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A MAINTENANCE MODEL FOR K-OUT-OF-N SUBSYSTEMS
ABOARD A FLEET OF ADVANCED COMMERCIAL AIRCRAFT

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A MAINTENANCE MODEL FOR K-OUT-OF-N SUBSYSTEMS ABOARD A FLEET OF ADVANCED COMMERCIAL AIRCRAFT

by

Douglas R. Miller

Proposed highly reliable fault-tolerant reconfigurable digital control systems for a future generation of commercial aircraft consist of several k-out-of-n subsystems. Each of these flight-critical subsystems will consist of n identical components, k of which must be functioning properly in order for the aircraft to be dispatched. Failed components are recoverable; they are repaired in a shop. Spares are inventoried at a main base where they may be substituted for failed components on planes during layovers. Penalties are assessed when failure of a k-out-of-n subsystem causes a dispatch cancellation or delay. A maintenance model for a fleet of aircraft with such control systems is presented. The goals are to demonstrate economic feasibility and to optimize. (This is a preliminary report of ongoing research.)

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A MAINTENANCE MODEL FOR K-OUT-OF-N SUBSYSTEMS ABOARD A FLEET OF ADVANCED COMMERCIAL AIRCRAFT

by

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1. Introduction

This memorandum is a preliminary report about ongoing research and development of a cost model for maintenance/maintainability of proposed highly reliable fault-tolerant reconfigurable digital control systems aboard a future generation of commercial aircraft.

Fault-tolerant digital control systems will allow for more efficient fuel usage, longer fatigue life, and bad weather operation on future aircraft. It is necessary to check that the savings so generated will not be offset by the increased initial costs, maintenance costs and penalties due to dispatch criticality of such control systems; in other words, is the system economically feasible? To this end, a model is developed to indicate, prior to actually using the system, the economic impact of maintaining such a control system. This paper presents such a maintenance model and suggests ways to analyze it.

The advanced avionics control systems under consideration are envisioned as consisting of several types of components, each replicated several times to achieve fault-tolerance and high system reliability. Components will include CPU's, memories, busses, input-output units, sensors, actuators, and possibly additional units, depending on the particular architecture under consideration. (Two such systems are currently in the development stage: the SIFT computer of Stanford Research Institute and the FTMP of Charles Stark Draper Laboratories.) In general, let us say there are s types of components: n_1 of type 1,..., n_s of type s. The minimum number of replicates of each type necessary to perform certain computing loads will be given by a performance model of the computer/control system. If the system consists of this number of replicates and suffers a failure, it will no longer be able to perform all its functions. Thus for system reliability considerations additional replicates of each type of component shall be included in the system. A reliability model in conjunction with the performance model will give, among other things, the minimum number of replicates of each type of component required for reliable operation of the system; a typical reliability constraint will be for the probability of catastrophic failure to be less than 10^{-9}. For example, it may require k_1 replicates of component type 1, k_2 of type 2, etc. Presumably the Federal Aviation Agency (FAA) will adopt these k-values for the minimum dispatch complement of functioning components in the system. One almost certain result of a maintenance model will be that an economically optimal system, from a maintenance point of view, will consist of more than the minimum dispatch complement of components, i.e., $n_i > k_i$. Thus in designing the system, maintainability must be considered in addition to meeting a minimum reliability constraint. This fact is illustrated by a greatly simplified example in Appendix A. An optimal system is obtained by adding extra replicates of some components but not others; the decision is dictated by the mean lifetime of the components, the cost of the components, and financing considerations.

In general, then, we have a system consisting of n_i replicates of component i, $i=1,2,\ldots,s$. If fewer than k_i are functioning properly, then the system cannot be dispatched. Thus, the system can be modelled as s k-out-of-n subsystems in series configuration, [1]. The goal of the maintenance operation is to minimize system failure (nondispatch) by ensuring that there are always at least k_i good copies of component type i, $i=1,2,\ldots,s$, in the system. Exponential lifetimes are assumed.

Maintenance of this system is supported by line maintenance centers located at some of the cities served by the airline; spares will be provisioned at these centers. Also supporting the maintenance operation will be one repair shop with a certain repair-rate capacity. Whenever enough failures accumulate to prevent dispatch, the plane is grounded and unscheduled maintenance must be performed; otherwise, the maintenance is performed during scheduled overnight stops at line maintenance centers. No preventive maintenance is performed.

Our goal is to predict the costs incurred in maintaining such an avionics control system. Once an entire system is specified, i.e., the detailed computer architecture and its maintenance support system, many of the costs can be incorporated into a total cost function in a straightforward way; for example, cost of spares, cost of repairmen, and overhead costs for the line maintenance center. Any such fixed costs are simply added together. However, costs which are incurred randomly due to events such as flight delays or cancellations, diversions, late arrivals, etc., cannot be included in the total cost of operating the system unless the long run average frequency of such events is known or can be predicted. Because such systems have never been flown, such frequencies are unknown and must be calculated from a probability model. This brings us to the purpose of this paper: namely, to find an analytically tractable model which approximately describes the stochastic behavior of the above control system and supporting maintenance activity.

The time-dependent behavior of the system is fairly complicated. The state space consists of billions of points. Analysis appears impossible. To circumvent this apparent difficulty, we shall break the large problem into a number of smaller ones which interface with one another. We shall "uncouple" in three ways.

1. We shall treat one type of component at a time. This gives s separate processes which correspond to k_i-out-of-n_i systems which have failures and undergo repair.

2. We shall break the total operation of one of these k-out-of-n systems into suboperations: for example, (i) operation of the

components over the airline route structure, (ii) provisioning of spares at line maintenance centers, (iii) operation of the repair shop, and (iv) acquisition and inventorying of nonrecoverable subcomponents. Each of these suboperations can be modelled by itself: (i) a birth-and-death process, (ii) an allocation problem, (iii) a cyclical machine repair operation, and (iv) an inventory problem. Steady-state behavior can be determined for each suboperation. We shall use this to approximate the steady-state behavior of the entire system by analysis of the interfaces between the suboperations. An example of uncoupling queueing networks is that of Jackson queues [7].

3. We shall consider economic penalties resulting from poor maintenance (nondispatchability) separately from economic penalties resulting from unreliability, i.e., problems arising on a plane which met dispatch requirements at the start of the flight leg.

Given a certain control system and its supporting maintenance system, we can then determine the approximate rate of occurrence of certain random events related to maintenance and reliability. The costs incurred by these events will generally depend upon where and when they occur in the airline route. However, over the long run the proportion of events which occur in one city or flight leg will be proportional to the length of time the plane spends in that city or flight leg; thus from the route structure and individual costs it will be possible to obtain average costs incurred for a particular random maintenance or reliability event. These can then be incorporated into general cost equations.

When a cost model is created, the next step will be optimization. There are various control variables which can be set to values that achieve economic optimality.

The specific model presented in this paper is a special case of spares provisioning and control with recoverable units. There exists some literature in this general area; see References [4], [5], [10], and [14]. The current situation, however, has two features which do not appear to have been considered previously in problems of this type: namely, the k-out-of-n feature and the scheduled routine maintenance times. Other general maintenance models are found in [1] and [11].

The model presented here focuses on maintenance. There are other factors which should be considered in a complete cost model that are not addressed here.

1. Costs of different hardware designs.
2. Costs of software and software maintenance.
3. Allocation of components into line replacement units (LRU's).
4. Detailed aspects of airline accounting procedures, tax situations, etc.
5. Savings due to increased fatigue life.
6. Juggling the route structure, e.g., "switching tail numbers."
7. Costs of test equipment.

The reader is reminded that the model has many simplifications in it. It is intended as a prototype to explore and develop methodologies and to gain insight into modelling one aspect of airline costs. If it is success l, a more realistic extended model can then be developed.

2. The Model

The following simplified situation is modelled in this paper. The airline consists of a fleet of N planes. Aboard each plane there is a k-out-of-n subsystem: for example, n identical CPU's in the flight control system, k of which are required to be operating for dispatch. It is assumed that the n units fail independently of one another; each has an exponential lifetime with failure rate λ. It is assumed that failures are detected as they occur; there are no latent failures. Each of the components is a line replacement unit (LRU); individual failed components are replaced when the opportunity arises.

All routine maintenance of the k-out-of-n system is performed at one main base. This base includes a line maintenance center, a repair shop, and a pool of spare components. Each plane in the fleet is scheduled for an overnight stop at the main base at intervals of D days (or equivalently, T flight hours). At this time failed components are replaced by good components from the spares pool; the failed components are sent to the repair shop where they are repaired and then added to the spares pool (see Figure 1). There are s spare units; thus at any time there will be a total of s units in the repair shop and spares pool. There are r repairmen or repair channels in the repair shop; the repairmen work an eight-hour day from 8:00 a.m. to 5:00 p.m., say. The repair rate per day per repairman is μ, i.e., the number of units repaired by a repairman in one day has a Poisson distribution with mean μ, provided he is kept busy all day. Equivalently, the repair time for one unit is exponential with rate μ.

A spares strategy is specified for dealing with the possibility that there are insufficient spares in the spares pool to meet demand. Any demand from a plane whose system has dropped below k functioning units is considered an "emergency" demand because such a plane cannot be dispatched unless a replacement is made. Thus, whenever such a demand is made, a number of spares sufficient to restore the system to k functioning units is released if there are any spares available in the spares pool. If such an event occurs while the plane is away from the main base, the plane is grounded there until the spare parts are flown out to it; it then continues on its scheduled flight plan. It greatly simplifies the analysis if only minimal repairs (restore system to k operating components) are allowed away from the main base; therefore we shall make that assumption in this simplified model. The demands made on the spares pool by planes during their scheduled overnight stop at the main base will be considered "routine." When the spares pool is low it may not be optimal to fill all such demands. In general, if a system requests d replacement components and y units are available in the spares pool, $f(d,y)$ units will be released from the spares pool. We shall consider two special cases. First, a hoarding level h is set. All "routine" requests are filled which do not deplete the spares pool below h; other requests are denied: $f_0(d,y) = d$ if $y-d \geq h$, $= y-h$ if $h < y < h+d$, and $= 0$ if $y \leq h$. The second spares strategy will be more general; it is the same as above except that when

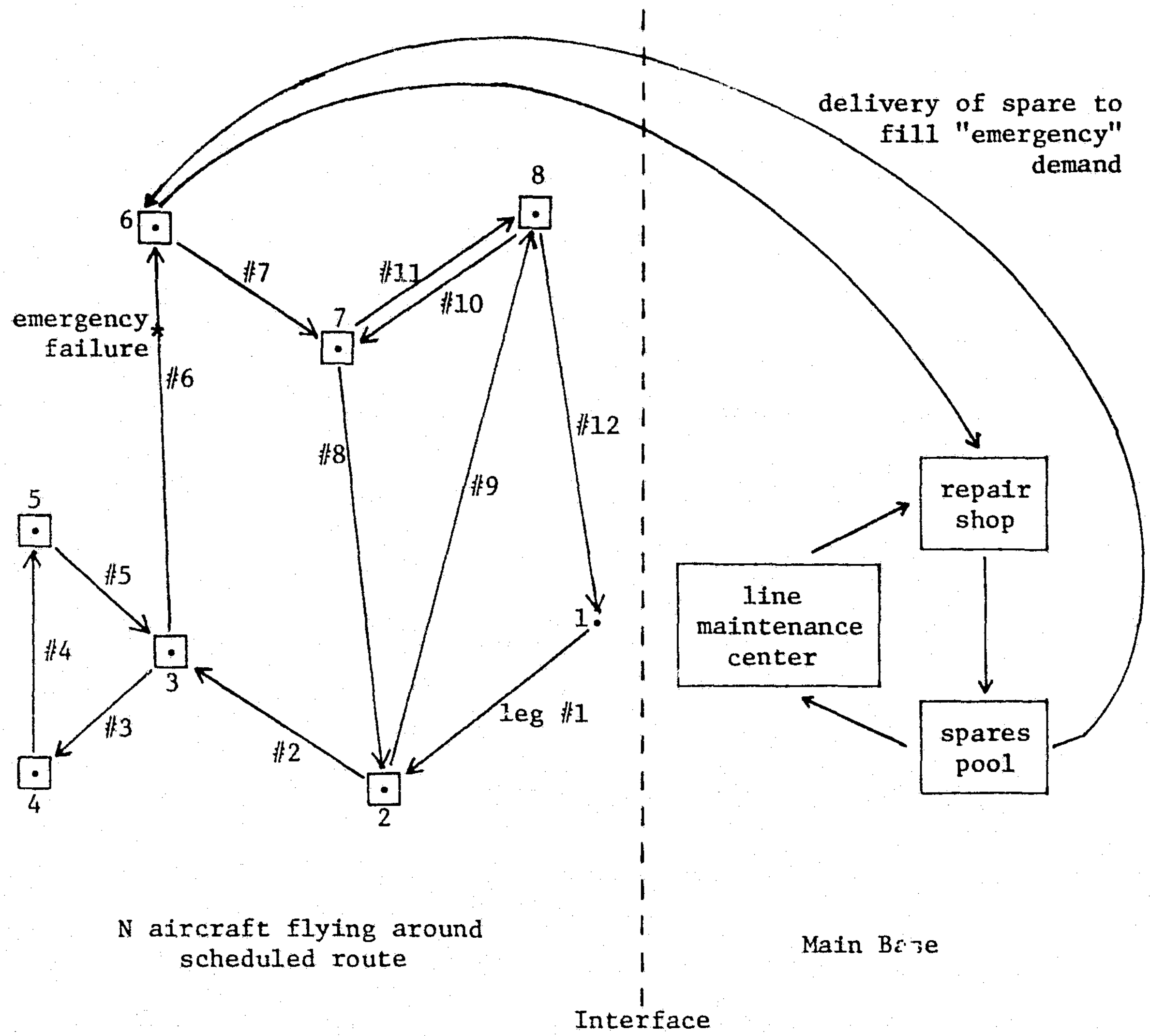

Figure 1.--Schematic of movement of aircraft and spares.

