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THE DEVELOPMENT OF THE POINCARÉ-SIMILAR ELEMENTS WITH THE TRUE ANOMALY AS THE INDEPENDENT VARIABLE

Advanced Mission Design Branch

MISSION PLANNING AND ANALYSIS DIVISION

National Aeronautics and Space Administration

LYNDON B. JOHNSON SPACE CENTER

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SHUTTLE PROGRAM

THE DEVELOPMENT OF THE POINCARÉ-SIMILAR ELEMENTS WITH THE TRUE ANOMALY AS THE INDEPENDENT VARIABLE

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1.0 INTRODUCTION

In reference 1, Scheifele established the Hamiltonian of the unperturbed two-body problem in extended phase space. Depending on the type of time transformation, eight canonical elements were developed with the true anomaly or the eccentric anomaly as the independent variable. These two new sets, DS(ϕ) and DS(u), however, contain singularities for small eccentricities and inclinations. In reference 2, these singularities are removed by a transformation from DS(u) to eight canonical PS(u) elements. As in the DS(u) variables, the PS(u) variables have the eccentric anomaly as the independent variable.

In reference 3, the DS(ϕ) variables are transformed to the PS(ϕ) elements to remove the singularities. However, no direct relation was established between the eight canonical PS(ϕ) elements and the Cartesian coordinates. It is the purpose of this report to establish those relations and to develop the perturbed equations of motion in the PS(ϕ) space. As will be seen, the relationships are not trivial to establish; however, once obtained, they are found to be rather simple expressions.

Lastly, this report will demonstrate the accuracy of this new set when it is applied to numerical orbit prediction problems.

2.0 TRANSFORMATION FROM DS(ϕ) TO PS(ϕ) VARIABLES

In reference 1, Scheifele presents the Delavnay-similar (DS) elements with the true anomaly as the independent variable - DS(ϕ). The angular variables are as follows.

ϕ = the true anomaly.

g = the argument of perifocus.

h = the argument of the ascending node.

ℓ = the time element.

The action variables are as follows.

Φ = conjugate to ϕ, related to two-body energy.

G = the total angular momentum.

H = the Z-component of the angular momentum.

L = the total energy (two-body energy plus perturbing potential).

The corresponding Hamiltonian with a perturbing potential V is given by

$$F = \Phi - \frac{\mu}{\sqrt{2L}} + \frac{r^2}{q} V \tag{1}$$

where $$q = G - \frac{1}{2}\Phi + \frac{\mu}{2\sqrt{2L}} \tag{2}$$

The equations of motion can be written as

$$\begin{aligned} \frac{d\ell'}{d\tau} &= \frac{\partial F}{\partial L'} \qquad \ell' = \phi,\ g,\ h,\ l \\ \frac{dL'}{d\tau} &= -\frac{\partial F}{\partial \ell'} \qquad L' = \Phi,\ G,\ H,\ L \end{aligned} \tag{3}$$

Note that in unperturbed motion ($V = 0$)

$$\begin{aligned} \phi &= \tau + \phi_o \\ \ell &= \frac{\mu}{(2L_o)^{3/2}} \tau + \ell_o \end{aligned} \tag{4}$$

where τ is the independent variable. All other variables remain constant.

The DS(ϕ) elements, however, become ill defined for vanishing eccentricities and inclinations. To remove these singularities, one should transform to a type of element similar to the Poincaré variables. These PS(ϕ) elements will be named $\underline{\sigma}$ (coordinates) and $\underline{\rho}$ (momenta). As in the DS(ϕ) elements, the PS(ϕ) elements have the true anomaly as the dependent variable. These elements should not be confused with the PS(u) variables presented in reference 2; however, in many aspects they are similar.

The PS(ϕ) elements can be defined by a canonical transformation from the DS(ϕ) elements by the generating function S.

$$S = \ell \rho_4 + (\phi + g + h)\rho_1 - \frac{1}{2}\rho_2^2 [\tan(g + h)] - \frac{1}{2}\rho_3^2 \tan(h) \tag{5}$$

where

$$\underline{\sigma} = \frac{\partial S}{\partial \underline{\rho}}, \quad L' = \frac{\partial S}{\partial \ell'}$$

and

$$\underline{\sigma}^T = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

$$\underline{\rho}^T = (\rho_1, \rho_2, \rho_3, \rho_4)$$

The transformation yields the coordinates,

$$\begin{aligned} \sigma_1 &= \phi + g + h \\ \sigma_2 &= -\sqrt{2(\phi - G)}\sin(g + h) \\ \sigma_3 &= -\sqrt{2(G - H)}\sin(h) \\ \sigma_4 &= \ell \end{aligned} \tag{6}$$

and the momentas,

$$\begin{aligned} \rho_1 &= \phi \\ \rho_2 &= \sqrt{2(\phi - G)}\cos(g + h) \\ \rho_3 &= \sqrt{2(G - H)}\cos(h) \\ \rho_4 &= L \end{aligned} \tag{7}$$

The Hamiltonian now written in PS(ϕ) elements reads

$$F = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} + \frac{r^2}{a} V \tag{8}$$

where

$$q = -\frac{1}{2}\left(\sigma_2{}^2 + \rho_2{}^2 - \rho_1 - \frac{\mu}{\sqrt{2\rho_4}}\right) \tag{9}$$

The equations of motion can be written as

$$\begin{aligned} \frac{d\underline{\sigma}}{d\tau} &= \frac{\partial F}{\partial \underline{\rho}} \\ \frac{d\underline{\rho}}{d\tau} &= -\frac{\partial F}{\partial \underline{\sigma}} \end{aligned} \tag{10}$$

For the solution in the unperturbed case ($V = 0$),

$$\begin{aligned} \sigma_1 &= \tau + \sigma_{10} \\ \sigma_4 &= \frac{\mu}{(2\rho_{40})^{3/2}}\,\tau + \sigma_{40} \end{aligned} \tag{11}$$

All other variables are constants.

3.0 EQUATIONS OF MOTION

In reference 4, a method is presented in which the equations of motion in extended phase space are extended to include forces that cannot be defined as derivatives of the potential. The equations of motions in Cartesian space may be written as

$$\ddot{\underline{x}} + \frac{\mu}{r^3} \underline{x} = - \frac{\partial V}{\partial \underline{x}} + \underline{P} = \underline{F} \tag{12}$$

where $\underline{F}$ is the perturbing force, V is the perturbing potential, and $\underline{P}$ is the force not expressed as a potential. If the unperturbed Hamiltonian is expressed as

$$F_o = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} \tag{13}$$

then the equations of motion in PS(ϕ) space now become

$$\frac{d\underline{\sigma}}{d\tau} = \frac{\partial F_o}{\partial \underline{\rho}} + V \frac{\partial (r^2/q)}{\partial \underline{\rho}} + \frac{r^2}{q} \left(\frac{\partial V}{\partial \underline{\rho}}\right) - \underline{R}$$

$$\frac{d\underline{\rho}}{d\tau} = V \frac{\partial (r^2/q)}{\partial \underline{\sigma}} + \frac{r^2}{q} \frac{\partial V}{\partial \underline{\sigma}} + \underline{S} \tag{14}$$

where $\underline{S}$ and $\underline{R}$ are the canonical forces and are defined as follows.

