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On the Tidal Oscillations of the Liquid Core of the Earth

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ABSTRACT

An important goal of a tidal theory is the improvement of nutational amplitude and of the parameters of the Earth's elastic response. With this goal in mind, a theory of tidal oscillations inside a rotating elliptical Earth is developed, with special emphasis on tides in the liquid core. The Molodensky and Kramer theory of the resonance effect, as caused by the proximity of the frequency of the free diurnal wobble of the liquid core to the frequency of K_1 astronomical tide, was amended to include the effect of the possible deviation of the liquid core from the state of neutral stability. Coupling effects between the toroidal and spheroidal oscillations, as caused by the Coriolis force, are taken into consideration.

CONTENTS

BASIC NOTATIONS

$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Basic unit vectors of a rotating system of coordinates with the origin and directions rigidly fixed relative to the "initial" (mean) position of the mantle
$I = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$	Idemfactor
$\mathbf{r}$	Position vector of the Earth particle in its "initial" (mean) state, P_0
ρ	Initial density at P_0
λ, μ	Lamé elastic parameters at P_0
S	Initial stress at P_0
p	Hydrostatic pressure at P_0
P	Density at the displaced point in the moment t
$\mathbf{R}$	Position vector of the displaced point, P, at the moment t
$\mathbf{w}$	Absolute acceleration of the displaced point, P, at the moment t
$\mathbf{u} = \mathbf{R} - \mathbf{r}$	Elastic displacement of P_0 at the moment t
r, θ, φ	Spherical coordinates, where θ is the colatitude and φ is the longitude
$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$	Unit vectors along the coordinate lines in the spherical system
$S_n^m = e^{im\varphi} P_n^m (\cos\theta)$	Scalar surface spherical harmonic
$\mathbf{A}_n^m, \mathbf{B}_n^m, \mathbf{C}_n^m$	Vectorial spherical harmonics
$\phi \wedge \psi$	External product of two linear differential forms

τ	Stress in the Earth as induced by the lunisolar tidal forces
T	Total stress
$\epsilon = \frac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u})$	Strain dyadic (At a later stage, ϵ designates the ellipticity.)
$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u}$	Local vorticity, $I \times \boldsymbol{\omega} = \boldsymbol{\omega} \times I = \frac{1}{2}(\mathbf{u}\nabla - \nabla\mathbf{u})$
G	Constant of gravitation
$U(\mathbf{r})$	Initial force function of self-gravitation (per unit of mass) at P_0 (Force function of the centrifugal force is not included.)
$\phi_0 = \frac{1}{2}\mathbf{r} \cdot (I - \mathbf{k}\,\mathbf{k}) \cdot \mathbf{r}$	"Centrifugal" force function
$V(r) = U + \phi_0$	Total gravitational force function
W	Force function of the tidal attraction
ϕ	Tidal variation in geopotential, $\psi = \phi + W$
$\Omega = \Omega_0\ (\mathbf{k}+\mathbf{N})$	Instantaneous angular speed of rotation of the Earth
$\Omega_0\mathbf{k}$	Constant part of Ω
$\Omega_0\mathbf{N}$	Effect of nutation in Ω
$\epsilon(r)$	Ellipticity of the interior equipotential surface (considered as an ellipsoid of rotation)
a	Mean radius of the interior equipotential surface
β	Pekeris index of stability of the outer core

ON THE TIDAL OSCILLATIONS OF THE LIQUID CORE OF THE EARTH

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INTRODUCTION

From a parametrized model of the Earth, as obtained from seismic information, a model of the tides in the interior of the Earth can be determined. The understanding of the internal tidal processes is important from the astronomical and geophysical standpoints because they influence the amplitudes of nutation and the parameters of the Earth's elastic response (Love numbers), as observed on the Earth's surface. Thus, an important goal of a tidal theory is the improvement of the nutational amplitudes and of the Earth's elastic response—pure, uncontaminated by the effects of the free diurnal wobble of the liquid core. With this goal in mind, a theory of tidal oscillations inside a rotating elliptical Earth was developed, with emphasis on tides in the liquid core. It is planned to apply this theory to the actual computation of the tides, particularly in the outer core.

The present work can be considered as an extension (or a modification) of tidal theories by Molodensky (References 1 and 2), Molodensky and Kramer (Reference 3), and Shen and Mansinha (Reference 4).

Some of the final equations are identical to the final equations of Shen and Mansinha, and some are not. The differential equations for tidal amplitudes were deduced. Some of the equations by Shen and Mansinha clearly represent linear combinations of the equations for amplitudes.

In the exposition the vectorial and dyadic symbolism is being used, partly to obtain the differential equations in a compact form and partly to clarify the kinematics of numerous couplings between the tides of different degrees, as well as between the spheroidal and toroidal tides. The basic equations include the effect of the Earth's deviation from the hydrostatic equilibrium in the initial state and the influence of the vorticity on the stress tensor, making use of Biot's (Reference 5) and Dahlen's (Reference 6) formulations of the elasticity theory.

Satellite observations indicate the existence of a small deviatory prestress. Biot's and Dahlen's works pave the way for its inclusion when it is considered opportune.

The tides and tidal forces were expanded into a series of complex vectorial harmonics, both spheroidal and toroidal (References 7 and 8). This expansion presents no numerical complications because modern electronic machines are now able to handle functions of complex arguments without difficulty. Some relationships were also established between vectorial harmonics of different degrees. These relationships facilitate the expansion of the Coriolis force into a series of vectorial harmonics and, consequently, the formulations of the differential equations for tides. Further, they eliminate the customary use of 3j-Wigner symbols, which are not convenient for this problem. As in the work of Shen and Mansinha (Reference 4), the original Molodensky and Kramer equations (Reference 3) were amended by including the perturbative effects of the ellipticity of equipotential surfaces and the deviation of the liquid core from the neutral stability.

The established equations indicate precisely when and why the terms that contain the ellipticity as a factor can be neglected. Although the influence of the ellipticity on the astronomical tides is relatively small, it affects the free diurnal wobble of the liquid core. It must also be taken into account in the transfer of the normal component from the liquid core to the mantle through the ellispoidal core-mantle interface. The period of free diurnal oscillations of the liquid core is only 3 minutes short of the period of the Earth's rotation (Reference 4). This proximity of periods causes a strong resonance of the free diurnal oscillations of the liquid core with the diurnal astronomical tide K_1 and, to a lesser degree, with the diurnal astronomical tide, P_1 (References 9 and 10). The influence of this strong resonance contaminates the observed diurnal Love numbers. The removal of this contamination is an important geophysical problem because it helps to obtain the Earth's elastic response pure to the astronomical tidal forces only.

The resonance effect is one possible cause of the dependence of diurnal Love numbers on frequency. Observations with horizontal pendulums and gravimeters lead to different values of Love numbers for different diurnal tidal constituents (Reference 10). Recent values of Love numbers for the "whole Earth," as deduced from laser observations of GEOS-3, also display a marked dependence on frequency (References 11 and 12). Haardeng-Pedersen's computations (Reference 13), based on the model of a rotating Earth with the effects of nutation and diurnal wobble of the liquid core excluded, also show the dependence of Love and Shida numbers on frequency.

Poincaré found the existence of the free diurnal wobble of the elliptical liquid core (Reference 14), and Jeffreys discussed the geophysical implications of it (Reference 15). In their theory of the wobble, Molodensky and Kramer (Reference 3) assumed the neutral stability of the outer core and the validity of the Adams-Williamson condition (Reference 16). Shen and Mansinha, Haardeng-Pedersen, Pekeris and Accad, and Crossley (References 4, 13, 17, and 18) recently studied the dynamical consequences of the deviation of the liquid core from the neutral stability.

In the present work, Molodensky and Kramer's theory of resonance (Reference 3) was amended by including the perturbative effects proportional to the index of stability of the liquid core. Perturbations in the geopotential and gravity are controlled by the Poisson equation. If the Earth is spherically symmetric, then only spheroidal tidal components enter into the Poisson equation and produce a change in gravity and, because of the absence of lateral inhomogeneities and dilatational changes, toroidal oscillations do not perturb the gravity. In the elliptic liquid core, however, there are lateral variations in density, and toroidal amplitudes enter into the Poisson equation with the ellipticity as a factor. Thus, the toroidal oscillations can also produce the perturbations in geopotential and in gravity if they are sufficiently large and if there are marked lateral inhomogeneities in the structure of the Earth.

Computation of tides in the mantle and in the liquid core are essentially two different problems. In the mantle, tides are basically static, and only spheroidal tidal oscillations are easily observable. By comparison, the toroidal tidal oscillations in the mantle are small and are usually neglected. In the liquid core, the rigidity vanishes or is very small. As a result, the rotation of the Earth (Reference 19) and the ellipticity of equipotential surfaces induce toroidal tidal oscillations, as well as numerous couplings.

The differential equations given in this work contain the effects of the Coriolis force, of the ellipticity of equipotential surfaces, and of the couplings between the toroidal and spheroidal tidal oscillations, as well as the influence of the possible departure of the outer core from the neutral stability.

It is of interest to determine the perturbative influence of viscosity of the liquid core on tides and on amplitudes of nutation. However, this influence cannot be easily estimated. Information on the viscosity in the outer core is incomplete, and the estimates of the kinematic viscosity coefficient, ν, vary in a large interval. At the present time, the building of tidal models for different values of ν can only be attempted.

RECURSIVE FORMULAS FOR ASSOCIATED LEGENDRE FUNCTIONS AND THEIR DERIVATIVES

This section contains some recursive relations between the associated Legendre functions and their derivatives. These relations are used in performing the expansion of tidal oscillations and tidal forces into a series in vectorial harmonics.

Beginning with

$$xP_n^m(x) = \frac{n - m + 1}{2n + 1} P_{n+1}^m(x) + \frac{n + m}{2n + 1} P_{n-1}^m(x) \tag{1}$$

the following is derived:

$$x^2 P_n^m(x) = \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} P_{n+2}^m(x)$$

$$+ \frac{2n^2 - 2m^2 + 2n - 1}{(2n-1)(2n+3)} P_n^m(x) \tag{2}$$

$$+ \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} P_{n-2}^m(x)$$

and

$$P_2(x) P_n^m(x) = + \frac{3}{2} \cdot \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} P_{n+2}^m(x)$$

$$+ \frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} P_n^m(x) \tag{3}$$

$$+ \frac{3}{2} \cdot \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} P_{n-2}^m(x)$$

$$(1 - x^2) P_n^m(x) = - \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} P_{n+2}^m(x)$$

$$+ \frac{2n^2 + 2m^2 + 2n - 2}{(2n-1)(2n+3)} P_n^m(x) \tag{4}$$

$$- \frac{(n+m-1)(n+m)}{(2n+1)(2n-1)} P_{n-2}^m(x)$$

From

$$(1 - x^2) \frac{d P_n^m(x)}{dx} = (n+1) x P_n^m(x) - (n-m+1) P_{n+1}^m(x) \tag{5}$$

and, taking equation 1 into account,

$$(1 - x^2) \frac{d P_n^m (x)}{dx} = - \frac{n (n - m + 1)}{2n + 1} P_{n+1}^m (x) + \frac{(n + 1)(n + m)}{2n + 1} P_{n-1}^m (x) \tag{6}$$

Equations 1, 2, and 5 yield:

$$x (1 - x^2) \frac{d P_n^m (x)}{dx} = - \frac{n (n - m + 1)(n - m + 2)}{(2n + 1)(2n + 3)} P_{n+2}^m (x) + \frac{n^2 - 3m^2 + n}{(2n - 1)(2n + 3)} P_n^m (x) + \frac{(n + 1)(n + m)(n + m - 1)}{(2n + 1)(2n - 1)} P_{n-2}^m (x) \tag{7}$$

From the normalization conditions,

$$\int_{-1}^{+1} P_n^m (x) P_\nu^m (x)\, dx = \alpha_n \delta_{n\nu} \tag{8}$$

where

$$\alpha_n = \frac{2}{2n + 1} \cdot \frac{(n + m)!}{(n - m)!} \tag{9}$$

and $\delta_{n\nu}$ are Kronecker's deltas, and, taking equation 1 into consideration,

$$\int_{-1}^{+1} x\, P_n^m(x)\, P_\nu^m(x)\, dx = + \frac{n - m + 1}{2n + 1} \alpha_{n+1} \delta_{\nu, n+1} + \frac{n + m}{2n + 1} \alpha_{n-1} \delta_{\nu, n-1} \tag{10}$$

Substituting

$$A = P_n^m(x), \; B = P_\nu^m(x)$$

into the identity,

$$A \frac{d}{dx}\left[(1 - x^2) \frac{dB}{dx}\right] + B \frac{d}{dx}\left[(1 - x^2) \frac{dA}{dx}\right]$$

$$= \frac{d^2}{dx^2}\left[(1 - x^2) AB\right] - 2(1 - x^2) \frac{dA}{dx} \frac{dB}{dx} + 2 \frac{d}{dx}(xAB) \tag{11}$$

and taking into account

$$\frac{d}{dx}\left[(1 - x^2) \frac{d P_n^m(x)}{dx}\right] + \left[n(n + 1) - \frac{m^2}{1 - x^2}\right] P_n^m(x) = 0$$

$$\frac{d}{dx}\left[(1 - x^2) \frac{d P_\nu^m(x)}{dx}\right] + \left[\nu(\nu + 1) - \frac{m^2}{1 - x^2}\right] P_\nu^m(x) = 0$$

yields:

$$(1 - x^2) \frac{d P_n^m(x)}{dx} \cdot \frac{d P_\nu^m(x)}{dx} + \frac{m^2}{1 - x^2} P_n^m(x) P_\nu^m(x)$$

$$= \frac{1}{2} \frac{d^2}{dx^2}\left[(1 - x^2) P_n^m(x) P_\nu^m(x)\right] + \frac{1}{2}\left[n(n + 1) + \nu(\nu + 1)\right] P_n^m(x) P_\nu^m(x) \tag{12}$$

$$+ \frac{d}{dx}\left[x P_n^m(x) P_\nu^m(x)\right]$$

Multiplying equation 12 by x and integrating by parts yields:

$$\int_{-1}^{+1} x\left[(1-x^2)\frac{d P_n^m(x)}{dx}\cdot\frac{d P_\nu^m(x)}{dx}+\frac{m^2}{1-x^2}P_n^m(x)P_\nu^m(x)\right]dx$$

$$=+\frac{1}{2}\left[n(n+1)+\nu(\nu+1)-2\right]\int_{-1}^{+1} x\,P_n^m(x)P_\nu^m(x)\,dx$$

and, taking equation 10 into account,

$$\int_{-1}^{+1} x\left[(1-x^2)\frac{d P_n^m(x)}{dx}\cdot\frac{d P_\nu^m(x)}{dx}+\frac{m^2}{1-x^2}P_n^m(x)P_\nu^m(x)\right]dx$$

$$=\frac{n(n-m+1)(n+2)}{2n+1}\alpha_{n+1}\delta_{\nu,n+1} \tag{13a}$$

$$+\frac{(n+m)(n^2-1)}{2n+1}\alpha_{n-1}\delta_{\nu,n-1}$$

From equation 12, taking equation 8 into account,

$$\int_{-1}^{+1}\left[(1-x^2)\frac{d P_n^m(x)}{dx}\cdot\frac{d P_\nu^m(x)}{dx}+\frac{m^2}{1-x^2}P_n^m(x)P_\nu^m(x)\right]dx \tag{13b}$$

$$=n(n+1)\alpha_n\delta_{n\nu}$$

From equations 6 and 8, it follows that

$$\int_{-1}^{+1} (1 - x^2) \frac{d P_\nu^m (x)}{dx} P_n^m (x) = + \frac{(n + 2)(n + m + 1)}{2n + 3} \alpha_n \delta_{\nu,n+1} - \frac{(n - 1)(n - m)}{2n - 1} \alpha_n \delta_{\nu,n-1} \tag{14}$$

SPHERICAL VECTORIAL HARMONICS

In this work, the tidal oscillations and the tidal forces are expanded into a series in vectorial harmonics (References 3, 7, 8, 20, and 21):

$$\mathbf{A}_n^m (\theta, \varphi) = \mathbf{e}_r S_n^m = \mathbf{e}_r e^{+im\varphi} P_n^m (x) \tag{15}$$

$$\mathbf{B}_n^m (\theta, \varphi) = r \nabla S_n^m \tag{16}$$

$$= e^{+im\varphi} \left[- \mathbf{e}_\theta (1 - x^2)^{1/2} \frac{dP_n^m (x)}{dx} + im (1 - x^2)^{-1/2} \mathbf{e}_\varphi P_n^m (x) \right]$$

$$\mathbf{C}_n^m (\theta, \varphi) = - \mathbf{r} \times \nabla S_n^m \tag{17}$$

$$= e^{+im\varphi} \left[+ im (1 - x^2)^{-1/2} \mathbf{e}_\theta P_n^m (x) + \mathbf{e}_\varphi (1 - x^2)^{1/2} \frac{d P_n^m (x)}{dx} \right]$$

where

$$S_n^m = e^{+im\varphi} P_n^m (x)$$

$$x = \cos \theta \tag{18}$$

$$\mathbf{e}_r = \mathbf{i} \sin \theta \cos \varphi + \mathbf{j} \sin \theta \sin \varphi + \mathbf{k} \cos \theta$$

$$\mathbf{e}_\theta = + \mathbf{i} \cos \theta \cos \varphi + \mathbf{j} \cos \theta \sin \varphi - \mathbf{k} \sin \theta$$

$$\mathbf{e}_\varphi = - \mathbf{i} \sin \varphi + \mathbf{j} \cos \varphi$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\varphi, \mathbf{e}_\theta \times \mathbf{e}_\varphi = \mathbf{e}_r, \mathbf{e}_\varphi \times \mathbf{e}_r = \mathbf{e}_\theta \tag{19}$$

and

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \tag{20}$$

In the frame of the present (linearized) theory, there is no mutual influence between vectorial harmonics of different orders. For this reason, the index m in the exposition will be omitted in all cases in which ambiguity does not arise.