$d = n-k$ at least one spare will be released if available: $f_1(d,y) = d$ if $y-d \geq h$, $= y-h$ if $h < y < h+d$, $= 1$ if $d = n-k$ and $1 \leq y \leq h$, and $= 0$ if $d < n-k$ and $y \leq h$. Let δ be a control variable (equal to 0 or 1) which corresponds to the two cases above, then

$$f_\delta(d,y) = \begin{cases} d, & \text{if } y-d \geq h \\ y-h, & \text{if } h < y < h+d \\ \delta, & \text{if } d = n-k,\ 1 \leq y \leq h \\ 0, & \text{if } d < n-k,\ 1 \leq y \leq h \\ 0, & \text{if } y < 1. \end{cases}$$

It is possible to let δ equal other values, but restricting it to 0 or 1 will illustrate the method of analysis. Recall that demand from a system with fewer than k operating components is considered an "emergency" and thus in the above equation for "routine" demands we do not consider $d > n-k$.

Another consideration in this model is the route structure of the airline. It is assumed that each of the N planes flies the same schedule staggered by a one-day time delay so that, in effect, the planes are following each other on a path throughout the system, making a complete cycle in vT hours or vD days, where v is the number of planes scheduled for overnight stops at the main base each night. We assume that v is an integer; for example, for the United 747 fleet, $v = 1$. In addition to the main base, the route will consist of several primary bases and secondary bases. While flying the route a plane cannot be dispatched if fewer than k components in the k-out-of-n system are functioning. If the system drops below k functioning units in the course of a flight this fact is noted and the plane is grounded at its destination until a replacement is made, bringing the system back to k or more functioning components. The penalty for grounding at a primary base reflects lost time and the cost of sending a spare from the main base. The penalty for grounding at a secondary base includes these factors plus the cost of flying in a maintenance man to make the replacement. The penalty costs are developed more fully in Sections 4 and 5 of this treatise; it is assumed there that groundings are distributed statistically over the route according to the schedule. Thus it is necessary in the present analysis to keep track only of how many occur. (This is an example of uncoupling.)

The state of the airline and its maintenance support at any time is described by the number of components functioning in the k-out-of-n subsystem on each plane and the number of spares in the spares pool; it is an $N+1$ dimensional vector. Consider the number of units functioning in a single k-out-of-n system over the interval $[0,T]$, i.e., the time interval to complete one cycle of the route, main base to main base. Let $Z(t)$ equal the number of components of the k-out-of-n system functioning at time t (accumulated flight time since departure from the main base). A representative sample path of Z is depicted in Figure 2a, and the corresponding approximation to be used in modelling is given in Figure 2b. It will be much easier to analyze the approximate behavior depicted in Figure 2b.

The paths in Figures 2a and 2b correspond to the following behavior. The system is dispatched from the main base with $k+3$ components of the k-out-of-n system functioning. The first three failures have no effect on dispatchability. At time t_1 a failure occurs reducing the system to $k-1$ functioning components; it is therefore grounded at the completion of the flight leg, time t_2. It loses $t_4 - t_2$ flight hours waiting for a spare and resumes its flight plan at time t_4. The approximate model (Figure 2b) assumes instantaneous repair although a penalty will be assessed to correspond to the flight time lost in Figure 2a. This approximation will be conservative because it exposes the system to more risk, indeed it is possible for the approximate system to suffer another failure (at time t_3) while the true system's operation is suspended. At accumulated flight time t_5 another failure occurs and before completion of the leg (at t_7) a second failure occurs (at t_6); after a delay the flight is resumed at t_8. In the approximation the two failures are assumed to be repaired instantly; a penalty is assessed for each separately. This will be a conservative approximation because the penalty for two separate events will be greater than the penalty for one event of two failures; with reliable components the failure of two in one flight leg will be very unlikely, therefore only a small error is introduced by this approximation. Another failure occurs (at t_9)

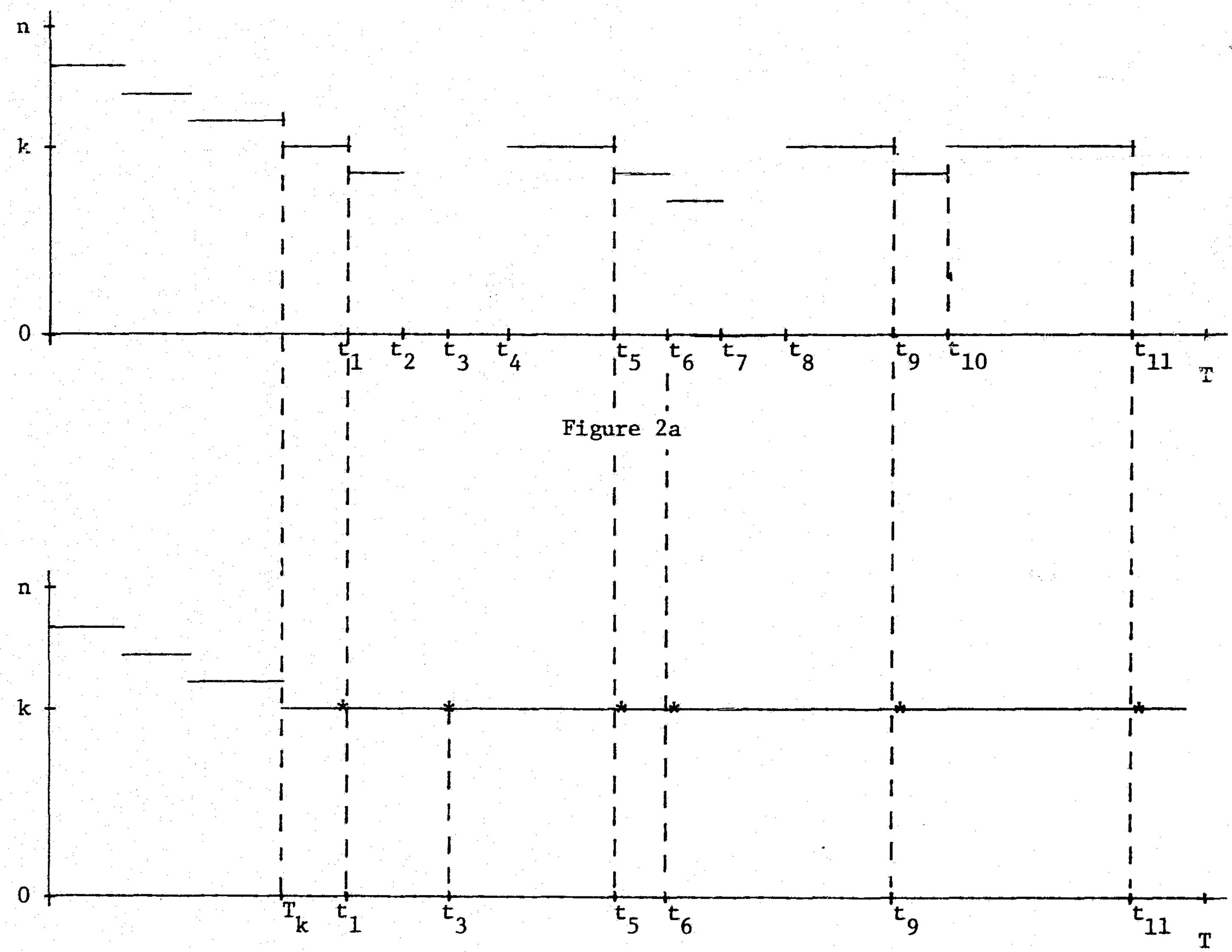

Figure 2a

Figure 2b

Figure 2.--A sample of the state of the k-out-of-n system process and approximation Z.

on a flight leg which ends at t_{10} at an overnight stop where it is possible to fly in a spare and make the replacement without losing any flight time. The failure is noted in the approximation in Figure 2b; the fact of no lost time will be reflected in the statistical distribution of failures over the route structure that is presented in Section 4. Similarly, a failure occurs at time t_{11} on the leg terminating at the main base. This failure occurs in the approximation in Figure 2b; the fact that there is no penalty for lost time or grounding will again be accounted for by statistically distributing the failures over the route structure in order to assess penalties in Section 4. The approximation in Figure 2b should result in only a very slightly higher cost due to the small likelihood of its differing greatly from the true behavior in Figure 2a. The process in Figure 2a is a rather complicated birth-and-death process which is difficult to treat analytically; however, the process in Figure 2b is much simpler. It is a pure-death process with absorbing state k ; after being absorbed at time T_k , its failures occur according to a Poisson process. The state of the ith plane will be denoted Z_i , $i=1, 2,\ldots,N$.

The number of spares available in the spares pool will be denoted by Y . There are three distinct activities related to the spares pool: restocking with output from the repair shop, depletion due to "emergency" demand (units required to restore subsystems to k operating units), and depletion due to "routine" demand (requests from planes during scheduled overnight stops at the main base to increase their number of functioning units from k or more). Roughly speaking, the first activity takes place during the day, the second later in the day, and the third at night. It will be convenient for modelling purposes to apportion the three activities into nonoverlapping time intervals of morning until evening, evening until night, and night until the following morning. The size of the spares pool at these times will be denoted Y^M , Y^E , and Y^N , respectively. A representative sample path over several days is depicted in Figure 3, where a negative value of Y corresponds to unfilled "emergency" demand.

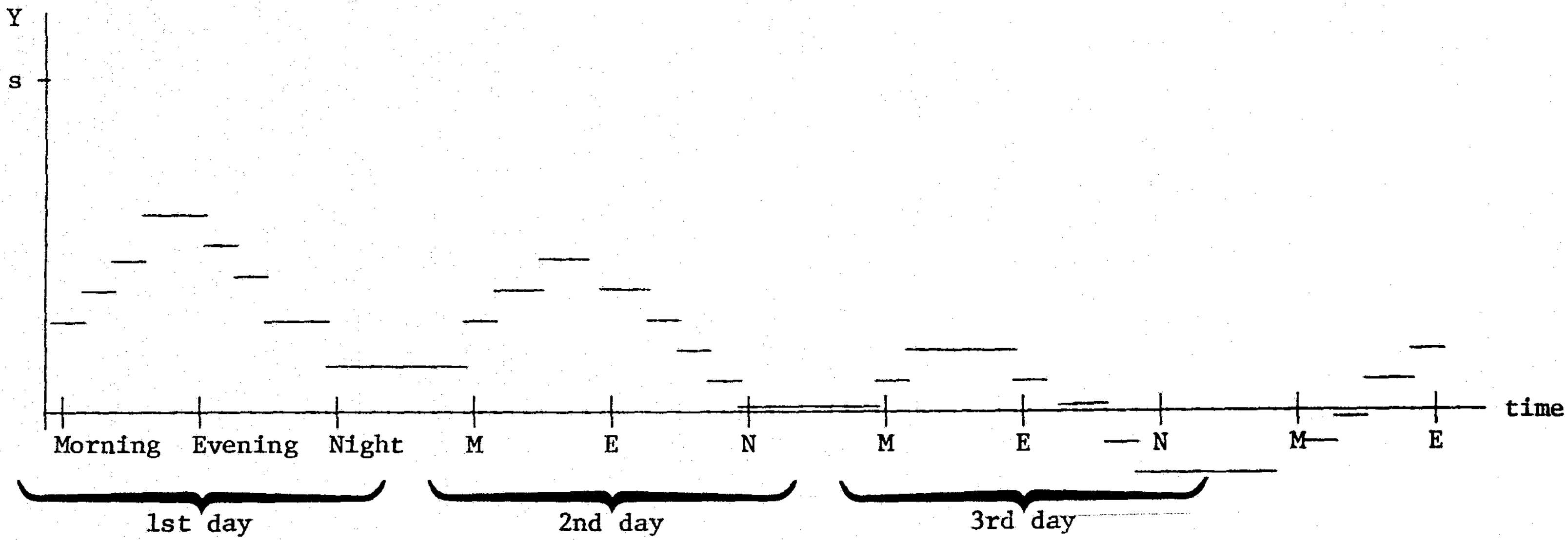

Figure 3.--A sample path of the spares pool process Y.