$$\underline{R} = \frac{r^2}{q} A^T \underline{P} , \quad A = \frac{\partial \underline{x}}{\partial \underline{\rho}}$$

$$\underline{S} = \frac{r^2}{q} B^T \underline{P} , \quad B = \frac{\partial \underline{x}}{\partial \underline{\sigma}} \tag{15}$$

$$\frac{\partial V}{\partial \underline{\rho}} = A^T \frac{\partial V}{\partial \underline{x}}$$

$$\frac{\partial V}{\partial \underline{\sigma}} = B^T \frac{\partial V}{\partial \underline{x}}$$

where

$$\underline{X} = (x_1,\ x_2,\ x_3,\ t = \text{time})$$

$$\underline{P}^T = (P_1,\ P_2,\ P_3,\ P_4)$$

$$P_4 = -\left(\dot{x}_1 \cdot P_1 + \dot{x}_2 \cdot P_2 + \dot{x}_3 \cdot P_3\right)$$

$$\dot{x}_1,\ \dot{x}_2,\ \dot{x}_3 = \text{velocity vector}$$

Equation 14 can now be written as

$$\frac{d\underline{\sigma}}{d\tau} = \frac{\partial F_o}{\partial \underline{\rho}} + V \frac{\partial\left(\frac{r^2}{q}\right)}{\partial \underline{\rho}} + \frac{r^2}{q} A^T \left(\frac{\partial V}{\partial \underline{X}} - \underline{P}\right)$$

$$\frac{d\underline{\rho}}{d\tau} = -V \frac{\partial\left(\frac{r^2}{q}\right)}{\partial \underline{\sigma}} - \frac{r^2}{q} B^T \left(\frac{\partial V}{\partial \underline{X}} - \underline{P}\right) \qquad (16)$$

The matrices A and B can be found from the equations relating the Cartesian coordinates and time in terms of $\underline{\rho}$ and $\underline{\sigma}$. Here, also, it is worthy to note that

$$\frac{dt}{d\tau} = \frac{r^2}{q} \qquad (16a)$$

is the time transformation equation. <u>However, this equation is not used to integrate to find the time.</u>

4.0 CARTESIAN COORDINATES (x_1, x_2, x_3, t, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$) IN TERMS OF PS(ϕ) ELEMENTS ($\underline{\rho}$, $\underline{\sigma}$)

Although the PS(ϕ) elements are uniquely defined by equations (6) and (7), their relations with the Cartesian state vector must be given directly, without making use of the DS(ϕ) elements, since ϕ and g are numerically ill defined for near-circular orbits and h is ill defined for small inclination orbits. However, the DS(ϕ) elements G and H may be used occasionally as abbreviations since they are always well defined.

From reference 1, the coordinates and the time can be obtained from the DS(ϕ) elements as

$$\begin{aligned}
x_1 &= r\,[\cos(\phi + g)\cos h - \sin(\phi + g)\sin h \cos I] \\
x_2 &= r\,[\cos(\phi + g)\sin h + \sin(\phi + g)\cos h \cos I] \\
x_3 &= r \sin(\phi + g)\sin I \\
x_4 &= t = \ell + \frac{\mu}{(2L)^{3/2}} \left(E - \phi - \frac{r}{p} \sqrt{1 - e^2}\, e \sin \phi\right)
\end{aligned} \tag{17}$$

where

$$r = \frac{p}{1 + e \cos \phi} \tag{18}$$

$$p = \frac{1}{\mu}\left(G - \phi + \frac{\mu}{2L}\right)^2 \tag{19}$$

$$\sqrt{1 - e^2} = \sqrt{\frac{2L}{\mu}\, p} \tag{20}$$

$$\cos I = \frac{H}{G} \tag{21}$$

$$E = 2 \arc\tan\left(\sqrt{\frac{1 - e}{1 + e}} \tan \frac{\phi}{2}\right) \tag{22}$$

Note here that these notations are used as abbreviations in the DS(ϕ) theory to transform from the coordinates to DS(ϕ) elements. Although they are similar in form to Keplerian elements, they are the same only when the perturbing potential $V = 0$.

By trigonometric identities, equations (17) may be rewritten as

$$x_1 = r \sin^2 \frac{I}{2} \cos\,[h - (\phi + g)] + r \cos^2 \frac{I}{2} \cos(\phi + g + h)$$

$$x_2 = r \sin^2 \frac{I}{2} \sin [h - (\phi + g)] + r \cos^2 \frac{I}{2} \sin(\phi + g + h) \tag{23}$$

$$x_3 = 2 r \cos \frac{I}{2} \sin \frac{I}{2} \sin (\phi + g)$$

From equations (5) and (6), it is easily shown that

$$\phi - G = \frac{1}{2} (\rho_2{}^2 + \sigma_2{}^2)$$

and (24)

$$G - H = \frac{1}{2} (\rho_3{}^2 + \sigma_3{}^2)$$

Then, using equations (21) and (24), we find

$$\sqrt{2(G - H)} = 2 \sqrt{G} \sin \frac{I}{2} = \sqrt{\rho_3{}^2 + \sigma_3{}^2} \tag{25}$$

Equation 19 may be written as

$$P = \frac{1}{\mu} \left[- \frac{1}{2} (\rho_2{}^2 + \sigma_2{}^2) + \frac{\mu}{\sqrt{2\rho_4}} \right]^2 \tag{26}$$

From the identity

$$\sin \sigma_1 \sin (g + h) + \cos \sigma_1 \cos(g + h) = \cos \phi$$

we find by multiplication with $\sqrt{2(\phi - G)}$

$$- \sigma_2 \sin \sigma_1 + \rho_2 \cos \sigma_1 = \sqrt{2(\phi - G)} \cos \phi \tag{27}$$

Similarly, one can derive

$$\rho_2 \sin \sigma_1 + \sigma_2 \cos \sigma_1 = \sqrt{2(\phi - G)} \sin \phi \tag{28}$$

From the identity

$$\sin \sigma_1 \sin h + \cos \sigma_1 \cos h = \cos (\phi + g)$$

one can derive

$$-\sigma_3 \sin \sigma_1 + \rho_3 \cos \sigma_1 = 2 \sqrt{G} \sin \frac{I}{2} \cos(\phi + g) \tag{29}$$

and, similarly,

$$\rho_3 \sin \sigma_1 + \sigma_3 \cos \sigma_1 = 2\sqrt{G} \sin \frac{I}{2} \sin(\phi + g) \tag{30}$$

Although ϕ is ill defined for small eccentricities, the right-hand sides of equations (27) and (28) remain well defined because multiplication factor $\sqrt{2(\phi - G)}$ becomes quite small. The same argument applies for equations (28) and 29) if the inclination is small where $|(\phi + g)|$ is ill defined.