From equations 15 through 17, taking into account

$$\nabla^2 S_n = - n(n + 1) r^{-2} S_n \tag{21}$$

$$\mathbf{r} \cdot \nabla\nabla S_n = - \nabla S_n \tag{22}$$

yields

$$\nabla \cdot \mathbf{A}_n = + 2r^{-1} S_n \tag{23}$$

$$\nabla \times \mathbf{A}_n = + r^{-1} \mathbf{C}_n \tag{24}$$

$$\nabla\nabla \cdot \mathbf{A}_n = + 2r^{-2} (\mathbf{B}_n - \mathbf{A}_n) \tag{25}$$

$$\nabla \times \nabla \times \mathbf{A}_n = + n(n + 1) r^{-2} \mathbf{A}_n \tag{26}$$

$$\nabla^2 \mathbf{A}_n = + 2r^{-2} \mathbf{B}_n - (n^2 + n + 2) r^{-2} \mathbf{A}_n \tag{27}$$

$$\nabla \cdot \mathbf{B}_n = - n(n + 1) r^{-1} S_n \tag{28}$$

$$\nabla \times \mathbf{B}_n = - r^{-1} \mathbf{C}_n \tag{29}$$

$$\nabla\nabla \cdot \mathbf{B}_n = + n(n + 1) r^{-2} (\mathbf{A}_n - \mathbf{B}_n) \tag{30}$$

$$\nabla \times \nabla \times \mathbf{B}_n = - n(n + 1) r^{-2} \mathbf{A}_n \tag{31}$$

$$\nabla^2 \mathbf{B}_n = + n(n + 1) r^{-2} (2\mathbf{A}_n - \mathbf{B}_n) \tag{32}$$

$$\nabla \cdot \mathbf{C}_n = 0 \tag{33}$$

$$\nabla \times \mathbf{C}_n = r^{-1} [n(n+1)\mathbf{A}_n + \mathbf{B}_n] \tag{34}$$

$$\nabla\nabla \cdot \mathbf{C}_n = 0 \tag{35}$$

$$\nabla \times \nabla \times \mathbf{C}_n = + r^{-2} n(n+1)\mathbf{C}_n \tag{36}$$

$$\nabla^2 \mathbf{C}_n = - r^{-2} n(n+1)\mathbf{C}_n \tag{37}$$

Making use of

$$\mathbf{k} = + \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta$$

the following is derived from equations 15 through 17:

$$\mathbf{k} \cdot \mathbf{A}_n \times \mathbf{A}^*_{n'} = 0 \tag{38}$$

$$\mathbf{k} \cdot \mathbf{A}_n \times \mathbf{B}^*_{n'} = - im P^m_n(x) P^m_{n'}(x) \tag{39}$$

$$\mathbf{k} \cdot \mathbf{A}_n \times \mathbf{C}^*_{n'} = + (1 - x^2) \frac{d P^m_{n'}(x)}{dx} P^m_n(x) \tag{40}$$

$$\mathbf{k} \cdot \mathbf{B}_n \times \mathbf{A}^*_{n'} = - im P^m_n(x) P^m_{n'}(x) \tag{41}$$

$$\mathbf{k} \cdot \mathbf{B}_n \times \mathbf{B}^*_{n'} = + imx \frac{d}{dx} \left[P^m_n(x) P^m_{n'}(x) \right] \tag{42}$$

$$\mathbf{k} \cdot \mathbf{B}_n \times \mathbf{C}^*_{n'} = - x \left[(1 - x^2) \frac{d P^m_n(x)}{dx} \cdot \frac{d P^m_{n'}(x)}{dx} + \frac{m^2}{1 - x^2} P^m_n(x) P^m_{n'}(x) \right] \tag{43}$$

$$\mathbf{k} \cdot \mathbf{C}_n \times \mathbf{A}^*_{n'} = - (1 - x^2) \frac{dP^m_n(x)}{dx} P^m_{n'}(x) \tag{44}$$

$$\mathbf{k} \cdot \mathbf{C}_n \times \mathbf{B}^*_{n'} = + x \left[(1 - x^2) \frac{d P^m_n(x)}{dx} \cdot \frac{d P^m_{n'}(x)}{dx} + \frac{m^2}{1 - x^2} P^m_n(x) P^m_{n'}(x) \right] \tag{45}$$

$$\mathbf{k} \cdot \mathbf{C}_n \times \mathbf{C}^*_{n'} = + im x \frac{d}{dx} [P^m_n(x) P^m_{n'}(x)] \tag{46}$$

The asterisks designate complex conjugates. The use of equations 38 through 46 facilitates the expansion of the Coriolis force into a series of vectorial harmonics. In performing the expansions of tides and tidal forces into a series of vectorial harmonics, the inner product is used:

$$(\mathbf{F}, \mathbf{G}) = \int_{-1}^{+1} \mathbf{F} \cdot \mathbf{G}^* \, dx \tag{47}$$

where **F** and **G** must be of the same order. The dot designates the standard scalar product. From equations 13b and 15 through 17, the following normalization and orthogonality conditions are derived:

$$(\mathbf{A}_n, \mathbf{A}_n^*) = \alpha_n \, \delta_{nn'} \tag{48}$$

$$(\mathbf{B}_n, \mathbf{B}_{n'}) = (\mathbf{C}_n, \mathbf{C}_{n'}) = n(n+1)\alpha_n \, \delta_{nn'} \tag{49}$$

$$(\mathbf{A}_n, \mathbf{B}_{n'}) = (\mathbf{B}_n, \mathbf{C}_{n'}) = (\mathbf{C}_n, \mathbf{A}_{n'}) = 0 \tag{50}$$

and from equations 38 through 46, 8, 10, and 12 through 14, the following relations are derived:

$$(\mathbf{k} \times \mathbf{A}_n, \mathbf{A}_{n'}) = 0 \tag{51}$$

$$(\mathbf{k} \times \mathbf{A}_n, \mathbf{B}_{n'}) = -\, im \, \alpha_n \, \delta_{nn'} \tag{52}$$

$$\begin{aligned}(\mathbf{k} \times \mathbf{A}_n, \mathbf{C}_{n'}) = &+ \frac{(n+2)(n+m+1)}{2n+3} \alpha_n \, \delta_{n',n+1} \\ &- \frac{(n-1)(n-m)}{2n-1} \alpha_n \delta_{n',\, n-1}\end{aligned} \tag{53}$$

$$(\mathbf{k} \times \mathbf{B}_n, \mathbf{A}_{n'}) = -\, im \, \alpha_n \, \delta_{nn'} \tag{54}$$

$$(\mathbf{k} \times \mathbf{B}_n, \mathbf{B}_{n'}) = -\, im \, \alpha_n \, \delta_{nn'} \tag{55}$$

$$(\mathbf{k} \times \mathbf{B}_n, \mathbf{C}_{n'}) = - \frac{n(n-m+1)(n+2)}{2n+1} \alpha_{n+1} \delta_{n',n+1} - \frac{(n+m)(n^2-1)}{2n+1} \alpha_{n-1} \delta_{n',n-1} \tag{56}$$

$$(\mathbf{k} \times \mathbf{C}_n, \mathbf{A}_{n'}) = - \frac{(n+1)(n+m)}{2n+1} \alpha_{n-1} \delta_{n',n-1} + \frac{n(n-m+1)}{2n+1} \alpha_{n+1} \delta_{n',n+1} \tag{57}$$

$$(\mathbf{k} \times \mathbf{C}_n, \mathbf{B}_{n'}) = + \frac{n(n-m+1)(n+2)}{2n+1} \alpha_{n+1} \delta_{n',n+1} + \frac{(n+m)(n^2-1)}{2n+1} \alpha_{n-1} \delta_{n',n-1} \tag{58}$$

$$(\mathbf{k} \times \mathbf{C}_n, \mathbf{C}_{n'}) = - im \alpha_n \delta_{nn'} \tag{59}$$

which are useful in the process of expanding the geostrophic force.

The expansion of a vector, **F**, in terms of spherical vectorial harmonics has the form:

$$\mathbf{F} = \sum_n \frac{1}{\alpha_n} \left\{ (\mathbf{F}, \mathbf{A}_n) \mathbf{A}_n + \frac{1}{n(n+1)} \left[(\mathbf{F}, \mathbf{B}_n) \mathbf{B}_n + (\mathbf{F}, \mathbf{C}_n) \mathbf{C}_n \right] \right\} \tag{60}$$

where all vectors are of the same order. On several occasions, the series on the right-hand side is finite, and it will be simply a finite linear combination of vectorial harmonics. In particular, taking equations 51 through 59 into account,

$$\mathbf{k} \times \mathbf{A}_n = - \frac{im}{n(n+1)} \mathbf{B}_n - \frac{n+m}{n(2n+1)} \mathbf{C}_{n-1} + \frac{n-m+1}{(n+1)(2n+1)} \mathbf{C}_{n+1} \tag{61}$$

$$\mathbf{k} \times \mathbf{B}_n = -\, im\,\mathbf{A}_n - \frac{im}{n(n+1)}\,\mathbf{B}_n$$

$$- \frac{(n+1)(n+m)}{n(2n+1)}\,\mathbf{C}_{n-1} - \frac{n(n-m+1)}{(n+1)(2n+1)}\,\mathbf{C}_{n+1} \tag{62}$$

$$\mathbf{k} \times \mathbf{C}_n = - \frac{(n+1)(n+m)}{2n+1}\,\mathbf{A}_{n-1} + \frac{n(n-m+1)}{2n+1}\,\mathbf{A}_{n+1}$$

$$+ \frac{(n+1)(n+m)}{n(2n+1)}\,\mathbf{B}_{n-1} + \frac{n(n-m+1)}{(n+1)(2n+1)}\,\mathbf{B}_{n+1} - \frac{im}{n(n+1)}\,\mathbf{C}_n \tag{63}$$

DIFFERENTIAL EQUATIONS OF TIDAL OSCILLATIONS

Let v be the volume of a portion of the Earth in the "initial" state. Assume that in v and on its boundary surface, a, the density and elastic parameters are continuous functions of the position vector, **r**.

Let dv be the element of v, and let **da** be the oriented surface element of **a**. At the moment, t, under the influence of tidal forces, these quantities become V, A, P, **R**, dV, and d**A**, respectively. Assume that the tidal displacement

$$\mathbf{u} = \mathbf{R} - \mathbf{r}$$

is small and that all quantities of the second order in **u** can be neglected. From

$$d\mathbf{A} = \frac{1}{2}\, d\mathbf{R} \wedge d\mathbf{R}$$

$$= \frac{\partial \mathbf{R}}{\partial y} \times \frac{\partial \mathbf{R}}{\partial z}\, dy\, dz + \frac{\partial \mathbf{R}}{\partial z} \times \frac{\partial \mathbf{R}}{\partial x}\, dz\, dx + \frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y}\, dx\, dy \tag{64}$$

and

$$d\mathbf{a} = \frac{1}{2}\, d\mathbf{r} \wedge d\mathbf{r} = \mathbf{i}\, dy\, dz + \mathbf{j}\, dz\, dx + \mathbf{k}\, dx\, dy \tag{65}$$

the following is derived:

$$d\mathbf{A} = d\mathbf{a} \cdot \Lambda \tag{66}$$

where the dyadic Λ has the form:

$$\Lambda = \mathbf{i}\,\frac{\partial \mathbf{R}}{\partial y} \times \frac{\partial \mathbf{R}}{\partial z} + \mathbf{j}\,\frac{\partial \mathbf{R}}{\partial z} \times \frac{\partial \mathbf{R}}{\partial x} + \mathbf{k}\,\frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y} \tag{67}$$

Substituting

$$\mathbf{R} = \mathbf{r} + \mathbf{u}$$

into equation 67 and taking the identities,

$$I \times \mathbf{i} = \mathbf{kj} - \mathbf{jk}, I \times \mathbf{j} = \mathbf{ik} - \mathbf{ki}, I \times \mathbf{k} = \mathbf{ji} - \mathbf{ij}$$

into account, then neglecting the terms of the second order in $\mathbf{u}$,

$$\Lambda = I - (I \times \nabla) \times \mathbf{u} \tag{68}$$

Taking

$$(I \times \nabla) \times \mathbf{u} = \mathbf{u}\nabla - I\nabla \cdot \mathbf{u}$$

and

$$\mathbf{u}\nabla = \epsilon + I \times \boldsymbol{\omega}$$

into consideration,

$$\Lambda = (1 + \theta) I - \epsilon - I \times \boldsymbol{\omega} \tag{69}$$

Under the influence of vorticity, the prestress, S, will receive at the moment, t, the increment (Reference 5)

$$\boldsymbol{\omega} \times S - S \times \boldsymbol{\omega}$$

and the total-stress tensor becomes:

$$T = \tau + S + \boldsymbol{\omega} \times S - S \times \boldsymbol{\omega} \tag{70}$$

where τ is the stress induced by lunisolar tidal forces. Let U (**R**) be the force function of self-gravitation (with centrifugal effects *excluded*), and let W be the tidal-force function. Let ϕ be the change in the gravitational-force function as caused by a small redistribution of matter in the Earth's interior under the influence of tidal forces.

In this case, the principle of D'Alembert takes the form:

$$\int_V \left\{\nabla\,[U(\mathbf{R}) + \psi] - \mathbf{w}\right\} P\,dV + \int_A d\mathbf{A} \cdot T = 0 \tag{71}$$

where

$$\psi = W + \phi$$

$$\mathbf{w} = \frac{\partial^2 \mathbf{R}}{\partial t^2} + 2\Omega \times \frac{\partial \mathbf{R}}{\partial t} + \frac{d\Omega}{dt} \times \mathbf{R} + \Omega \times (\Omega \times \mathbf{R}) \tag{72}$$

is the absolute acceleration, and $\mathbf{\Omega}$ is the instantaneous speed of rotation. The gradients are taken relative to **R**. Taking

$$P\,dV = \rho\,dv$$

and equation 66 into account, equation 71 becomes:

$$\int_v \rho\,[\nabla U(\mathbf{r}) + \mathbf{u} \cdot \nabla\nabla U(\mathbf{r}) + \nabla\psi - \mathbf{w}]\,dv + \int_a d\mathbf{a} \cdot \Lambda \cdot T = 0 \tag{73}$$

in which the terms of higher order are neglected. By applying Gauss' theorem to equation 73, the differential equation of tidal oscillations takes the form:

$$\rho\mathbf{w} = \rho(\nabla U + \nabla\psi + \mathbf{u} \cdot \nabla\nabla U) + \nabla \cdot (\Lambda \cdot T) \tag{74}$$

which contains the influence of the deviatory prestress. From equations 69 and 70,

$$\Lambda \cdot T = \tau + (1 + \theta)S - S \times \boldsymbol{\omega} - \epsilon \cdot S \tag{75}$$

and

$$\nabla \cdot (\Lambda \cdot T) = \nabla \cdot \tau + (1 + \theta) \nabla \cdot S - (\nabla \cdot \epsilon - \nabla \theta) \cdot S$$

$$- (S \cdot \nabla) \times \omega - (\nabla \cdot S) \times \omega - \epsilon \cdot\cdot \nabla S$$

or taking into account

$$\nabla \cdot \epsilon = \nabla\theta - \nabla \times \omega$$

so that

$$\nabla \cdot (\Lambda \cdot T) = \nabla \cdot \tau + (1 + \theta) \nabla \cdot S + \omega \times \nabla \cdot S$$

$$+ (\nabla \times \omega) \cdot S - (S \cdot \nabla) \times \omega - \epsilon \cdot\cdot \nabla S$$

equation 74 becomes:

$$\rho \mathbf{w} = \rho \nabla (U + \psi) + \rho \mathbf{u} \cdot \nabla\nabla U + \nabla \cdot \tau + (1 + \theta) \nabla \cdot S$$
$$+ \omega \times (\nabla \cdot S) + (\nabla \times \omega) \cdot S - (S \cdot \nabla) \times \omega - \epsilon \cdot\cdot \nabla S \qquad (76)$$

Note the existence of a coupling between the prestress and the vorticity. This coupling disappears if the prestress is reduced to hydrostatic pressure. Thus

$$\Omega = \Omega_o (\mathbf{k} + \mathbf{N}) \qquad (77)$$

where Ω_o is a constant and $\Omega_o \mathbf{N}$ are the perturbations in the speed of rotation. Our primary interest is in diurnal and semidiurnal tides. Then $\Omega_o \mathbf{N}$ represents the nutation, astronomical and free. The typical term in $\mathbf{N}$ can be written in the form:

$$\mathbf{N} = \varepsilon (\mathbf{i} \cos \sigma t - \mathbf{j} \sin \sigma t)$$

thus,

$$\mathbf{k} \cdot \mathbf{N} = 0 \qquad (78)$$

In the system rotating with the Earth, the astronomical nutation is represented by a trigonometrical series with the same arguments as in diurnal Earth tides (Reference 10).

