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A RECURSIVELY FORMULATED FIRST-ORDER SEMIANALYTIC ARTIFICIAL SATELLITE THEORY BASED ON THE GENERALIZED METHOD OF AVERAGING

VOLUME 1
THE GENERALIZED METHOD OF AVERAGING APPLIED TO THE ARTIFICIAL SATELLITE PROBLEM

Prepared For
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A RECURSIVELY FORMULATED FIRST-ORDER SEMIANALYTIC ARTIFICIAL SATELLITE THEORY BASED ON THE GENERALIZED METHOD OF AVERAGING

Volume I. The Generalized Method of Averaging Applied to the Artificial Satellite Problem

Prepared by

COMPUTER SCIENCES CORPORATION

For

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Prepared by:

Wayne D. McClain 11/21/77
W. D. McClain Date

Reviewed by:

A. C. Long 11/21/77
A. C. Long Date
Section Manager

Approved by:

K. G. Nickerson 11/21/77
K. G. Nickerson Date
Quality Assurance Reviewer

R. L. Taylor 11/21/77
R. L. Taylor Date
Project Manager-Mission Support

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ABSTRACT

This report presents, in two volumes, a recursively formulated, first-order, semianalytic artificial satellite theory, based on the generalized method of averaging. Volume I comprehensively discusses the theory of the generalized method of averaging applied to the artificial satellite problem. Volume II (to be published in early 1978) presents the explicit development in the nonsingular equinoctial elements of the first-order averaged equations of motion. The recursive algorithms used to evaluate the first-order averaged equations of motion are also presented in Volume II.

This semianalytic theory is, in principle, valid for a term of arbitrary degree in the expansion of the third-body disturbing function (nonresonant cases only) and for a term of arbitrary degree and order in the expansion of the nonspherical gravitational potential function. This theory has been implemented in the Goddard Trajectory Determination System (GTDS) Research and Development (R&D) version.

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SECTION 1 - INTRODUCTION

In the past, considerable attention was focused on the formulation of the equations of motion for complex dynamical problems and on the method of solution to insure that a sufficiently accurate result, meeting the investigators requirements, was obtained with an economy of effort. Without such careful consideration, the most prominent problem of classical mechanics, i.e., the motion of planets about the Sun, would probably not have been solved with anywhere near the accuracy actually obtained. It is a testimony to the ability of men such as Lagrange, Gauss, Leverrier, Hill, Hansen, and others that not only ingenious formulations of the equations of motion were obtained but that the thousands of arithmetic operations required to evaluate the solution were organized in such a manner as to minimize the number of these operations and considerably reduce the probability of undetected accidental errors.

The advent of the high-speed electronic computer has relaxed this consideration by making brute-force, error-free calculations possible. However, the competition for computer access has grown rapidly within the last decade. As a result of this overload on computer resources, current problems of interest should be formulated in a manner that not only fulfills the investigator's requirements but also minimizes computational cost.

One of the more computationally expensive dynamical problems today is the prediction and definitive determination of artificial satellite orbits. Maintaining reasonably accurate ephemerides for the ever-increasing number of artificial satellites (which include active scientific, defense, communication, and weather satellites as well as defunct satellites, launch vehicles, and other debris) requires a considerable expenditure in terms of computing time. Also, prelaunch mission analysis requires that several hundred satellite trajectories over periods of up to several years be generated for the purposes of lifetime and geometry constraint analysis. In addition, mission feasibility studies consume an inordinate amount of computer time. Generally, these applications do not require the extremely accurate high-precision orbit generation techniques which rely on the expensive process of numerically integrating Newton's equations of motion or some equivalent set of differential equations.

1.1 REVIEW OF ORBIT GENERATION TECHNIQUES

Another approach to the artificial satellite problem is provided by the purely analytical methods of solution in which analytical formulas for the coordinates or orbital elements are usually obtained to first or second order in a small parameter. A standard approach is to separate the short-period, long-period, and secular components of the motion through a series of canonical transformations (Reference 1). The secular contributions to the motion are evaluated at a given time, and the canonical transformation used to remove the long-period component of motion is inverted to provide the long-period motion in terms of the secular elements. Finally, the transformation to remove the short-period terms is inverted and evaluated with the secular and long-period contributions to the elements, thus obtaining the short-period contributions to the motion.

Although computationally efficient analytical satellite theories have been developed,[1] many of these theories suffer from severely restricted perturbation models. Several theories are limited to the lower degree zonal harmonic terms in the nonspherical gravitational model of the central body. The third-body perturbation, when included, is usually restricted to the cases of very close-Earth satellites. Also, many of these theories are restricted further by the use of the small eccentricity and/or small inclination approximations. In addition, the use of Keplerian elements in these formulations introduces singularities caused by vanishing eccentricity and/or inclination. Some of these limitations are accounted for by the fact that many of these analytical theories were developed manually. The tremendous amount of necessary algebraic manipulation required that these theories be severely restricted.

In the last decade, the appearance of machine automated algebraic processors has facilitated the development of analytical satellite theories with more sophisticated perturbation models. All that is required is sufficient computer time

[1] Y. Hagihara (Reference 2) gives an extensive list of references to the work in artificial satellite theory.

and storage. However, a reasonably general first-order analytical satellite theory can comprise tens of thousands of terms which require a prohibitive storage capacity. The only way to reduce the storage requirements for an explicit analytical theory is to restrict the theory itself.[1]

Finally, although several attempts to incorporate atmospheric drag in analytical satellite theories have been made, they have proven less than adequate for producing reasonably accurate ephemerides over extended time intervals. This is not surprising in view of the fact that even high-precision numerical techniques which use sophisticated atmospheric models have difficulty predicting ephemerides of strongly drag perturbed satellites over periods of several weeks (Reference 3).

