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A bounded operator A on a Hilbert space H is positive provided $\langle Ax,x \rangle \geqq 0$ for all $x \in H$. These operators are symmetric, and as such constitute a natural generalization of non-negative real diagonal matrices. The following result is thus both well known and not surprising:

Theorem: A positive operator has a unique positive square root (under operator composition).

This may be established by integration of the correct function, invoking the spectral theorem for self-adjoint operators. A more accessible argument for those not acquainted with the mysteries of spectral measures may be found in [1,p.317].

While square roots and their iterates seem to provide a sufficient analytic tool for most purposes, it is also a (folk) theorem that positive operators possess unique positive n^{th} roots for every positive integer n. As in the $n = 2$ case, existence follows from an application of the spectral theorem; however, we give an argument in the spirit of [1]. The purpose in so doing is not to exercise the reader's knowledge of induction, but rather to illustrate another use of the Law of the Mean as a motivational instrument.

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Let I be the identity operator on H, and let $B(H)$ denote the set of bounded operators on H. We will need the following properties of positive operators:

(1) the relation on positive operators defined by $A \leq B$ if and only if $B - A$ is positive, is reflexive, transitive, and consistent with the notation $0 \leq A$ for any positive A; moreover, this relation is preserved by operator addition and positive real scalar multiplication, and reversed by negative scalar multiplication.

(2) If A and B are positive and if $AB = BA$, then AB is positive.

(3) If $0 \leq A \leq I$, then $0 \leq I-A \leq I$.

(4) If $0 \leq A$, then $A \leq ||A||I$, so that $(||A||)^{-1}A \leq I$, if $A \neq 0$.

(5) If $0 \leq A \leq I$, then $A^n \leq A$ for all positive integers n.

We also require:

<u>Lemma</u>. If $\{S_n\}$ is a sequence in $B(H)$ such that $0 \leq S_n \leq S_{n+1} \leq I$, then there exists $S \in B(H)$ such that $\{S_n u\} \to Su$ for all $u \in H$.

All of the conclusions above are verified by straightforward arguments in [1,pp. 317-320].

<u>Theorem</u>: Let $A \in B(H)$, $0 \leq A$, and let k be a positive integer. Then there exists a unique positive operator B such that $B^k = A$.

<u>Proof</u>: By (4) above, we need only consider the case in which $A \leq I$.

We first prove the existence of B. Since the theorem is a tautology for all operators when $k = 1$, we assume the existence of positive $(k-1)$-st roots for all positive operators.

Under the momentary supposition that B exists, let $R = I - A$ and $S = I - B$. Then $(I - S)^k = I - R$, so that

$$(*) \qquad S = (1/k)\left[R + \sum_{r=2}^{k} \binom{k}{r} (-1)^r S^r\right] .$$

Clearly the existence of a positive operator satisfying this implicit relation is necessary and sufficient to establish the existence of the desired operator B. To this end, we define a sequence of operators by $S_o = 0$, $S_{n+1} = (1/k)\left[R + \sum_{r=2}^{k} \binom{k}{r} (-1)^r S_n^r\right]$. In order to show $S_n \leq S_{n+1}$ it suffices to show, under the assumption $0 \leq S_{n-1} \leq S_n \leq I$, that $0 \leq S_{n+1} - S_n =$ $(1/k)\left[\sum_{r=2}^{k} \binom{k}{r} (-1)^r (S_n^r - S_{n-1}^r)\right]$.

To accomplish this, we digress to a consideration of the polynomial $f(x) = \sum_{r=2}^{k} \binom{k}{r} (-1)^r x^r = (1-x)^k + kx - 1$. Since Since $f'(x) = k\left[1 - (1 - x)^{k-1}\right] \geq 0$ on $[0,1]$, clearly f is increasing on this interval. To translate this to operators, it is necessary to examine the situation more carefully. By the Mean Value Theorem, given $0 \leq y < z \leq 1$, there exists a (unique) number $c \in (y,z)$ such that

$$(**) \qquad f(z) - f(y) = f'(c)(z - y) .$$

Upon solving, $c = 1 - \left[(1/k) \sum_{r=0}^{k-1} (1 - y)^{k-r-1}(1 - z)^r\right]^{1/(k-1)}$