Substituting

$$\frac{\partial \mathbf{R}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t}$$

into equation 72 and making use of equation 77 yields:

$$\mathbf{w} = \frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} - \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt}$$
$$- \nabla [\phi_o - \Omega_o^2 (\mathbf{r} \times \mathbf{k}) \cdot (\mathbf{N} \times \mathbf{r})] - \mathbf{u} \cdot \nabla\nabla\phi_o \tag{79}$$

where

$$\phi_0 = \frac{1}{2} \mathbf{r} \cdot (\mathrm{I} - \mathbf{k}\mathbf{k}) \cdot \mathbf{r} \tag{80}$$

is the centrifugal-force function. Assuming the static equilibrium in the "initial" state,

$$\nabla \cdot \mathrm{S} + \rho \nabla \mathrm{V} = 0 \tag{81}$$

where

$$\mathrm{V} = \mathrm{U} + \phi_o \tag{82}$$

is the total gravitational force function. Using equations 78 and 81, differential equation 76 of tidal oscillations becomes:

$$\rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} \right) = - \rho \theta \nabla \mathrm{V} + \rho (\mathbf{u} \cdot \nabla\nabla \mathrm{V} - \boldsymbol{\omega} \times \nabla \mathrm{V})$$
$$+ \rho \nabla (\psi - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r}) + \nabla \cdot \tau + \rho \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} \tag{83}$$
$$+ (\nabla \times \boldsymbol{\omega}) \cdot \mathrm{S} - (\mathrm{S} \cdot \nabla) \times \boldsymbol{\omega} - \epsilon \cdot\cdot \nabla \mathrm{S}$$

In the liquid core, the relation between the stress and strain tensors is assumed to be either Hooke's law,

$$\tau = \lambda_e I \theta + 2\mu_e \epsilon = \lambda_e I \theta + \mu_e (\mathbf{u}\nabla + \nabla \mathbf{u}) \tag{84}$$

or, more generally, the Kelvin-Voigt law,

$$\tau = \lambda I \theta + 2\mu\epsilon = \lambda I \theta + \mu(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{85}$$

where

$$\lambda = \lambda_e + \lambda_v \frac{\partial}{\partial t}, \mu = \mu_e + \mu_v \frac{\partial}{\partial t} \tag{86}$$

For the oscillations of a given frequency, σ,

$$\lambda = \lambda_e - i\sigma\lambda_v, \mu = \mu_e - i\sigma\mu_v \tag{87}$$

Generally speaking, λ_e, μ_e, λ_v, and μ_v are functions of $\mathbf{r}$. The values of λ_e and μ_e are supplied by Earth's models. Knowledge of the viscosity of the liquid core is incomplete (Reference 22). Estimates of the kinematic viscosity coefficient of the liquid core vary between $10^{-7}\, m^2 s^{-1}$ and $10^3\, m^2 s^{-1}$ (References 21 and 23). In its main characteristics, the motion in the interior of the liquid core follows the laws of classical hydrodynamics (Reference 21). For this reason and to obtain at least preliminary information on the influence of the viscosity on the tidal oscillations, the numerical integration can be performed under the Stokesian approximation:

$$\frac{\mu_v}{\rho} = \nu \qquad \frac{\lambda_v}{\rho} = -\frac{2}{3}\nu$$

where ν is a constant.

Equation 83 is the basic. Any other equation of tidal oscillations that appears in the present exposition is either a consequence or a paraphrase of equation 83. Of special interest are transformations that bring equation 83 closer to a form associated with the hydrostatic equilibrium. For example, making use of the identity,

$$\mathbf{u} \cdot \nabla\nabla V = \boldsymbol{\omega} \times \nabla V + \nabla(\mathbf{u} \cdot \nabla V) - \epsilon \cdot \nabla V$$

yields:

$$\rho\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t}\right) = \nabla \cdot \tau - \rho\theta\nabla V \tag{88}$$

$$+ \rho \nabla (\psi + \mathbf{u} \cdot \nabla V - \Omega_o^2 \mathbf{r} \cdot \mathbf{N k} \cdot \mathbf{r})$$ (88 continued)

$$+ \rho \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} - (\rho \epsilon \cdot \nabla V + \epsilon \cdot\cdot \nabla S)$$

$$+ (\nabla \times \boldsymbol{\omega}) \cdot S - (S \cdot \nabla) \times \boldsymbol{\omega}$$

or, taking the identity,

$$- \rho \theta \nabla V + \rho \nabla (\mathbf{u} \cdot \nabla V) = - \nabla \cdot (\rho \mathbf{u}) \nabla V + \nabla (\rho \mathbf{u} \cdot \nabla V)$$

$$- \mathbf{u} \times (\nabla \rho \times \nabla V)$$

into account yields:

$$\rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} \right) = \nabla \cdot \tau - \nabla \cdot (\rho \mathbf{u}) \nabla V$$

$$+ \rho \nabla (\psi - \Omega_o^2 \mathbf{r} \cdot \mathbf{N k} \cdot \mathbf{r}) + \nabla (\rho \mathbf{u} \cdot \nabla V) - \mathbf{u} \times (\nabla \rho \times \nabla V) \tag{89}$$

$$+ \rho \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} - (\rho \epsilon \cdot \nabla V + \epsilon \cdot\cdot \nabla S) + (\nabla \times \boldsymbol{\omega}) \cdot S - (S \cdot \nabla) \times \boldsymbol{\omega}$$

Considerable simplification can be achieved if the prestress is reduced to hydrostatic pressure,

$$S = - pI$$

then,

$$\rho \nabla V - \nabla p = 0$$

and, as a consequence,

$$\nabla \rho \times \nabla V = 0 \qquad \nabla p \times \nabla V = 0$$

i.e., in the initial state, the equipotential surfaces—the surfaces of equal pressure and equal density—coincide. It is assumed that, in the initial state, the equipotential surfaces also coincide with the surfaces of equal compressibility and rigidity. As a consequence of the hydrostatic hypothesis,

$$\rho \epsilon \cdot \nabla V + \epsilon \cdot\cdot \nabla S = 0$$

$$(\nabla \times \omega) \cdot S - (S \cdot \nabla) \times \omega = 0$$

and equations 88 and 89 become, respectively,

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho} \nabla \cdot \tau - \theta \nabla V$$
$$+ \nabla (\psi + \mathbf{u} \cdot \nabla V - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r}) + \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} \tag{90}$$

$$\rho \left(\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} \right) = \nabla \cdot \tau - \nabla \cdot (\rho \mathbf{u}) \nabla V$$
$$+ \nabla (\rho \mathbf{u} \cdot \nabla V) + \rho \nabla (\psi - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r}) + \rho \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} \tag{91}$$

Equation 91 represents a generalization, with the effects of nutation and geostrophic force added, of the equation used by Alterman et al. (Reference 7) for computing free oscillations of the nonrotating spherically symmetric Earth. Equation 90 can serve as the foundation of the Molodensky theory of tides (Reference 3). Equation 90 is transformed here to a different shape, more comparable to the classical Molodensky equation. As previously stated, because of hydrostatic equilibrium, the equipotential surfaces—the surfaces of equal pressure and of equal density—coincide. It was also assumed that the surfaces of equal λ and μ remain coincident with the equipotential surfaces. Consequently,

$$\nabla \lambda = \frac{\lambda'}{g} \nabla V, \nabla \mu = \frac{\mu'}{g} \nabla V, \nabla \rho = \frac{\rho'}{g} \nabla V \tag{92}$$

where λ', μ', and ρ' are the derivatives in the direction of gravity ∇V, and $g = | \nabla V |$. Using equations 92 and 85 and after some easy transformations:

$$\frac{1}{\rho} \nabla \cdot \tau = \frac{\lambda \rho'}{g \rho^2} \theta \nabla V + \nabla \left(\frac{\lambda \theta}{\rho} \right) + \frac{\mu}{\rho} (\nabla^2 \mathbf{u} + \nabla \theta) \tag{93}$$

$$+ \frac{\mu'}{g\rho} [g\,\mathbf{u}' + (\nabla \mathbf{u}) \cdot \nabla V] \qquad \text{(93 continued)}$$

By substituting equation 93 into equation 90, the Molodensky equation (Reference 3) is obtained:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} = \nabla (\psi + \mathbf{u} \cdot \nabla V + \frac{\lambda\theta}{\rho} - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r})$$

$$- \beta \theta \nabla V + \frac{\mu}{\rho} (\nabla^2 \mathbf{u} + \Delta \theta) + \frac{\mu'}{g\rho} [g\,\mathbf{u}' + (\nabla \mathbf{u}) \cdot \nabla V] \tag{94}$$

$$+ \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt}$$

where

$$\beta = 1 - \frac{\lambda \rho'}{g\rho^2} \tag{95}$$

is the core stability parameter (Reference 17). If the viscosity in the liquid core is neglected, equation 94 becomes:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} = \nabla (\psi + \mathbf{u} \cdot \nabla V + \frac{\lambda\theta}{\rho} - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r})$$

$$- \beta \theta \nabla V + \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt} \tag{96}$$

Finally, if the liquid core is in neutral equilibrium (i.e., the Adams-Williamson relation, $\beta = 0$, is satisfied (Reference 16)), the equation takes a very simple form:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\Omega_o \mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} = \nabla (\psi + \mathbf{u} \cdot \nabla V + \frac{\lambda\theta}{\rho} - \Omega_o^2 \mathbf{r} \cdot \mathbf{N} \mathbf{k} \cdot \mathbf{r}) \tag{97}$$

$$+ \Omega_o \mathbf{r} \times \frac{d\mathbf{N}}{dt}$$

If only one nutational term is selected, complete identity with Molodensky can be achieved (except for notations):

$$\mathbf{N} = \varepsilon\,(\mathbf{i}\cos\sigma t - \mathbf{j}\sin\sigma t) \tag{98}$$

From equation 98,

$$\frac{d\mathbf{N}}{dt} = -\,\sigma\mathbf{k} \times \mathbf{N} \tag{99}$$

and, after some easy vectorial transformation,

$$-\nabla\,(\Omega_o^2\,\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}) + \Omega_o\,\mathbf{r} \times \frac{d\mathbf{N}}{dt} = -\nabla\,[\Omega_o\,(\Omega_o - \sigma)\,\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}] - 2\Omega_o\sigma\,\mathbf{k}\,\mathbf{r}\cdot\mathbf{N} \tag{100}$$

Then, for example, tidal equations 90, 91, and 94 become, respectively:

$$\begin{aligned}\frac{\partial^2\mathbf{u}}{\partial t^2} + 2\Omega_o\,\mathbf{k} \times \frac{\partial\mathbf{u}}{\partial t} &= \frac{1}{\rho}\nabla\cdot\tau - \theta\;V \\ &+ \nabla\,[\psi + \mathbf{u}\cdot\nabla V - \Omega_o\,(\Omega_o - \sigma)\,\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}] - 2\Omega_o\,\sigma\,\mathbf{k}\,\mathbf{r}\cdot\mathbf{N}\end{aligned} \tag{101}$$

$$\begin{aligned}\rho\left(\frac{\partial^2\mathbf{u}}{\partial t^2} + 2\Omega_o\,\mathbf{k} \times \frac{\partial\mathbf{u}}{\partial t}\right) &= \nabla\cdot\tau - \nabla\cdot(\rho\mathbf{u})\nabla V \\ &+ \nabla\,(\rho\mathbf{u}\cdot\nabla V) + \rho\nabla\,[\psi - \Omega_o\,(\Omega_o - \sigma)\,\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}] \\ &- 2\Omega_o\,\sigma\,\mathbf{k}\,\mathbf{r}\cdot\mathbf{N}\end{aligned} \tag{102}$$

and

$$\begin{aligned}\frac{\partial^2\mathbf{u}}{\partial t^2} + 2\Omega_o\,\mathbf{k} \times \frac{\partial\mathbf{u}}{\partial t} &= \nabla\left[\psi + \mathbf{u}\cdot\nabla V + \frac{\lambda\theta}{\rho} - \Omega_o\,(\Omega_o - \sigma)\,\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}\right] \\ &- \beta\,\theta\,\nabla V + \frac{\mu}{\rho}\,(\nabla^2\mathbf{u} + \nabla\,\theta) + \frac{\mu'}{g\rho}\,[\,g\mathbf{u}' + (\nabla\mathbf{u})\cdot\nabla V] \\ &- 2\Omega_o\,\sigma\,\mathbf{k}\,\mathbf{r}\cdot\mathbf{N}.\end{aligned} \tag{103}$$

TRANSFORMATION OF NUTATIONAL TERM

From

$$
\begin{aligned}
\mathbf{N} &= \varepsilon\,(+\,\mathbf{i}\cos\sigma t\;-\mathbf{j}\sin\sigma t)\\
\mathbf{e}_r &= +\,\mathbf{i}\sin\theta\cos\varphi\;+\;\mathbf{j}\sin\theta\sin\varphi\;+\;\mathbf{k}\cos\theta\\
\mathbf{k} &= +\;\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta
\end{aligned}
\tag{104}
$$

and taking

$$
\begin{aligned}
S_1^{+1} &= -e^{+i\varphi}\sin\theta & S_1^{-1} &= +e^{-i\varphi}\sin\theta\\
S_2^{+1} &= -3e^{+i\varphi}\sin\theta\cos\theta & S_1^{-1} &= +3e^{-i\varphi}\sin\theta\cos\theta.
\end{aligned}
\tag{105}
$$

into consideration,

$$
\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r} = +\frac{1}{6}\,\varepsilon r^2\,(-e^{+i\sigma t}\,S_2^{+1}\;+e^{-i\sigma t}\,S_2^{-1}) \tag{106}
$$

and, taking equations 15 and 16 into account,

$$
\nabla\,(\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}) = -\varepsilon r\left(+\frac{1}{3}\,\mathbf{A}_2^{+1}\;+\;\frac{1}{6}\,\mathbf{B}_2^{+1}\right)e^{+i\sigma t}\;+\;\varepsilon r\left(+\frac{1}{3}\,\mathbf{A}_2^{-1}\;+\frac{1}{6}\,\mathbf{B}_2^{-1}\right)e^{-i\sigma t} \tag{107}
$$

The following is obtained in a similar manner:

$$
\mathbf{k}\,\mathbf{N}\cdot\mathbf{r} = \;\varepsilon r\;\;(\mathbf{e}_r\sin\theta\cos\theta\;-\;\mathbf{e}_\theta\,\sin^2\theta)\cos(\sigma t\;+\;\varphi) \tag{108}
$$

and, taking equation 105 into account,

$$
\mathbf{e}_r\sin\theta\cos\theta\cos(\sigma t\;+\;\varphi) = -\frac{1}{6}\,e^{+i\sigma t}\,\mathbf{A}_2^{+1}\;+\;\frac{1}{6}\,e^{-i\sigma t}\,\mathbf{A}_2^{-1} \tag{109}
$$

Making use of equation 60 and equation 8,

$$
\begin{aligned}
\mathbf{e}_\theta\sin^2\theta\cos(\sigma t\;+\;\varphi) = {}&\left(+\frac{1}{12}\,\mathbf{B}_2^{+1}\;+\;\frac{1}{4}\,i\,\mathbf{C}_1^{+1}\right)e^{+i\sigma t}\\
&+\left(-\frac{1}{12}\,\mathbf{B}_2^{-1}\;+\frac{1}{4}\,i\,\mathbf{C}_1^{-1}\right)\,e^{-i\sigma t}
\end{aligned}
\tag{110}
$$

Substituting equations 109 and 110 into 108,

$$\begin{aligned} \mathbf{k}\,\mathbf{N}\cdot\mathbf{r} = & -\varepsilon r\left(+\frac{1}{6}\mathbf{A}_2^{+1} + \frac{1}{12}\mathbf{B}_2^{+1} + \frac{1}{4}\,i\,\mathbf{C}_1^{+1}\right)e^{+i\sigma t} \\ & +\varepsilon r\left(+\frac{1}{6}\mathbf{A}_2^{-1} + \frac{1}{12}\mathbf{B}_2^{-1} - \frac{1}{4}\,i\,\mathbf{C}_1^{-1}\right)e^{-i\sigma t} \end{aligned} \quad (111)$$

The following representations of the nutational term is obtained from equations 107 and 111:

$$\begin{aligned} & -\Omega_o(\Omega_o - \sigma)\,\nabla\,(\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}) - 2\Omega_o\,\sigma\,\mathbf{k}\,\mathbf{N}\cdot\mathbf{r} \\ = & +\varepsilon r\left(+\frac{1}{3}\,\Omega_o^2\,\mathbf{A}_2^{+1} + \frac{1}{6}\,\Omega_o^2\,\mathbf{B}_2^{+1} - \frac{1}{2}\,i\,\Omega_o\,\sigma\,\mathbf{C}_1^{+1}\right)e^{+i\sigma t} \\ & -\varepsilon r\left(+\frac{1}{3}\,\Omega_o^2\,\mathbf{A}_2^{-1} + \frac{1}{6}\,\Omega_o^2\,\mathbf{B}_2^{-1} + \frac{1}{2}\,i\,\Omega_o\sigma\,\mathbf{C}_1^{-1}\right)e^{-i\sigma t} \end{aligned} \quad (112)$$

By introducing the tesseral-force function:

$$\Pi_2^1 = +\frac{1}{6}\,\varepsilon\,\Omega_o^2\,r^2\,(S_2^{+1}\,e^{+i\sigma t} - S_2^{-1}\,e^{-i\sigma t}) \quad (113)$$

which arises from the variation of latitude, the nutational term can be written in the form:

$$-\Omega_o(\Omega_o - \sigma)\,\nabla\,(\mathbf{r}\cdot\mathbf{N}\,\mathbf{k}\cdot\mathbf{r}) - 2\Omega_o\,\sigma\,\mathbf{k}\,\mathbf{N}\cdot\mathbf{r} = \nabla\Pi_2^1 + \frac{1}{2}i\,\varepsilon\,r\,\Omega_o\,\sigma\,(\mathbf{C}_1^{+1}\,e^{+i\sigma t} + \mathbf{C}_1^{-1}\,e^{-i\sigma t}) \quad (114)$$

Thus, the nutation, forced or free, modifies the lunisolar tidal potential and initiates the toroidal oscillations. For the astronomical nutation, the force function, Π_2^1, is similar in form to the corresponding term in the lunisolar tidal-force function. It is possible to fuse both force functions together. As a result, small terms added to W would be expected to modify the coefficients of the Earth's tidal elastic response slightly. Each particular frequency has its own unique Love and Shida numbers.