The method of averaging offers another approach to the artificial satellite problem that has been shown to be more computationally efficient by several orders of magnitude than the high-precision techniques (Reference 4). In addition, the method is very flexible with respect to the perturbation models and suffers fewer restrictions than purely analytical satellite theories. Although not as accurate as the high-precision techniques, this technique produces results sufficiently accurate for all but the highest accuracy requirements, e.g., maneuvers, etc. More specifically, an application to first order of the method of averaging produces the long-period and secular motion of a satellite extremely accurately in most cases (Reference 4) and provides for the recovery of probably 90 to 95 percent of the short-period motion (Reference 5). Consequently, this approach provides a low-cost, long-term orbit prediction capability for the following:

- Mission feasibility studies
- Mission analysis (lifetime and geometric constraints)

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[1] For certain applications where one particular type of satellite is encountered, e.g., circular geosynchronous satellites, a restricted theory is not only acceptable but advisable. If, however, a single theory is to be used for several different types of satellites, a general theory is required.

- Tracking station acquisition schedules
- Dynamic modeling in definitive orbit determination procedures where either extended data intervals or extended data gaps are encountered
- Dynamic modeling required for differential correction (DC) procedures used to solve for dynamical parameters, e.g., high-order geopotential coefficients

The motivation for using the method of averaging procedure is as follows. The maximum step size which can be used in the numerical integration of a set of differential equations is constrained by the highest significant frequency contained therein. The method of averaging is used to remove high-frequency components from the equations of motion. The resulting averaged equations of motion are integrated numerically but with a significantly greater step size than can be used with the high-precision equations. The long-period and secular components of the satellite motion are thus obtained. The short-period component of the motion can be computed either numerically (Reference 5) or from analytical formulas which are presented in Volume II of this report. In most cases, the computational savings achieved by the larger step size (which results in fewer force evaluations) far outwieghs the increased cost of the derivative evaluation, thereby effecting a significant decrease in the overall computational costs.

The technique of removing the high-frequency terms from the equations of motion was first used by Lagrange in his investigations of the planetary motion. Because of a particular formulation of the equations of motion developed by Lagrange, the high-frequency terms, in the case of conservative perturbing forces, could be isolated more or less by inspection. However, a rigorous mathematical foundation for this technique was not provided until the relatively recent work by Krylov and Bogoliubov (Reference 6) on asymptotic methods for nonlinear oscillations.

Two approaches are available for the application of the method of averaging. The high-frequency components of the equations of motion can be removed numerically by application of a quadrature around an appropriate formulation of the high-precision equations of motion. This procedure is known as the numerical averaging approach. If the perturbing forces are conservative, the equations of motion can be expressed using Lagrange's formulation, and the averaging quadrature can be performed analytically. Under certain assumptions,[1] this method produces the same result as that obtained by inspection. This semianalytical procedure of numerically integrating the analytically averaged equations of motion is referred to as the analytical averaging approach.

[1] The assumptions arise when either the Greenwich Hour Angle, i.e., the Earth's rotation, or the fast variable of the disturbing third body appear in the perturbation models. Specifically, these quantities are assumed to be completely independent of the satellite fast variable, both explicitly and implicitly through the time.

1.2 THE NUMERICAL AVERAGING APPROACH

Recently, the numerical averaging method has been successfully applied to the artificial satellite problem (References 4, 5, and 7). This technique is particularly flexible in that it can be applied to any perturbation which can be deterministically modeled. It is also quite attractive because of the ease with which different element sets can be accommodated (Reference 8) and because of the ease of modifying the first-order assumption (References 5 and 6). Numerical averaging is also appealing because the implementation of the procedure seems to be rather straightforward.

However, the implementation, and more importantly, the proper use of numerical (as well as analytical) averaging techniques depend on the understanding of several basic concepts, many of which are addressed in this report and in References 4 and 9. Furthermore, Early (Reference 10) demonstrated that a straightforward application of the numerical averaging technique is not well suited to cases where the perturbing force varies by several orders of magnitude over a short arc in the orbit while remaining essentially negligible outside that interval.

Notwithstanding, numerical averaging has been shown (Reference 4) to be an effective procedure for generating the long-period and secular motion of a satellite for a wide variety of cases and to be considerably more efficient than the high-precision techniques. Consequently, the numerical averaging approach has been used either wholly (References 4 and 5) or in part (References 11, 12, 13, 14, and 15) in the development of several averaged orbit generator programs.

1.3 THE ANALYTICAL AVERAGING APPROACH

The method of analytical averaging is attractive because it is not only significantly more computationally efficient than high-precision techaniques but also is usually an order of magnitude more efficient than numerical averaging techniques (Reference 9). This computational advantage is accounted for by the fact that the analytically averaged perturbation models, although more complex than the high-precision perturbation models, are evaluated only once per integration step. The numerical averaging approach requires that the high-precision perturbation models be evaluated once at each abscissa of the quadrature. Thus, the method of numerical averaging requires between 12 and 96 force evaluations to compute the averaged element rates (Reference 10). In addition to the greater computational efficiency, the analytical averaging method offers greater precision with respect to computation of the element rates and therefore should be used whenever possible (Reference 8).

The analytical averaging method has been used in the development of several averaged orbit generator programs (References 11, 12, 13, 14, 15, and 16). These programs suffer from one or more limitations, however. In particular, most programs are based on theories formulated in terms of the Keplerian elements, which produce singularities in the equations of motion for vanishing eccentricities or inclinations.[1] Dallas and Khan (Reference 14) modified the element set to remove the small eccentricity problem; however, the small inclination problem remains. The Earth Satellite Mission Analysis Program (ESMAP) initiated by Cefola (Reference 11) is formulated in a completely nonsingular element set but is severely restricted in its perturbation models as are the programs described in Reference 15 and 16.

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[1]This is, of course, not peculiar to the averaging method but rather to the form of the high-precision equations of motion.

The program developed by Wagner (Reference 13) is based on general expressions for the analytically averaged perturbation models developed by Kaula (References 17 and 18) which are formulated in terms of singular Keplerian elements. Cook (Reference 16) implemented Kaula's perturbation models using Allan's recursive algorithm for the inclination functions and a recursive algorithm for the Hanson coefficients based on the recursive properites of Legendre polynomials. Unfortunately, Cook's program is based on the singular Keplerian element set, and the nonspherical gravitational potential is restricted to the zonal harmonics.