We desire more than one possibility for analyzing this system; thus, in addition to considering Z_i as a process over $[0,T]$, we shall also divide it up according to time of day. Let Z_i^M, Z_i^E, and Z_i^N represent, respectively, the state of the subsystem on the ith plane when it is dispatched in the morning, when it completes service for the day (number of components operating plus the number of emergency demands), and after emergency repair is made. Note that Z^N equals Z^M for the next day except for a plane remaining overnight at the main base. Let G_i equal the number of groundings (or emergency demands) made during the day by the ith plane.

The basic probability laws governing the above processes will be presented in Section 3 and the solution for steady-state probabilities discussed in Section 4. Before proceeding we shall summarize the notation introduced up to this point.

N	planes in the fleet
1	k-out-of-n subsystem aboard each plane
n	components in the subsystem
k	components must be functioning for dispatch
λ	failure rate per hour per component
1	central line maintenance center (at main base)
T	flight hours per plane between scheduled overnight stops at main base
D	days per plane between scheduled overnight stops at main base
v	planes visit main base each night for overhaul
1	repairshop
r	repairmen at shop
s	spare units
μ	repair rate per day per repairman
h	hoarding level of spares pool
δ	minimum number of replacements for system with $n-k$ failures
$Z_i(t)$	number of components functioning aboard ith plane at time t
Y^M	size of spares pool in early morning
Y^E	size of spares pool after restock before emergency demand

Y^N size of spares pool after emergency demand before routine demand

Z_i^M state of ith plane at first dispatch in morning

Z_i^E state of ith plane at end of flying day

G_i number of groundings of ith plane

Note that $vD = N$ and $N/v = D$, so the average number of hours flown each day is $T/D = vT/N$ per plane. Actually, the model can easily incorporate different flight times each day; $t_1 + t_2 + t_3 + \ldots + t_D = T$. For the time being we shall consider only the case where $D = N$ and $v = 1$.

In the above list of variables, n, r, s, h, and δ are control variables. They will be set in order to optimize some economic objective function. In addition to these variables, in Section 5 we shall introduce the possibility of maintaining buffer spares pools, whose sizes can be controlled, at the primary bases.

3. The Underlying Probability Laws

In this section various transition probabilities and other stochastic relationships are derived for the Y and Z processes. Various subsets of these probabilities can be used for different analytic approaches. For the appropriate background material in probability and stochastic processes, see References [2], [3], [8], and [13].

3.1 Q^0: $Z(0) \to Z(T)$

The quantity $Z(0)$ equals the number of functioning components aboard a plane when it is dispatched after its overnight stay at the main base, and $Z(T)$ equals the number of components working when it returns for its next overnight/maintenance visit to the main base. Thus the transition $Z(0) \to Z(T)$ reflects the deterioration of the system in $[0,T]$, recalling that k is an absorbing state. Let Q^0 be the matrix of these transition probabilities:

$$Q^0_{ij} = P\{Z(T)=j \mid Z(0)=i\} .$$

Thus

$$Q^0_{ij} = \begin{cases} 0 & , \quad \text{if } j > i \\ P\{i\text{-}j \text{ failures out of } i \text{ in } [0,T]\} , & \text{if } i \geq j > k \\ P\{i\text{-}j \text{ or more failures in } [0,T]\} , & \text{if } i \geq j = k . \end{cases}$$

Using the fact that a component with failure rate λ will fail with probability $1 - \exp(-\lambda T)$ in the interval $[0,T]$ yields

$$Q^0_{ij} = \begin{cases} 0 & , \quad j > i \\ \binom{i}{i-j}\left(1 - e^{-\lambda T}\right)^{i-j}\left(e^{-\lambda T}\right)^{j} & , \quad i \geq j > k \\ \sum_{m=0}^{k} \binom{i}{m}\left(1 - e^{-\lambda T}\right)^{i-m}\left(e^{-\lambda T}\right)^{m} , & \quad i \geq j = k . \end{cases}$$

3.2 Q^T: $Z(T) \rightarrow Z(0)$

The transition $Z(T) \rightarrow Z(0)$ corresponds to "routine" replacements of failed components during the scheduled overnight visit to the main base. If x failed components are replaced, then $Z(0) = Z(T) + x$. The number x of failed components replaced depends on three things: the number of failures in the k-out-of-n system, $n - Z(T)$; the number of spares available in the spares pool, Y^N ; and the strategy for releasing spares from the spares pool, f_δ . Thus the transition probabilities for $Z(T)$ to $Z(0)$ may depend on the value of Y^N . Define

$$Q^T_{ij|y} = P\{Z(0)=j \mid Z(T)=i, Y^N=y\} .$$

It follows that this is actually a deterministic transition

$$Q^T_{ij|y} = \begin{cases} 1 , & \text{if } f_\delta(n-i,y) = j-i \\ 0 , & \text{otherwise.} \end{cases}$$

In some instances it may be possible to approximate the transition probabilities of $Z(T)$ to $Z(0)$ by assuming that $Z(T)$ and Y^N are independent when steady state is achieved. In "light traffic," i.e., little demand on or depletion of the spares pool, this approximation of independence might be reasonable. (It is discussed in Section 4.) Let

$$p^N(y) = P\{Y^N=y\}$$

be the steady-state marginal probabilities of Y^N. Assuming independence of $Z(T)$ and Y^N (i.e., the demands of one plane have little influence on the distribution of the number of units in the spares pool) gives

$$\begin{aligned} Q^T_{ij} &= P\{Z(0)=j \mid Z(T)=i\} \\ &= \sum_y Q^T_{ij|y}\, p^N(y) \,. \end{aligned}$$

3.3 Q^M: $Z^M \to (Z^E, G)$

Suppose a plane starts its flight day with Z^M good components. It flies for t hours, then Z^E is the number of good components at the end of the flight day (t hours). It is assumed that emergency replacements prevent the value of Z^E from falling below k ; G equals the number of such emergency replacements. (Note that we are bending the rule about emergency demands depleting the spares pool at the end of the day, after it has been restocked by the repair shop. It is believed that this will not have a significant effect, as it is only a small perturbation along the time axis.)

The possible values taken by (Z^E,G) are depicted in Figure 4. Note that G is nonzero only when $Z^E = k$. Let Q^M be the matrix of transition probabilities for $Z^M \to (Z^E,G)$,

$$Q^M_{i,(j,g)} = P\{(Z^E,G)=(j,g) \mid Z^M=i\} \,.$$

First consider the case $i \geq j \geq k$, $G = 0$

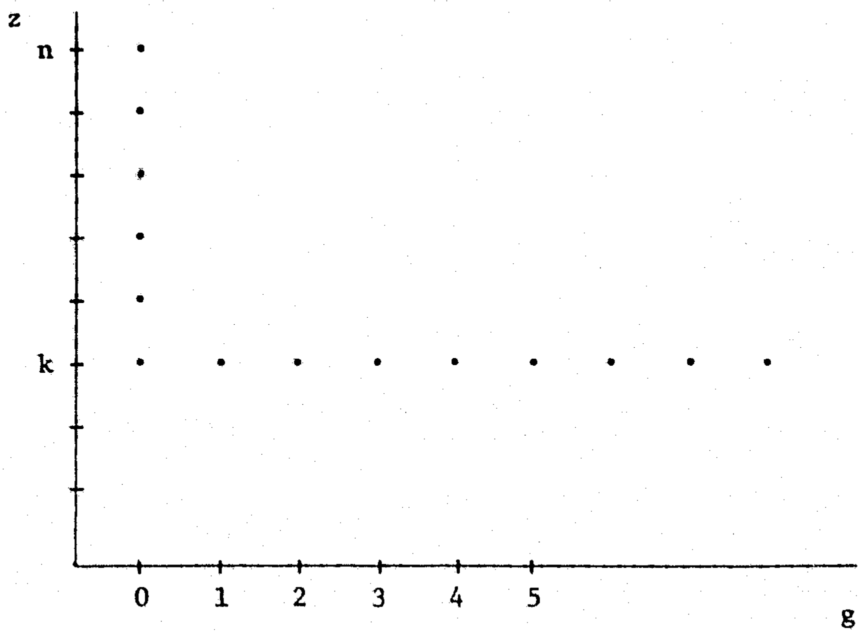

Figure 4.--State space of (Z^E,G) .

$$
\begin{aligned}
Q^M_{i,(j,0)} &= P\{Z^E=j,\ G=0 \mid Z^M=i\} \\
&= P\{i\text{-}j \text{ failures out of } i \text{ in } [0,t]\} \\
&= P\{j \text{ survivors out of } i \text{ in } [0,t]\} \\
&= \binom{i}{j}\left(e^{-\lambda t}\right)^j \left(1 - e^{-\lambda t}\right)^{i-j} .
\end{aligned}
$$

Now consider the case $i > j = k$. In this case we shall compute $Q^M_{i,(k,g)}$ by conditioning over T_k , the absorption time by state k given that $Z^M = i$.

$$T_k = \inf\left(t : Z(t)=k\right) .$$

The distribution of $T_{k|i}$ is derived by noting that it is the time until the (i-k)th failure, thus

$$
\begin{aligned}
F_{T_k|Z^M=i}(t) &= P\{T_k \le t \mid Z^M=i\} \\
&= P\{k \text{ or fewer survivors in } [0,t]\} \\
&= \sum_{j=0}^{k} \binom{i}{j}\left(e^{-\lambda t}\right)^j \left(1 - e^{-\lambda t}\right)^{i-j} .
\end{aligned}
$$

The conditional density of T_k given $Z^M=i$ is

$$f_{T_k|Z^M=i}(t) = \sum_{j=0}^{k} \binom{i}{j}\lambda e^{-j\lambda t}\left(1 - e^{-\lambda t}\right)^{i-j-1}\left(ie^{-\lambda t} - j\right) .$$

Returning to Q^M and using a conditional probability argument: for $g \geq 1$, $i > k$,

$$
\begin{aligned}
Q^M_{i,(k,g)} &= P\{Z^E=k,\ G=g \mid Z^M=i\} \\
&= \int_0^t P\{G=g \mid T_k=s\} f_{T_k|Z^M=i}(s)ds \\
&= \int_0^t e^{-k\lambda(t-s)} \frac{[k\lambda(t-s)]^g}{g!} f_{T_k|Z^M=i}(s)ds \\
&= \sum_{j=0}^{k} \lambda \binom{i}{j} e^{-k\lambda t} \frac{(k\lambda)^g}{g!} \sum_{m=0}^{i-j-1} (-1)^m \\
&\quad \times \left\{ i \int_0^t (t-s)^g\, e^{\lambda s(k-j-m-1)} ds - j \int_0^t (t-s)^g\, e^{\lambda s(k-j-m)} ds \right\} .
\end{aligned}
$$

The above is obtained via substitution and the Binomial theorem. This expression can be evaluated by noting that

$$
\int_0^t (t-s)^g\, e^{\alpha s} ds = t^{g+1}\, e^{\alpha t} \int_0^1 u^g\, e^{-(\alpha t)u} du ,
$$

and that

$$
\int_0^1 u^g\, e^{-\beta u} du = g!\, e^{-\beta} \sum_{i=1}^{\infty} \frac{\beta^{i-1}}{(g+i)!} .
$$

In actual applications it will be possible to truncate the values of g to perhaps $g \leq 2$ or 3 because with reliable components it will be very unlikely to have more than one or two failures in one day's flight time.