EXPANSION OF THE CORIOLIS FORCE

In this section, the Coriolis force is expanded into spheroidal and toroidal components. In the mantle, the tidal oscillations are predominantly spheroidal. In the liquid core, however, the Coriolis force and the ellipticity produce both toroidal and spheroidal tides and the numerous couplings between spheroidal and toroidal tidal constituents.

By assuming that the tidal oscillations can be expanded into a sum of terms of the form:

$$\mathbf{u}_n = U_n(r,t)\,\mathbf{A}_n + V_n(r,t)\,\mathbf{B}_n + i\,T_n(r,t)\,\mathbf{C}_n \tag{115}$$

and by taking equations 61 through 63 into account, the following is obtained for the typical term in the expansion of the Coriolis force:

$$\begin{aligned}
2\Omega_o\,\mathbf{k}\times\frac{\partial\mathbf{u}}{\partial t} = 2\Omega_o \Bigg\{ & -i\,\frac{(n+1)(n+m)}{2n+1}\cdot\frac{\partial T_n}{\partial t}\,\mathbf{A}_{n-1} \\
& + i\,\frac{(n+1)(n+m)}{n(2n+1)}\cdot\frac{\partial T_n}{\partial t}\,\mathbf{B}_{n-1} - \frac{n+m}{n(2n+1)}\left[\frac{\partial U_n}{\partial t} + (n+1)\frac{\partial V_n}{\partial t}\right]\mathbf{C}_{n-1} \\
& - i\,m\,\frac{\partial V_n}{\partial t}\,\mathbf{A}_n - \frac{im}{n(n+1)}\left(\frac{\partial U_n}{\partial t} + \frac{\partial V_n}{\partial t}\right)\mathbf{B}_n \\
& + \frac{m}{n(n+1)}\,\frac{\partial T_n}{\partial t}\,\mathbf{C}_n + i\,\frac{n(n-m+1)}{2n+1}\,\frac{\partial T_n}{\partial t}\,\mathbf{A}_{n+1} \\
& + i\,\frac{n(n-m+1)}{(n+1)(2n+1)}\,\frac{\partial T_n}{\partial t}\,\mathbf{B}_{n+1} + \frac{n-m+1}{(n+1)(2n+1)}\left(\frac{\partial U_n}{\partial t} - n\,\frac{\partial V_n}{\partial t}\right)\mathbf{C}_{n+1} \Bigg\}
\end{aligned} \tag{116}$$

Under the influence of the Earth's rotation, the spheroidal (toroidal) oscillations produce toroidal (spheroidal) terms in the expansion of the Coriolis force. There is also a mutual influence between the tides of the $(n-1)^{th}$, n^{th}, and $(n+1)^{th}$ degrees.

Substituting first n + 1, then n - 1, for n in equation 116 and retaining only the terms of the n^{th} degree,

$$2\Omega_o\,\mathbf{k}\times\frac{\partial\mathbf{u}}{\partial t} = 2\Omega_o \Bigg\{ i\left[+ \frac{(n-1)(n-m)}{2n-1}\cdot\frac{\partial T_{n-1}}{\partial t} - m\,\frac{\partial V_n}{\partial t} \right. \tag{117}$$

$$- \frac{(n+2)(n+m+1)}{2n+3} \cdot \frac{\partial T_{n+1}}{\partial t} \Bigg] \mathbf{A}_n \qquad \text{(117 continued)}$$

$$+ i \Bigg[+ \frac{(n-1)(n-m)}{n(2n-1)} \cdot \frac{\partial T_{n-1}}{\partial t} - \frac{m}{n(n+1)} \cdot \frac{\partial U_n}{\partial t} - \frac{m}{n(n+1)} \cdot \frac{\partial V_n}{\partial t}$$

$$+ \frac{(n+2)(n+m+1)}{(2n+3)(n+1)} \cdot \frac{\partial T_{n+1}}{\partial t} \Bigg] \mathbf{B}_n$$

$$+ \Bigg[+ \frac{n-m}{n(2n-1)} \cdot \frac{\partial U_{n-1}}{\partial t} - \frac{(n-1)(n-m)}{n(2n-1)} \cdot \frac{\partial V_{n-1}}{\partial t} + \frac{m}{n(n+1)} \cdot \frac{\partial T_n}{\partial t}$$

$$- \frac{n+m+1}{(n+1)(2n+3)} \cdot \frac{\partial U_{n+1}}{\partial t} - \frac{(n+2)(n+m+1)}{(n+1)(2n+3)} \cdot \frac{\partial V_{n+1}}{\partial t} \Bigg] \mathbf{C}_n \Bigg\}$$

This equation shows the manner in which the tides of the n^{th} degree are being influenced by the tides of $(n-1)^{th}$ and $(n+1)^{th}$ degrees through the Coriolis force (References 6, 18, 24, and 25). There is no coupling, however, between the tides of different orders. In particular, for oscillations of the first degree and order,

$$2\Omega_o \, k \times \frac{\partial \mathbf{u}}{\partial t} = 2\Omega_o \Bigg[i \left(- \frac{\partial V_1^1}{\partial t} - \frac{9}{5} \frac{\partial T_2^1}{\partial t} \right) \mathbf{A}_1^1$$

$$+ i \left(- \frac{1}{2} \frac{\partial U_1^1}{\partial t} - \frac{1}{2} \frac{\partial V_1^1}{\partial t} + \frac{9}{10} \frac{\partial T_2^1}{\partial t} \right) \mathbf{B}_1^1 \qquad (118)$$

$$+ \left(+ \frac{1}{2} \frac{\partial T_1^1}{\partial t} - \frac{3}{10} \frac{\partial U_2^1}{\partial t} - \frac{9}{10} \frac{\partial V_2^1}{\partial t} \right) \mathbf{C}_1^1 \Bigg]$$

For diurnal tides of the second degree,

$$2\Omega_o \, k \times \frac{\partial \mathbf{u}}{\partial t} = 2\Omega_o \Bigg[i \left(+ \frac{1}{3} \frac{\partial T_1^1}{\partial t} - \frac{\partial V_2^1}{\partial t} - \frac{16}{7} \frac{\partial T_3^1}{\partial t} \right) \mathbf{A}_2^1 \qquad (119)$$

$$+ i\left(+\frac{1}{6}\frac{\partial T_1^1}{\partial t}-\frac{1}{6}\frac{\partial U_2^1}{\partial t}-\frac{1}{6}\frac{\partial V_2^1}{\partial t}+\frac{16}{21}\frac{\partial T_3^1}{\partial t}\right)\mathbf{B}_2^1 \qquad \text{(119 continued)}$$

$$\left.+\left(+\frac{1}{6}\frac{\partial U_1^1}{\partial t}-\frac{1}{6}\frac{\partial V_1^1}{\partial t}+\frac{1}{6}\frac{\partial T_2^1}{\partial t}-\frac{4}{21}\frac{\partial U_3^1}{\partial t}-\frac{16}{21}\frac{\partial V_3^1}{\partial t}\right)\mathbf{C}_2^1\right]$$

For diurnal tides of the third degree,

$$2\Omega_o\, k \times \frac{\partial \mathbf{u}}{\partial t} = 2\Omega_o\left\{ i\left(+\frac{4}{5}\frac{\partial T_2^1}{\partial t}-\frac{\partial V_3^1}{\partial t}-\frac{25}{9}\frac{\partial T_4^1}{\partial t}\right)\mathbf{A}_3^1\right.$$

$$+ i\left(+\frac{4}{15}\frac{\partial T_2^1}{\partial t}-\frac{1}{12}\frac{\partial U_3^1}{\partial t}-\frac{1}{12}\frac{\partial V_3^1}{\partial t}+\frac{25}{36}\frac{\partial T_4^1}{\partial t}\right)\mathbf{B}_3^1 \qquad (120)$$

$$\left.+\left(+\frac{2}{15}\frac{\partial U_2^1}{\partial t}-\frac{4}{15}\frac{\partial V_2^1}{\partial t}+\frac{1}{12}\frac{\partial T_3^1}{\partial t}-\frac{5}{36}\frac{\partial U_4^1}{\partial t}-\frac{25}{36}\frac{\partial V_4^1}{\partial t}\right)\mathbf{C}_3^1\right\}$$

For the most important semidiurnal effects,

$$2\Omega_o\, k \times \frac{\partial \mathbf{u}}{\partial t} = 2\Omega_o\left\{ i\left(-2\frac{\partial V_2^2}{\partial t}-\frac{20}{7}\frac{\partial T_3^2}{\partial t}\right)\mathbf{A}_2^2\right.$$

$$+ i\left(-\frac{1}{3}\frac{\partial U_2^2}{\partial t}-\frac{1}{3}\frac{\partial V_2^2}{\partial t}+\frac{20}{21}\frac{\partial T_3^2}{\partial t}\right)\mathbf{B}_2^2 \qquad (121)$$

$$\left.+\left(+\frac{1}{3}\frac{\partial T_2^2}{\partial t}-\frac{5}{21}\frac{\partial U_3^2}{\partial t}-\frac{20}{21}\frac{\partial V_3^2}{\partial t}\right)\mathbf{C}_2^2\right\}$$

and for n = 3, m = 2,

$$2\Omega_\theta\, k \times \frac{\partial \mathbf{u}}{\partial t} = 2\Omega_o\left\{ i\left(+\frac{2}{5}\frac{\partial T_2^2}{\partial t}-2\frac{\partial V_3^2}{\partial t}-\frac{10}{3}\frac{\partial T_4^2}{\partial t}\right)\mathbf{A}_3^2\right. \qquad (122)$$

$$+ i \left(+ \frac{2}{15} \frac{\partial T_2^2}{\partial t} - \frac{1}{6} \frac{\partial U_3^2}{\partial t} - \frac{1}{6} \frac{\partial V_3^2}{\partial t} + \frac{5}{6} \frac{\partial T_4^2}{\partial t} \right) \mathbf{B}_3^2 \qquad \text{(122 continued)}$$

$$+ \left(+ \frac{1}{15} \frac{\partial U_2^2}{\partial t} - \frac{2}{15} \frac{\partial V_2^2}{\partial t} + \frac{1}{6} \frac{\partial T_3^2}{\partial t} - \frac{1}{6} \frac{\partial U_4^2}{\partial t} - \frac{5}{6} \frac{\partial V_4^1}{\partial t} \right) \mathbf{C}_3^2 \Bigg\}$$

The influence of free nutation on semidiurnal tides is considerably smaller than on diurnal tides. In the liquid core, the Coriolis force produces numerous couplings between the semidiurnal tides of different degrees.

If tides of a given frequency σ only are of interest, then by replacing

$$U_m \text{ by } e^{+i\sigma t}\, U_m,\ V_m \text{ by } e^{+i\sigma t}\ V_m,\ \text{etc.}$$

in equation 118 we obtain

$$\begin{aligned}
2 i \Omega_o \sigma \mathbf{k} \times \mathbf{u}_n = - 2\Omega_o \sigma \Bigg\{ & \left[+ \frac{(n-1)(n-m)}{2n-1} T_{n-1} - mV_n \right. \\
& \left. - \frac{(n+2)(n+m+1)}{2n+3} T_{n+1} \right] \mathbf{A}_n \\
& + \left[+ \frac{(n-1)(n-m)}{n(2n-1)} T_{n-1} - \frac{m}{n(n+1)} U_n - \frac{m}{n(n+1)} V_n \right. \\
& \left. + \frac{(n+2)(n+m+1)}{(2n+3)(n+1)} T_{n+1} \right] \mathbf{B}_n \\
& - i \left[+ \frac{n-m}{n(2n-1)} U_{n-1} - \frac{(n-1)(n-m)}{n(2n-1)} V_{n-1} + \frac{m}{n(n+1)} T_n \right. \\
& \left. - \frac{n+m+1}{(n+1)(2n+3)} U_{n+1} - \frac{(n+2)(n+m+1)}{(n+1)(2n+3)} V_{n+1} \right] \mathbf{C}_n \Bigg\}
\end{aligned} \qquad (123)$$

In particular, for diurnal and semidiurnal tides up to the third degree, as a paraphrase of equations 118 through 122,

$$2i\Omega_o\sigma\,\mathbf{k}\times\mathbf{u} = -2\Omega_o\sigma\Bigg[\left(-V_1^1 - \frac{9}{5}T_2^1\right)\mathbf{A}_1^1$$

$$+\left(-\frac{1}{2}U_1^1 - \frac{1}{2}V_1^1 + \frac{9}{10}T_2^1\right)\mathbf{B}_1^1 - i\left(+\frac{1}{2}T_1^1 - \frac{3}{10}U_2^1 - \frac{9}{10}V_2^1\right)\mathbf{C}_1^1\Bigg] \tag{124}$$

$$2i\Omega_o\sigma\,\mathbf{k}\times\mathbf{u} = -2\Omega_o\sigma\Bigg[\left(+\frac{1}{3}T_1^1 - V_2^1 - \frac{16}{7}T_3^1\right)\mathbf{A}_2^1$$

$$+\left(+\frac{1}{6}T_1^1 - \frac{1}{6}U_2^1 - \frac{1}{6}V_2^1 + \frac{16}{21}T_3^1\right)\mathbf{B}_2^1 \tag{125}$$

$$-i\left(+\frac{1}{6}U_1^1 - \frac{1}{6}V_1^1 + \frac{1}{6}T_2^1 - \frac{4}{21}U_3^1 - \frac{16}{21}V_3^1\right)\mathbf{C}_2^1\Bigg]$$

$$2i\Omega_o\sigma\,\mathbf{k}\times\mathbf{u} = -2\Omega_o\sigma\Bigg\{\left(+\frac{4}{5}T_2^1 - V_3^1 - \frac{25}{9}T_4^1\right)\mathbf{A}_3^1$$

$$+\left(+\frac{4}{15}T_2^1 - \frac{1}{12}U_3^1 - \frac{1}{12}V_3^1 + \frac{25}{36}T_4^1\right)\mathbf{B}_3^1 \tag{126}$$

$$-i\left(+\frac{2}{15}U_2^1 - \frac{4}{15}V_2^1 + \frac{1}{12}T_3^1 - \frac{5}{36}U_4^1 - \frac{25}{36}V_4^1\right)\mathbf{C}_3^1\Bigg\}$$

$$2i\Omega_o\sigma\,\mathbf{k}\times\mathbf{u} = -2\Omega_o\sigma\Bigg\{\left(-2V_2^2 - \frac{20}{7}T_3^2\right)\mathbf{A}_2^2$$

$$+\left(-\frac{1}{3}U_2^2 - \frac{1}{3}V_2^2 + \frac{20}{21}T_3^2\right)\mathbf{B}_2^2 \tag{127}$$

$$-i\left(+\frac{1}{3}T_2^2 - \frac{5}{21}U_3^2 - \frac{20}{21}V_3^2\right)\mathbf{C}_2^2\Bigg\}$$

$$2 i \Omega_o \sigma \mathbf{k} \times \mathbf{u} = -2\Omega_o \sigma \left\{ \left(+ \frac{2}{5} T_2^2 - 2V_3^2 - \frac{10}{3} T_4^2 \right) \mathbf{A}_3^2 \right.$$

$$+ \left(+ \frac{2}{15} T_2^2 - \frac{1}{6} U_3^2 - \frac{1}{6} V_3^2 + \frac{5}{6} T_4^2 \right) \mathbf{B}_3^2 \qquad (128)$$

$$\left. - i \left(+ \frac{1}{15} U_2^2 - \frac{2}{15} V_2^2 + \frac{1}{6} T_3^2 - \frac{1}{6} U_4^2 - \frac{5}{6} V_4^2 \right) \mathbf{C}_3^2 \right\}$$