By using equations (20), (27), and (28), one can derive expressions

$$e \cos \phi = Z_1 Q \tag{31}$$

$$e \sin \phi = Z_2 Q \tag{32}$$

where

$$Z_1 = \rho_2 \cos \sigma_1 - \sigma_2 \sin \sigma_1 \tag{33}$$

$$Z_2 = \rho_2 \sin \sigma_1 + \sigma_2 \cos \sigma_1 \tag{34}$$

and

$$Q = \frac{1}{\mu} \left[\rho_4 \left(\frac{2\mu}{\sqrt{2\rho_4}} - \frac{1}{2}\left(\sigma_2^2 + \rho_2^2\right)\right)\right]^{1/2} \tag{35}$$

There is no numerical problem in computing Q except in the case where

$$e \approx 1 \quad \text{or} \quad \frac{2\mu}{\sqrt{2\rho_4}} \approx \frac{1}{2} (\sigma_2^2 + \rho_2^2)$$

We can now make use of equations (6), (7), (29), and (30) to substitute in equations (23).

$$\begin{aligned} x_1 &= -R^* \sigma_3 + r \cos \sigma_1 \\ x_2 &= -R^* \rho_3 + r \sin \sigma_1 \\ x_3 &= R^* \sqrt{2(G + H)} \end{aligned} \tag{36}$$

where

$$R^* = \frac{rR}{2G} \tag{37}$$

and

$$R = \rho_3 \sin \sigma_1 + \sigma_3 \cos \sigma_1 \tag{38}$$

Note that the second terms in x_1 and x_2 are dominant for small eccentricities and inclinations.

The radius,

$$r = \frac{p}{1 + e \cos \phi} \tag{39}$$

is evaluated by using equations (26) and (31).

The time equation,

$$x_4 = t = \sigma_4 + \frac{\mu}{(2\rho_4)^{3/2}} \left(E - \phi - \frac{r}{p} \sqrt{1 - e^2}\, e \sin \phi\right) \tag{40}$$

is evaluated by using equations (20) and (32) and by

$$E - \phi = -2 \arctan \left(\frac{e \sin \phi}{1 + \sqrt{1 - e^2} + e \cos \phi}\right) \tag{41}$$

We may now proceed to the computation of the velocity components. In the case of the DS(ϕ) elements, the velocity components were preferably computed by forming the time derivatives of the transformation equations for the coordinates. The reason was to avoid the numerical difficulties that are encountered at some points of the orbit. These numerical difficulties were caused by the intermediate step of computing the spherical coordinates.

The time derivatives of the DS(ϕ) elements are different from zero and depend on the current value of the perturbing potential, V_c. We denote the "unperturbed orbit" that is obtained from the Hamiltonian,

$$F_{DS} = \phi - \frac{\mu}{\sqrt{2L}} + \frac{r^2}{G} V_c \tag{42}$$

where V_c is a constant, the "DS orbit."

Because g, h, ℓ, and H do not appear in the Hamiltonian, the DS(ϕ) orbit has the following properties.

$$G = \text{const.}, \quad H = \text{const.}, \quad L = \text{const.}, \quad h = \text{const.} \tag{43}$$

This means that the orbital plane and the total energy remain unchanged or that G, H, L, and h are osculating elements. By examination of equation (17), one must conclude that, in fact, the quantity $\phi + g$ is osculating. Thus, from the angular momentum law,

$$\frac{d\omega}{dt} = \frac{d(\phi + g)}{dt} = \frac{G}{r^2} \tag{44}$$

If these facts are taken into account in the PS(ϕ) element system, we can conclude from equations (6) and (7) that

$$\dot{\rho}_3 = 0, \quad \dot{\sigma}_3 = 0, \quad \dot{\sigma}_1 = \frac{G}{r^2}, \quad \dot{\rho}_4 = 0 \tag{45}$$

where $(\dot{\ }) = \frac{d(\ )}{dt}$.

The time derivatives of the remaining PS(ϕ) elements can be computed by establishing the canonical equations to the Hamiltonian.

$$F_{PS} = \rho_1 - \frac{\mu}{\sqrt{2\rho_4}} + \frac{r^2}{a} V_c \tag{46}$$

$$q = -\frac{1}{2}\left(\sigma_2{}^2 + \rho_2{}^2 - \rho_1 - \frac{\mu}{\sqrt{2\rho_4}}\right) \tag{47}$$

First, the equations for ρ_2 and σ_2 are established.

$$\frac{d\rho_2}{d\tau} = \frac{\partial F_{PS}}{\partial\sigma_2} = -V_c \frac{\partial\left(\frac{r^2}{a}\right)}{\partial\sigma_2} \tag{48}$$

$$\frac{d\sigma_2}{d\tau} = \frac{\partial F_{PS}}{\partial\rho_2} = V_c \frac{\partial\left(\frac{r^2}{a}\right)}{\partial\rho_2} \tag{49}$$

Noting equation (10a), we find

$$\dot{\rho}_2 = \frac{d\rho_2}{d\tau} \cdot \frac{d\tau}{dt} = -V_c \frac{a}{r^2} \frac{\partial\left(\frac{r^2}{a}\right)}{\partial\sigma_2} \tag{50}$$

$$\dot{\sigma}_2 = \frac{d\sigma_2}{d\tau} \cdot \frac{d\tau}{dt} = V_c \frac{a}{r^2} \frac{\partial\left(\frac{r^2}{a}\right)}{\partial\sigma_2} \tag{51}$$

However, the time derivatives of ρ_2 and σ_2 are not needed directly but as combinations - as are

$$\sigma_2 \dot{\sigma}_2 + \rho_2 \dot{\rho}_2 \tag{52}$$

Before we proceed to the computation of expression (52), we will observe that if ξ is a function of the expression $\sigma_2{}^2 + \rho_2{}^2$ -

$$\xi = \xi(\sigma_2{}^2 + \rho_2{}^2) \tag{53}$$

and does not depend on σ_2 or ρ_2 in any other fashion, then

$$\sigma_2 \frac{\partial\xi}{\partial\rho_2} - \rho_2 \frac{\partial\xi}{\partial\sigma_2} = 0 \tag{54}$$

for symmetrical reasons

Observe that p, q, and Q are functions of the type found in equation (53).

Thus,

$$\sigma_2\dot{\sigma}_2 + \rho_2\dot{\rho}_2 = V_c \frac{q}{r^2} \left[\sigma_2 \frac{\partial\left(\frac{r^2}{q}\right)}{\partial \rho_2} - \rho_2 \frac{\partial\left(\frac{r^2}{q}\right)}{\partial \sigma_2}\right]$$

$$= V_c \frac{q}{r^2} \left[2 \frac{r}{q} \left(\sigma_2 \frac{\partial r}{\partial \rho_2} - \rho_2 \frac{\partial r}{\partial \sigma_2}\right) - \frac{r^2}{q} \left(\underbrace{\sigma_2 \frac{\partial q}{\partial \rho_2} - \rho^2 \frac{\partial q}{\partial \sigma_2}}_{0}\right)\right]$$

Noting equations (32) and (39), we find

$$\sigma_2\dot{\sigma}_2 + \rho_2\dot{\rho}_2 = V_c \frac{2}{r} \left\{\frac{r}{p} \left(\underbrace{\sigma_2 \frac{\partial p}{\partial \rho_2} - \rho_2 \frac{\partial p}{\partial \sigma_2}}_{0}\right) - \frac{r^2}{p} \left[\sigma_2 \frac{\partial (e \cos \phi)}{\partial \rho_2} - \rho_2 \frac{\partial (e \cos \phi)}{\partial \rho_2}\right]\right\}$$

$$= -V_c \frac{r}{p} \left[Z_1 \left(\underbrace{\sigma_2 \frac{\partial Q}{\partial \rho_2} - \rho_2 \frac{\partial Q}{\partial \sigma_2}}_{0}\right) + Q \left(\sigma_2 \cos \sigma_1 + \rho_2 \sin \sigma_1\right)\right]$$

$$= -\frac{2r}{p} V_c Q \left(\sigma_2 \cos \sigma_1 + \rho_2 \sin \sigma_1\right) \qquad (55)$$