Finally consider the case $i = k$:

$$Q^M_{k,(k,g)} = P\{Z^E=k,\ G=g \mid Z^M=k\}$$

$$= P\{g \text{ events of Poisson process in } [0,t]\}$$

$$= e^{-k\lambda t} \frac{(k\lambda t)^g}{g!} .$$

3.4 $r^T_{k,z} = E\left(R^T_k \mid Z(0)=z\right)$

Let R^T_k be the random variable equal to the proportion of time during $[0,T]$ that $Z(t)$ equals k . This is the time when the k-out-of-n system is susceptible to a failure which causes an "emergency."

$$R^T_k = (T-T_k)^+/T ,$$

where T_k is the passage time defined in 3.3 and $x^+ = \max(0,x)$.

$$r^T_{k,z} = E\left(R^T_k \mid Z(0)=z\right)$$

$$= \frac{1}{T} E\left((T-T_k)^+ \mid Z(0)=z\right)$$

$$= \frac{1}{T} \int_0^T (T-t)dF_{T_k|Z(0)=z}(t)$$

$$= \frac{1}{T} \int_0^T F_{T_k|Z(0)=z}(t)dt ,$$

integrating by parts. Using the formula for $F_{T_k|Z(0)}$ in 3.3 gives

$$r_{k,z}^{T} = \frac{1}{T}\int_0^T \sum_{m=0}^{k} \binom{z}{m}\left(e^{-\lambda t}\right)^m\left(1 - e^{-\lambda t}\right)^{z-m} dt$$

$$= \frac{1}{T}\sum_{m=0}^{k} \binom{z}{m} \int_0^T \left(e^{-\lambda t}\right)^m\left(1 - e^{-\lambda t}\right)^{z-m} dt$$

$$= \frac{1}{T}\sum_{m=0}^{k} \binom{z}{m} \int_0^T \sum_{i=0}^{z-m} (-1)^i e^{-\lambda t(m+i)} dt \binom{z-m}{i}$$

$$= \frac{1}{T}\sum_{m=0}^{k} \binom{z}{m} \sum_{i=0}^{z-m} (-1)^i \frac{1 - e^{-\lambda T(m+i)}}{\lambda(m+i)} \binom{z-m}{i}.$$

The steady-state expected proportion of time the k-out-of-n system is in state k will depend on the steady-state distribution of $Z(0)$. Let

$$q^0(i) = P\{Z(0)=i\}$$

when the system has reached steady state. Then

$$r_k^T = E\left(R_k^T\right) = E\left((T-T_k)^+\right)/T = \sum_z r_{k,z}^T q^0(z) .$$

This quantity will be of interest because if we are dealing with components which fail infrequently, the point process of "emergency" failures will be approximately a Poisson process with rate proportional to r_k .

3.5 $m_z(t) = E\left(G(t) \mid Z(0)=z\right)$

Let $G(t)$ be the cumulative number of "emergency" failures of the Z process in the time interval. Calculation of the conditional expected value of $G(t)$ is similar to the analysis in 3.4. Conditional on $T_k = s < t$, $G(t)$ is a Poisson random variable with mean $\lambda k(t-s)$; if $T_k > t$, $G(t)$ is identically 0 , a degenerate form of the Poisson distribution. Thus $G(t)$ is a mixture of Poisson random variables, the mixing distribution being $F_{T_k|Z(0)=z}$, truncated at t . Thus the expected value of $G(t)$ will be the same mixture of expected values

$$m_z(t) = \int_0^t \lambda k(t-s)dF_{T_k|Z(0)=z}(s)$$

$$= \lambda k \; t \; r^t_{k,z} \; ,$$

from 3.4. Similarly, if the system has reached a steady-state behavior, the cumulative expected number of failures will be

$$m(t) = E\big(G(t)\big) = \sum_z m_z(t)q^0(z) = \lambda k \; t \; r^t_k \; .$$

This quantity will be useful in statistically distributing the "emergency" failures over the route structure in order to compute the average cost of an emergency failure.

3.6 $P^M : Y^M \rightarrow Y^E$

By an assumption of the model, any change in the spares pool during the period morning-to-evening is a result of restocking from the repair shop. Let P^M be the matrix of transition probabilities, for $i,j \geq 0$,

$$\begin{aligned} P^M_{ij} &= P\{Y^E=j \mid Y^M=i\} \\ &= P\{\text{repair shop output} = j-i \mid s-i \text{ units in shop}\} \\ &= R_{s-i,s-j} \; , \end{aligned}$$

where R is the matrix of transition probabilities for the repair shop: $R_{\ell,m}$ equals the probability that m units will remain in the shop at the end of the day given that ℓ were in the shop at the beginning of the day. Let $L(t)$ equal the number in the shop at time t (time is measured in days). Then

$$R_{\ell,m} = P\{L(1^-)=m \mid L(0)=\ell\} \; .$$

The process $\{L(t), \; 0 \leq t < 1\}$ is a pure-death process with transition intensities, for $i \geq 1$,

$$P\{L(t+\Delta t) = i-1 \mid L(t)=i\} = \min(i,r)\mu \; \Delta t \; ,$$

where time is measured as fraction of working day. Consider the following cases.

Suppose $i > r$; then define $T_{i-1}, T_{i-2}, \ldots, T_r$ as the first passage times to states $i-1, i-2, \ldots, r$, respectively:

$$T_{i-1} = \inf(t : L(t) = i-1) \text{ , etc.}$$

If $L(0) = i$ and $i > r$, then T_{i-1} has an exponential distribution with failure rate $r\mu$. Similarly, $T_\ell - T_{\ell+1}$ has an exponential distribution with rate $r\mu$ provided $\ell \geq r$. For $\ell \geq r$, T_ℓ has a gamma $(r\mu, i-\ell)$ distribution and

$$P\{L(1)>\ell \mid L(0)=i\} = P\{T_\ell > 1\} .$$

This is the same probability as for a Poisson process with rate $r\mu$; thus R_{ij} must be the same for $j \geq r$ as for a Poisson process, i.e.,

$$R_{i,j} = \text{l}\{\text{Poisson } (r\mu) \text{ process has } i-j \text{ events in } [0,1]\}$$

$$= e^{-r\mu} \frac{(r\mu)^{i-j}}{(i-j)!} , \qquad i \geq j \geq r .$$

Now consider the case $i > r, j < r$,

$$R_{i,j} = \int_0^1 P\{L(1)=j \mid L(0)=i, T_r=t\} f_{T_r|L(0)=i}(t)dt$$

$$= \int_0^1 P\{L(1-t)=j \mid L(0)=r\} f_{T_r|L(0)=i}(t)dt ,$$

where T_r is the first passage time to state r . The T_r has a Gamma $(r\mu, i-r)$ distribution with density

$$f(t) = \frac{1}{(i-r-1)!} (r\mu)^{i-r} t^{i-r-1} e^{-r\mu t} , \quad t > 0 .$$

Also

$$P\{L(1-t)=j \mid L(0)=r\} = \binom{r}{j}\left(e^{-\mu(1-t)}\right)^j \left(1 - e^{-\mu(1-t)}\right)^{r-j} .$$

Substituting these expressions into the equation for $R_{i,j}$ and integrating, using the equality

$$\int_0^1 e^{-\alpha t} t^{\nu} dt = \frac{\nu!}{\alpha^{\nu+1}} \left[1 - e^{-\alpha} \sum_{r=0}^{\nu} \frac{\alpha^r}{r!} \right],$$

yields

$$R_{i,j} = \binom{r}{j} (r\mu)^{i-r} \sum_{m=0}^{r-j-1} \frac{\binom{r-j}{m} (-1)^m}{[(r-m-j)\mu]^{i-r}} \left[e^{-\mu(j+m)} - e^{-\mu r} \sum_{n=0}^{i-r-1} \frac{[(r-m-j)\mu]^n}{n!} \right]$$

$$+ (-1)^{r-j} \frac{\binom{r}{j} (r\mu)^{i-r}}{(i-r)!} e^{-\mu r} .$$

Finally, consider the case $i \leq r$; then

$$R_{i,j} = P\left(L(1)=j \mid L(0)=i\right)$$

$$= \binom{i}{j} \left(e^{-\mu}\right)^j \left(1 - e^{-\mu}\right)^{i-j} .$$

We have now given $R_{i,j}$ for $0 \leq j \leq i$, the only nonzero values. Returning to P_{ij}^M , it should be noted that Y^M is allowed to take negative values to reflect unfilled emergency demand. We follow the convention of not returning a failed component to the repair shop before it is replaced by a good component from the spares pool, thus $Y^M < 0$ implies exactly s spares in the repair shop and

$$P_{i,j}^M = \begin{cases} R_{s-i,s-j} , & \text{if } i \geq 0 \\ R_{s,s-j+i} , & \text{if } i < 0 . \end{cases}$$

3.7 $P^E : Y^E \rightarrow Y^N$

The transitions of the spares pool in the evening-to-night period is assumed due to emergency demand. It is assumed that the system is optimized in such a way that emergency demands are relatively infrequent; this suggests a Poisson distribution of the number of emergency demands. As mentioned in 3.4, the expected number of such demands will be proportional to the proportion of time the planes spend in the "danger" state k, namely r_k^T. More specifically, the mean number of emergency demands in a day will be proportional to the number of flight hours spent in state k by the fleet, namely $N(vT/N)r_k^T$. Since k components with failure rate λ are subject to failure under this circumstance, the mean number of emergency demands in one day is

$$\nu = vTr_k^T\lambda k .$$

Thus, letting P^E be the matrix of transition probabilities,

$$\begin{aligned} P_{ij}^E &= P\{Y^N=j \mid Y^E=i\} \\ &= P\{\text{emergency demand} = i-j\} \\ &= \begin{cases} 0 & , \quad j > i \\ \dfrac{e^{-\nu}\,\nu^{i-j}}{(i-j)!} & , \quad j \le i . \end{cases} \end{aligned}$$

3.8 $P^N : Y^N \to Y^M$

The transitions in the spares pool during the time interval from night to the next morning are assumed due to replacements made on the plane (or planes) visiting the main base for their scheduled overnight maintenance. Let P^N be the matrix of transition probabilities of depletion of the spares pool by one such visitor. If there is more than one visitor then this transition is iterated once for each visitor and $(P^N)^V : Y^N \to Y^M$. The number of spares taken from the spares pool depends upon the number requested, $n - Z(T)$. Thus as in 3.2, there exists a dependence between Y and Z. As before, we shall assume that the dependence can be uncoupled in the steady state,

$$\begin{aligned} P_{ij|Z(T)=z}^N &= P\{Y^M=j \mid Y^E=i,\ Z(T)=z\} \\ &= \begin{cases} 1 , & \text{if } j-i = f_\delta(n-z,i) \\ 0 , & \text{otherwise} \end{cases} \end{aligned}$$

and if the probabilities of $Z(T)$ in the steady state are

$$q_z^T = P\{Z(T)=z\} ,$$

then steady-state transitions are approximately

$$\begin{aligned} P_{ij}^N &= P\{Y^M=j \mid Y^E=i\} \\ &= \sum_z P_{ij|Z(T)=z}^N q_z^T . \end{aligned}$$

3.9 Joint Probability Laws for Z and Y

In Sections 3.1 through 3.8 only marginal distributions of Y and Z are presented. Any dependence between Y and Z was assumed to be weak enough that any joint probabilistic behavior could be approximated by mixing over the marginals; for example, the transitions in 3.2 of Z depend on Y so the conditional transitions were mixed, weighted by the distribution of Y when the system (airline operation) is in steady state. Analysis in 3.8 is similar. This approach of attempting to uncouple Y and Z in this manner is quite unconventional and some justification for it must be given.