INFLUENCE OF ELLIPTICITY OF EQUIPOTENTIAL SURFACES

In the frame of the present theory, it is assumed that the interior equipotential surfaces are ellipsoids:

$$r \left[1 + \frac{2}{3} \epsilon(a) P_2(\cos\theta) \right] = a$$

$$P_2(\cos\theta) = \frac{3}{2} \cos^2\theta - \frac{1}{2} \qquad (129)$$

where a is the ellipsoid's mean radius, and ϵ is its ellipticity. In the expansions only the first power of ϵ is retained. By assuming that U (r) is the force function of self-gravitation associated with a properly selected (or assumed) spherical model of the Earth, the following is obtained (Reference 6) for the potential of the ellipsoidal model with the centrifugal part included:

$$V = U(r) + \frac{1}{3} \Omega_o^2 r^2 - c P_2(\cos\theta) \qquad (130)$$

where

$$c = + \frac{2}{3} r \epsilon(r) g(r) + \frac{1}{3} \Omega_o^2 r^2$$

$$g = - \frac{dU}{dr} \qquad (131)$$

From equation 130, for the undisturbed force of self-gravitation,

$$\nabla V = - \left[\gamma + \frac{dc}{dr} P_2 (\cos\theta) \right] \mathbf{e}_r + \frac{3c}{r} \mathbf{e}_\theta \sin\theta \cos\theta \tag{132}$$

where

$$\gamma = g - \frac{2}{3} \Omega_o^2 r \tag{133}$$

In the further exposition, it is assumed that the stability parameter, β, with sufficient accuracy can be considered either a constant (the case of the uniform stability) or a function of r. Taking into account

$$\theta_n = \nabla \cdot U_n = X_n S_n \tag{134}$$

where

$$X_n = \frac{dU_n}{dr} + \frac{2}{r} U_n - \frac{n(n+1)}{r} V_n \tag{135}$$

yields:

$$- \beta \theta_n \nabla V = \beta X_n \left[\gamma \mathbf{A}_n + \frac{dc}{dr} \mathbf{e}_r P_2 (x) S_n + \frac{3c}{r} \mathbf{Q}_n \right] \tag{136}$$

where

$$\mathbf{Q}_n = - \mathbf{e}_\theta S_n x \sqrt{1 - x^2} \tag{137}$$

$$x = \cos\theta$$

From equation 3,

$$\mathbf{e}_r S_n P_2 (x) = + \frac{3}{2} \cdot \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \mathbf{A}_{n+2} \tag{138}$$

$$+ \frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} A_n \qquad (138 \text{ continued})$$

$$+ \frac{3}{2} \cdot \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} A_{n-2}$$

From equations 16, 17, and 7, and taking the normalization conditions into account,

$$\frac{1}{\nu(\nu+1)\alpha_\nu} (Q_n, B_\nu) = - \frac{(n+m-1)(n+m)}{(n-1)(2n-1)(2n+1)} \delta_{\nu,n-2}$$

$$+ \frac{n^2 - 3m^2 + n}{n(n+1)(2n-1)(2n+3)} \delta_{\nu,n} \qquad (139)$$

$$+ \frac{(n-m+1)(n-m+2)}{(n+2)(2n+3)(2n+1)} \delta_{\nu,n+2}$$

and

$$\frac{1}{\nu(\nu+1)\alpha_\nu} (Q_n, C_\nu) = + \frac{im(n-m+1)}{(n+1)(n+2)(2n+1)} \delta_{\nu,n+1}$$

$$+ \frac{im(n+m)}{n(n-1)(2n+1)} \delta_{\nu,n-1} \qquad (140)$$

and, making use of equation 60, the following expansion is obtained:

$$Q_n = - \frac{(n+m-1)(n+m)}{(n-1)(2n-1)(2n+1)} B_{n-2} + \frac{n^2 - 3m^2 + n}{n(n+1)(2n-1)(2n+3)} B_n$$

$$+ \frac{(n-m+1)(n-m+2)}{(n+2)(2n+3)(2n+1)} B_{n+2} + \frac{im(n+m)}{(n-1)n(2n+1)} C_{n-1} \qquad (141)$$

$$+ \frac{im(n-m+1)}{(n+1)(n+2)(2n+1)} C_{n+1}$$

By substituting equations 138 and 141 into equation 136,

$$
\begin{aligned}
-\beta\theta_n \nabla V = \beta X_n \gamma \mathbf{A}_n &+ \beta \frac{dc}{dr}\left[+\frac{3}{2} \cdot \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} \mathbf{A}_{n-2}\right. \\
&+ \left.\frac{n^2-3m^2+n}{(2n-1)(2n+3)} \mathbf{A}_n + \frac{3}{2} \cdot \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} \mathbf{A}_{n+2}\right] X_n \\
&+ \beta \frac{3c}{r}\left[-\frac{(n+m-1)(n+m)}{(n-1)(2n-1)(2n+1)} \mathbf{B}_{n-2} + \frac{n^2-3m^2+n}{n(n+1)(2n-1)(2n+3)} \mathbf{B}_n\right. \\
&+ \frac{(n-m+1)(n-m+2)}{(n+2)(2n+3)(2n+1)} \mathbf{B}_{n+2} \\
&+ \left.\frac{im(n+m)}{(n-1)n(2n+1)} \mathbf{C}_{n-1} + \frac{im(n-m+1)}{(n+1)(n+2)(2n+1)} \mathbf{C}_{n+1}\right] X_n
\end{aligned}
\tag{142}
$$

By substituting n - 2, n - 1, n, n + 1, n + 2 for n in equation 142 and by retaining only the terms of the n^{th} degree,

$$
\begin{aligned}
-\beta\theta \nabla V &= +\beta X_n \gamma \mathbf{A}_n \\
&+ \beta \frac{dc}{dr}\left[+\frac{3}{2} X_{n-2} \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)}\right. \\
&+ X_n \frac{n^2-3m^2+n}{(2n-1)(2n+3)} \\
&+ \left.\frac{3}{2} X_{n+2} \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)}\right] \mathbf{A}_n \\
&+ \beta \frac{3c}{r}\left[+ X_{n-2} \frac{(n-m-1)(n-m)}{n(2n-1)(2n-3)}\right. \\
&+ X_n \frac{n^2-3m^2+n}{n(n+1)(2n-1)(2n+3)}
\end{aligned}
\tag{143}
$$

$$- X_{n+2} \frac{(n+m+1)(n+m+2)}{(n+1)(2n+3)(2n+5)}\Bigg] \mathbf{B}_n \qquad \text{(143 continued)}$$

$$+ i\beta \frac{3cm}{r} \left[+ X_{n-1} \frac{n-m}{n(n+1)(2n-1)} + X_{n+1} \frac{n+m+1}{n(n+1)(2n+3)}\right] \mathbf{C}_n$$

In particular, for diurnal and semidiurnal tides up to the third degree,

$$n = 1, m = 1,$$

$$- \beta\theta\nabla V = + \beta\gamma X_1^1 \mathbf{A}_1^1 + \beta \frac{dc}{dr} \left(- \frac{1}{5} X_1^1 + \frac{18}{35} X_3^1\right) \mathbf{A}_1^1 \tag{144}$$

$$- \frac{3\beta c}{r} \left(+ \frac{1}{10} X_1^1 + \frac{6}{35} X_3^1\right) \mathbf{B}_1^1 + \frac{9}{10} i\beta \frac{c}{r} X_2^1 \mathbf{C}_1^1$$

$$n = 2, m = 1$$

$$- \beta\theta\nabla V = + \beta\gamma X_2^1 \mathbf{A}_2^1 + \beta \frac{dc}{dr} \left(+ \frac{1}{7} X_2^1 + \frac{10}{21} X_4^1\right) \mathbf{A}_2^1 \tag{145}$$

$$+ \beta \frac{c}{r} \left(+ \frac{1}{14} X_2^1 - \frac{20}{63} X_4^1\right) \mathbf{B}_2^1 + i\beta \frac{c}{r} \left(+ \frac{1}{6} X_1^1 + \frac{2}{7} X_3^1\right) \mathbf{C}_2^1$$

$$n = 3, m = 1$$

$$- \beta\theta\nabla V = \gamma\beta X_3^1 \mathbf{A}_3^1 + \beta \frac{dc}{dr} \left(+ \frac{1}{5} X_1^1 + \frac{1}{5} X_3^1 + \frac{5}{11} X_5^1\right) \mathbf{A}_3^1 \tag{146}$$

$$+ \beta \frac{c}{r} \left(+ \frac{2}{15} X_1^1 + \frac{1}{20} X_3^1 - \frac{5}{22} X_5^1\right) \mathbf{B}_3^1 + i\beta \frac{c}{r} \left(+ \frac{1}{10} X_2^1 + \frac{5}{36} X_4^1\right) \mathbf{C}_3^1$$

$$n = 2, m = 2$$

$$- \beta\theta\nabla V = + \beta\gamma X_2^2 \mathbf{A}_2^2 + \beta \frac{dc}{dr} \left(- \frac{2}{7} X_2^2 + \frac{5}{7} X_4^2\right) \mathbf{A}_2^2 \tag{147}$$

$$- \beta \frac{c}{r} \left(+ \frac{1}{7} X_2^2 + \frac{10}{21} X_4^2\right) \mathbf{B}_2^2 + i\beta \frac{c}{r} \cdot \frac{5}{7} X_3^2 \mathbf{C}_2^2 \tag{148}$$

$$n = 3, m = 2$$

$$- \beta \theta \Delta V = + \beta \gamma X_3^2 A_3^2 + \beta \frac{7}{11} X_5^2 \left(+ \frac{dc}{dr} A_3^2 - \frac{1}{2} \frac{c}{r} B_3^2 \right) \tag{149}$$

$$+ i \beta \frac{c}{r} \left(+ \frac{1}{10} X_2^2 + \frac{1}{3} X_4^2 \right) C_3^2$$

Because the spheriodal terms in equation 149 as produced by the ellipticity are extremely small, only the toroidal perturbative part might be significant.

From equations 115 and 132, and taking equations 15 through 17 into consideration,

$$\eta_n = \mathbf{u}_n \cdot \nabla V = - \gamma U_n S_n - \frac{dc}{dr} U_n P_2 (x) P_n^m (x) e^{+im\varphi}$$

$$- \left[\frac{3c}{r} V_n x (1 - x^2) \frac{dP_n^m (x)}{dx} + m T_n x P_n^m (x) \right] e^{+im\varphi} \tag{150}$$

By taking equations 1, 2, and 7 into account, some easy transformations yield:

$$\eta_n = - \gamma U_n S_n$$

$$- \frac{(n + m - 1)(n + m)}{(2n - 1)(2n + 1)} \left[+ \frac{3}{2} \frac{dc}{dr} U_n + (n + 1) \frac{3c}{r} V_n \right] S_{n-2}$$

$$- \frac{3c}{r} \cdot \frac{m (n + m)}{2n + 1} T_n S_{n-1}$$

$$- \frac{n^2 - 3m^2 + n}{(2n - 1)(2n + 3)} \left(+ \frac{dc}{dr} U_n + \frac{3c}{r} V_n \right) S_n \tag{151}$$

$$- \frac{3c}{r} \cdot \frac{m (n - m + 1)}{2n + 1} T_n S_{n+1}$$

$$- \frac{(n - m + 1)(n - m + 2)}{(2n + 1)(2n + 3)} \left(+ \frac{3}{2} \frac{dc}{dr} U_n - \frac{3c}{r} n V_n \right) S_{n+2}$$

Replacing n in equation 151 by n + 2, n + 1, n, n - 1, n - 2, and keeping only the essential terms (i.e., the terms of the degree n) yields:

$$\eta = \mathbf{u} \cdot \nabla V = - \gamma U_n S_n$$

$$+ \left\{ - \frac{dc}{dr} \left[+ \frac{3}{2} \cdot \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} U_{n+2} \right. \right.$$

$$+ \frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} U_n$$

$$\left. + \frac{3}{2} \cdot \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} U_{n-2} \right]$$

$$- \frac{3c}{r} \left[+ \frac{(n+m+1)(n+m+2)(n+3)}{(2n+3)(2n+5)} V_{n+2} \right. \tag{152}$$

$$+ \frac{m(n+m+1)}{2n+3} T_{n+1}$$

$$+ \frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} V_n$$

$$+ \frac{m(n-m)}{2n-1} T_{n-1}$$

$$\left. \left. - \frac{(n-m-1)(n-m)(n-2)}{(2n-3)(2n-1)} V_{n-2} \right] \right\} S_n$$

In particular, for diurnal and semidiurnal tides up to the third degree,

for n = 1, m = 1 ,

$$\eta_1^1 = - \gamma U_1^1 S_1^1 - \left[\frac{dc}{dr} \left(+ \frac{18}{35} U_3^1 - \frac{1}{5} U_1^1 \right) \right. \tag{153}$$

$$+ \frac{c}{r}\left(+ \frac{144}{35} V_3^1 + \frac{9}{5} T_2^1 - \frac{3}{5} V_1^1\right)\Bigg] S_1^1 \qquad \text{(153 continued)}$$

for n = 2, m = 1,

$$\eta_2^1 = -\gamma U_2^1 S_2^1 - \Bigg[+ \frac{dc}{dr}\left(+ \frac{10}{21} U_4^1 + \frac{1}{7} U_2^1\right) \tag{154}$$

$$+ \frac{c}{r}\left(+ \frac{100}{21} V_4^1 + \frac{12}{7} T_3^1 + \frac{3}{7} V_2^1 + T_1^1\right)\Bigg] S_2^1$$

for n = 3, m = 1,

$$\eta_3^1 = -\gamma U_3^1 S_3^1 - \Bigg[\frac{dc}{dr}\left(+ \frac{5}{11} U_5^1 + \frac{1}{5} U_3^1 + \frac{1}{5} U_1^1\right) \tag{155}$$

$$+ \frac{c}{r}\left(+ \frac{60}{11} V_5^1 + \frac{5}{3} T_4^1 + \frac{3}{5} V_3^1 + \frac{6}{5} T_2^1 - \frac{2}{5} V_1^1\right)\Bigg] S_3^1$$

and for n = 2, m = 2,

$$\eta_2^2 = -\gamma U_2^2 S_2^2 - \Bigg[\frac{dc}{dr}\left(+ \frac{5}{7} U_4^2 - \frac{2}{7} U_2^2\right) \tag{156}$$

$$+ \frac{c}{r}\left(+ \frac{50}{7} V_4^2 + \frac{30}{7} T_3^2 - \frac{6}{7} V_2^2\right)\Bigg] S_2^2$$

Let λ, μ, and ρ be the values of the elastic parameters and density for an ellipsoidal model of the Earth, and let λ_o, μ_o, and ρ_o be their values for the corresponding spherical model. If the viscosity of the core is neglected, then λ_o, μ_o, and ρ_o are functions of r. With the viscosity considered, they become the differential operators with respect to time:

$$\begin{aligned} \lambda &= \lambda_o + \frac{2}{3} \lambda_o' \, r \, \epsilon(r) P_2(x) \\ \mu &= \mu_o + \frac{2}{3} \mu_o' \, r \, \epsilon(r) P_2(x) \end{aligned} \tag{157}$$

$$\rho = \rho_o + \frac{2}{3}\rho_o' \, r \, \epsilon(r) P_2(x) \qquad \text{(157 continued)}$$

where

$$\lambda_o' = \frac{d\lambda_o}{dr}, \quad \mu_o' = \frac{d\mu_o}{dr}, \quad \rho_o' = \frac{d\rho_o}{dr}, \quad x = \cos\theta$$

From equations 157, 134, and 135,

$$\frac{\lambda}{\rho}\theta = \left[\frac{\lambda_o}{\rho_o} + s P_2(x)\right] X_n S_n \qquad (158)$$

where

$$s = \frac{\lambda_o}{\rho_o} r \left(\frac{\lambda_o'}{\lambda_o} - \frac{\rho_o'}{\rho_o}\right) \epsilon(r) \qquad (159)$$

Taking equation 3 into consideration,

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_n S_n + s \left[+ \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} S_{n+2} \right.$$

$$\left. + \frac{2}{3}\frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} S_n + \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} S_{n-2}\right] \cdot X_n \qquad (160)$$

Replacing n in the last equation by n – 2, n, n + 2 and keeping only the terms of the n^{th} degree,

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_n S_n + s \left[+ \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} X_{n-2} \right.$$

$$\left. + \frac{2}{3}\frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} X_n + \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} X_{n+2}\right] S_n \qquad (161)$$

In particular, from equation 161 for diurnal and semidiurnal tides up to the third degree,

$$n = 1, m = 1,$$

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_1^1 S_1^1 + s\left(-\frac{2}{15} X_1^1 + \frac{12}{35} X_3^1\right) S_1^1 \tag{162}$$

$$n = 2, m = 1,$$

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_2^1 S_2^1 + s\left(+\frac{2}{21} X_2^1 + \frac{20}{63} X_4^1\right) S_2^1 \tag{163}$$

$$n = 3, m = 1,$$

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_3^1 S_3^1 + s\left(+\frac{2}{15} X_1^1 + \frac{2}{15} X_3^1 + \frac{10}{33} X_5^1\right) S_3^1 \tag{164}$$

$$n = 2, m = 2,$$

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_2^2 S_2^2 + s\left(-\frac{4}{21} X_2^2 + \frac{10}{21} X_4^2\right) S_2^2 \tag{165}$$

$$n = 3, m = 2,$$

$$\frac{\lambda}{\rho}\theta = \frac{\lambda_o}{\rho_o} X_3^2 S_3^2 + \frac{14}{33} s X_5^2 S_3^2 \tag{166}$$

