastic couplings generated by the flexbeam elastic deflections, the pitch-control system, and the angular offsets of the blade and flexbeam. The linearized equations are written in the nonrotating system retaining only the cyclic rotor modes; thus they comprise a system of homogeneous ordinary differential equations with constant coefficients. All contributions to the linearized perturbation equations from inertia, gravity, quasi-steady aerodynamics, and the flexbeam equilibrium deflections are retained exactly. Part II describes a computer program based on these equations of motion.

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