In contrast to this tentative "uncoupling" approach, we now present the exact probability laws for a fleet of N aircraft flying an N-day route pattern, i.e., one visitor per night at the main base. A schematic of the stochastic behavior of this system from one morning until the next morning is presented in Figure 5. Referring to that figure, note the following: On a given day the planes are numbered $1,2,\ldots,N$, corresponding to the day of the route pattern the plane is on; for example, Z_1^M equals the number of functioning components in the k-out-of-n subsystem on the plane which is starting the first day of the route, i.e., the plane that has just completed its scheduled overnight at the main base. There are t_i flight hours in the ith day of the route pattern. The transition of the spares pool from evening to night, $Y^E \to Y^N$, is actually a <u>deterministic</u>

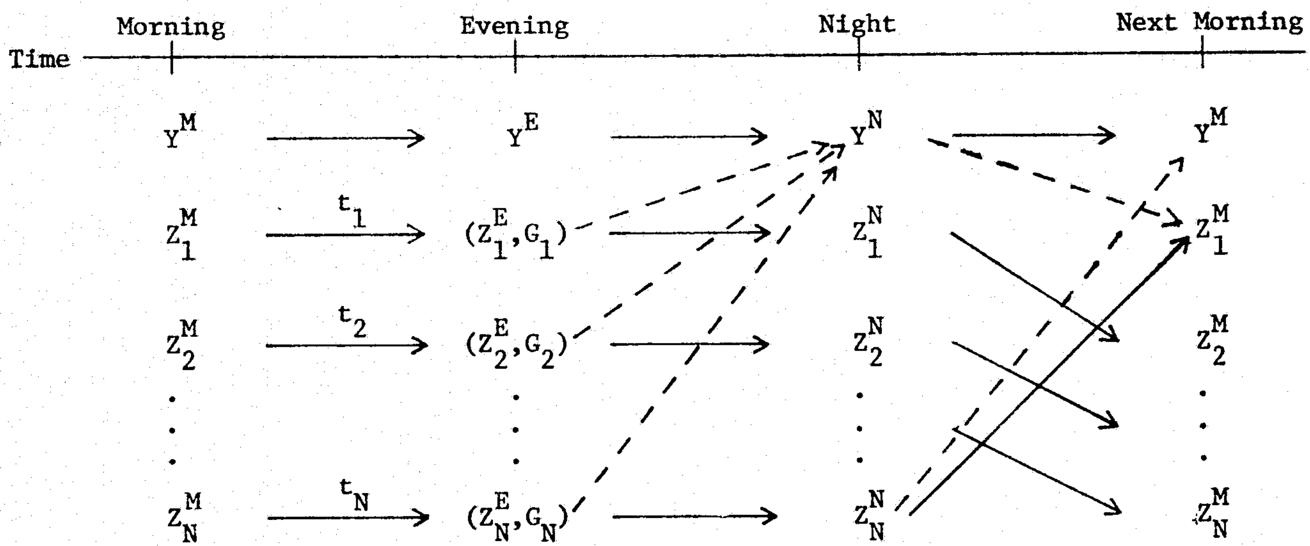

Figure 5.--Schematic of exact transitions for fleet of N planes flying an N day cycle. Arrows signify transitions; broken arrows indicate joint probability effect in transitions.

transition: $Y^N = Y^E - \sum_{i=1}^{N} G_i$. The broken arrows in Figure 5 denote the fact that Y^N also depends on the G_i's ; also $Z_i^N = Z_i^E$, $i=1,2,\ldots,N$. The transition of the spares pool between evening and the next morning are joint with Z_N^N , in particular $(Y^N, Z_N^N) \to (Y^M, Z_1^M)$ is given by $Y^M = Y^N - x$, $Z_1^M = Z_N^N + x$, where $x = f_\delta(n-Z_N^N, Y^N)$; this transition is also deterministic. Finally $Z_{i+1}^M = Z_i^N$, $i=1,2,\ldots,N-1$. The only stochastic transitions occur during the time interval from morning until evening; the transition probabilities for these are given in 3.6 and 3.3. The remaining transitions are deterministic as described above.

4. Steady-state Behavior of the Airline System

The goal of this study is to determine long-run average costs of maintaining the k-out-of-n systems on the fleet of aircraft over the lifetime of the fleet, approximately 20 years. We shall assume that the system reaches a steady state quickly relative to this 20-year length of time, and therefore the long-run average behavior of the system starting from new can be accurately approximated using steady-state probabilities. Thus these probabilities must be obtained. This section is devoted to some approximate or tentative approaches to this problem. Future work will include more thorough investigation of these and other possible approaches.

4.1 Uncoupling Y and Z

One possible approach to solving for steady-state probabilities is to assume that the dependence between Y and Z can be uncoupled. More specifically, when the system is in steady state we assume that for Z transitions, which are conditional upon the state of Y , it suffices to use the weighted sum of these conditional transition probabilities, weighted by the steady-state marginal distribution of Y ; and vice versa for Y transitions. Such weighted sums occur in the transition matrices Q^T and P^N , in 3.2 and 3.8, respectively.

The uncoupled analysis is shown in Figure 6. The Z process is analyzed separately from Y ; it only uses a steady-state distribution of Y^N as an input. Let $p^N(i) = P\{Y^N=i\}$ be a steady-state distribution of Y^N . For purposes of deriving the steady-state distribution of Z it is assumed that p^N is given and fixed independently of the stochastic behavior of Z ; the transition matrix Q^T is actually a function of p^N : $Q^T = Q^T(p^N)$, Section 3.2. If the steady-state probabilities of $Z(0)$ and $Z(T)$ are q^0 and q^T , respectively,

$$q^0(z) = P\{Z(0)=z\}$$

$$q^T(z) = P\{Z(T)=z\} ,$$

then they must satisfy

$$\begin{aligned} q^T &= q^0 Q^0 \\ q^0 &= q^T Q^T(p^N) . \end{aligned} \tag{4.1.1}$$

Thus in order to obtain steady-state probabilities of the Z process, we first must supply a steady-state distribution for Y^N and then solve the system of equations (4.1.1). Quantities of interest arising from the Z process (see Figure 6) are emergency demand and routine demand for spares. As discussed in Section 4.4, the emergency demand when the system is in steady state may be approximately a Poisson process with rate proportional to r_k . Routine demand equals $n - Z(T)$ and therefore its distribution is obtained from q^T for the system in steady state.

The Y process requires steady-state distribution of emergency and routine demands for spares as input. Therefore assume that the emergency demands form a Poisson process with rate ν . As in the previous paragraph, it is assumed that this input process is independent of Y . The transition matrix P^E actually depends on ν , $P^E = P^E(\nu)$, see Section 3.7. Next we assume that the demand process for routine replacement of failed parts is

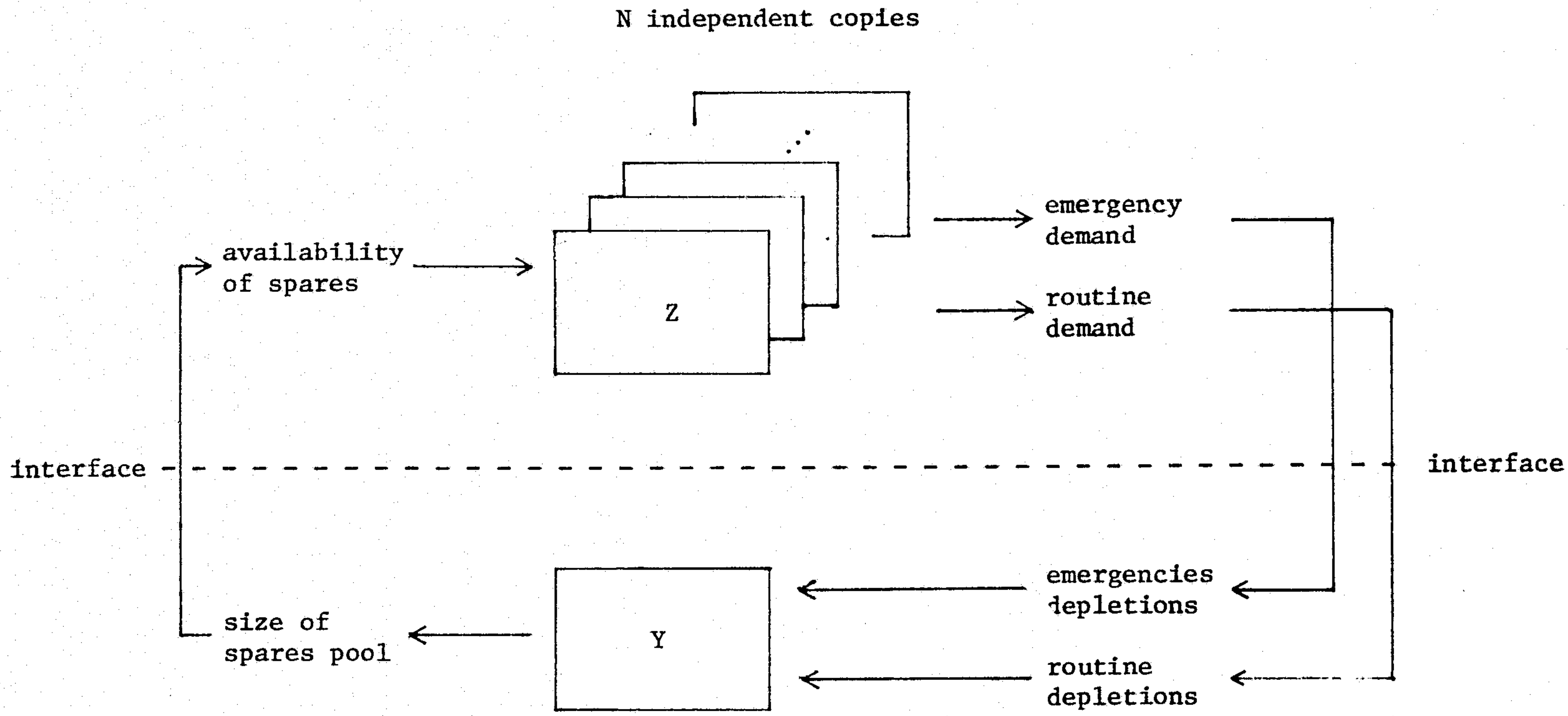

Figure 6.--Schematic of coupling of Z and Y processes.

a sequence of independent demands $D_1^r, D_2^r, \ldots$, where the steady-state distribution is

$$P\{D_i^r = d\} = q^T(n-d) .$$

Thus the transition matrix P^N actually depends on q^T, $P^N = P^N(q^T)$; see Section 3.8. If the steady-state probabilities of Y^M, Y^E, and Y^N are p^M, p^E, and p^N, respectively,

$$p^M(y) = P\{Y^M = y\}$$

$$p^E(y) = P\{Y^E = y\}$$

$$p^N(y) = P\{Y^N = y\} ,$$

then they must satisfy

$$\begin{aligned} p^E &= p^M P^M \\ p^N &= p^E P^E(\nu) \qquad (4.1.2) \\ p^M &= p^N P^N(q^T) . \end{aligned}$$

Thus, in order to obtain steady-state probabilities for the Y process, it is necessary to provide steady-state distributions of spares demand.

The above analysis of the Z process corresponds to the upper half of Figure 6. It is assumed that the input processes are given and controlled so that behavior of the Z process creates no feedback which can affect the distribution of these inputs. Similarly, the above analysis of the Y process assumes no feedback to the input process. These analyses of the processes separately as described are valid; however, now we suggest the questionable technique of analyzing them simultaneously with the above methods. In particular the output of Z becomes the input of Y and vice versa, creating a feedback loop that violates the previous assumptions. It is hoped that in some cases of interest this will not be a serious violation and approximate steady-state probabilities can be obtained.

Combining the separate analyses of Z and Y gives the following system of equations:

$$
\begin{aligned}
q^T &= q^0 Q^0 \\
q^0 &= q^T Q^T(p^N) \\
r_k^T &= \sum_z r_{k,z}^T \, q^0(z) \\
p^E &= p^M P^M \\
p^N &= p^E P^E(\nu) \\
p^M &= p^N P^N(q^T) \\
\nu &= T r_k^T \lambda k \ .
\end{aligned}
\qquad (4.1.3)
$$

This system can probably be solved iteratively, or perhaps by using a matrix inversion computer algorithm.

The uncoupled analysis described in this section is intended as an approximation. It is highly unlikely that it will produce exact steady-state probabilities for any system of interest. The approximation might be good in "light traffic," i.e., excess spares or repair facilities. It also might be a good approximation in the case where the mean repair time is much shorter than D , the number of days in one cycle of the route structure; this case would tend to have little dependence between the size of the spares pool at a particular time and the number of spares requested by an overnight visitor at the main base.

Future work will include a closer look at the above technique. In Appendix B a simple example with some numerical results is presented which illustrates the fact that the above uncoupling technique may give incorrect answers.

4.2 Partial Uncoupling of Y and Z

The uncoupled analysis in Section 4.1 amounts, in some sense, to ignoring the joint distributions of the different components of the vector-valued process and just allowing marginal distributions to enter into the analysis. In Figure 6 the only information allowed to cross the interface (the dotted line in Figure 6) between the Y and Z processes are the marginal distributions. We can extend this idea of uncoupling by allowing more but not all information to cross the interface, in particular some of the joint probabilities between Y and Z, but not all. This idea is only in the formative stages and is not precise. The following example may help.

Let Y^M, Y^E, Y^N, $Z(0)$, $Z(T)$, and R_k be defined as in previous sections. Define the following joint transitions of (Y,Z). In the morning-to-evening time period $(Y^M, Z(0)) \rightarrow (Y^E, Z(T), G)$, according to the transitions given in 3.3 and 3.6. In the evening-to-night period, $(Y^E, Z(T), G) \rightarrow (Y^N, Z(T))$, where $Y^N = Y^E - G$, a deterministic transition. Finally in the night-to-morning period we have the transition $(Y^N, Z(T)) \rightarrow (Y^M, Z(0))$, where $Y^M = Y^N - x$ and $Z(0) = Z(T) + x$, x being $f_\delta(n - Z(T), Y^N)$. The three transition matrices can be multiplied to give the transition of $(Y^M, Z(0))$ from one morning to the next, and then steady-state joint probabilities can be solved for from the eigenvector equation.