Next, let us compute $\dot{\rho}_1$, where we note that σ_1 occurs in r only by $e \cos \phi$ in equation (31).

$$\dot{\rho}_1 = \frac{q}{r^2} \frac{\partial F_{PS}}{\partial \sigma_1} = -\frac{q}{r^2} V_c \frac{\partial}{\partial \sigma_1} \left(\frac{r^2}{q}\right) = \frac{-2}{r} V_c \frac{\partial r}{\partial \sigma_1}$$

$$= \frac{2r}{p} V_c \frac{\partial (e \cos \phi)}{\partial \sigma_1} = \frac{-2r}{p} V_c Q \left(\rho_2 \sin \sigma_1 + \sigma_2 \cos \sigma_1\right) \qquad (56)$$

Finally, we proceed to the computation of the time derivative of the distance, $\dot{r}$, on a "PS orbit." For this purpose, it is convenient to give the derivative of $\frac{r^2}{q}$, which is

$$\left(\dot{\overline{\frac{r^2}{q}}}\right) = \frac{2r}{q}\,\dot{r} - \frac{r^2}{q^2}\,\dot{q} \tag{57}$$

$$\left(\dot{\overline{\frac{r^2}{q}}}\right) = \frac{q}{r^2}\,\frac{d\left(\frac{r^2}{q}\right)}{d\tau} = \frac{q}{r^2}\sum_{k=1}^{4}\left[\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_k}\frac{d\sigma_k}{d\tau} + \frac{\partial\left(\frac{r^2}{q}\right)}{\partial\rho_k}\frac{d\rho_k}{d\tau}\right]$$

$$= V_c\,\frac{q}{r^2}\sum_{k=1}^{4}\left[\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_k}\underbrace{\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\rho_k} - \frac{\partial\left(\frac{r^2}{q}\right)}{\partial\rho_k}}_{0}\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_k}\right]$$

$$+ \frac{q}{r^2}\,\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_1} + \underbrace{\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_4}}_{0}\,\frac{\mu}{(2\rho_4)^{3/2}}$$

$$= \frac{2^r}{p}\,Q\,(\rho_2 \sin\sigma_1 + \sigma_2\cos\sigma_1) \tag{58}$$

To obtain $\dot{q}$, we use equations (55) and (56).

$$\dot{q} = \sum_{k=1}^{4}\left(\frac{\partial q}{\partial\rho_k}\,\dot{\rho}_k + \frac{\partial q}{\partial\sigma_k}\,\dot{\sigma}_k\right) = \frac{1}{2}\,\dot{\rho} - (\sigma_2\dot{\sigma}_2 + \rho_2\dot{\rho}_2)$$

$$= \frac{r}{p}\,V_c Q\,(\rho_2 \sin\sigma_1 + \sigma_2\cos\sigma_1) \tag{59}$$

Finally, equation (58) is solved for $\dot{r}$.

$$\dot{r} = \frac{r}{2q}\,\dot{q} + \frac{q}{2r}\left(\dot{\overline{\frac{r^2}{q}}}\right) \tag{60}$$

and expressions (59) and (60) are inserted, with the result that

$$\dot{r} = \frac{Q}{p}\left(V_c\,\frac{r^2}{q} + q\right)\left(\sigma_2\cos\sigma_1 + \rho_2\sin\sigma_1\right)$$

Noting equations (31) and (33), we find

$$\dot{r} = \frac{e \sin \phi}{P} \left(V_c \frac{r^2}{2q} + q \right) \tag{61}$$

In DS theory (and, accordingly, PS theory), one requires that the Hamiltonian vanish on the DS (or PS) orbit. Noting equation (46), one can see that for the Hamiltonian to vanish,

$$\frac{r^2}{q} V_c = \frac{\mu}{\sqrt{2\rho_4}} - \rho_1 \tag{62}$$

But from equation (47),

$$2q = \rho_1 + \frac{\mu}{\sqrt{2\rho_4}} - \left(\sigma_2^2 + \rho_2^2 \right)$$

$$2(q - \rho_1) + \sigma_2^2 + \rho_2^2 = \frac{\mu}{\sqrt{2\rho_4}} - \rho_1 \tag{63}$$

Inserting equations (62) and (63) into equation (61), we obtain the final form for $\dot{r}$.

$$\dot{r} = \frac{e \sin \phi}{P} \left[2q - \rho_1 + \frac{1}{2} \left(\sigma_2^2 + \rho_2^2 \right) \right] \tag{64}$$

We now can obtain the velocities by differentiating equations (36).

$$\begin{aligned} \dot{x}_1 &= -\dot{R}^* \sigma_3 + \dot{r} \cos \sigma_1 - r\dot{\sigma}_1 \sin \sigma_1 \\ \dot{x}_2 &= -\dot{R}^* \rho_3 + \dot{r} \sin \sigma_1 + r\dot{\sigma}_1 \cos \sigma_1 \\ \dot{x}_3 &= \dot{R}^* \sqrt{2(G + H)} \end{aligned} \tag{65}$$

where

$$\dot{R}^* = \frac{\dot{r}R + \dot{R}r}{2G}, \quad \dot{R} = \dot{\sigma}_1 \left(\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1 \right)$$

Noting equation (45) and grouping terms, we arrive at the final form of the velocities.

$$\dot{x}_1 = -\dot{R}^{*}\sigma_3 + \dot{r}\cos\sigma_1 - \frac{G}{r}\sin\sigma_1 \tag{66}$$

$$\dot{x}_2 = -\dot{R}^{*}\rho_3 + \dot{r}\sin\sigma_1 + \frac{G}{r}\cos\sigma_1 \tag{67}$$

$$\dot{x}_3 = \dot{R}^{*}\sqrt{2(G + H)} \tag{68}$$

$$\dot{R}^{*} = \frac{\dot{r}R + \dot{R}r}{2G}, \quad \dot{R} = \frac{G}{r^2}\left(\rho_3\cos\sigma_1 - \sigma_3\sin\sigma_1\right) \tag{69}$$

5.0 PS(ϕ) ELEMENTS ($\bar{\rho}$, $\bar{\sigma}$) IN TERMS OF THE CARTESIAN COORDINATES (x_1, x_2, x_3, t, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$)

From reference 3, we obtain

$$G_1 = x_2\dot{x}_3 - x_3\dot{x}_2 \tag{70}$$

$$G_2 = x_3\dot{x}_1 - x_1\dot{x}_3 \tag{71}$$

$$G_3 = x_1\dot{x}_2 - x_2\dot{x}_1 \tag{72}$$

$$G = \sqrt{G_1^2 + G_2^2 + G_3^2} \tag{73}$$

$$H = G_3 \tag{74}$$

$$L = \frac{\mu}{r} - \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - V \tag{75}$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{76}$$