Examples of computer-generated, explicit analytically averaged perturbation models are given by Sridharan and Renard (Reference 19) for the long-period, disturbing third-body model using the potentially singular Keplerian elements and by Collins (Reference 20) for a restricted 2:1 resonant geopotential model using the nonsingular equinoctial elements.

1.4 RECENT DEVELOPMENTS IN ANALYTICAL AVERAGING THEORY

Very recently, several authors have investigated general, analytically averaged perturbation models for the third-body and nonspherical gravitational perturbations in terms of nonsingular element sets. Cefola and Broucke (Reference 21) developed recursively formulated models for the nonresonant third-body and zonal harmonic perturbations based on the equinoctial elements. The development of the zonal harmonic model is similar to that of Cook's model, with the exception that the inclination function is developed in terms of associated Legendre polynomials and their derivatives and certain complex polynomials. Cefola's third-body model is developed in terms of the direction cosines of the disturbing third-body position vector, which proves computationally efficient but is limited to nonresonant cases. Cefola outlined an extension of his zonal harmonic model to include the nonresonant tesseral harmonic terms (Reference 22) and later completed and extended the model to include resonant phenomena (Reference 23). Giacaglia (Reference 24) reformulated Kaula's perturbation models (using Allan's inclination function) in a nonsingular element set and provided a set of recursive algorithms for computational purposes. Finally, Nacozy and Dallas (Reference 25) also reformulated the Kaula geopotential model (using Allan's inclination function) in terms of a nonsingular element set. No recursive algorithms were provided.

The relatively simple recursive algorithms of Cook, Cefola, and Giacaglia are appealing in view of the alternative of evaluating the complicated polynomials found in the work of Nacozy and Dallas. However, the brute-force implementation of recursive algorithms can contribute to computational inefficiency and can possibly introduce artificial singularities (not in the equations of motion, but in the model evaluation). To insure against this possibility, careful consideration must be given to the ordering of the terms in the models such that the recursion formulas proceed in the proper direction to avoid small divisors and the amount of recomputation and storage requirements are minimized.[1] The

[1] Cefola has considered the question of the efficient implementation of his theory (Reference 21).

alternatives of computation and recomputation of all quantities as needed while storing nothing or the computation of all distinct quantities once and storing of each are costly in terms of machine processing time and storage, respectively.

1.5 SUMMARY

This report is an outgrowth of a series of task assignments with the objective of implementing in the Goddard Trajectory Determination System (GTDS) Research and Development (R&D) version a set of recursively formulated, first-order analytically averaged equations of motion for an artificial satellite perturbed by nonresonant third-body and nonspherical gravitational perturbations. This analytical averaging capability enhances the GTDS numerical averaging capability (Reference 4) and provides for optimal averaged perturbation models for each specific type of perturbation (with the possible exception of third-body resonance cases, which were not considered). Partial results obtained for some of the optimal averaged perturbation models in GTDS have been presented in Reference 9.

Cefola's averaged perturbation models are adopted for the nonresonant third-body and zonal harmonic perturbations. The tesseral harmonic model was developed using the approach outlined by Cefola in Reference 22. The models developed were generalized to handle retrograde as well a direct equinoctial elements (see Appendix A).

As part of this investigation, a fairly detailed comparison of the theories of Cefola and Giacaglia was performed. Briefly stated, the theories were found to be basically equivalent. Minor differences in the theories include different nonsingular element sets and different computational procedures for the inclination function. Arguments can be made concerning the relative advantages and disadvantages of these nonsingular element sets, but in regard to the removal of the singularities from the equations of motion, both are acceptable. Giacaglia computes the entire inclination function recursively, requiring a more complicated recursion relation with more back values of the function. Cefola uses recursion formulas for several quantities comprising the inclination function. The recursion relations are simpler, requiring fewer back values, but more recursion formulas are needed.

Regarding the implementation in the GTDS R&D version of the resonant tesseral harmonic model, it was felt that this capability should be very flexible with respect to the specific resonant harmonic terms used. The existence of a resonance dictates which terms in the potential expansion are significant to the long-period motion. Knowledge of the common characteristics of these terms and the proper use of the recursive algorithms would have provided a means for further optimization of this model. However, the procedure would have been automatic, with the program expecting a certain set of terms. Therefore, for the purposes of flexibility and at some additional computational costs, the contributions from each spherical harmonic term are computed entirely independently from all other terms.[1]

Due to the extensive new software for the analytical averaging capability as well as to the extensive modifications required to the previously implemented averaging software (particularly the input processor and initialization procedures and the attendant added complexity of executing the GTDS R&D averaging capability), it was decided that a system description and user's guide for the GTDS R&D averaging capability would be issued under a separate cover. In addition, a document extending the numerical results beyond those presented in Reference 9 is also in preparation. This document will discuss the computational costs in terms of machine processing time, the accuracy of the analytical averaging methods, and the procedure and algorithms used to develop an automatic truncation capability to further optimize the perturbation models for each particular case.

The current report consists of two volumes. The theory of the method of averaging is discussed in Volume I. Volume II presents the explicit development of a semianalytical artificial satellite theory based on the method of averaging.

Volume I presents a fairly comprehensive discussion of the application of the generalized method of averaging to the artificial satellite problem and the resulting formulation of the averaged equations of motion. In Section 2, a discussion

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[1]The capability to automatically select the resonant terms was implemented in the GTDS R&D version. However, no special relationship among them is assumed.

of the Variation of Parameters (VOP) formulation of the equations of motion, upon which the method of averaging is based, is presented. Section 3 discusses the application of the method of averaging to the VOP equations of motion. The criterion for the selection of short-period terms is discussed in Section 3.1, and the generalized method of averaging is applied to the VOP equations for the case of a single perturbing function in Section 3.2. A discussion of the application of the method of averaging to the case of two or more perturbing functions is presented in Section 3.3, followed by a description of the modification required for the application of the method of averaging to cases involving resonance phenomena in Section 3.4. Next, Section 3.5 addresses the application of higher order averaging theories. Finally, a discussion of the first-order short-period variations in the elements and their application to osculating-to-mean and mean-to-osculating element conversions is given in Section 4.