We note that, while the uncoupling in 4.1 ignored dependence and thus may have removed some of the interaction and reduced some of the fluctuations in the behavior of the system, the analysis in this section may have the opposite effect, thus providing an approximation with higher operating cost than the one in 4.1, which may have a lower operating cost. The reason for increased fluctuations is the fact that the above analysis is equivalent to making one plane fly the whole N-day route in one day instead of having N planes staggered out on the route. Having all the emergency failures of a plane charged on the day it is dispatched from the main base aggravates any shortage in the spares pool; if the pool was low the plane was dispatched in less than perfect condition, thus stochastically increasing the number of emergency failures the plane will suffer in the next N days. If these failures are all charged to the spares pool on the first day, it will increase the chance of a stockout over the case of charging them on the day they occur because the repair shop will have less time to catch up on its work.

Another possible treatment of the interface between the Y and Z processes would be to allow only information on the total number of bad units among those in planes and in spares to cross the interface, then to assume a statistical distribution of the number of bad in each plane and at the base, conditional on the total number of bad units in the entire system.

4.3 Sparse Matrix Techniques

Consider the transitions of $(Y,Z_1,\ldots,Z_N)$ described in Section 3.9. The state space of this process will be huge. As an example, consider the United 747 fleet of 18 aircraft with CSDL microprocessor aboard; this has 10 CPU's. Suppose that 6 (2 triads) must be functioning for dispatch, giving a 6-out-of-10 system; suppose there are 10 spares; then ignoring backorders, the state space consists of $(10+1)(10-6+1)^{18} = 4.2 \times 10^{13}$ points. Any straightforward Markov analysis is clearly infeasible; however, as is pointed out in Section 3.9, there are actually a much, much lower number of transitions within this huge state space than can occur in general. This suggests an analysis using sparse matrix techniques. The example in Appendix B should be amenable to such an approach; see the end of Appendix B. A recent reference is [12].

4.4 Monte Carlo Simulation

In spite of the size of the state space it is probably feasible to perform a Monte Carlo simulation of the system in Section 3.9. This is because stochastic transitions of individual components are independent of each other and all joint transitions are deterministic. Another fact favoring success of a Monte Carlo approach is that we are interested only in a subset of marginal probabilities and a subset of expected values to be used as inputs in the cost equations (see Sections 5 and 6). The first fact suggests that implementation is tractable; the second that run times until reaching equilibrium of quantities of interest will be of manageable length.

It will be of interest to run a Monte Carlo simulation starting with a perfect new system and to note the length of time required for the system to reach equilibrium. If the equivalent real operating time is 10 years or so (a significant part of the lifetime of the system), this obviates the need for a steady-state analysis of the system. In that case, transients must be considered.

Finally, Monte Carlo simulation of the exact system (in Section 3.9) can be used to evaluate the accuracy of the approximate solution techniques proposed in Sections 4.1 and 4.2. If the accuracy is questionable, it might be efficient to use the approximate solution techniques to obtain initial vectors for a Monte Carlo simulation. Conversely, if the approximate solution probability vector is used as an initial vector in the simulation and after several iterations the marginal probabilities have not changed appreciably, this constitutes verification of the approximate solution approach.

5. Interface with Reliability and Cost Models

In Sections 2 through 4 the underlying stochastic behavior of a fleet of aircraft with k-out-of-n subsystems is described along with proposed methods of analysis. This underlying stochastic behavior generates events that have consequences in the realm of reliability penalty costs and maintenance penalty costs. We distinguish between these two penalty situations as follows. A maintenance penalty event consists of dispatch cancellations or delay of a plane due to the k-out-of-n system dropping below k functioning units on the previous flight leg and inability of line maintenance men to replace failed component(s) prior to the next dispatch time. A reliability penalty event occurs when a plane is dispatched on a flight leg with k or more units functioning at dispatch time but during the course of the flight something happens which prevents the aircraft from completing the flight according to flight plan. Reliability penalty events include abortions, diversions, delayed arrivals, excess fuel consumption, crashes, etc. The penalties incurred will depend upon where in the route pattern the particular event occurs. To compute the average penalty costs incurred by operating the airline in steady state during a time interval, we shall compute a weighted average of average cost per penalty event occurring at each point of the route pattern, weighted by the expected number of events at that point during the time interval.

5.1 The Route Schedule and Penalty Costs

The route pattern of a D-day, T-hour route can be described by a $4 \times m$ array in which m equals the number of legs in one cycle through the route:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ d_1 & d_2 & d_3 & \cdots & d_m \\ t_1 & t_2 & t_3 & \cdots & t_m \\ g_1 & g_2 & g_3 & \cdots & g_m \end{pmatrix},$$

where a_i is the ith city visited, d_i the day of the D-day cycle, t_i the accumulated number of flight hours when landing at the ith city, and g_i the amount of ground time at the ith city. For example, consider the route depicted in Figure 1. A possible route schedule for this situation is:

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 3 & 6 & 7 & 2 & 8 & 7 & 8 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 \\ 2.0 & 4.5 & 6.0 & 7.3 & 8.3 & 11.3 & 12.9 & 16.0 & 19.7 & 21.2 & 22.7 & 25.7 \\ 1.0 & 2.5 & 14.5 & .9 & 1.5 & 2.0 & 12.7 & 2.0 & 1.7 & 12.0 & 3.5 & 16.0 \end{pmatrix}.$$

Labelling the main base as City #1, in the above array we have 3 legs on the first day of the 4-day cycle: $1 \to 2$, $2 \to 3$, $3 \to 4$, with cumulative flight time equal to 6.0 hours. The second day starts with a leg $4 \to 5$, etc.

Now we wish to assign costs to reliability and maintenance penalty events which can occur in the course of flying the route. These costs will be a function of where the event occurs in the route. First consider maintenance penalty events; the only such event is an emergency failure, i.e., the k-out-of-n subsystem dropping below k functioning components. As an example, suppose an emergency failure occurs at cumulative flight hour 10.0 in the above route schedule; this means that during the second day on the leg from City #3 to City #6 the k-out-of-n subsystem drops to $k-1$ functioning components. It is assumed that the dispatch minimum level of k working components is conservative and that the plane can complete the flight leg as planned with only $k-1$ units; however, a failed unit must be replaced at City #6 before the plane can be dispatched on the next leg of the route. We assume that all spare units are inventoried at the main base and thus a replacement must be flown out to City #6; furthermore, if City #6 is a secondary base, a maintenance man must be sent to make the replacement. Time required to get the spare and make the replacement may exceed the 2-hour ground time at City #6, interfering with dispatch of the next leg, $6 \rightarrow 7$. Consequently this leg may be cancelled or delayed. If it is cancelled the cost may be computed as proportional to the length multiplied by the size of the plane; depending upon the amount of airline information available it should, in principle, be possible to compute the cost to any desired precision. Delay costs can also be calculated from airline experience. If a spare is available from the main base inventory the problem should be fixed in time for the plane to get back on schedule to start day #3. If the last leg of day #2 is cancelled, a ferrying cost will be incurred in getting the plane from City #6 to City #2. All the penalty costs resulting from an emergency failure on the sixth leg, i.e., $3 \rightarrow 6$, are deterministic and in theory can be calculated to give a cost C_6 . Likewise similar costs can be computed for an emergency failure on any leg. This information can then be used to give the penalty cost resulting from an emergency failure at time t flight hours; denote it $C_E(t)$. A possible example of $C_E(t)$ for the above 4-day route schedule is plotted in Figure 7. The steady-state expected cumulative number of emergency demands in $(0,t]$ is calculated in Section 3.5: $m(t) = E(G(t))$. Therefore the average cost due to emergency failures on one plane in $[0,T]$ for an airline operating in steady

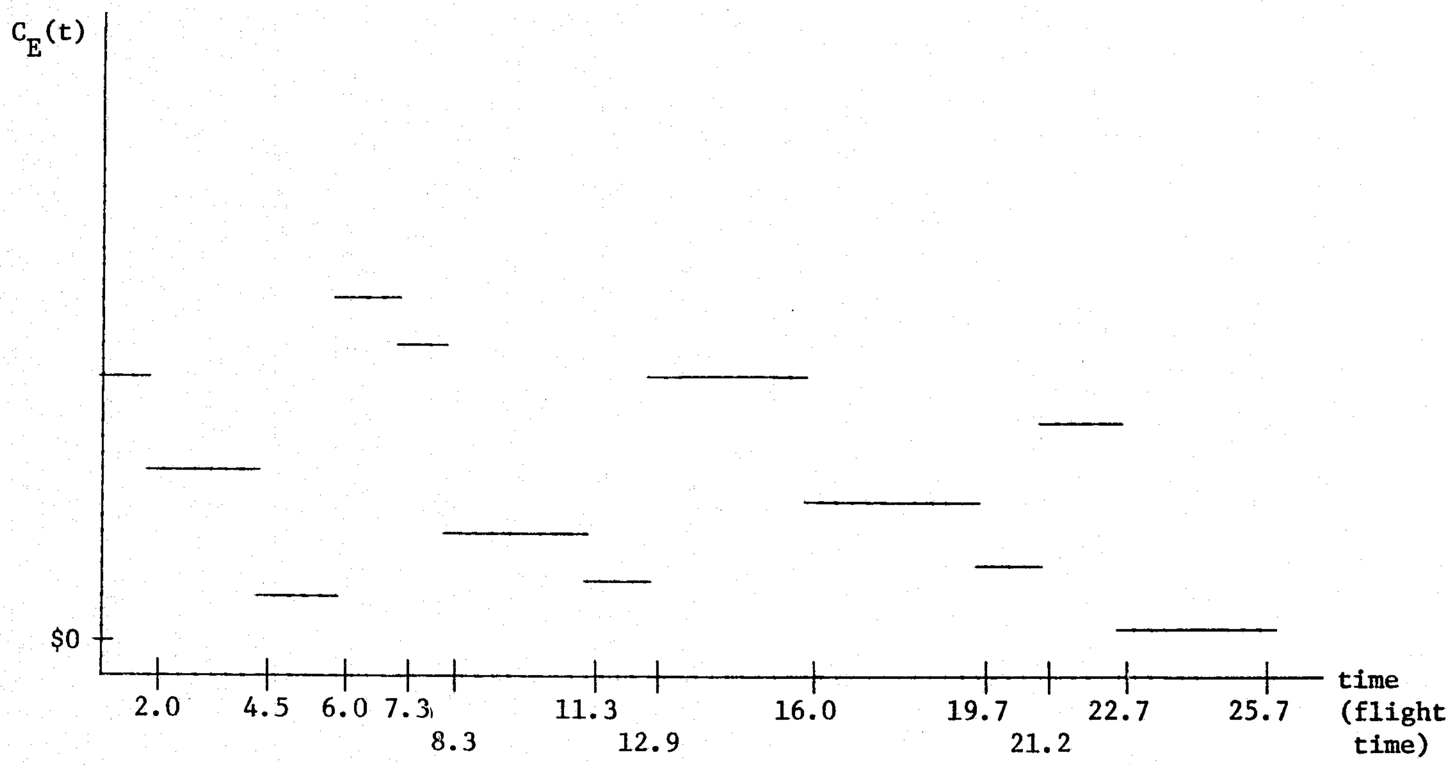

Figure 7.--Average penalty costs for emergency failures as a function of their times of occurrence.

state is a weighted sum of the costs, weighted by the expected number of times each cost is incurred. Let C_E equal this average cost; then

$$C_E = \int_0^T C_E(t)dm(t) .$$

Now consider the reliability penalties incurred by a plane flying a scheduled route. A reliability penalty event can be any of several possibilities, each with a different penalty cost; thus we shall assume we know the expected reliability penalty cost incurred when a plane is dispatched on the ith leg of the route schedule with z components in the k-out-of-n subsystem working correctly. Denote this quantity as $\overline{C}_R(z,i)$. Let $q^i(z)$ equal the steady-state probability of the subsystem having z good units at time t_{i-1} . Then the expected penalty costs charged to reliability in the course of a plane flying the route for an airline in steady state is

$$C_R = \sum_{i=1}^{m} \sum_{z=k}^{n} \overline{C}_R(z,i)q^i(z) .$$

The probability vector q^i is calculated from q^0 , using the techniques of Section 3.1. The expected cost function $\overline{C}_R$ is related to work supported by a National Aeronautics and Space Administration (NASA) grant with the University of Michigan (John Meyer, Principal Investigator); thus we shall not pursue it here.

Any further discussion of reliability is beyond the scope of this memorandum. However, there is a rather compelling reason for not considering it in this preliminary effort at a cost model: namely, if the 10^{-9} reliability constraint is met, it is very difficult to imagine such a control system causing great economic penalties due to unreliability. Consider that the chances of loss of control are 10^{-9} in a 10-hour flight; thus approximately 10^{-10} in a 1-hour flight. If we take the extremely conservative view that 1 in 100,000 cases of reliability failure events will involve loss of control that still implies at most 10^{-5} probability of some sort of reliability problem in 1 hour. For a fleet of 100 planes operating 10 hours per day, 365 days per year, this results in 3.65 expected reliability penalty events each year. The expected number of cases involving loss of control will be 3.65×10^{-5}. The economic impact of such rare events will be small in the average day-to-day costs of running the plane. The extremely rare catastrophic event can be incorporated by simply estimating the cost per event and multiplying by the expected frequency, e.g., $\$100,000,000 \times 3.65 \times 10^{-5} = \$3650$, and adding this figure to the annual cost. (The above reasoning is rather imprecise, but it should justify ignoring the economic impact of reliability penalty costs for now; we shall assume that k is fixed at a level for which the 10^{-9} probability constraint is met.)