Returning to our operator problem, we wish to apply this information to the sequence $\{S_n\}$. Since all members of this family are polynomials in $R = I - A$, any two of them commute. This is a property sufficient to permit imitation of equation (**) with operators; let $z = S_n$, $y = S_{n-1}$. In this format, we use C to represent the operator $I - J$, where J is (any) positive (k-1)st root of the operator $(1/k) \sum_{r=0}^{k-1} (I - S_{n-1})^{k-r-1}(I - S_n)^r$. The following chain of equalities is easily calculated:

$$S_{n+1} - S_n = (1/k)\cdot(f(S_n) - f(S_{n-1}))$$

$$= (1/k)\{k[I - (I - C)^{k-1}]\}\cdot(S_n - S_{n-1})$$

$$= [I - (I - C)^{k-1}]\cdot(S_n - S_{n-1})$$

$$= [I - J^{k-1}]\cdot(S_n - S_{n-1})$$

$$= [I - \{(1/k) \sum_{r=0}^{k-1} (I-S_{n-1})^{k-r-1}(I-S_n)^r\}]\cdot(S_n - S_{n-1})$$

By application of remarks (2), (3) and (5), the assumption of existence of (k-1)st roots, and the inductive hypothesis $S_{n-1} \leqq S_n$, the latter operator product exists and is positive. Hence $S_n \leqq S_{n+1}$, and the sequence $\{S_n\}$ is increasing. Of course, the Law of the Mean is not applicable in this setting, nor is it used other than to motivate the choice of C. Indeed, the discerning reader will note that the extremes of the chain above may be shown to be equal without the introduction of C. However, the rather unusual factorization of $S_{n+1} - S_n$ would be more difficult to discover without the example furni[illegible]d by the derivative in the real function situation.

To invoke the Lemma and complete the proof of existence of $k^{\underline{th}}$ roots, it remains to show $S_n \leqq I$ for all n. Assuming $0 \leqq S_m \leqq I$, we have $kS_{m+1} = R + \sum_{r=2}^{k} \binom{k}{r} (-1)^r S_m^r = R - I + kS_m + (I - S_m)^k$. By remark (5), $(I - S_m)^k \leqq I - S_m$; therefore

$$R + kS_m - I + (I - S_m)^k \leqq R + kS_m - I + I - S_m$$
$$\leqq I + (k-1)S_m \leqq kI.$$

Hence $kS_{m+1} \leqq kI$ and $S_{m+1} \leqq I$, as desired. Thus, the Lemma gives an operator as in (*), and $I - S = B$ is a $k^{\underline{th}}$ root of A.

In order to prove the uniqueness of a positive $k^{\underline{th}}$ root of A, we first observe that if T is any positive $k^{\underline{th}}$ root of A, then T must perforce commute with A, hence with $I - A = R$, hence with each S_n, and thus with S and $I - S = B$. Let $u \in H$, $v = (B-T)u$. Then $0 = \langle (B^k - T^k)u, v \rangle = \langle (\sum_{r=0}^{k-1} B^{k-r-1}T^r)(B-T)u, v \rangle = \sum_{r=0}^{k-1} \langle B^{k-r-1}T^r v, v \rangle$. Since B and T commute, $0 \leqq B^{k-r-1}T^r$, whence $\langle B^{k-r-1}T^r v, v \rangle = 0$, $r = 0,1,\ldots,k-1$. Let F_r be any positive (hence symmetric) square root of $B^{k-r-1}T^r$. Then $||F_r v||^2 = \langle F_r v, F_r v \rangle = \langle F_r^2 v, v \rangle = 0$, so that $F_r v = 0$ and $B^{k-r-1}T^r v = F_r^2 v = 0$. Therefore $B^{k-r-1}T^r(B-T)u = 0$, or $B^{k-r}T^r u = B^{k-r-1}T^{r+1}u$, $r = 0,1,\ldots k-1$. In particular, for $r = k-1$, $BT^{k-1} = T^k$. Multiplying by T, we have $B^{k+1} = BA = BT^k = T^{k+1}$.

If $k = 2$, the argument above shows $Bv = 0 = Tv$, whence $||(B-T)u||^2 = \langle (B-T)^2 u, u \rangle = \langle (B-T)v, u \rangle = 0$. Hence $Bu = Tu$ for all $u \in H$, and B is thus unique. Now assume all positive roots, of order less than k, for positive operators are unique. If $k = 2j$, then $(B^j)^2 = B^{2j} = B^k = T^k = (T^j)^2$, whence $B^j = T^j$ and thus $B = T$. If k is odd, we have shown above that $B^{k+1} = T^{k+1}$, so, by the even exponent argument, again $B = T$. This completes the proof.

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