SCALAR FORM OF DIFFERENTIAL EQUATIONS OF TIDAL OSCILLATIONS

Substituting the expansions given in the three previous sections into the differential equation of tidal oscillations (e.g., into equation 103) and setting $\mu = 0$ for the liquid core, for the general core of tidal oscillations of degree n:

$$\frac{\partial^2 U_n}{\partial t^2} + 2\Omega_o i\left[+\frac{(n-1)(n-m)}{2n-1}\frac{\partial T_{n-1}}{\partial t} - m\frac{\partial V_n}{\partial t}\right.$$

$$\left. - \frac{(n+2)(n+m+1)}{2n+3}\frac{\partial T_{n+1}}{\partial t}\right] \tag{167}$$

$$= +\beta\gamma X_n + \beta\frac{dc}{dr}\left[+\frac{3}{2}\cdot\frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} X_{n-2}\right.$$

$$+\frac{n^2-3m^2+n}{(2n-1)(2n+3)} X_n + \frac{3}{2} \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} X_{n+2} \Bigg] + \frac{\partial F_n}{\partial r} \qquad \text{(167 continued)}$$

$$\frac{\partial^2 V_n}{\partial t^2} + 2\Omega_o i \Bigg[- \frac{m}{n(n+1)} \left(\frac{\partial U_n}{\partial t} + \frac{\partial V_n}{\partial t} \right)$$

$$+ \frac{(n-1)(n-m)}{n(2n-1)} \frac{\partial T_{n-1}}{\partial t} + \frac{(n+2)(n+m+1)}{(2n+3)(n+1)} \frac{\partial T_{n+1}}{\partial t} \Bigg]$$

$$= \quad + \beta \frac{3c}{r} \Bigg[+ \frac{(n-m-1)(n-m)}{n(2n-1)(2n-3)} X_{n-2} + \frac{n^2-3m^2+n}{n(n+1)(2n-1)(2n+3)} X_n \qquad (168)$$

$$- \frac{(n+m+1)(n+m+2)}{(n+1)(2n+3)(2n+5)} X_{n+2} \Bigg] + \frac{1}{r} F_n$$

where

$$F_n = \psi_n - \gamma U_n - \frac{dc}{dr} \Bigg[+ \frac{3}{2} \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} U_{n-2}$$

$$+ \frac{n^2-3m^2+n}{(2n-1)(2n+3)} U_n + \frac{3}{2} \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} U_{n+2} \Bigg]$$

$$- \frac{3c}{r} \Bigg[- \frac{(n-m-1)(n-m)(n-2)}{(2n-3)(2n-1)} V_{n-2} + \frac{n^2-3m^2+n}{(2n-1)(2n+3)} V_n \qquad (169)$$

$$+ \frac{(n+m+1)(n+m+2)(n+3)}{(2n+3)(2n+5)} V_{n+2} \Bigg]$$

$$- \frac{3c}{r} \Bigg[+ \frac{m(n-m)}{2n-1} T_{n-1} + \frac{m(n+m+1)}{2n+3} T_{n+1} \Bigg]$$

$$+ \frac{\lambda_o}{\rho_o} X_n + s \left[+ \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} X_{n-2} \right. \qquad \text{(169 continued)}$$

$$\left. + \frac{2}{3} \frac{n^2 - 3m^2 + n}{(2n-1)(2n+3)} X_n + \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} X_{n+2} \right]$$

The nutational term

$$\Pi_2^1 = + \frac{1}{6} \varepsilon \Omega_0^2 r^2 e^{+i\sigma t}$$

must be added to the right-hand side of equation 169 for n = 2, m = 1,

$$\frac{\partial^2 T_n}{\partial t^2} - 2\Omega_o i \left[+ \frac{n-m}{n(2n-1)} \frac{\partial U_{n-1}}{\partial t} - \frac{(n-1)(n-m)}{n(2n-1)} \frac{\partial V_{n-1}}{\partial t} \right.$$

$$\left. - \frac{n+m+1}{(n+1)(2n+3)} \frac{\partial U_{n+1}}{\partial t} - \frac{(n+2)(n+m+1)}{(n+1)(2n+3)} \frac{\partial V_{n+1}}{\partial t} + \frac{m}{n(n+1)} \frac{\partial T_n}{\partial t} \right] \qquad (170)$$

$$= + \beta \frac{3c}{r} m \left[+ \frac{n-m}{n(n+1)(2n-1)} X_{n-1} + \frac{n+m+1}{n(n+1)(2n+3)} X_{n+1} \right]$$

The nutational term,

$$+ \frac{1}{2} \varepsilon r \Omega_o \sigma e^{+i\sigma t}$$

must be added to the right-hand side of equation 170 if n = 1. If interest is in tides of a given frequency only, taking equation 123 into account yields:

$$- \sigma^2 U_n - 2 \Omega_o \sigma \left[+ \frac{(n-1)(n-m)}{2n-1} T_{n-1} - m V_n \right.$$

$$\left. - \frac{(n+2)(n+m+1)}{2n+3} T_{n+1} \right] \qquad (171)$$

$$= + \beta \gamma X_n + \beta \frac{dc}{dr} \left[+ \frac{3}{2} \frac{(n-m-1)(n-m)}{(2n-3)(2n-1)} X_{n-2} \right.$$

$$
+\frac{n^2-3m^2+n}{(2n-1)(2n+3)} X_n + \frac{3}{2} \cdot \frac{(n+m+1)(n+m+2)}{(2n+3)(2n+5)} X_{n+2}\Bigg]
$$

(171 continued)

$$
+\frac{\partial F_n}{\partial r}
$$

$$
-\sigma^2 V_n - 2\,\Omega_o\,\sigma \Bigg[-\frac{m}{n(n+1)} (U_n + V_n)
$$

$$
+\frac{(n-1)(n-m)}{n(2n-1)} T_{n-1} + \frac{(n+2)(n+m+1)}{(2n+3)(n+1)} T_{n+1}\Bigg]
$$

$$
= \quad +\beta\frac{3c}{r}\Bigg[+\frac{(n-m-1)(n-m)}{n(2n-1)(2n-3)} X_{n-2} + \frac{n^2-3m^2+n}{n(n+1)(2n-1)(2n+3)} X_n
$$

(172)

$$
-\frac{(n+m+1)(n+m+2)}{(n+1)(2n+3)(2n+5)} X_{n+2}\Bigg] + \frac{1}{r} F_n
$$

$$
-\sigma^2 T_n + 2\,\Omega_o\,\sigma \Bigg[+\frac{n-m}{n(2n-1)} U_{n-1} - \frac{(n-1)(n-m)}{n(2n-1)} V_{n-1}
$$

$$
-\frac{n+m+1}{(n+1)(2n+3)} U_{n+1} - \frac{(n+2)(n+m+1)}{(n+1)(2n+3)} V_{n+1} + \frac{m}{n(n+1)} T_n\Bigg] \qquad (173)
$$

$$
= \quad \beta\frac{3c}{r} m \Bigg[+\frac{n-m}{n(n+1)(2n-1)} X_{n-1} + \frac{n+m+1}{n(n+1)(2n+3)} X_{n+1}\Bigg]
$$

Equations 167 through 173 show that there is a strong mutual interaction between U_n, V_n, T_{n-1}, and T_{n+1}. The corresponding differential equations constitute a separate group.

Besides U_n, V_n, T_{n-1}, and T_{n+1}, some other terms enter into these differential equations, but they represent the effects of higher order.

For example, for diurnal tides,

$$\frac{\partial^2 U_2^1}{\partial t^2} + 2\,\Omega_o\, i \left(+ \frac{1}{3} \frac{\partial T_1^1}{\partial t} - \frac{\partial V_2^1}{\partial t} - \frac{16}{7} \frac{\partial T_3^1}{\partial t} \right)$$
$$= +\beta \gamma X_2^1 + \beta \frac{dc}{dr} \left(+ \frac{1}{7} X_2^1 + \frac{10}{21} X_4^1 \right) + \frac{\partial F_2^1}{\partial r} \tag{174}$$

$$\frac{\partial^2 V_2^1}{\partial t^2} + 2\,\Omega_o\, i \left(+ \frac{1}{6} \frac{\partial T_1^1}{\partial t} - \frac{1}{6} \frac{\partial U_2^1}{\partial t} - \frac{1}{6} \frac{\partial V_2^1}{\partial t} + \frac{16}{21} \frac{\partial T_3^1}{\partial t} \right)$$
$$= +\beta \frac{c}{r} \left(+ \frac{1}{14} X_2^1 - \frac{20}{63} X_4^1 \right) + \frac{1}{r} F_2^1 \tag{175}$$

$$\frac{\partial^2 T_1^1}{\partial t^2} - 2\,\Omega_o\, i \left(+ \frac{1}{2} \frac{\partial T_1^1}{\partial t} - \frac{3}{10} \frac{\partial U_2^1}{\partial t} - \frac{9}{10} \frac{\partial V_2^1}{\partial t} \right)$$
$$= + \frac{9}{10} \beta \frac{c}{r} X_2^1 + \frac{1}{2} \epsilon\, r\, \Omega_o\, \sigma\, e^{+i\sigma t} \tag{176}$$

$$\frac{\partial^2 T_3^1}{\partial t^2} - 2\,\Omega_o\, i \left(+ \frac{2}{15} \frac{\partial U_2^1}{\partial t} - \frac{4}{15} \frac{\partial V_2^1}{\partial t} + \frac{1}{12} \frac{\partial T_3^1}{\partial t} - \frac{5}{36} \frac{\partial U_4^1}{\partial t} - \frac{25}{36} \frac{\partial V_4^1}{\partial t} \right) \tag{177}$$
$$= + \beta \frac{c}{r} \left(+ \frac{1}{10} X_2^1 + \frac{5}{36} X_4^1 \right)$$

$$F_2^1 = \psi_2^1 - \gamma U_2^1 - \frac{dc}{dr} \left(+ \frac{1}{7} U_2^1 + \frac{10}{21} U_4^1 \right) \tag{178}$$

$$-\frac{c}{r}\left(T_1^1+\frac{3}{7}V_2^1+\frac{12}{7}T_3^1+\frac{100}{21}V_4^1\right)$$

(178 continued)

$$+\frac{\lambda_o}{\rho_o}X_2^1+s\left(+\frac{2}{21}X_2^1+\frac{20}{63}X_4^1\right)+\Pi_2^1$$

where

$$\Pi_2^1=+\frac{1}{6}\varepsilon\Omega_o^2\ r^2\ S_2^{+1}\ e^{+i\sigma t}$$

and

$$\frac{\partial^2 U_3^1}{\partial t^2}+2\Omega_o i\left(+\frac{4}{5}\frac{\partial T_2^1}{\partial t}-\frac{\partial V_3^1}{\partial t}-\frac{25}{9}\frac{\partial T_4^1}{\partial t}\right)$$

$$=+\beta\gamma X_3^1+\beta\frac{dc}{dr}\left(+\frac{1}{5}X_1^1+\frac{1}{5}X_3^1+\frac{5}{11}X_5^1\right)+\frac{\partial F_3^1}{\partial r} \qquad (179)$$

$$\frac{\partial^2 V_3^1}{\partial t^2}+2\Omega_o i\left(+\frac{4}{15}\frac{\partial T_2^1}{\partial t}-\frac{1}{12}\frac{\partial U_3^1}{\partial t}-\frac{1}{12}\frac{\partial V_3^1}{\partial t}+\frac{25}{36}\frac{\partial T_4^1}{\partial t}\right)$$

(180)

$$=+\beta\frac{c}{r}\left(+\frac{2}{15}X_1^1+\frac{1}{20}X_3^1-\frac{5}{22}X_5^1\right)+\frac{1}{r}F_3^1$$

$$\frac{\partial^2 T_2^1}{\partial t^2}-2\Omega_o i\left(+\frac{1}{6}\frac{\partial T_2^1}{\partial t}+\frac{1}{6}\frac{\partial U_1^1}{\partial t}-\frac{1}{6}\frac{\partial V_1^1}{\partial t}-\frac{4}{21}\frac{\partial U_3^1}{\partial t}-\frac{16}{21}\frac{\partial V_3^1}{\partial t}\right) \qquad (181)$$

$$=+\beta\frac{c}{r}\left(+\frac{1}{6}X_1^1+\frac{2}{7}X_3^1\right)$$

$$\frac{\partial^2 T_4^1}{\partial t^2} - 2\,\Omega_o\, i \left(+ \frac{1}{20}\,\frac{\partial T_4^1}{\partial t} + \frac{3}{28}\,\frac{\partial U_3^1}{\partial t} - \frac{9}{28}\,\frac{\partial V_3^1}{\partial t} - \frac{6}{55}\,\frac{\partial U_5^1}{\partial t} - \frac{36}{55}\,\frac{\partial V_5^1}{\partial t} \right) \tag{182}$$

$$= +\beta\frac{3c}{r} \left(+ \frac{3}{140}\, X_3^1 + \frac{3}{110}\, X_5^1 \right)$$

$$F_3^1 = \gamma\, U_3^1 - \frac{dc}{dr} \left(+ \frac{1}{5}\, U_1^1 + \frac{1}{5}\, U_3^1 + \frac{5}{11}\, U_5^1 \right) \tag{183}$$

$$- \frac{c}{r} \left(- \frac{2}{5}\, V_1^1 + \frac{3}{5}\, V_3^1 + \frac{60}{11}\, V_5^1 + \frac{6}{5}\, T_2^1 + \frac{5}{3}\, T_4^1 \right)$$

$$+ \frac{\lambda_o}{\rho_o}\, X_3^1 + s \left(+ \frac{2}{15}\, X_1^1 + \frac{2}{15}\, X_3^1 + \frac{10}{33}\, X_5^1 \right)$$

and, equations 121 and 122, 147 and 148 yield for main semidiurnal tides,

$$\frac{\partial^2 U_2^2}{\partial t^2} + 2\,\Omega_o\, i \left(- 2\,\frac{\partial V_2^2}{\partial t} - \frac{20}{7}\,\frac{\partial T_3^2}{\partial t} \right) \tag{184}$$

$$= +\beta\,\gamma\, X_2^2 + \beta\,\frac{dc}{dr} \left(- \frac{2}{7}\, X_2^2 + \frac{5}{7}\, X_4^2 \right) + \frac{\partial F_2^2}{\partial r}$$

$$\frac{\partial^2 V_2^2}{\partial t^2} + 2\,\Omega_o\, i \left(- \frac{1}{3}\,\frac{\partial U_2^2}{\partial t} - \frac{1}{3}\,\frac{\partial V_2^2}{\partial t} + \frac{20}{21}\,\frac{\partial T_3^2}{\partial t} \right) \tag{185}$$

$$= -\beta\frac{c}{r} \left(+ \frac{1}{7}\, X_2^2 + \frac{10}{21}\, X_4^2 \right) + \frac{1}{r}\, F_2^2$$

$$\frac{\partial^2 T_3^2}{\partial t^2} - 2\,\Omega_o\, i \left(+ \frac{1}{15} \frac{\partial U_2^2}{\partial t} - \frac{2}{15} \frac{\partial V_2^2}{\partial t} + \frac{1}{6} \frac{\partial T_3^2}{\partial t} - \frac{1}{6} \frac{\partial U_4^2}{\partial t} - \frac{5}{6} \frac{\partial V_4^2}{\partial t} \right) \tag{186}$$

$$= + \beta \frac{c}{r} \left(+ \frac{1}{10} X_2^2 + \frac{1}{3} X_4^2 \right)$$

$$F_2^2 = \psi_2^2 - \gamma U_2^2 - \frac{dc}{dr} \left(- \frac{2}{7} U_2^2 + \frac{5}{7} V_4^2 \right) - \frac{c}{r} \left(- \frac{6}{7} V_2^2 + \frac{30}{7} T_3^2 + \frac{50}{7} V_4^2 \right) \tag{187}$$

$$+ \frac{\lambda_o}{\rho_o} X_2^2 + s \left(- \frac{4}{21} X_2^2 + \frac{10}{21} X_4^2 \right)$$