Since the value of the value of the homogeneous Hamiltonian (eq.(1)) must vanish, one finds that, in DS(ϕ) theory,

$$\phi = G - \sqrt{G^2 - 2r^2 V} + \frac{\mu}{\sqrt{2L}} \tag{77}$$

From equation (7),

$$\rho_1 = \phi, \quad \rho_4 = L \tag{78}$$

Inserting these values in equations (2) and (19), one arrives at

$$q = G - \frac{1}{2}\phi + \frac{\mu}{2\sqrt{2L}} \tag{79}$$

$$p = \frac{1}{\mu}\left(G - \phi + \frac{\mu}{\sqrt{2L}}\right)^2 \tag{80}$$

And noting equation (24), we find that equation (35) can be expressed as

$$Q = \frac{1}{\mu}\left[\rho_4\left(\frac{\partial\mu}{\sqrt{2\rho_4}} + H - G\right)\right]^{1/2} \tag{81}$$

From expressions (31) and (39), one obtains

$$Z_1 = \frac{(P/r - 1)}{Q} \tag{82}$$

and if we note that

$$G = \rho_1 - \frac{1}{2}(\sigma_2{}^2 + \rho_2{}^2) \tag{83}$$

then one obtains from expressions (32) and (64)

$$Z_2 = \frac{\dot{r}P}{Q(2a - G)} \tag{84}$$

where

$$\dot{r} = \frac{\underline{x} \cdot \underline{\dot{x}}}{r} \tag{85}$$

By the definition of the angular momentum, its components can be written as

$$\begin{aligned} G_1 &= G \sin I \sin h \\ G_2 &= -G \sin I \cos h \\ G_3 &= G \cos I \end{aligned} \tag{36}$$

From expressions (6), (7), (25), and (86), one can derive

$$\sigma_3 = \frac{-2G_1}{[2(G + H)]^{1/2}} \tag{87}$$

$$\rho_3 = \frac{-2G_2}{[2(G + H)]^{1/2}} \tag{88}$$

By equations (36) and (37), we obtain

$$R = \frac{2G(x_3/r)}{[2(G + H)]^{1/2}} \tag{89}$$

$$r \cos \sigma_1 = x_1 + R^* \sigma_3 \tag{90}$$

$$r \sin \sigma_1 = x_2 + R^* \sigma_3 \tag{91}$$

and thus

$$\sigma_1 = \text{arc tan}\left(\frac{r \sin \sigma_1}{r \cos \sigma_1}\right) \tag{92}$$

Solving equations (33) and (34) for σ_2 and ρ_2, we find

$$\sigma_2 = Z_2 \cos \sigma_1 - Z_1 \sin \sigma_1 \tag{93}$$

$$\rho_2 = Z_2 \sin \sigma_1 + Z_1 \cos \sigma_1 \tag{94}$$

Noting equations (20), (31), (32), (40), and (41), one obtains the final element

$$\sigma_4 = t - \frac{\mu}{(2\rho_4)^{3/2}} \left(E - \phi - \frac{r}{p} Z_2 Q \sqrt{1 - e^2} \right) \tag{95}$$

where

$$E - \phi = -2 \text{ arc tan} \left[\frac{Z_2 \cdot Q}{(1 + \sqrt{1 - e^2} + Z_1 \cdot Q)} \right] \tag{96}$$

and

$$\sqrt{1 - e^2} = \sqrt{\frac{2L}{\mu}}\, P \tag{97}$$

A sequence of the computations that give the transformations to and from the elements and coordinates will be definitely outlined in the appendix.

6.0 MATRICES A, B, $\frac{\partial\left(\frac{r^2}{q}\right)}{\partial \underline{\rho}}$, AND $\frac{\partial\left(\frac{r^2}{q}\right)}{\partial \underline{\sigma}}$

To solve the equations of motion (eqs.(16)), one must evaluate the derivatives of the Cartesian position vector (eqs.(36)), time (eqs.(40)), (and the time transformation (eqs.(16a)) with respect to the PS(ϕ) variables $\underline{\sigma}$ and $\underline{\rho}$. The algebra involved in taking the partial derivatives is lengthy but straightforward and so will be omitted. The results are

$$A = \frac{\partial \underline{x}}{\partial \underline{\rho}}$$

$$a_{11} = \frac{\partial x_1}{\partial \rho_1} = \frac{\sigma_3 R^*}{G}$$

$$a_{12} = \frac{\partial x_1}{\partial \rho_2} = \frac{x_1}{r} \cdot \frac{\partial r}{\partial \rho_2} - \sigma_3 \rho_2 \frac{R^*}{G}$$

$$a_{13} = \frac{\partial x_1}{\partial \rho_3} = \frac{-\sigma_3 r}{2G} \sin \sigma_1$$

$$a_{14} = \frac{\partial x_1}{\partial \rho_4} = \frac{x_1}{r} \frac{2r}{2\rho_4}$$

$$a_{21} = \frac{\partial x_2}{\partial \rho_1} = \frac{\rho_3 R^*}{G}$$

$$a_{22} = \frac{\partial x_2}{\partial \rho_2} = \frac{x_2}{r} \cdot \frac{\partial r}{\partial \rho_2} - \frac{R^*}{G} \rho_3 \rho_2$$

$$a_{23} = \frac{\partial x_2}{\partial \rho_3} = - R^* - \frac{\rho_3 r}{2G} \sin \sigma_1 \qquad (98)$$

$$a_{24} = \frac{\partial x_2}{\partial \rho_4} = \frac{x_2}{r} \cdot \frac{\partial r}{\partial \rho_4}$$

$$a_{31} = \frac{\partial x_3}{\partial \rho_1} = -x_3 \cdot \frac{H}{G(G + H)}$$

$$a_{32} = \frac{\partial x_3}{\partial \rho_2} = \rho_2 x_3 \frac{H}{G(G + H)} + \frac{x_3}{r} \frac{\partial r}{\partial \rho_2}$$

$$a_{33} = \frac{\partial x_3}{\partial \rho_3} = \frac{\rho_3 x_3}{2G(G + H)} + x_2 \frac{\sqrt{2(G + H)}}{2G}$$

$$a_{34} = \frac{\partial x_3}{\partial \rho_4} = \frac{x_3}{r} \cdot \frac{\partial r}{\partial \rho_4}$$

$$a_{4k} = \frac{\partial x_4}{\partial \rho_k} = \frac{\mu}{(2\rho_4)^{3/2}} \left[\frac{\partial (E - \phi)}{\partial \rho_k} - \frac{r}{p} \left\{ \sqrt{1 - e^2} \ \frac{\partial (e \sin \phi)}{\partial \rho_k} \right. \right.$$

$$\left. \left. + e \sin \phi \frac{\partial \sqrt{1 - e^2}}{\partial \rho_k} + \frac{r}{p} \frac{\partial (e \cos \phi)}{\partial \rho_k} \sqrt{1 - e^2} \, e \sin \phi \right\} \right]$$

$$k = 1, 2, 3, 4$$

$$a_{44} = a_{44} - \frac{3}{2} \left(\frac{x_4 - \sigma_4}{\rho_4} \right)$$

$$B = \frac{\partial \underline{x}}{\partial \underline{\sigma}}$$

$$b_{11} = \frac{\partial x_1}{\partial \sigma_1} = \frac{x_1}{r} \frac{\partial r}{\partial \sigma_1} - r \sin \sigma_1 - \frac{\sigma_3 r}{2G} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1)$$

$$b_{12} = \frac{\partial x_1}{\partial \sigma_2} = \frac{x_1}{r} \frac{\partial r}{\partial \sigma_2} - \sigma_3 \sigma_2 \frac{R^*}{G}$$