Volume II presents the mathematical formulation of the nonspherical gravitational and nonresonant third-body models required for the first-order averaged equations of motion. In this volume, the nonspherical gravitational potential is developed in the nonsingular equinoctial element set, and the zonal harmonic model, the combined zonal and nonresonant tesseral harmonic model, and the resonant tesseral harmonic model are isolated. The nonresonant third-body disturbing function is also developed in equinoctial elements and in the direction cosines of the third body. All models are presented in what is considered to be an optimal form, taking into account the minimization of the combined computational and storage costs while avoiding computational singularities. It is this final form of the models that was implemented in the GTDS R&D version.

SECTION 2 - THE VARIATION OF PARAMETERS (VOP) EQUATIONS

Classically, the Variation of Parameters (VOP) formulation of the equations of motion was used to investigate the long-period and secular motion of the planets. The VOP formulation was introduced by Euler while investigating the mutual perturbations of Jupiter and Saturn and was later generalized and completed by Lagrange (Reference 26). Since the primary objective of the current investigation is the development of an efficient orbit generation method for the prediction of the long-period and secular motion of artificial satellites, the VOP formulation was used.

In this section, a derivation of the basic VOP equations is presented in an attempt to provide some background information to the reader who is not already familiar with the method. Although the derivation presented is not the most elegant, it serves the purpose of explaining the basic principles of the method and provides a logical foundation for the form of the VOP equations used in this investigation.

2.1 PRINCIPLES OF THE VOP FORMULATION

The VOP formulation of the equations of motion for a perturbed dynamical system requires that the solution for the corresponding unperturbed system be known. The unperturbed dynamical system associated with the artificial satellite problem is the classical two-problem of celestial mechanics. As a starting point in the development of the VOP formulation, the differential equation of Newton describing the perturbed motion of a satellite relative to the central body is considered, i.e.,

$$\ddot{\vec{r}} + k^2(m + m_s)\frac{\vec{r}}{r^3} = \vec{Q}(\vec{r}, \dot{\vec{r}}, t) \tag{2-1}$$

where $\vec{r}$ and r denote the satellite position vector and its magnitude, $\dot{\vec{r}}$ is the velocity vector, k is the Gaussian constant, m and m_s are the masses of the central body and satellite, respectively, $\vec{Q}$ is the perturbing acceleration vector caused by conservative and/or nonconservative perturbing forces, and t is the time. For $m_s \ll m$, the satellite mass can be neglected.

For the unperturbed problem where $\vec{Q} \equiv \vec{0}$, Equation (2-1) reduces to

$$\ddot{\vec{r}} + k^2 m \frac{\vec{r}}{r^3} = \vec{0} \tag{2-2}$$

A solution of this system of equations requires six constants of integration. These constants are denoted by a_i (where $i = 1, 2, \ldots, 6$) or by the vector $\vec{a}$. The constants are identically the components of the initial position and velocity vectors or any set of six independent functions of the initial position and velocity. The solution of Equation (2-2) is denoted by the vector function $\vec{\phi}_0(a, t)$. The method used to obtain this solution is discussed in References 27 and 28. The solution $\vec{\phi}_0$ describes the motion of a point on an ellipse at a particular spatial orientation with the central body located at one of the foci.

In the VOP formulation, the perturbed two-body problem represented by Equation (2-1) is assumed to possess a solution ϕ of the same form as the function $\vec{\phi}_0$ with the single exception that the constants of the unperturbed motion, a_i, vary with time. Solving Equations (2-1) then reduces to determining this time dependence.

The VOP equations of motion consist of a set of six first-order differential equations as follows:

$$\frac{da_k}{dt} = G_k(\vec{a}, t) \qquad (k = 1, 2, \ldots, 6) \qquad (2\text{-}3)$$

where the constants of the unperturbed motion, referred to as elements, are treated as time-dependent parameters. This system of equations can be obtained directly by transformation of Equations (2-1). Expressing the three coordinate variables in Equations (2-1) formally in terms of the six elements and the time results in the three equations

$$x_i = f_i(\vec{a}, t) \qquad (i = 1, 2, 3) \qquad (2\text{-}4)$$

involving six unknowns a_k. Consequently, three arbitrary relations or constraints may be imposed on the six elements. These relations may be specified implicitly and are usually chosen such that the following equations are satisfied:

$$\frac{dx_i}{dt} = \frac{\partial f_i}{\partial t} \qquad (i = 1, 2, 3) \qquad (2\text{-}5)$$

which requires that

$$\sum_{k=1}^{6} \frac{\partial f_i}{\partial a_k} \frac{da_k}{dt} = 0 \qquad (i = 1, 2, 3) \qquad (2\text{-}6)$$

The motivation for this particular choice is discussed below.

The implicit relations between the position and velocity and the six unknowns a_k specified by Equations (2-4) and (2-5) will be used to transform Equation (2-1) into Equations (2-3). Differentiating Equations (2-5) with respect to the time yields

$$\frac{d^2 x_i}{dt^2} = \frac{\partial^2 f_i}{\partial t^2} + \sum_{k=1}^{6} \frac{\partial^2 f_i}{\partial a_k \partial t} \frac{da_k}{dt} \qquad (i = 1, 2, 3) \qquad (2\text{-}7)$$

Substituting the right-hand sides of Equations (2-7), (2-5), and (2-4) into Equations (2-1) yields the following three first-order differential equations in the six unknowns a_k :

$$\frac{\partial^2 f_i}{\partial t^2} + \sum_{k=1}^{6} \frac{\partial^2 f_i}{\partial a_k \partial t} \frac{da_k}{dt} + k^2 m \frac{f_i}{\left(\sum_{j=1}^{3} f_j^2\right)^{3/2}} = Q_i\left(f_j, \frac{\partial f_j}{\partial t}, t\right) \qquad (2\text{-}8)$$

$$(i = 1, 2, 3; \quad j = 1, 2, 3)$$

Equations (2-6) provide the three other first-order differential equations required to determine the system.