If only terms of a given diurnal or semidiurnal frequency, σ, are of interest, then replacing

$$U_n \text{ by } U_n\, e^{+i\sigma t},\ V_n \text{ by } V_n\, e^{+i\sigma t}, \text{etc.}$$

in equations 174 through 187 yields:

$$- \sigma^2 U_2^1 + \Omega_o \sigma \left(- \frac{2}{3} T_1^1 + \frac{32}{7} T_3^1 + 2 V_2^1 \right)$$

$$= + \beta \gamma X_2^1 + \beta \frac{dc}{dr} \left(+ \frac{1}{7} X_2^1 + \frac{10}{21} X_4^1 \right) + \frac{dF_2^1}{dr} \tag{188}$$

$$\left(- \sigma^2 + \frac{1}{3} \Omega_o \sigma \right) V_2^1 + \Omega_o \sigma \left(- \frac{1}{3} T_1^1 - \frac{32}{21} T_3^1 + \frac{1}{3} U_2^1 \right) \tag{189}$$

$$= + \frac{1}{14} \beta \frac{c}{r} \left(X_2^1 - \frac{40}{9} X_4^1 \right) + \frac{1}{r} F_2^1$$

$$(- \sigma^2 + \Omega_o \sigma) T_1^1 + \Omega_o \sigma \left(- \frac{3}{5} U_2^1 - \frac{9}{5} V_2^1 \right) \tag{190}$$

$$= +\frac{9}{10}\beta\frac{c}{r}X_2^1 + \frac{1}{2}\varepsilon\Omega_o\sigma\tau \qquad \text{(190 continued)}$$

$$(-\sigma^2 + \frac{1}{6}\Omega_o\sigma)T_3^1 + \Omega_o\sigma\left(+\frac{4}{15}U_2^1 - \frac{8}{15}V_2^1\right)$$

$$= +\beta\frac{c}{r}\left(+\frac{1}{10}X_2^1 + \frac{5}{36}X_4^1\right) \qquad (191)$$

$$-\sigma^2 U_3^1 + \Omega_o\sigma\left(-\frac{8}{5}T_2^1 + \frac{50}{9}T_4^1 + 2V_3^1\right) \qquad (192)$$

$$= +\beta\gamma X_3^1 + \beta\frac{dc}{dr}\left(+\frac{1}{5}X_1^1 + \frac{1}{5}X_3^1 + \frac{5}{11}X_5^1\right) + \frac{dF_3^1}{dr}$$

$$\left(-\sigma^2 + \frac{1}{6}\Omega_o\sigma\right)V_3^1 + \Omega_o\sigma\left(-\frac{8}{15}T_2^1 - \frac{25}{18}T_4^1 + \frac{1}{6}U_3^1\right) \qquad (193)$$

$$= +\beta\frac{c}{r}\left(+\frac{2}{15}X_1^1 + \frac{1}{20}X_3^1 - \frac{5}{22}X_5^1\right) + \frac{1}{r}F_3^1$$

$$\left(-\sigma^2 + \frac{1}{3}\Omega_o\sigma\right)T_2^1 + \Omega_o\sigma\left(+\frac{1}{3}U_1^1 - \frac{1}{3}V_1^1 - \frac{8}{21}U_3^1 - \frac{32}{21}V_3^1\right) \qquad (194)$$

$$= +\beta\frac{c}{r}\left(+\frac{1}{6}X_1^1 + \frac{2}{7}X_3^1\right)$$

$$\left(-\sigma^2 + \frac{1}{10}\Omega_o\sigma\right)T_4^1 + \Omega_o\sigma\left(+\frac{3}{14}U_3^1 - \frac{9}{14}V_3^1 - \frac{12}{55}U_5^1 - \frac{72}{55}V_5^1\right) \qquad (195)$$

$$= +\frac{9}{10}\beta\frac{c}{r}\left(+\frac{1}{14}X_3^1 + \frac{1}{11}X_5^1\right)$$

and, for the main semidiurnal tide:

$$- \sigma^2 U_2^2 + \Omega_o \sigma \left(+ \frac{40}{7} T_3^2 + 4 V_2^2\right)$$

$$= + \beta \gamma X_2^2 + \beta \frac{dc}{dr} \left(- \frac{2}{7} X_2^2 + \frac{5}{7} X_4^2\right) + \frac{d F_2^2}{dr} \tag{196}$$

$$\left(- \sigma^2 + \frac{2}{3} \Omega_o \sigma\right) V_2^2 + \Omega_o \sigma \left(- \frac{40}{21} T_3^2 + \frac{2}{3} U_2^2\right)$$

$$= - \beta \frac{c}{r} \left(+ \frac{1}{7} X_2^2 + \frac{10}{21} X_4^2\right) + \frac{1}{r} F_2^2 \tag{197}$$

$$\left(- \sigma^2 + \frac{1}{3} \Omega_o \sigma\right) T_3^2 + \Omega_o \sigma \left(+ \frac{2}{15} U_2^2 - \frac{4}{15} V_2^2 - \frac{1}{3} U_4^2 - \frac{5}{3} V_4^2\right)$$

$$= + \beta \frac{c}{r} \left(+ \frac{1}{10} X_2^2 + \frac{1}{3} X_4^2\right) \tag{198}$$

The lunisolar tides, U_1^1, V_1^1, do not exist, and it appears that the influence of tides U_4^1, V_4^1, U_5^1, V_5^1, U_4^2, and V_4^2 in equations 188 through 198 is negligible. Of course, this assumption requires a full confirmation by numerical integration, but at least it can be used at the beginning of the computational process. Differential equations 188 through 198 are then split into three separate groups that can be solved independently.

TRANSFORMATION OF THE POISSON EQUATION

The tidal forces cause a small redistribution of matter inside the Earth. This results in a small increment ψ of the interior geopotential that satisfies the Poisson equation:

$$\nabla^2 \psi = + 4\pi G \nabla \cdot (\rho \mathbf{u}) \tag{199}$$

By assuming the expansion,

$$\psi = \sum \psi_n^m S_n^m \tag{200}$$

equation 199 can be reduced to an equivalent set of ordinary differential equations for ψ_n^m that must be solved with the corresponding set (equations 167, 168, and 170) for the tidal displacements and ψ_n^m.

By substituting

$$\rho = \rho_o(r) + q(r) P_2(x)$$

$$q = \frac{2}{3} r \frac{d\rho}{dr} \epsilon(r), \qquad x = \cos\theta \tag{157}$$

and

$$\mathbf{u} = U_n \mathbf{A}_n + V_n \mathbf{B}_n + i T_n \mathbf{C}_n \tag{116}$$

into $\nabla \cdot (\rho\, \mathbf{u})$ and by taking equations 15 through 18 into account, some easy vectorial transformations result in:

$$\nabla \cdot (\rho \mathbf{u}) = \left\{ \left[\frac{d\rho_o}{dr} U_n + \rho_o X_n + \left(\frac{dq}{dr} U_n + q X_n \right) P_2(x) \right.\right.$$

$$\left. + 3 \frac{q}{r} m x T_n \right] P_n^m(x)$$

$$\left. + 3 \frac{q}{r} V_n x (1 - x^2) \frac{d P_n^m}{dx} \right\} e^{+im\varphi}$$

By taking equations 1, 3, and 7 into account, the foregoing equation becomes:

$$\nabla \cdot (\rho \mathbf{u}) = \left(\frac{d\rho_o}{dr} U_n + \rho_o X_n \right) S_n$$

$$+ \frac{3}{2} \cdot \frac{(n+m)(n+m-1)}{(2n+1)(2n-1)} \left[\frac{dq}{dr} U_n + q X_n + 2(n+1) \frac{q}{r} V_n \right] S_{n-2} \tag{201}$$

$$+ 3 \frac{m(n+m)}{2n+1} \frac{q}{r} T_n S_{n-1}$$

$$+\frac{n^2-3m^2+n}{(2n-1)(2n+3)}\left(\frac{dq}{dr}U_n+qX_n+3\frac{q}{r}V_n\right)S_n \qquad \text{(201 continued)}$$

$$+\;3\,\frac{m(n-m+1)}{2n+1}\,\frac{q}{r}\,T_n\,S_{n+1}$$

$$+\frac{3}{2}\cdot\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\left(\frac{dq}{dr}U_n+qX_n-2n\frac{q}{r}V_n\right)S_{n+2}$$

In equation 201, replacing n by n + 2, n + 1, n – 2, and keeping only the terms of the n^{th} degree yields:

$$\nabla\cdot(\rho u)=\left\{\left(\frac{d\rho_o}{dr}U_n+\rho_o X_n\right)\right.$$

$$+\frac{3}{2}\cdot\frac{(n+m+2)(n+m+1)}{(2n+5)(2n+3)}\left[\frac{dq}{dr}U_{n+2}+qX_{n+2}+2(n+3)\frac{q}{r}V_{n+2}\right]$$

$$+\;3\cdot\frac{m(n+m+1)}{2n+3}\,\frac{q}{r}\,T_{n+1}$$

$$+\frac{n^2-3m^2+n}{(2n-1)(2n+3)}\left(\frac{dq}{dr}U_n+qX_n+3\frac{q}{r}V_n\right) \qquad (202)$$

$$+\;3\cdot\frac{m(n-m)}{2n-1}\,\frac{q}{r}\,T_{n-1}$$

$$\left.+\frac{3}{2}\cdot\frac{(n-m-1)(n-m)}{(2n-3)(2n-1)}\left[\frac{dq}{dr}U_{n-2}+qX_{n-2}-2(n-2)\frac{q}{r}V_{n-2}\right]\right\}S_n$$

Substituting equations 201 and 202 into equation 199 yields the differential equations satisfied by ψ_n^m:

$$\frac{d^2\psi_n}{dr^2}+\frac{2}{r}\frac{d\psi_n}{dr}-\frac{n(n+1)}{r^2}\psi_n=+4\pi G\left\{\left(\frac{d\rho_o}{dr}U_n+\rho_o X_n\right)\right. \qquad (203)$$

$$+\frac{3}{2}\cdot\frac{(n+m+2)(n+m+1)}{(2n+5)\ (2n+3)}\left[\frac{dq}{dr}U_{n+2}+qX_{n+2}+2(n+3)\ \frac{q}{r}V_{n+2}\right]$$

$$+\ 3\cdot\frac{m(n+m+1)}{2n+3}\ \frac{q}{r}\ T_{n+1}+\frac{n^2-3m^2+n}{(2n-1)(2n+3)}\left(\frac{dq}{dr}U_n+qX_n+3\frac{q}{r}V_n\right)$$

$$+\ 3\cdot\frac{m(n-m)}{2n-1}\ \frac{q}{r}\ T_{n-1} \qquad \text{(203 continued)}$$

$$\left.+\frac{3}{2}\cdot\frac{(n-m-1)(n-m)}{(2n-3)(2n-1)}\left[\frac{dq}{dr}U_{n-2}+qX_{n-2}-2(n-2)\frac{q}{r}V_{n-2}\right]\right\}$$

If the ellipticity of equipotential surfaces is neglected, the standard equation for a spherical model is:

$$\frac{d^2\psi_n}{dr^2}+\frac{2}{r}\frac{d\psi_n}{dr}-\frac{n(n+1)}{r^2}\psi_n=+4\pi G\left(\frac{d\rho_o}{dr}U_n+\rho_o X_n\right) \tag{204}$$

Some interesting conclusions can be derived from equations 203 and 204. It is clear that, if the Earth is assumed to be spherically symmetric and originally in hydrostatic equilibrium, then toroidal oscillations do not produce any change in the geopotential. This is a well-known fact, confirmed by the gravimetric observations on the surface of the Earth. In the mantle, the influence of the ellipticity of equipotential surfaces on the oscillations is very small, and the tides are predominantly spheroidal. The introduction of Love and Shida parameters as functions of r takes care of these two facts in the mantle.

However, in the liquid core, the ellipticity and the dependence of the density on the latitude (i.e., the lateral inhomogeneity) cause the toroidal terms to be present in equation 203, and the toroidal oscillations *do* influence the variations of the geopotential if they are sufficiently large. In general, with the introduction of lateral inhomogeneities into the Earth's model, their influence on the tidal oscillations can be expected to increase.

For the main diurnal tides, the equation for the variation of geopotential becomes:

$$\frac{d^2\psi_2^1}{dr^2}+\frac{2}{r}\frac{d\psi_2^1}{dr}-\frac{6}{r^2}\psi_2^1=+4\pi G\left[\left(\frac{d\rho_o}{dr}U_2^1+\rho_o X_2^1\right)\right. \tag{205}$$

$$
+\frac{1}{7}\left(\frac{dq}{dr}\,U_2^1 + q\,X_2^1 + 3\,\frac{q}{r}\,V_2^1\right) + \frac{q}{r}\,T_1^1 + \frac{12}{7}\,\frac{q}{r}\,T_3^1
$$

(205 continued)

$$
+\frac{10}{21}\left(\frac{dq}{dr}\,U_4^1 + q\,X_4^1 + 10\,\frac{q}{r}\,V_4^1\right)\Bigg]
$$

If the influence of U_4^1 and V_4^1 is neglected, then equations 205 and 188 through 191 constitute a separate system to be solved for U_2^1, V_2^1, T_1^1, T_3^1, and ψ_2^1.

For the main semidiurnal tides, equation 203 becomes:

$$
\frac{d^2\psi_2^2}{dr^2} + \frac{2}{r}\,\frac{d\psi_2^2}{dr} - \frac{6}{r^2}\,\psi_2^2 = +4\pi G\Bigg[\left(\frac{d\rho_o}{dr}\,U_2^2 + \rho_o\,X_2^2\right)
$$

$$
-\frac{2}{7}\left(\frac{dq}{dr}\,U_2^2 + q\,X_2^2 + 3\,\frac{q}{r}\,V_2^2\right) + \frac{30}{7}\,\frac{q}{r}\,T_3^2 \tag{206}
$$

$$
+\frac{5}{7}\left(\frac{dq}{dr}\,U_4^2 + q\,X_4^2 + 10\,\frac{q}{r}\,V_4^2\right)\Bigg]
$$

Neglecting the influence of U_4^2 and V_4^2 results in a complete system consisting of equations 206 and 196 through 198 to be solved for tides U_2^2, V_2^2, and T_3^1 and for ψ_2^2.

GENERALIZATION OF SOME MOLODENSKY EQUATIONS IN THE RESONANCE THEORY

The close proximity of the period of the diurnal wobble of the liquid core to the sidereal day produces a resonance effect that influences the amplitudes of nutation and the value of the diurnal Love number.

Molodensky and Kramer (Reference 3) have developed the theory of the resonance effect, assuming the neutral equilibrium of the liquid core and the validity of the Adams-Williamson condition (Reference 16).

Several works recently appeared (References 13, 17, and 18) that consider the dynamical consequences of the departure of the liquid core from the neutral stability. In this section the Molodensky and Kramer (Reference 3) theory is extended and the model of the liquid core is considered, which departs from the neutral stability and, thus, does not satisfy the condition of Adams-Williamson (Reference 16). The theory is of the first order (i.e., the influence of the ellipticity on the spheroidal tidal oscillations can be neglected), but the eccentricity shall be retained as a factor multiplying large toroidal oscillations and, in particular, in the boundary conditions on the core-mantle interface.

Molodensky and Kramer differential equations (Reference 3) are amended by perturbative terms proportional to the Pekeris and Accad index of stability of the outer core (Reference 17).

If $\sigma \sim \Omega_o$, then the toroidal oscillation, T_1^1, becomes large and, because of the small factor, $\Omega_o - \sigma$, equation 190 is no longer suitable for determining tidal amplitudes. Equations 188 through 191 must be replaced by a different combination of equations. Multiplying equation 190 by -1/3 and adding the result to equation 189 yields:

$$+\frac{1}{3}\sigma(\sigma - 2\Omega_o)\, r\, T_1 + \frac{1}{6}\varepsilon\,\Omega_o\,\sigma r^2 = F_2 + \Phi \tag{207}$$

where

$$\begin{aligned} r^{-1}\Phi = &+\frac{32}{21}\Omega_o\,\sigma\, T_3 - \frac{8}{15}\Omega_o\,\sigma\, U_2 + \left(\sigma^2 - \frac{14}{15}\Omega_o\,\sigma\right) V_2 \\ &- \frac{8}{35}\beta\frac{c}{r}X_2 - \frac{20}{63}\beta\frac{c}{r}X_4 \end{aligned} \tag{208}$$

Equation 188 can now be rewritten in the form:

$$-\frac{2}{3}\Omega_o\,\sigma\, T_1 + \Psi = \frac{d F_2}{dr} \tag{209}$$

where

$$\Psi = -\sigma^2 U_2 + \Omega_o\,\sigma\left(+\frac{32}{7}T_3 + 2V_2\right) - \beta\gamma X_2 - \beta\frac{dc}{dr}\left(+\frac{1}{7}X_2 + \frac{10}{21}X_4\right) \tag{210}$$

For brevity and because the ambiguity does not arise, the superscripts and the factor, S_2^1, are omitted in the exposition.

Eliminating F_2 from equation 209 by means of equation 207 yields:

$$\sigma(\sigma - 2\Omega_o)\frac{d}{dr}(r T_1) + 2\Omega_o \sigma T_1 + \varepsilon \Omega_o \sigma r = 3\left(\Psi + \frac{d\Phi}{dr}\right) \tag{211}$$

In this case, the following estimates of orders of magnitude are obtained:

$$\epsilon \sim \frac{\Omega_o - \sigma}{\Omega_o}, \quad \epsilon g \sim \Omega_o^2 r \sim \frac{c}{r} \tag{212}$$

and, from equation 190,

$$U_2, V_2, T_3 \sim \frac{\Omega_o - \sigma}{\Omega_o} T_1 \sim \epsilon T_1 \tag{213}$$

and, from equation 135,

$$X_2 \sim \epsilon \frac{T_1}{r} \tag{214}$$

Consequently,

$$\frac{1}{r}\Phi, \frac{d\Phi}{dr}, \Psi \sim \epsilon \Omega_o^2 T_1 \tag{215}$$

These estimates show that the terms on the right-hand side of equation 211 are less significant than the terms on the left-hand side. Furthermore, the form of equation 211 suggests the estimate,

$$T_1 - \frac{\alpha}{2} r \sim \epsilon T_1 \tag{216}$$

where α is a large constant (Molodensky and Kramer resonant parameter (Reference 3)). From equation 211,

$$\alpha \sim \frac{\epsilon \Omega_o}{\Omega_o - \sigma} \tag{217}$$

Setting

$$2\nu = \epsilon - \frac{\Omega_o - \sigma}{\Omega_o} \alpha \tag{218}$$

and keeping only the essential terms in equation 154 yields:

$$U_2 = -\frac{1}{g} \left(\eta_2 + \frac{c}{r} T_1 \right) \tag{219}$$

and the estimate

$$\eta_2 \sim \epsilon g T_1 \tag{220}$$

From equation 178 the following can be derived with the same accuracy as that given previously:

$$F_2 = \psi_2 + \Pi_2 + \eta_2 + \frac{\lambda}{\rho} X_2 \tag{221}$$

and, from equation 190

$$V_2 = -\frac{1}{3} U_2 + \frac{5}{9} \left(\frac{\Omega_o - \sigma}{\Omega_o} T_1 - \frac{1}{2} \epsilon r \right) - \frac{1}{2} \frac{\beta}{\Omega_o \sigma} \frac{c}{r} X_2 \tag{222}$$

and taking into account equations 216 and 218,

$$V_2 = -\frac{1}{3} U_2 - \frac{5}{9} \nu r - \frac{1}{2} \beta \frac{c}{\Omega_o \sigma r} X_2 \tag{223}$$

It follows from equations 212 and 214 that the last term on the right-hand side of equation 223 is of the order of $\beta \epsilon T_1$, and it must be retained if the influence of the departure of the core from the neutral stability is considered.