$$b_{13} = \frac{\partial x_1}{\partial \sigma_3} = -R^* - \frac{\sigma_3 r}{2G} \cos \sigma_1$$

$$b_{14} = \frac{\partial x_1}{\partial \sigma_4} = 0$$

$$b_{21} = \frac{\partial x_2}{\partial \sigma_1} = \frac{x_2}{r} \frac{\partial r}{\partial \sigma_1} + r \cos \sigma_1 - \frac{r \rho_3}{2G} (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1)$$

$$b_{22} = \frac{\partial x_2}{\partial \rho_2} = \frac{x_2}{r} \cdot \frac{\partial r}{\partial \sigma_2} - \sigma_2 \cdot a_{21}$$

$$b_{23} = \frac{\partial x_2}{\partial \sigma_3} = \frac{-r}{2G} \rho_3 \cos \sigma_1 \qquad (99)$$

$$b_{24} = \frac{\partial x_2}{\partial \sigma_4} = 0$$

$$b_{31} = \frac{\partial x_3}{\partial \sigma_1} = \frac{r}{2G} \sqrt{2(G + H)} \; (\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1) + \frac{x_3}{r} \frac{\partial r}{\partial \sigma_1}$$

$$b_{32} = \frac{\partial x_3}{\partial \sigma_2} = -\sigma_2 \; a_{31} + \frac{x_3}{r} \; \frac{\partial r}{\partial \sigma_2}$$

$$b_{33} = \frac{\partial x_3}{\partial \sigma_2} = \frac{-\sigma_3}{2} \; a_{31} + \frac{x_1}{2G} \sqrt{2(G + H)}$$

$$b_{34} = \frac{\partial x_3}{\partial \sigma_4} = 0$$

$$b_{4k} = \frac{\partial x_4}{\partial \sigma_4} = \frac{\mu}{(2\rho_4)^{3/2}} \left[\frac{\partial (E - \phi)}{\partial \sigma_k} - \frac{r}{p} \left\{ \sqrt{1 - e^2} \; \frac{\partial (e \sin \phi)}{\partial \sigma_k} \right. \right.$$

$$\left. + \; e \sin \phi \; \frac{\partial \sqrt{1 - e^2}}{\partial \sigma_k} + \frac{r}{p} \; \frac{\partial (e \cos \phi)}{\partial \sigma_k} \; \sqrt{1 - e^2} \; e \sin \phi \right]$$

$$k = 1, 2, 3, 4$$

$$b_{44} = b_{44} + 1$$

The derivatives of the time transformation $T = \frac{r^2}{q}$ are

$$\frac{\partial T}{\partial \rho_1} = -\frac{1}{2}\left(\frac{r}{q}\right)^2$$

$$\frac{\partial T}{\partial \rho_2} = \frac{2r}{q} \; \frac{\partial r}{\partial \rho_2} + \left(\frac{r}{q}\right)^2 \rho_2$$

$$\frac{\partial T}{\partial \rho_3} = 0$$

$$\frac{\partial T}{\partial \rho_4} = \frac{2r}{q} \; \frac{\partial r}{\partial \rho_4} + \frac{\mu}{2(2\rho_4)^{3/2}} - \left(\frac{r}{q}\right)^2 \qquad (100)$$

$$\frac{\partial T}{\partial \sigma_1} = \frac{2r}{q} \; \frac{\partial r}{\partial \sigma_1}$$

$$\frac{\partial T}{\partial \sigma_2} = \frac{2r}{q} \; \frac{\partial r}{\partial \sigma_2} + \left(\frac{r}{q}\right)^2 \sigma_2$$

$$\frac{\partial T}{\partial \sigma_k} = 0 \qquad k = 3, 4$$

The derivatives $\frac{\partial r}{\partial \sigma_k}$ and $\frac{\partial r}{\partial \rho_k}$ are determined from equation (39).

$$\frac{\partial r}{\partial \rho_k} = 0 \qquad k = 1 \text{ and } 3$$

$$\frac{\partial r}{\partial \rho_2} = \frac{r}{P}\left[-2\rho_2 \sqrt{\frac{P}{\mu}} - r\left(Q \cos \sigma_1 - \frac{Z_1 \rho_4}{2\mu^2 Q} \rho_2\right)\right]$$

$$\frac{\partial r}{\partial \rho_4} = \frac{r}{P}\left[\frac{-2\mu}{(2\rho_4)^{3/2}} \sqrt{\frac{P}{\mu}} - rZ_1\left(\frac{Q}{2\rho_4} - \frac{Z_1}{2\mu Q(2\rho_4)^{1/2}}\right)\right] \tag{101}$$

$$\frac{\partial r}{\partial \sigma_k} = 0, \qquad k = 3 \text{ and } 4$$

$$\frac{\partial r}{\partial \sigma_1} = \frac{r^2}{P} Z_2 Q$$

$$\frac{\partial r}{\partial \sigma_2} = \frac{r}{P}\left[-2\sigma_2 \sqrt{\frac{P}{\mu}} - r\left(Q \cos \sigma_1 - \frac{Z_1 \rho_4 \sigma_2}{2\mu^2 Q}\right)\right]$$

The derivatives $\frac{\partial(E - \phi)}{\partial \sigma_k}$ and $\frac{\partial(E - \phi)}{\partial \rho_k}$ are determined from equation (41).

$$\frac{\partial(E - \phi)}{\partial \sigma_k} = \frac{-r/P}{(1 + \sqrt{1 - e^2})}\left\{(1 + \sqrt{1 - e^2} + e \cos \phi)\frac{\partial(e \sin \phi)}{\partial \sigma_k} - e \sin \phi\left[\frac{\partial \sqrt{1 - e^2}}{\partial \sigma_k} + \frac{\partial(e \cos \phi)}{\partial \sigma_k}\right]\right\}$$

$$\frac{\partial(E - \phi)}{\partial \rho_k} = \frac{-r/P}{(1 + \sqrt{1 - e^2})}\left\{(1 + \sqrt{1 - e^2} + e \cos \phi)\frac{\partial(e \sin \phi)}{\partial \rho_k} - e \sin \phi\left[\frac{\partial \sqrt{1 - e^2}}{\partial \rho_k} + \frac{\partial(e \cos \phi)}{\partial \rho_k}\right]\right\} \tag{102}$$

$$k = 1, 2, 3, 4$$

The derivatives $\frac{\partial(e \cos \phi)}{\partial \sigma_k}$, $\frac{\partial(e \sin \phi)}{\partial \sigma_k}$, $\frac{\partial(e \cos \phi)}{\partial \rho_k}$, and

$\frac{\partial(e \sin \phi)}{\partial \rho_k}$ are derived from equations (31), and (32).