The function f_i, representing the ith component of the position vector, is determined from the formulas for elliptic (unperturbed) motion, i.e., through $\vec{\phi}_0$, which relate an instantaneous position to a set of instantaneous elements (in fact, infinitely many). It is not immediately obvious from Equation (2-4) alone that the perturbed velocity vector can be related to the same set of instantaneous elements through these formulas. However, Equation (2-5) indicates that the velocity components are determined by differentiating the position functions, f_i, while holding the elements constant, which is exactly the requirement for unperturbed motion. As a result, at any time t, the perturbed elements always correspond to a set of unperturbed elements. Such elements are referred to as osculating elements. The three constraints imposed on the elements by Equations (2-5) are not the only set possible, but they are the only set that allow both position and velocity to be related to these perturbed elements through the formulas for elliptic motion.

In Equations (2-3), five elements can be chosen such that they completely specify the osculating ellipse in space. The sixth element, a_6, in conjunction with the time t specifies the position of the object on the osculating ellipse at time t. The function $G_k(\bar{a}, t)$ represents the time rate of change of the ith osculating element caused by the perturbing force. In most cases, the perturbations are small compared with the central force, and, therefore, the magnitude of the function G_k is small. Consequently, in most problems the elements a_k are slowly varying.

For conservative perturbing forces, the osculating element rates can be represented in terms of the partial derivatives of a disturbing function. The disturbing function is the negative of the potential function, hence the restriction to conservative perturbing forces. To obtain a formulation dependent only on the elements, the disturbing function is developed in terms of the elements through a formal Fourier series expansion. Also, the Fourier series representation permits isolation of specific frequencies in the motion by inspection. If the series expansion is developed literally, Equations (2-3) can be integrated term by term using the method of successive approximations to obtain an analytical approximation to the solution (Reference 2). This approach is known as the method of general perturbations.

Under the category of special perturbation methods, several numerical techniques have been developed for evaluating the osculating element rates given by Equations (2-3). A particular solution for these equations is then generated using a numerical integration procedure. There are essentially two formulations of the special perturbation technique associated wtih the VOP formulation of the equations of motion. One formulation, associated with the name of Gauss, uses closed form expressions for the osculating element rates, i.e., the functions G_k are formulated in terms of the components of the acceleration. The other formulation is based on a Fourier series expansion for the disturbing function as used in the general perturbation method except that the coefficients are generated by some numerical scheme.

2.2 THE GAUSSIAN VOP EQUATIONS

This particular form of the VOP equations is easily obtained as follows. First, Equation (2-5) is substituted into Equations (2-8) to yield

$$\frac{\partial^2 x_i}{\partial t^2} + \sum_{h=1}^{6} \frac{\partial \dot{x}_i}{\partial a_h} \dot{a}_h + k^2 m \frac{x_i}{\left(\sum_{j=1}^{3} x_j^2\right)^{3/2}} = Q_i \quad (i = 1,2,3) \qquad (2\text{-}9)$$

Clearly, the corresponding equation for the unperturbed motion is

$$\frac{\partial^2 x_i}{\partial t^2} + k^2 m \frac{x_i}{\left(\sum_{j=1}^{3} x_j^2\right)^{3/2}} = 0 \quad (i = 1,2,3) \qquad (2\text{-}10)$$

Subtracting Equation (2-10) from Equation (2-9) gives

$$\sum_{h=1}^{6} \frac{\partial \dot{x}_i}{\partial a_h} \dot{a}_h = Q_i \quad (i = 1,2,3) \qquad (2\text{-}11)$$

Multiplying both sides of Equations (2-11) by $\partial a_j / \partial \dot{x}_i$ and summing over the index i yields

$$\sum_{i=1}^{3} \sum_{h=1}^{6} \frac{\partial a_j}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial a_h} \dot{a}_h = \sum_{i=1}^{3} \frac{\partial a_j}{\partial \dot{x}_i} Q_i \quad (j = 1,2,\ldots,6) \qquad (2\text{-}12)$$

But

$$\sum_{i=1}^{3} \frac{\partial a_j}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial a_k} = \delta_{j,k} \qquad (j = 1,2,\ldots,6) \qquad (2\text{-}13)$$

where $\delta_{j,k}$ is the classical Kronecker delta function since the elements a_k are mutually independent. Consequently, Equation (2-12) takes the form

$$\sum_{k=1}^{6} \delta_{j,k} \dot{a}_k = \sum_{i=1}^{3} \frac{\partial a_j}{\partial \dot{x}_i} Q_i \qquad (2\text{-}14)$$

or, more simply,

$$\dot{a}_j = \sum_{i=1}^{3} \frac{\partial a_j}{\partial \dot{x}_i} Q_i \qquad (j = 1,2,\ldots,6) \qquad (2\text{-}15)$$

This result is known as the Gaussian form of the VOP equations of motion.

The right-hand side of these equations can also be formulated in cylindrical coordinates where the radial, transverse, and normal components of the acceleration are used. This particular form of the equations can be found in most celestial mechanics references (e.g., Reference 29). The Gaussian formulation is particularly attractive becuase it is appropriate for both conservative and nonconservative perturbations. However, because most accelerations are best formulated in terms of position or position and velocity rather than as a Fourier series expansion, periodic phenomena cannot be isolated from the acceleration model by selecting the appropriate terms by inspection. Therefore, a numerical procedure must be used for isolating specific frequencies in the motion.

Because of the flexibility and relative ease of implementation, the Gaussian formulation has been used in the development of numerical first-order averaging procedures (References 4, 5, 11, 12, 13, and 14). This formulation has the disadvantage that conversions from the elements to position and velocity must be applied whenever the element rates are evaluated, i.e., at every integration step. In the Lagrangian formulation, this particular disadvantage is avoided at the possible expense of the closed-form expressions for the equations of motion.