From equations 135 and 223,

$$\left(1 - \frac{3\beta c}{\Omega_o \sigma r^2}\right) X_2 = \frac{1}{r^4} \frac{d}{dr}(r^4 U_2) + \frac{10}{3} \nu \tag{224}$$

In agreement with Molodensky and Kramer (Reference 3), setting

$$F_2 = \psi_2 + \Pi_2 - K \tag{225}$$

yields from equation 221,

$$X_2 = -\frac{\rho}{\lambda} (\eta_2 + K) \tag{226}$$

and, substituting

$$-\frac{\rho}{\lambda} = \frac{1}{1-\beta} \frac{1}{g\rho} \frac{d\rho}{dr} \tag{227}$$

into equation 226,

$$X_2 = \frac{1}{1-\beta} \frac{1}{g\rho} \frac{d\rho}{dr} (\eta_2 + K) \tag{228}$$

Eliminating U_2 and X_2 from equation 223 yields:

$$V_2 = +\frac{1}{3g}\left(\eta_2 + \frac{c}{r} T_1\right) - \frac{5}{9}\nu r$$
$$- \frac{\beta}{1-\beta} \cdot \frac{c}{2\Omega_o \sigma r} \cdot \frac{1}{g\rho}\frac{d\rho}{dr}(\eta_2 + K) \tag{229}$$

Combining equations 228 and 219 yields:

$$(1-\beta)\rho X_2 + \frac{d\rho}{dr} U_2 = \frac{1}{g}\frac{d\rho}{dr}\left(K - \frac{c}{r} T_1\right)$$

and, from this equation and equations 219, 224, and 228,

$$\frac{d}{dr}\left[\frac{\rho r^4}{g}\left(\eta_2 + \frac{c}{r} T_1\right)\right] - \frac{10}{3}\nu\rho r^4 + \frac{r^4}{g}\frac{d\rho}{dr}\left(K - \frac{c}{r} T_1\right)$$
$$= + \frac{r^4}{g}\frac{d\rho}{dr} \cdot \frac{2\beta}{1-\beta} \cdot \frac{\epsilon g}{\Omega_o^2 r}(\eta_2 + K) \tag{230}$$

which is one of two basic Molodensky equations (in a slightly modified form) with the influence of the departure of the liquid core from the neutral stability included.

Approximation

$$T_1 = \frac{1}{2}\alpha r$$

can be subtituted into equation 230.

Neglecting the terms of the order $\epsilon\, U_2 \sim \epsilon^2\, T_1$ in equation 205 yields:

$$\frac{d^2\psi_2}{dr^2} + \frac{2}{r}\frac{d\psi_2}{dr} - \frac{6}{r^2}\psi_2 = +4\pi G\left(\frac{d\rho}{dr} U_2 + \rho X_2 + \frac{q}{r} T_1\right) \tag{231}$$

Substituting equation 219 into equation 231 for U_2 and into equation 228 for X_2, some simple transformations result in:

$$\frac{d^2\psi_2}{dr^2} + \frac{2}{r}\frac{d\psi_2}{dr} - \frac{6}{r^2}\psi_2 = + \frac{4\pi G}{g}\frac{d\rho}{dr}\left(\frac{\beta\eta_2 + K}{1 - \beta} - \frac{1}{3}\Omega_o^2 r T_1\right) \tag{232}$$

and, eliminating ψ_2 in favor of K by means of equation 225 yields:

$$\frac{d^2K}{dr^2} + \frac{2}{r}\frac{dK}{dr} - \left(\frac{4\pi G}{g}\frac{d\rho}{dr} + \frac{6}{r^2}\right)K$$
$$+ \left(\frac{d^2F_2}{dr^2} + \frac{2}{r}\frac{dF_2}{dr} - \frac{6}{r^2}F_2\right) \tag{233}$$
$$= + \frac{4\pi G}{g}\frac{d\rho}{dr}\left[\frac{\beta}{1 - \beta}(\eta_2 + K) - \frac{1}{3}\Omega_o^2 r\, T_1\right]$$

in which the approximation, $T_1 = \alpha\, r/2$, can be used. To simplify equation 233, it is more convenient to introduce the new variable:

$$Q = K - \frac{1}{3}\Omega_o^2 r T_1 \tag{234}$$

rather than K so that

$$\frac{d^2Q}{dr^2} + \frac{2}{r}\frac{dQ}{dr} - \left(\frac{4\pi G}{g}\frac{d\rho}{dr} + \frac{6}{r^2}\right)Q$$
$$+ \left(\frac{d^2F_2}{dr^2} + \frac{2}{r}\frac{dF_2}{dr} - \frac{6}{r^2}F_2\right) \tag{235}$$
$$= + \frac{4\pi G}{g} \cdot \frac{d\rho}{dr} \cdot \frac{\beta}{1 - \beta}\left(\eta_2 + Q + \frac{1}{3}\Omega_o^2 r\, T_1\right)$$

From equations 207 and 218,

$$F_2 - \frac{1}{3}\sigma \left[(\sigma - 2\Omega_o)\, rT_1 + \frac{1}{2}\,\varepsilon\,\Omega_o r^2\right] \sim \epsilon\Omega_o^2 r T_1$$

From equations 212, 220, 221, and 228,

$$Q \sim K \sim \epsilon g\, T_1 \sim \Omega_o^2 r\, T_1$$

and, consequently,

$$\frac{d^2F_2}{dr^2} + \frac{2}{r}\frac{dF_2}{dr} - \frac{6}{r^2}F_2 \sim \frac{\epsilon Q}{r^2}$$

and it can be neglected. Then,

$$\frac{d^2Q}{dr^2} + \frac{2}{r}\frac{dQ}{dr} - \left(\frac{4\pi G}{g}\frac{d\rho}{dr} + \frac{6}{r^2}\right) Q \tag{236}$$

$$= + \frac{4\pi G}{g} \bullet \frac{d\rho}{dr} \bullet \frac{\beta}{1 - \beta}\left(\eta_2 + Q + \frac{1}{3}\Omega_o^2 r\, T_1\right)$$

or, in a slightly different form,

$$\frac{d}{dr}\left[r^6 \frac{d}{dr}(r^{-2}Q)\right] - \frac{4\pi G}{g}\frac{d\rho}{dr} r^4\, Q \tag{237}$$

$$= + \frac{4\pi G}{g} r^4 \frac{d\rho}{dr}\frac{\beta}{1 - \beta}\left(\eta_2 + Q + \frac{1}{3}\Omega_o^2 r\, T_1\right)$$

If the departure from the neutral stability is neglected, equations 236 and 237 become corresponding Molodensky equations.

Equations 219, 228 through 230, 234, and 236 constitute a complete system for the resonance case. It can be integrated only numerically. Solutions must be determined so as to satisfy the conditions on the boundary between the inner and outer cores and between the outer core and the mantle.

Because ϵ satisfies Clairaut equation

$$\frac{d^2\epsilon}{dr^2} + \frac{8\pi G\rho}{g}\frac{d\epsilon}{dr} + \left(\frac{8\pi G\rho}{gr} - \frac{6}{r^2}\right)\epsilon = 0 \tag{238}$$

it can be shown that c satisfies the differential equation:

$$\frac{1}{4\pi G}\frac{d}{dr}\left(r^6\frac{d}{dr}\frac{c}{r^2}\right) - \frac{r^4}{g}\frac{d\rho}{dr}c + \frac{1}{3}\frac{r^4}{g}\frac{d\rho}{dr}\Omega_o^2 r^2 = 0 \tag{239}$$

From equations 237 and 239,

$$\begin{aligned} &\frac{1}{4\pi G}\frac{d}{dr}\left[r^6\frac{d}{dr}\frac{1}{r^2}\left(Q - \frac{1}{2}c\,\alpha\right)\right] - \frac{r^4}{g}\frac{d\rho}{dr}\left(Q - \frac{1}{2}c\,\alpha\right) \\ &= \frac{r^4}{g}\frac{d\rho}{dr}\left[\frac{\beta}{1-\beta}(\eta_2 + K) + \frac{1}{6}\Omega_o^2 r^2\alpha\right] \end{aligned} \tag{240}$$

and combining equations 240 and 230 yields:

$$\begin{aligned} &\frac{d}{dr}\left\{\frac{\rho r^4}{g}\left(\eta_2 + \frac{1}{2}c\alpha\right) + \frac{r^6}{4\pi G}\frac{d}{dr}\frac{1}{r^2}\left(Q - \frac{1}{2}c\alpha\right)\right\} \\ &- \frac{10}{3}\nu\rho r^4 = \frac{r^4}{g}\frac{d\rho}{dr}\cdot\frac{\beta}{1-\beta}(\eta_2 + K)\left(\frac{2\epsilon g}{\Omega_o^2 r} + 1\right) \end{aligned} \tag{241}$$

which represents a generalization of Molodensky and Kramer equation 37 (Reference 3) to the case in which the liquid core deviates from the neutral stability. Integrating equation 241 over the liquid core gives the extension of equation 39 of Molodensky and Kramer:

$$\begin{aligned} &\left[\frac{\rho r^4}{g}\left(\eta_2 + \frac{1}{2}c\alpha\right) + \frac{r^6}{4\pi G}\frac{d}{dr}\frac{1}{r^2}\left(Q - \frac{1}{2}c\alpha\right)\right]_c^b \\ &= \frac{10}{3}\nu\int_c \rho r^4\,dr + \int_c^b \frac{r^4}{g}\cdot\frac{d\rho}{dr}\cdot\frac{\beta}{1-\beta}\left(\frac{2\epsilon g}{\Omega_o^2 r} + 1\right)(\eta_2 + K)\,dr \end{aligned} \tag{242}$$

Equation 242 connects the resonance parameter, the ellipticity, and the index of stability with the boundary conditions. Euler's equation for the angular momentum supplies an additional connection between the amplitude of nutation and the change in the tensor in inertia of the whole Earth, as produced by the small redistribution of mass (Reference 4). These two equations, together with the boundary conditions, constitute a complete set for determining necessary parameters of the theory, once the model of the Earth is given.

CONCLUSIONS

The present work provides the possibility of including the influence of the ellipticity and of the Coriolis force into the computations of tides in the Earth's interior. It also deals with the mutual effects between the toroidal and spheroidal tides of different degrees. The Molodensky theory of resonance between the diurnal wobble and the diurnal astronomical tides of the liquid core is amended by including the possible deviation of the liquid core from the state of neutral stability.

Plans for future work include the transformation of the boundary conditions and the construction of a model of tides in the outer core of the Earth.

This model shall be tested against the variation of latitudes and positions of the pole and against the values of parameters of the Earth's tidal elastic response (Love numbers) on the surface of the Earth. The analysis of the variation of latitudes and search for resonant frequencies was performed recently by Popov and Yatskiv (Reference 26) and Debarbat (References 27 and 28). It shall be repeated, making use of the observations of variation of latitudes and pole positions by modern technology such as laser ranging (Reference 11) and VLBI (Reference 29). The VLBI observations can also provide the values of Love number at each station and thus help to determine the inhomogeneities of the Earth's tidal elastic response (Reference 29).

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REFERENCES

1. Molodensky, M. S., "Displacements Caused by Tides in an Elastic Earth with the Coriolis Force Taken into Account," *Izv., Earth Physics,* No. 4, 1970, pp. 102-107.

2. Molodensky, M. S., "Tides in an Elastic Earth if Terms Representing the Compression Are Taken into Consideration," *Izv., Earth Physics,* No. 1, 1974, pp. 3-8.

3. Molodensky, M. S., and M. V. Kramer, "Zemnie Prilivy i Nutaciya Zemli," *Izd. Akad. Nauk,* Moscow, 1961, 44 pp. (in Russian).

4. Shen, Po-Yu, and L. Mansinha, "Oscillation, Nutation and Wobble of an Elliptical Rotating Earth with Liquid Outer Core," *Geophys. J. Roy. Astr. Soc.,* **46**, 1976, pp. 467-496.

5. Biot, Maurice A., *Mechanics of Incremental Deformations,* John Wiley and Sons, Inc., New York, 1965.

6. Dahlen, F. A., "The Normal Modes of a Rotating, Elliptical Earth," *Geophys. J. Roy. Astr. Soc.,* **16**, 1968, pp. 329-367.

7. Alterman, Z., H. Jarosch, and C. L. Pekeris, "Oscillations of the Earth," *Proc. Roy. Soc.,* London, **A252**, 1959, pp. 80-95.

8. Backus, G., "A Class of Self-Sustaining Dissipative Spherical Dynamos," *Ann. Phys.,* **4**, 1958, p. 381.

9. Jeffreys, Harold, and R. O. Vicente, "The Theory of Nutation and the Variation of Latitude," *Monthly Notices, Roy. Astr. Soc.,* **117** (2), 1957, pp. 142-173.

10. Melchior, Paul, and Baudouin Georis, "Earth Tides, Precession-Nutation and the Secular Retardation of Earth's Rotation," *Phys. Earth Planet. Interiors,* **1**, 1968, pp. 267-287.

11. Smith, D. E., R. Kolenkiewicz, P. J. Dunn, and M. Torrence, "Whole-Earth Tidal Numbers from GEOS-3," *EOS,* **58** (6), 1977, p. 370.

12. Rubincam, D. P., "Tidal Parameters Derived from the Perturbations in the Orbital Inclinations of BE-C, GEOS-1, and GEOS-2 Satellites," NASA TM X-71118, 1976.

13. Haardeng-Pedersen, G. P., *Studies on the Dynamics of the Rotating Earth,* thesis, Memorial University of Newfoundland, 1975.

14. Poincaré, H., "Sur la Précession des Corps Deformables," *Bull. Astr.,* **27**, 1910, pp. 321-356.

15. Jeffreys, Harold, "The Earth's Core and the Lunar Nutation," *Monthly Notices, Roy. Astr. Soc.*, **108**, 1948, pp. 206-209.

16. Williamson, E. D., and L. H. Adams, "Density Distribution in the Earth," *J. Wash. Acad. Sci.*, **13**, 1923, pp. 413-428.

17. Pekeris, C. L., and Y. Accad, "Dynamics of the Liquid Core of the Earth," *Phil. Trans. Roy. Soc. London*, **A273**, 1972, pp. 237-260.

18. Crossley, D. J., "Core Undertones with Rotation," *Geophys. J. Roy. Astr. Soc.*, **42**, 1975, pp. 477-488.

19. MacDonald, Gordon J. F., and Norman F. Ness, "A Study of the Free Oscillations of the Earth," *J. Geophys. Res.*, **66** (6), 1961, pp. 1865-1911.

20. Morse, P. M., and H. Feshback, *Methods of Theoretical Physics, Vol. 2*, McGraw-Hill, 1953.

21. Jeffreys, H., *The Earth*, Cambridge University Press, 1970.

22. Hide, R., "On Planetary Atmospheres and Interiors, Math. Problems in the Geophys. Sciences," 1971, p. 229.

23. Bullard E. C., and H. Gellman, "Homogeneous Dynamos and Terrestrial Magnetism," *Phil. Trans. Roy. Soc. London*, **A247**, 1954, pp. 213-278.

24. Smith, Martin L., "The Scalar Equations of Infinitesimal Elastic-Gravitational Motion for a Rotating Slightly Elliptical Earth," *Geophys. J. Roy. Astr. Soc.*, **37**, 1974, pp. 491-526.

25. Shen, Po-Yu, *Dynamics of the Liquid Outer Core of the Earth*, thesis, University of Western Ontario, London, Canada, 1975.

26. Popov, N. A., and Ya. S. Yatskiv, "Amplitude Variations in the Free Diurnal Nutation of the Earth," *Sov. Astr. A. J.*, **14**, 1971, pp. 1057-1059.

27. Debarbat, S., "Nearly Diurnal Nutation from Time Measurements," *Astron. Astrophys.*, **14**, 1971, pp. 306-310.

28. Debarbat, S., "Nutation presque diurne et termes périodiques des coordonnies locales," *Astron. Astrophys.*, **1**, 1969, pp. 334-355.

29. Robertson, D. S., *Geodetic and Astrometric Measurements with Very-Long-Baseline Interferometry*, thesis, Massachusetts Institute of Technology, Cambridge, 1975.

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16. Abstract

An important goal of a tidal theory is the improvement of nutational amplitude and of the parameters of the Earth's elastic response. With this goal in mind, a theory of tidal oscillations inside a rotating elliptical Earth is developed, with special emphasis on tides in the liquid core. The Molodensky and Kramer theory of the resonance effect, as caused by the proximity of the frequency of the free diurnal wobble of the liquid core to the frequency of K_1 astronomical tide, was amended to include the effect of the possible deviation of the liquid core from the state of neutral stability. Coupling effects between the toroidal and spheroidal oscillations, as caused by the Coriolis force, are taken into consideration.

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