$$
\begin{aligned}
\frac{\partial(e \sin \phi)}{\partial \rho_k} &= 0, \qquad k = 1 \text{ and } 3 \\
\frac{\partial(e \sin \phi)}{\partial \rho_2} &= Q \sin \sigma_1 - \frac{Z_2 \rho_4 \rho_2}{2\mu^2 Q} \\
\frac{\partial(e \sin \phi)}{\partial \rho_4} &= Z_2 \left[\frac{Q}{2\rho_4} - \frac{1}{2\mu Q (a\rho_4)^{1/2}} \right] \\
\frac{\partial(e \sin \phi)}{\partial \sigma_k} &= 0, \qquad k = 3 \text{ and } 4 \\
\frac{\partial(e \sin \phi)}{\partial \sigma_1} &= e \cos \phi \\
\frac{\partial(e \sin \phi)}{\partial \sigma_2} &= Q \cos \sigma_1 - Z_2 \frac{\rho_4 \sigma_2}{2\mu^2 Q} \\
\frac{\partial(e \cos \phi)}{\partial \rho_k} &= 0, \qquad k = 1 \text{ and } 3 \\
\frac{\partial(e \cos \phi)}{\partial \rho_2} &= Q \cos \sigma_1 - \frac{Z_1 \rho_4 \rho_2}{2\mu^2 Q} \\
\frac{\partial(e \cos \phi)}{\partial \rho_4} &= Z_1 \left[\frac{Q}{2\rho_4} - \frac{1}{2\mu Q (a\rho_4)^{1/2}} \right] \\
\frac{\partial(e \cos \phi)}{\partial \sigma_k} &= 0, \qquad k = 3 \text{ and } 4 \\
\frac{\partial(e \cos \phi)}{\partial \sigma_1} &= -e \sin \phi \\
\frac{\partial(e \cos \phi)}{\partial \sigma_2} &= -\left(Q \sin \sigma_1 + \frac{Z_1 \rho_4 \sigma_2}{2\mu^2 Q} \right)
\end{aligned}
\tag{103}
$$

And, finally, the derivatives $\frac{\partial \sqrt{1 - e^2}}{\partial \sigma_k}$ and $\frac{\partial \sqrt{1 - e^2}}{\partial \rho_k}$ are derived from equation (20).

$$\frac{\partial\sqrt{1 - e^2}}{\partial \sigma_k} = 0, \qquad k = 1,\ 3,\ \text{and}\ 4$$

$$\frac{\partial\sqrt{1 - e^2}}{\partial \sigma_2} = \frac{-\sigma_2}{\mu} \sqrt{2\rho_4}$$

$$\frac{\partial\sqrt{1 - e^2}}{\partial \rho_k} = 0, \qquad k = 1 \text{ and } 3 \tag{104}$$

$$\frac{\partial\sqrt{1 - e^2}}{\partial \rho_2} = \frac{-\rho_2}{\mu} \sqrt{2\rho_4}$$

$$\frac{\partial\sqrt{1 - e^2}}{\partial \rho_4} = \sqrt{\frac{P}{2\mu\rho_4}} - \frac{1}{\partial \rho_4}$$

7.0 NUMERICAL APPLICATIONS

In reference 2, a number of numerical experiments were performed, comparing the PS(u) formulation with the KS formulation described in reference 4. In reference 2, a complete description of the force models and the initial conditions for different orbits were given. These same experiments were performed except that the orbits were integrated with the use of PS(ϕ) elements.

7.1 Example 1

The first example is a highly eccentric orbit ($e = 0.95$) about the Earth. The satellite is subject to the perturbing conservative potential of the Earth's oblateness (J_2) and the perturbing forces due to the Moon.

As in reference 2, the PS(ϕ) elements were integrated with an RK-45 fixed-step method. The PS(u), KS, and PS(ϕ) solutions, at the end of 50 revolutions, were compared to an extremely accurate reference solution found in reference 4.

The problem description for the first example is as follows.

Coordinate system: X_1 X_2 fixed in Earth equatorial plane.

X_3 perpendicular to Earth equatorial plane.

Initial conditions: $x_1 = 0.0$ km, $x_2 = -5888.9727$ km

$x_3 = -3400.0$ km

$\dot{x}_1 = 10.691338$ km/sec

$\dot{x}_2 = 0$ km/sec, $\dot{x}_3 = 0$ km/sec

The time of comparison is at 288.12768941 days. The Earth oblateness and lunar perturbation models are described in section 9.0, parts a and b of reference 2.

TABLE I.- EXAMPLE 1

	x_1, km	x_2, km	x_3, km	Energy check	Steps/rev
Ref.	-24 219.0503	227 962.1064	129 753.4424		
PSu	8.2157	.0072	.3843	0.21×10^{-6}	40
KS	8.2221	.0079	.3883	$.20 \times 10^{-6}$	40
PS ϕ	43.7435	99.6752	74.1071	0.33×10^{-7}	40
PSu	.0422	.1066	.4421	$.22 \times 10^{-8}$	80
KS	.0422	.1070	.4420	$.11 \times 10^{-8}$	80
PS ϕ	21.2402	3.4532	4.2147	$.98 \times 10^{-9}$	80
PSu	.0499	.1064	.4424	$.18 \times 10^{-9}$	160
KS	.0499	.1064	.4424	$.14 \times 10^{-9}$	160
PS ϕ	.1738	.1694	.4783	$.42 \times 10^{-10}$	160

Note: The values shown are identical to the corresponding reference value except for the digits shown in the table. The energy check is the difference between the integrated value for the total energy and the total energy computed from position, velocity, and perturbing potential, divided by the integrated energy.

As one can see, the PS(u) and KS solutions produce similar results, whereas the PS(ϕ) solutions produces results that are not as accurate for the same step size. The reason for this is clear. When a highly eccentric satellite approaches the apogee point, the lunar perturbations become very pronounced. The PS(u) and KS both have independent variables that are similar to the eccentric anomaly, whereas the PS(ϕ) has the true anomaly as the independent variable. Therefore, near the apogee, the PS(u) and KS formulations are evaluating the strong lunar perturbations more often than the PS(ϕ) formulation for the simple reason of geometry.

7.2 Example 2

In the second example, a satellite is subject to the J_2 perturbations and the drag perturbations of the form described in reference 3, section 9.0 a and b. The satellite is in orbit about the Earth, with an eccentricity of $e = 0.174$. Again, the experiments were performed in the same manner as described in the reference. One can see from Table II that all formulations integrate this example easily.

The problem description for the second example is as follows.

Coordinate system: same as that for example 1

Initial conditions: x_1 = 0.0 km, x_2 = -5888.9727 km, x_3 = -3400.0 km

$\dot{x}_1$ = 8.3 km/sec, $\dot{x}_2$ = 0.0 km/sec, $\dot{x}_3$ = 0.0 km/sec

The time of comparison is at 4.30899150 days, after approximately 50 revolutions. The oblateness and drag perturbations were computed from the equations in parts a and c of section 9, reference 2.

TABLE II.- EXAMPLE 2

	x_1, km	x_2, km	x_3, km	Energy	Steps/rev
Ref.	819.9225	-5960.6168	-3175.3500	0.70×10^{-12}	160
PSu	.9225	.6168	.3500	$.27 \times 10^{-8}$	30
KS	.9225	.6168	.3500	$.21 \times 10^{-8}$	30
PS ϕ	.9225	.6168	.3500	$.74 \times 10^{-9}$	30
PSu	.9225	.6168	.3500	$.22 \times 10^{-9}$	50
KS	.9225	.6168	.3500	$.18 \times 10^{-9}$	50
PS ϕ	.9225	.6168	.3500	$.70 \times 10^{-10}$	50

7.3 Example 3

In this last example, a particle is placed at the libration point in the Earth-Moon system. If the Earth and the Moon are assumed to move in circular orbits about their center of mass and if an infinitesimal mass is placed 60° ahead of or behind the Moon at the same radius and circular speed as the Moon, then from the theory of the restricted three-body problem, the mass will remain at either of these points indefinitely. If the particle is placed there with a small initial position error, then the particle will librate about that point. The motion of such a particle is computed by the PS(ϕ), PS(u), and KS methods. As before, the experiment is the same as described in reference 2, section 9.3.