2.3 THE LAGRANGE PLANETARY EQUATIONS

The derivation of the Lagrange VOP equations of motion (referred to as the Lagrange Planetary Equations) is identical to the Gaussian formulation through Equations (2-11), with the exception that the perturbing function or acceleration component, Q_i, is restricted to depend only on the position and can then be expressed as the gradient of the disturbing function, $R(x_1, x_2, x_3)$, i.e.,

$$Q_i = \frac{\partial R}{\partial x_i} \qquad (i = 1,2,3) \qquad (2\text{-}16)$$

Equations (2-11) then take the form

$$\sum_{k=1}^{6} \frac{\partial \dot{x}_i}{\partial a_k} \dot{a}_k = \frac{\partial R}{\partial x_i} \qquad (i = 1,2,3) \qquad (2\text{-}17)$$

Multiplying Equation (2-17) by $\partial x_i / \partial a_j$ and summing over i yields

$$\sum_{i=1}^{3} \sum_{k=1}^{6} \frac{\partial x_i}{\partial a_j} \frac{\partial \dot{x}_i}{\partial a_k} \dot{a}_k = \sum_{i=1}^{3} \frac{\partial x_i}{\partial a_j} \frac{\partial R}{\partial x_i} = \frac{\partial R}{\partial a_j} \qquad (j = 1,2,\ldots,6) \qquad (2\text{-}18)$$

Similarly, multiplying Equation (2-6) by $\partial \dot{x}_i / \partial a_j$ and summing over i yields

$$\sum_{i=1}^{3} \sum_{k=1}^{6} \frac{\partial \dot{x}_i}{\partial a_j} \frac{\partial x_i}{\partial a_k} \dot{a}_k = 0 \qquad (2\text{-}19)$$

(It should be noted that x_i has been substituted for f_i in Equations (2-6).)

Subtracting Equation (2-19) from Equations (2-18) yields

$$\sum_{k=1}^{6} \left[a_j, a_k \right] \dot{a}_k = \frac{\partial R}{\partial a_j} \qquad (j = 1, 2, \ldots, 6) \qquad (2\text{-}20)$$

where

$$\left[a_j, a_k \right] = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial a_j} \frac{\partial \dot{x}_i}{\partial a_k} - \frac{\partial \dot{x}_i}{\partial a_j} \frac{\partial x_i}{\partial a_k} \right) \qquad (2\text{-}21)$$

is called the Lagrange Bracket.

Although there are a total of 36 Lagrange Brackets required for the complete set of equations specified by Equation (2-20), at most only fifteen must be determined because

$$\left[a_j, a_j \right] = 0 \qquad (2\text{-}22a)$$

and

$$\left[a_j, a_k \right] = - \left[a_k, a_j \right] \qquad (2\text{-}22b)$$

These conditions follow from inspection of the definition given by Equations (2-21). It should be pointed out that the Lagrange Brackets depend only on the formulas for elliptic motion because

$$\frac{\partial x_i}{\partial a_k} \equiv \frac{\partial f_i}{\partial a_k} \quad \text{and} \quad \frac{\partial \dot{x}_i}{\partial a_k} \equiv \frac{\partial}{\partial a_k} \frac{\partial f_i}{\partial t}$$

The fifteen necessary Lagrange Brackets required for Equations (2-20) can be evaluated explicitly in terms of the elements and the system of equations inverted to yield $\dot{a}_k$. An explanation of the evaluation of these quantities is presented in Reference 29.

An alternate derivation of the Lagrange Planetary Equations can be obtained with the aid of the following relation given by Broucke (Reference 30):

$$\frac{\partial a_k}{\partial \dot{x}_i} = -\sum_{j=1}^{6} (a_k, a_j) \frac{\partial x_i}{\partial a_j} \tag{2-23}$$

where the quantity (a_j, a_k) is the well-known Poisson bracket and is defined in Cartesian coordinates by

$$(a_k, a_j) = \sum_{i=1}^{3} \left(\frac{\partial a_k}{\partial x_i} \frac{\partial a_j}{\partial \dot{x}_i} - \frac{\partial a_k}{\partial \dot{x}_i} \frac{\partial a_j}{\partial x_i} \right) \tag{2-24}$$

The Poisson Brackets also share the properites of the Lagrange Brackets, i.e.,

$$(a_k, a_k) = 0 \tag{2-25a}$$

$$(a_k, a_j) = -(a_j, a_k) \tag{2-25b}$$

Equation (2-23) is immediately verified by direct substitution of the Poisson Bracket definition.

Expressing the Gaussian VOP equations (Equations (2-15)) in terms of the disturbing function yields

$$\dot{a}_k = \sum_{i=1}^{3} \frac{\partial a_k}{\partial \dot{x}_i} \frac{\partial R}{\partial \dot{x}_i} \qquad (k = 1, 2, \ldots, 6) \qquad (2\text{-}26)$$

Substituting the expression for $\partial a_k / \partial \dot{x}_i$ in Equation (2-23) into Equation (2-26) immediately yields

$$\dot{a}_k = -\sum_{j=1}^{6} (a_k, a_j) \sum_{i=1}^{3} \frac{\partial R}{\partial x_i} \frac{\partial x_i}{\partial a_j} \qquad (k = 1, 2, \ldots, 6) \qquad (2\text{-}27)$$

or simply

$$\dot{a}_k = -\sum_{j=1}^{6} (a_k, a_j) \frac{\partial R}{\partial a_j} \qquad (k = 1, 2, \ldots, 6) \qquad (2\text{-}28)$$

Equations (2-28) are the Poisson Bracket representation of the Lagrange Planetary Equations.

The relationship between the Lagrange and Poisson brackets is immediately obtained by substituting Equation (2-28) into Equation (2-20). The result is

$$-\sum_{k=1}^{6} \sum_{j=1}^{6} [a_i, a_k] (a_k, a_j) \frac{\partial R}{\partial a_j} = \frac{\partial R}{\partial a_i} \qquad (2\text{-}29)$$

which requires the condition

$$\sum_{k=1}^{6} \left[a_i, a_k\right](a_j, a_k) = \delta_{ij} \qquad (2\text{-}30)$$

(Equation (2-25) was used to remove the negative sign in Equation (2-30).)