5.2 Buffer Spares Pools at Additional Airports

One of the simplifying assumptions in the model in this paper is the existence of a single solitary spares pool located at the main base. It is assumed that when a plane's k-out-of-n system drops below k good components away from the main base that a spare is flown out to it from the main base. In actual airline operations some spares are also distributed at certain primary bases in addition to the main base. In this section we show how to incorporate such a feature into the model of Sections 2 and 3 without complicating the analysis of the underlying stochastic behavior. This approach is intended to apply to very small spares pools at only a very limited number of additional airports. In effect these additional spares pools will act as buffers for emergency spare demand. If a plane is grounded at an airport with a buffer spares pool, a spare is used from that pool to replace a failed component (which is assumed returned to the repair shop by the next morning), then a demand is placed to the main base for a spare to restock the buffer pool. This demand is treated as an emergency demand by the main base and filled within one day if spares are available; if no spares are available at the main base, a backorder is recorded (the Y process goes negative). There is no transshipment between buffer spares pools. The spares in a buffer pool are used only for emergency replacements at the airport where the buffer pool is located. The only modification required in the model will be in the cost terms for penalty for stockout of the spares pool at the main base and delay cost. Most of the backorders arising from demands for spares from airports with buffer spares pools will not reflect a true emergency (a grounded plane with no spare available), but simply a shortage in the buffer pool. Thus the penalty for stockout of the main spares pool must be decreased by a factor proportional to the percentage of demands which do not reflect true emergencies. Thus we must compute the fraction of demands which come from an airport with a buffer pool when there is no plane waiting for a spare.

To illustrate the above situation, consider an airline route with one buffer spares pool consisting of at most one spare at a primary base. Presumably this will be a base with relatively heavy traffic. For example, in Figure 1 of Section 2, we shall put a buffer spares pool of size one at City #7. For an airline flying this route according to the schedule in Section 5.1 in steady state, the expected number of emergency demands placed by a plane at City #7 equals $\frac{dm(t)}{dt}$ integrated over flight legs 7 and 10,

$$m_7 = (m(12.9) - m(11.3)) + (m(21.2) - m(19.7))$$

If there are N planes flying a D-day schedule then the total expected demand at City #7 will be $m_7 N/D = M_7$. Since the buffer pool will be placed at an airport with a lot of traffic it is assumed that the daily emergency demand for spares can come from many planes; it is therefore reasonable to assume that the demand has a Poisson distribution with mean daily demand M_7. It takes one day to restock the buffer pool, thus we need the proportion of demands which find a spare present. This is a case of Palm's theorem [6]; it will equal the probability that no demand was made in the last 24-hour period. If the demand is Poisson with mean M_7 per day, then the distribution of time since last demand (LD) is exponential with rate M_7 and

$$P\{LD > 1\} = \exp(-M_7) .$$

Thus proportion $\exp(-M_7)$ of the demands arriving at the main spares pool from City #7 are not really emergencies and no penalty should be incurred for failure to fill them immediately. These constitute proportion

$$p^* = M_7 \exp(-M_7) / (N/D)\, m(T)$$

of the total expected demand on the spares pool at the main base. In Section 6, the term for the penalty for stockout of the main base spares pool should be decreased by a factor of p^*. Also, the penalty for an emergency failure at the airport with the spares buffer must be reduced, since less delay is involved in proportion $\exp(-M_7)$ of the cases.

6. Cost Equations

The average quantities derived from the stochastic behavior of the maintenance operation and costs associated with them can now be combined with other costs to form a fairly general model of the cost of acquiring and maintaining the system and its backup for a fleet of aircraft.

We first consider fixed initial costs. The initial purchase cost of units plus spares is

$$C_p = N(C_{p,0} + C_{p,1}n) + C_{p,2}s .$$

The cost of setting up the repair shop is

$$C_s = C_{s,0} + C_{s,1}r .$$

Certification and training costs for repairmen and line maintenance men is

$$C_c = C_{c,0} + C_{c,1}r + C_{c,2}\ (\#\text{ primary airports}) .$$

The total initial cost will be

$$C_i = C_p + C_s + C_c .$$

Next consider fixed continuing costs. The shop overhead is

$$C_0 = C_{0,0} + C_{0,1} r .$$

The salaries of repairmen are

$$C_r = r C_{r,1} .$$

The retraining/recertification cost of repairmen is

$$C_t = C_{t,0} + C_{t,1} r .$$

The overhead cost of the line maintenance center at the main base and the line maintenance centers at other bases which are charged to this subsystem are

$$C_m = C_{m,0} + C_{m,1} \text{ (\# primary)} + C_{m,2} \text{ (\# secondary)} .$$

The annual fixed costs are the sum

$$C_f = C_0 + C_r + C_t + C_m .$$

The present value of fixed costs when the system is new is

$$\overline{C}_f = C_i + \sum_{n=0}^{y-1} (1-d)^n C_f ,$$

where d is the annual discount factor and y is the lifetime of the system in years.

Finally, we consider the costs which are incurred daily in a random fashion during the life of the system due to the underlying stochastic nature of the system. The expected daily cost of emergency groundings for the fleet (due to repair and lost revenue) is

$$\overline{C}_E = (N/D) C_E ,$$

where C_E is defined in Section 5.1. The expected daily cost due to stockout of the spares pool is equivalent to one day of lost service for each item short for a day. If the average stockout is

$$\overline{N}_{SO} = E([Y^E]^-) = -\sum_{i=-N}^{-1} i p_i^E ,$$

and R_d is the average daily revenue per plane, then the expected daily penalty for stockout is

$$C_{SO} = \overline{N}_{SO} R_d .$$

(If a buffer spares pool consisting of one unit is added to the system as described in Section 5.2, then the following modifications must be made to the cost equations: $C^*_{SO} = (1-p^*)C_{SO}$. $\overline{C}_E$ must be adjusted to reflect lower grounding penalties at the airport with buffer. If $C_{b,0}$ is the expected daily penalty at this airport without a spare available and $C_{b,1}$ is the penalty if a spare is always immediately available, then the expected daily savings by having the buffer is $\Delta = \exp(-M_7)C_{b,1} + (1 - \exp(-M_7))C_{b,0}$ and $\overline{C}^*_E = \overline{C}_E - \Delta$; $C_{b,1}$ and $C_{b,0}$ can be computed as described in Section 5.1. Finally, the cost of the extra spare must be added to C_p , $C^*_p = C_p + C_{p,2}$.)

Line maintenance charges at the main base may be a linear function of the average number of bad units and the average number of replacements made on each aircraft visiting on a scheduled overnight stop. The average number of bad units on a plane is

$$\overline{B} = n - EZ(T) = n - \sum_{z=k}^{n} z q_z^T .$$

The average number of units replaced on a plane is

$$\overline{R} = E(Z(0)) - E(Z(T)) = \sum_{z=k}^{n} z q_z^0 - \sum_{z=k}^{n} z q_z^T .$$

Thus the daily expected line maintenance cost is

$$C_{LM} = C_{LM,0} + C_{LM,1} v\overline{B} + C_{LM,2} v\overline{R} .$$

There may be some feature where repairmen in the shop can work on other things if they are not busy repairing components for this particular system. The expected number of idle repairmen is

$$\bar{I} = \sum_{r=0}^{r-1} (r-i)p_{s-i}^{M} ,$$

and the credit for each is C_W , creating an expected credit

$$C_{IR} = C_W \bar{I} .$$

The daily expected costs due to stochastic behavior is

$$\bar{C}_{ST} = \bar{C}_E + C_{SO} + C_{LM} - C_{IR} .$$

The total cost is

$$C = C_i + \bar{C}_f + \sum_{n=0}^{y-1} (1-d)^n 365\bar{C}_{ST} .$$

There are undoubtedly other sources of cost but the above should include many of the major ones. (One thing we have ignored is the possibility of periodic--say, every 10,000 hours--extensive ground testing for latent faults.)

7. Conclusions

It is premature to make any definitive concluding remarks about the model and analysis discussed in this memorandum because the research is still in progress. However, we shall summarize some of the assumptions made in the model, all of which turn out to be of a conservative nature, and outline some future work.

7.1 Conservatism of the Model

There are numerous instances where the model only approximates the behavior of an actual airline maintenance operation as it exists in the real world. In these cases we have chosen the approximation so that it tends to overestimate the cost (or number of failures, etc.), thereby giving a lower bound on performance of the system for particular values of the control variables; that is, in the real world we are assured of doing as well or better. Some instances of these conservative approximations are:

(a) Only minimal emergency repair is allowed when the system drops below k functioning components away from the main base.

(b) There is only one spares pool (located at the main base).

(c) The spares strategy is special; a more general one might result in lower operating costs.

(d) Only one line maintenance center is allowed to do routine replacement.

(e) The system is required to complete all T flight hours even if some legs are cancelled, thus exposing the system to more time on test than it gets in the real world and thus increasing the chance of failures.

(f) There is no flexibility in the repair shop such as overtime work in order to reduce stockout.

(g) It is assumed that an entire day is lost if the replacement demand by a grounded plane is not met because of stockout.

(h) It is assumed that all units of the k-out-of-n subsystem have the same failure rate as a component in use, even though some components may be in a standby configuration.

(i) If the number of functioning units drops from k to k-2 during one flight leg, the event is counted as two groundings and costed as such in the cost equations.

(j) There is no dispatch flexibility; for example, airlines tend to allow a plane with k-1 good units to be dispatched to the main base. We do not.

There is at least one nonconservative aspect to our model; we ignore the possibility of component failures on the ground or in the stockroom.

7.2 Future Work

(a) An attempt will be made to solve the model with realistic parameter values numerically on the computer.

(b) Some theoretical work will focus on justification for interfacing stochastic processes in the way presented here, e.g., Sections 4.1 and 4.2.

(c) Some optimization will be attempted. The control variables are n, r, s, h, and δ. Economic feasibility of an optimal solution will be examined.

(d) If we get computer output for a sufficient number of different parameter values, we shall do some sensitivity analysis and possibly eliminate some of the model's complexity. One approach is Repro-modeling [9].

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APPENDIX A

A Simplified Trade-off Example

In this section we describe several variations of a flight control system and perform a greatly simplified economic analysis of the maintenance of each system to illustrate trade-offs between redundancy and performance.

Suppose that in order for a control system to control the aircraft it requires a minimum of 6 processors, 6 memories, 2 buses, 2 each of sensors A, B, and C, and 2 each of actuators 1, 2, 3, and 4. If the control system consists of this minimum set of components then a single failure causes loss of control. We shall assume that memories and processors have mean times between failures (MTBF's) of 10,000 hours and all the other components have MTBF's of 100,000 hours. We shall denote the above system as

$$\mathrm{Min}_{\mathrm{control}} = (6;\ 6;\ 2;\ 2,\ 2,\ 2;\ 2,\ 2,\ 2,\ 2)\ .$$

Now consider dispatchability. We use the requirement that the probability of loss of control in a 10-hour flight cannot exceed 10^{-9}. Using elementary calculations involving binomial probabilities we compute

$$P\{\text{drop to 5 or fewer processors in 10 hrs} \mid \text{start with 8}\}$$

$$\approx \binom{8}{5}\left(1 - 10^{-3}\right)^{5}\left(10^{-3}\right)^{3} = 56 \times 10^{-9} > 10^{-9}$$

$$P\{\text{drop to 1 or fewer buses in 10 hrs} \mid \text{start with 3}\}$$

$$\approx \binom{3}{1}\left(1 - 10^{-4}\right)^{1}\left(10^{-4}\right)^{2} = 30 \times 10^{-9} > 10^{-9}\ .$$

Thus if the control system consists of only 8 processors and memories and 3 each of buses, each sensor and each actuator, it cannot be dispatched.