The problem description is as follows.

Coordinate system: X_1 and X_2, in Earth-Moon plane; X_3, perpendicular

Initial conditions: x_1 = 192 300 km x_2 = 332 900.1652 km, x_3 = 0 km

$\dot{x}_1$ = -0.8872840638 km/sec, $\dot{x}_2$ = 0.5125402247 km/sec

$\dot{x}_3$ = 0 km/sec

The final time is 10 000 days. The lunar ephemeris used in this example is given in part d, section 9, of reference 2.

TABLE III.- EXAMPLE 3

	x_1, km	x_2, km	r, km	Energy check	Steps/rev
Ref.	-178 094.8956	-340 462.3726	384 229.6435	0.50×10^{-13}	80
PSu	5.4693	1.9627	.5463	$.12 \times 10^{-7}$	10
KS	6.4119	.8837	30.7993	$.72 \times 10^{-5}$	10
PSϕ	5.0474	.0933	.4664	$.12 \times 10^{-7}$	10
PSu	.9156	.3688	.6495	$.25 \times 10^{-9}$	20
KS	.9470	.3989	.6907	$-.24 \times 10^{-6}$	20
PSϕ	.9036	.3753	.6497	$.25 \times 10^{-9}$	20
PSu	.8963	.3726	.6439	$.36 \times 10^{-11}$	40
KS	.8972	.3735	.6451	$.75 \times 10^{-8}$	40
PSϕ	.8958	.3729	.6439	$.38 \times 10^{-9}$	40

From table III, one finds that the PS(ϕ) and PS(u) formulations give more accurate solutions than the KS method, for step sizes of 10, 20, and 40 steps/rev.

8.0 CONCLUSIONS

The differential equations for the perturbed two-body elliptical motion are defined in elements that are similar to the two-body classical canonical Poincaré elements, which have time as the independent variable. A new Hamiltonian is defined that enables the introduction of two more canonical variables, the total energy and the time element. Also, these new elements have as their independent variable, the generalized true anomaly.

The accuracy of the solution of this new set is equal to that of the PS(u) or KS formulations except for the highly eccentric case where the PS(ϕ) is at a disadvantage because of geometric reasons.

REFERENCES

1. Scheifele, G.: On Nonclassical Cononical Systems. Celestial Mechanics (Reidel Publishing Co., Dordrecht, Holland); vol. 2, 1970, pp. 296-310.

2. Bond, V. R.: Development of the Poincaré-Similar Elements with True Anomaly as the Independent Variable. JSC IN 75-FM-16, Mar. 1975.

3. Scheifele, G.; and Graf, O.: Analytical Satellite Theories Based On a New Set of Canonical Elements. AIAA Paper No. 74-838, Aug. 1974.

4. Stiefel, E.; and Scheifele, G.: Linear and Regular Celestial Mechanics. Springer-Verlag (New York, Heidelberg, Berlin), 1971.

APPENDIX

COMPUTATIONAL PROCEDURE

APPENDIX

COMPUTATIONAL PROCEDURE

In the numerical integration of a trajectory by means of the PS(ϕ) elements, one may follow the procedure described below. Initially, the position and velocity vectors are converted to the PS(ϕ) elements (COTOPS). The numerical integrator will evaluate the PS(ϕ) differential equations (PSDEEQ) and estimate the value of the PS(ϕ) elements at $\tau = \tau_0 + \Delta\tau$, the independent variable. In PSDEEQ, one must evaluate the value of the perturbing forces that are usually given as a function of the Cartesian state vector and time. Therefore, PSDEEO must convert the PS$^\phi$ elements into the coordinates (PSTOCO). The algorithms for COTOPS, PSDEEQ, and PSTOCO are outlined in the following sections. The left column gives the quantity to be computed, and the right column references the equation number in the text.

COTOPS

Given $\underline{x}$, $\dot{\underline{x}}$, and time, transform to $\underline{\sigma}$ and $\underline{\rho}$.

Evaluate the potential, V.

Then sequentially compute:		From equation
*	$\rho_4 = L$	(75)
	G_1, G_2, G_3	(70)-(72)
	G_3	(73)
	H	(74)
*	$\rho_1 = \phi$	(77)
	q	(79)
	P	(80)
	Q	(81)
*	σ_3	(87)
*	ρ_3	(88)
	R	(89)
	$r \cos \sigma_1$	(90)
	$r \sin \sigma_1$	(91)
*	σ_1	(92)
	$\dot{r}$	(85)

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Z_1	(82)
Z_2	(84)
* σ_2	(93)
* ρ_2	(94)
$E - \phi$	(96)
$\sqrt{1 - e^2}$	(97)
* σ_4	(95)

PSTOCO

Given $\underline{\sigma}$, $\underline{\rho}$ transform to x, $\dot{x}$, t = time.

Sequentially compute:	From equation
Z_1	(33)
Z_2	(34)
Q	(35)
$e \cos \phi$	(31)
$e \sin \phi$	(32)
P	(26)
r	(39)
R	(38)
G	(83)
$H = G - (G - H)$	(24)
x_1, x_2, x_3	(36)
$\sqrt{1 - e^2}$	(20)
$E - \phi$	(41)
t	(40)
q	(9)
$\dot{r}$	(64)
$\dot{R}$*	(69)
$\dot{x}_1, \dot{x}_2, \dot{x}_3$	(66)-(68)

Given $\underline{\sigma}$, $\underline{\rho}$, compute $\frac{d\sigma}{d\tau}$, $\frac{d\rho}{d\tau}$.

Proceed to PSTOCO.

Then evaluate $\frac{\partial V}{\partial \underline{x}}$, $\frac{\partial V}{\partial t}$, V, P_1, P_2, and P_3.

Compute sequentially: From equation

$\frac{\partial\sqrt{1 - e^2}}{\partial\sigma_k}$, $\frac{\partial\sqrt{1 - e^2}}{\partial\rho_k}$	(104)
$\frac{\partial(e \sin \phi)}{\partial\sigma_k}$, $\frac{\partial(e \sin \phi)}{\partial\rho_k}$, $\frac{\partial(e \cos \phi)}{\partial\sigma_k}$, $\frac{\partial(e \cos \phi)}{\partial\rho_k}$	(103)
$\frac{\partial(E - \phi)}{\partial\sigma_k}$, $\frac{\partial(E - \phi)}{\partial\rho_k}$	(103)
$\frac{\partial r}{\partial\rho_k}$, $\frac{\partial r}{\partial\sigma_k}$	(101)
$\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\rho_k}$, $\frac{\partial\left(\frac{r^2}{q}\right)}{\partial\sigma_k}$	(100)
$A = \frac{\partial \underline{x}}{\partial \underline{\rho}}$	(98)
$B = \frac{\partial \underline{x}}{\partial \underline{\sigma}}$	(99)
$\frac{\partial F_o}{\partial\rho_1}$, $\frac{\partial F_o}{\partial\rho_4}$	(13)
$\frac{d\underline{\sigma}}{d\tau}$, $\frac{d\underline{\rho}}{d\tau}$	(16)