The particular VOP formulation adopted for this report is a modified version of Lagrange's Planetary Equations and is given by

$$\frac{da_i}{dt} = -\sum_{j=1}^{6} (a_i, a_j) \frac{\partial R}{\partial a_j} \qquad (i = 1, 2, \ldots, 5) \qquad (2\text{-}31a)$$

$$\frac{d\ell}{dt} = n - \sum_{j=1}^{6} (\ell, a_j) \frac{\partial R}{\partial a_j} \qquad (2\text{-}31b)$$

where n is the mean motion and a_6 now denotes the variable ℓ under the summation. The variable ℓ, referred to variously as the fast variable or the rapidly rotating phase, is not a true slowly varying element but is a linear combination of the time with an element such that

$$\ell = nt + a_6 \qquad (2\text{-}32)$$

The parameter ℓ measures the angular distance of the satellite from some departure point in the orbit. This modification, which was made by Tisserand

(Reference 31), is necessary to avoid the presence of mixed secular terms in the equations of motion. A mixed secular term has the form

$$t^n \cos m\ell$$

$$t^n \sin m\ell$$

and quickly degrades the solution as time t increases. The appearance of such terms is not inherent to the problem but to the formulation of the problem. The mean motion, n, enters into Equations (2-31) through Equation (2-32). Use of the variable ℓ appears to have significantly changed the form of the Lagrange Planetary Equations. However, the original form of the equations given by Equations (2-28) is easily recovered by modifying the disturbing function with the addition of the negative of the total energy to the original disturbing function, i.e., if the semimajor axis is denoted by a, then

$$R' = R + \frac{\mu}{2a}$$

Equations (2-31) can then be expressed as

$$\frac{da_i}{dt} = -\sum_{j=1}^{6} (a_i, a_j) \frac{\partial R'}{\partial a_j} \tag{2-33}$$

where a_6 is understood to represent the variable ℓ. A more complete discussion of this question is presented by Plummer (Reference 32). This refinement is not necessary for the purpose of this investigation and, accordingly, will not be used.

2.4 DISCUSSION OF ORBITAL ELEMENT SETS

The preceding discussion of the VOP equations has made numerous references to the "elements" or "osculating elements." The question of which element set to use has not been addressed, and, in fact, a general discussion of the VOP formulation need not be concerned with any specific element set. However, the application of the VOP equations does require the selection of a set of elements.

There are several well-known element sets, the best known of which is the set of cl. sical or Keplerian elements. The VOP equations formulated in Keplerian elements contain the eccentricity, e, and the sine of the inclination as divisors and therefore are singular for vanishing eccentricity and/or inclination. There are several nonsingular element sets available, and the choice of a particular set is arbitrary insofar as removing the singularities from the equations of motion. However, some of these sets can present a slight computational advantage over other sets when converting from elements to position and velocity. For other applications, such as differential correction and error analysis procedures, the choice of the element set may no longer be quite so artibrary. According to Broucke and Cefola (Reference 33), the nonsingular set of elements, which are called equinoctial elements, can possess marked computational advantages over other nonsingular element sets.

The equinoctial element set, $a = (a, h, k, p, q, \lambda)$, is used in this investigation. It is defined in terms of the Keplerian elements by the following:

$$
\begin{aligned}
a &= a \\
h &= e \sin(\omega + I\Omega) \\
k &= e \cos(\omega + I\Omega) \\
p &= \tan^{I}(i/2) \sin\Omega \\
q &= \tan^{I}(i/2) \cos\Omega \\
\lambda &= \ell + \omega + I\Omega
\end{aligned}
$$

where I is the retrograde factor which takes on the values

$$I = 1 \quad (\text{for } 0 \leq i \leq (\pi/2))$$

$$I = -1 \quad (\text{for } (\pi/2) < i \leq \pi)$$

A more complete discussion of this element set, including the Lagrange and Poisson brackets and the conversion to position and velocity, is presented in Appendix A.

The VOP equations expressed in equinoctial elements take the form

$$\frac{da}{dt} = \frac{2a}{A}\frac{\partial R}{\partial \lambda} \tag{2-34a}$$

$$\frac{dh}{dt} = \frac{B}{A}\left(\frac{\partial R}{\partial k} - \frac{h}{1+B}\frac{\partial R}{\partial \lambda}\right) + \frac{kC}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \tag{2-34b}$$

$$\frac{dk}{dt} = -\frac{B}{A}\left(\frac{\partial R}{\partial h} + \frac{k}{1+B}\frac{\partial R}{\partial \lambda}\right) - \frac{hC}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \tag{2-34c}$$

$$\frac{dp}{dt} = -\frac{pC}{2AB}\left(k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial k} + \frac{\partial R}{\partial \lambda}\right) + \frac{IC^2}{4AB}\frac{\partial R}{\partial q} \tag{2-34d}$$

$$\frac{dq}{dt} = -\frac{qC}{2AB}\left(k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial k} + \frac{\partial R}{\partial \lambda}\right) - \frac{IC^2}{4AB}\frac{\partial R}{\partial p} \tag{2-34e}$$

$$\frac{d\lambda}{dt} = n - \frac{2a}{A}\frac{\partial R}{\partial a} + \frac{B}{A(1+B)}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right)$$

$$+ \frac{C}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \qquad (2\text{-}34f)$$

where

$$A = na^2$$

$$B = \sqrt{1-h^2-k^2}$$

$$C = 1 + p^2 + q^2$$

The disturbing functions presented in Volume II of this report are better expressed in terms of the direction cosines (α, β, γ) with respect to the equinoctial reference frame $(\hat{f}, \hat{g}, \hat{w})$ of either the equatorial $\hat{z}$ axis or the third-body position vector, rather than in terms of the equinoctial elements p and q. Consequently, expressions of the form