Control System	Penalty/Flight Hour (× 1000)
$\text{Min}_{\text{dispatch}}$	$8(4 \times 10^{-5}) + 2(9 \times 10^{-4}) = 2.12 \times 10^{-3}$
CS_{I}	$8(4 \times 10^{-5}) + 2(2.25 \times 10^{-5}) = 3.65 \times 10^{-4}$
CS_{II}	$8(5 \times 10^{-8}) + 2(9 \times 10^{-4}) = 1.80 \times 10^{-3}$
CS_{III}	$8(5 \times 10^{-8}) + 2(2.25 \times 10^{-5}) = 4.54 \times 10^{-5}$

Now consider the repair costs per flight hour. Let us assume that the average repair cost of a memory or a processor is $1,000 and the average repair cost for a bus, sensor, or actuator is $500. A 4-out-of-4 system with MTBF 10^5 for each component has failures at the rate of 4×10^{-5} per flight hour, a 4-out-of-5 system has failures at approximately the rate of 5×10^{-5} per flight hour. A 9-out-of-9 system with MTBF 10^4 for each component has failures at the rate of 9×10^{-4} per flight hour; a 9-out-of-10 system has failures at approximately the rate of 10×10^{-4} per flight hour. Thus the average expected repair costs per flight hour for the different systems is:

Control System	Repair Cost/Flight Hour (× 1000)
$\text{Min}_{\text{dispatch}}$	$8(4 \times 10^{-5}) \times .5 + 2(9 \times 10^{-4}) \times 1 = 1.96 \times 10^{-3}$
CS_{I}	$8(4 \times 10^{-5}) \times .5 + 2(10 \times 10^{-4}) \times 1 = 2.16 \times 10^{-3}$
CS_{II}	$8(5 \times 10^{-5}) \times .5 + 2(9 \times 10^{-4}) \times 1 = 2.00 \times 10^{-3}$
CS_{III}	$8(5 \times 10^{-5}) \times .5 + 2(10 \times 10^{-4}) \times 1 = 2.20 \times 10^{-3}$

Adding the above costs (grounding plus repair) gives the cost of maintenance per flight hour.

Control System	Maintenance Cost/Flight Hour ($)
$Min_{dispatch}$	4.08
CS_I	2.53
CS_{II}	3.80
CS_{III}	2.21

The last system is clearly cheaper to maintain, but it will be the most expensive. Assume that an extra memory and processor costs $5,000 and that adding an extra bus, 3 extra sensors, and 4 extra actuators costs $10,000. Then the extra cost is

Control System	Additional Cost
$Min_{dispatch}$	0
CS_I	5,000
CS_{II}	10,000
CS_{III}	15,000

If the additional cost is spread over 12 years with 10 flight hours per day (43,800 flight hours) these additional costs become

$.114 $.228 and $.342

per flight hour for CS_I, CS_{II}, and CS_{III}, respectively. These costs must be paid initially and thus it might be more accurate to amortize the initial investment over the 12-year period assuming an 8% interest rate. In this case the additional cost per flight hour is increased by a factor

$$\frac{r}{1-\left(\frac{1}{1+r}\right)^n} = \frac{.08}{1-\left(\frac{1}{1.08}\right)^{12}} = 1.59 ,$$

and the values become

$.182 $.364 and $.546

per flight hour for CS_I, CS_{II}, and CS_{III}, respectively.

Now consider the total difference in costs for the four systems (maintenance cost per flight hour plus initial investment per flight hour).

Control System	Δ Cost/Flight Hour ($)	
	Amortized	Not Amortized
$Min_{dispatch}$	4.08	4.08
CS_I	2.71	2.64
CS_{II}	4.16	4.03
CS_{III}	2.76	2.55

Thus CS_I is optimal if we consider the cost of financing or of having money invested in equipment; CS_{III} is optimal if we ignore this factor. This illustrates how accounting procedures could have a significant effect.

The above example is meant only for illustrative purposes. The analysis is too simple to be successful with a real control system.

APPENDIX B

Uncoupling and Sparse Matrices: An Example

A simple specific example of a stochastic process is presented which may lend itself to approximate uncoupled analyses and also to sparse matrix techniques. It is intended to be used as an example for developing methodologies in the above areas.

Consider a bivariate continuous-time process $\{X(t), Y(t), t \geq 0\}$; the process is Markovian. Let n be an integer time point. In the interval $[n, n+1)$ the transitions of X and Y are independent of each other. Here X is a pure-death process on $[n, n+1)$ with state space $\{0,1,2,\ldots,b\}$, and X can be thought of in connection with a b component system: $X(t)$ = number of functioning units in the system at time t . The transition intensities for X are

$$P\{X(t+\Delta t) = i-1 \mid X(t) = i\} = i\lambda\Delta t ,$$

for $n \leq t < t+\Delta t < n+1$; $i=1,2,\ldots,a$; 0 is an absorbing state during $[n, n+1)$.

Similarly let Y be a pure-birth process on $[n, n+1)$ with state space $\{0,1,2,\ldots,c\}$. Here Y can be thought of as the number of components in a spares pool. If the repair shop that fixed the failed components from the X-process has c repairmen, then the transition intensities of Y are

$$P\{Y(t+\Delta t) = i+1 \mid Y(t) = i\} = (c-i)\mu\Delta t ,$$

for $n \leq t < t+\Delta t < n+1$; $i=0,1,2,\ldots,c$.

At integer time points the system whose state is described by the X-process is repaired by replacing failed components with good components from the spares pool. This transition is completely deterministic.

$$\left(X(n^-),\ Y(n^-)\right) \to \left(X(n),\ Y(n)\right) ,$$

where

$$X(n) = X(n^-) + r \quad , \quad Y(n) = Y(n^-) - r$$

and

$$r = \min\left(b - X(n^-),\ Y(n^-)\right) .$$

The process (X,Y) incorporates some of the features of the more general processes in which we are interested and thus may be a useful example for studying the possibilities of uncoupled analysis. We consider a special case $(b=1,\ c=1)$ to show that uncoupled analysis gives incorrect answers. In future work it may be possible to determine the approximate size of the error introduced by this approach in more general situations.

Consider the special case of (X,Y) with $X = 0$ or 1 and $Y = 0$ or 1. At integer time the process can be in one of four states, $(0,0)$, $(0,1)$, $(1,0)$, or $(1,1)$. Suppose that during one unit of time a good unit in the system described by X has probability p of failing and a unit in the repair shop has probability q of being repaired. Then in the interval $[n, n+1)$ the following transition probabilities hold:

$$P\{X(n+1^-)=0 \mid X(n)=1\} = p$$

$$P\{X(n+1^-)=1 \mid X(n)=1\} = 1-p$$

$$P\{X(n+1^-)=0 \mid X(n)=0\} = 1$$

$$P\{X(n+1^-)=1 \mid X(n)=0\} = 0$$

and

$$P\{Y(n+1^-)=0 \mid Y(n)=1\} = 0$$

$$P\{Y(n+1^-)=1 \mid Y(n)=1\} = 1$$

$$P\{Y(n+1^-)=0 \mid Y(n)=0\} = 1-q$$

$$P\{Y(n+1^-)=1 \mid Y(n)=0\} = q .$$

The joint transition matrix for $\left(X(n),\ Y(n)\right)$ to $\left(X(n+1^-),\ Y(n+1)^-\right)$ is

$$P = \begin{pmatrix} 1-q & q & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p(1-q) & pq & (1-p)(1-q) & (1-p)q \\ 0 & p & 0 & 1-p \end{pmatrix}.$$

The joint transition matrix for $\left(X(n^-), Y(n^-)\right)$ to $\left(X(n), Y(n)\right)$ is

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The joint transition matrix for $\left(X(n), Y(n)\right)$ to $\left(X(n+1), Y(n+1)\right)$ is

$$PQ = \begin{pmatrix} 1-q & 0 & q & 0 \\ 0 & 0 & 1 & 0 \\ p(1-q) & 0 & pq+(1-p)(1-q) & (1-p)q \\ 0 & 0 & p & 1-p \end{pmatrix}.$$

The steady-state probability vector for the process $\left\{\left(X(n), Y(n)\right), n=0,1,2,\ldots\right\}$ is the solution of

$$\vec{r} = \vec{r}PQ .$$

For the special case $p = .1$ and $q = .5$, the steady-state joint distribution of $\left(X(n), Y(n)\right)$ is

$$P\{X(n)=0,\ Y(n)=0\} = \frac{1}{56}$$

$$P\{X(n)=0,\ Y(n)=1\} = 0$$

$$P\{X(n)=1,\ Y(n)=0\} = \frac{10}{56}$$

$$P\{X(n)=1,\ Y(n)=1\} = \frac{45}{56} .$$

If an "uncoupled" analysis is performed in order to approximate the marginal steady-state probabilities of X and Y (an analysis similar to that described in 4.1), the following is obtained. Let

$$p_x = P\{X(n)=1\}, \quad 1 - p_x = P\{X(n)=0\}$$

$$p_y = P\{Y(n)=1\}, \quad 1 - p_y = P\{Y(n)=0\}$$

be the notation for marginal probabilities in steady state. The conditional transition probabilities for $X(n)$ to $X(n+1)$ are

$$P\{X(n+1)=0 \mid X(n)=1, Y(n)=0\} = p(1-q)$$

$$P\{X(n+1)=0 \mid X(n)=1, Y(n)=1\} = 0,$$

and so on; thus, when mixing using the assumed steady-state distribution of $Y(n)$ we get the "unconditional" transition probabilities

$$P\{X(n+1)=0 \mid X(n)=1\} = (1-p_y)p(1-q) + p_y \cdot 0$$

$$P\{X(n+1)=1 \mid X(n)=1\} = (1-p_y)[(1-p)+pq] + p_y \cdot 1$$

$$P\{X(n+1)=0 \mid X(n)=0\} = (1-p_y)(1-q) + p_y \cdot 0$$

$$P\{X(n+1)=1 \mid X(n)=0\} = (1-p_y)q + p_y \cdot 1 .$$

Similar "unconditional" steady-state transition probabilities can be computed for Y :

$$P\{Y(n+1)=0 \mid Y(n)=1\} = (1-p_x)1 + p_x \cdot p$$

$$P\{Y(n+1)=1 \mid Y(n)=1\} = (1-p_x)0 + p_x(1-p)$$

$$P\{Y(n+1)=0 \mid Y(n)=0\} = (1-p_x)1 + p_x[1-q+pq]$$

$$P\{Y(n+1)=1 \mid Y(n)=0\} = (1-p_x)0 + p_x[(1-p)q] .$$

For the special case $p = .1$ and $q = .5$, these transition values become

$$R_X = \begin{pmatrix} .5 - .5\,p_y & .5 + .5\,p_y \\ .05 - .05p_y & .95 + .05p_y \end{pmatrix}$$

$$R_Y = \begin{pmatrix} 1 - .45p_x & .45p_x \\ 1 - .9\,p_x & .9\,p_x \end{pmatrix}$$

for the marginal X process and Y process, respectively. The steady-state probability vectors were thought to satisfy

$$(1-p_x,\ p_x) = (1-p_x,\ p_x)R_X$$

$$(1-p_y,\ p_y) = (1-p_y,\ p_y)R_Y \ .$$

Solving the above system gives

$$p_y = \frac{-4 + \sqrt{340}}{18} = .8022$$

$$p_x = \frac{20}{9}\,\frac{p_y}{1+p_y} = .9891 \ .$$

The exact marginals for steady-state behavior are

$$P\big(Y(n)=1\big) = \frac{45}{56} = .8036$$

$$P\big(X(n)=1\big) = \frac{55}{56} = .9821 \ .$$

The accuracy of the approximation raises a glimmer of hope that the uncoupling technique may be useful.

The process (X,Y) may also be a useful example for investigating the application of sparse matrix techniques to solving for steady-state distributions. Note that the state space of the process has cardinality $(b+1)(c+1)$. A straightforward analysis would involve solving a system of $(b+1)(c+1)$ linear equations in $(b+1)(c+1)$ unknowns: a tedious task for moderate values of b and c , an impossible task for large values of b and c . However, recalling the underlying stochastic behavior of the system, recall that there are $(b+1)(b+2)/2$ possible transitions of $X(n)$ to $X(n+1^-)$ and $(c+1)(c+2)/2$ possible transitions of $Y(n)$ to $Y(n+1^-)$, and that these transitions are independent of each other and therefore joint probabilities contain no more information than the marginals. The transition of $\big(X(n^-),\ Y(n^-)\big)$ to $\big(X(n),\ Y(n)\big)$ is deterministic and thus consists of $(b+1)(c+1)$ bits of information. Thus the transition probabilities of $\big(X(n),\ Y(n)\big)$ to $\big(X(n+1),\ Y(n+1)\big)$ actually are generated by

$$\frac{(b+1)(b+2)}{2} + \frac{(c+1)(c+2)}{2} + (b+1)(c+1) = \frac{(c+b+2)(c+b+3)}{2}$$

values, which is much smaller than the number of entries in the transition matrix,

$$[(b+1)(c+1)]^2 \ .$$

For example, if $b = c = 10$, we are comparing 253 with 14641. It seems plausible that some sparse matrix techniques might be used to exploit this characteristic.