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial \alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial R}{\partial \beta}\frac{\partial \beta}{\partial p} + \frac{\partial R}{\partial \gamma}\frac{\partial \gamma}{\partial p}$$

$$\frac{\partial R}{\partial q} = \frac{\partial R}{\partial \alpha}\frac{\partial \alpha}{\partial q} + \frac{\partial R}{\partial \beta}\frac{\partial \beta}{\partial q} + \frac{\partial R}{\partial \gamma}\frac{\partial \gamma}{\partial q}$$

will be used to modify Equations (2-34) in order to accommodate the particular form of the disturbing functions. The following results, presented here without proof, are demonstrated in Volume II:

$$\frac{\partial \alpha}{\partial p} = -\frac{2}{C}(q\beta I + \gamma) \qquad (2\text{-}35a)$$

$$\frac{\partial \alpha}{\partial q} = \frac{2p\beta I}{c} \tag{2-35b}$$

$$\frac{\partial \beta}{\partial p} = \frac{2q\alpha I}{c} \tag{2-35c}$$

$$\frac{\partial \beta}{\partial q} = -\frac{2I}{C}\left(p\alpha - \gamma\right) \tag{2-35d}$$

$$\frac{\partial \gamma}{\partial p} = \frac{2\alpha}{C} \tag{2-35e}$$

$$\frac{\partial \gamma}{\partial q} = -\frac{2\beta I}{C} \tag{2-35f}$$

$$\frac{\partial R}{\partial p} = \frac{2}{C}\left[\alpha \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \alpha} + qI\left(\alpha \frac{\partial R}{\partial \beta} - \beta \frac{\partial R}{\partial \alpha}\right)\right] \tag{2-36a}$$

$$\frac{\partial R}{\partial q} = -\frac{2I}{C}\left[\beta \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \beta} + p\left(\alpha \frac{\partial R}{\partial \beta} - \beta \frac{\partial R}{\partial \alpha}\right)\right] \tag{2-36b}$$

$$p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q} = \frac{2}{C}\left[p\left(\alpha \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \alpha}\right) - qI\left(\beta \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \beta}\right)\right] \tag{2-37}$$

where C is defined as before. Substituting these expressions into Equations (2-34) yields the final form of the VOP equations of motion used in the current investigation, i.e.,

$$\frac{da}{dt} = \frac{2a}{A}\frac{\partial R}{\partial \lambda} \tag{2-38a}$$

$$\frac{dh}{dt} = \frac{B}{A}\frac{\partial R}{\partial k} + \frac{k}{AB}\left(p R_{\alpha,\gamma} - I q R_{\beta,\gamma}\right) - \frac{hB}{A(1+B)}\frac{\partial R}{\partial \lambda} \tag{2-38b}$$

$$\frac{dk}{dt} = -\left[\frac{B}{A}\frac{\partial R}{\partial h} + \frac{h}{AB}\left(p R_{\alpha,\gamma} - I q R_{\beta,\gamma}\right) - \frac{kB}{A(1+B)}\frac{\partial R}{\partial \lambda}\right] \tag{2-38c}$$

$$\frac{dp}{dt} = \frac{C}{2AB}\left[p\left(R_{h,k} - R_{\alpha,\beta} - \frac{\partial R}{\partial \lambda}\right) - R_{\beta,\gamma}\right] \tag{2-38d}$$

$$\frac{dq}{dt} = \frac{C}{2AB}\left[q\left(R_{h,k} - R_{\alpha,\beta} - \frac{\partial R}{\partial \lambda}\right) - I R_{\alpha,\gamma}\right] \tag{2-38e}$$

$$\frac{d\lambda}{dt} = n - \frac{2a}{A}\frac{\partial R}{\partial a} + \frac{B}{A(1+B)} R_{h,k} + \frac{1}{AB}\left(p R_{\alpha,\gamma} - I q R_{\beta,\gamma}\right) \tag{2-38f}$$

where A, B, and C are defined as in Equation (2-34) and

$$R_{x,y} = x \frac{\partial R}{\partial y} - y \frac{\partial R}{\partial x}$$

for any two variables x and y .

It should be pointed out that a considerable simplification occurs for the non-resonant third-body and zonal harmonic perturbations where

$$R_{h,k} = R_{\alpha,\beta} \equiv 0$$

This simplification was first reported in Equations (5-57) of Reference 11 and will be demonstrated in Volume II of this report.

SECTION 3 - THE AVERAGED VOP EQUATIONS OF MOTION

Classically, in the investigation of the long-period and secular motion of the planets, the Lagrangian Variation of Parameters (VOP) equations (known as the Lagrangian Planetary Equations) were expanded in a literal Fourier series and, with the proper assumptions, the terms which contribute to the long-period and secular motion could be easily isolated by inspection. This technique produces excellent results when the perturbations are small, and it has been used extensively to investigate the planetary motions over long time intervals.

Alternatively, the long-period and secular contributions to the motion can be systematically isolated by applying the method of averaging to the VOP equations of motion to eliminate the short-period contributions. The solution of the resulting system of averaged equations is a set of parameters, usually referred to as mean[1] elements, that describe the long-period and secular deviations of the perturbed dynamical system from the unperturbed system.

The technique of eliminating the short-period terms from the equations of motion was without a mathematically rigorous foundation until the relatively recent work of Krylov and Bogoliubov (Reference 6) on asymptotic methods for nonlinear oscillations. The theory of the method of averaging is based on Poincare's theory of asymptotic expansions (Reference 34) and the introduction by Krylov and Bogoliubov of the concept of a near-identity transformation. The theory has been extended most notably by Mitropolsky (Reference 35).

Further elaboration and discussion of the theory has been contributed by several authors. Kruskal (Reference 36) remarked on the possibility of a recursively formulated general inversion of the near-identity transformation. Stern

[1]The mean elements are defined operationally as the solution to the averaged equations of motion. Consequently, the exact definition of a particular set of mean elements depends on the interval over which the equations of motion are averaged. This is discussed in more depth in Section 3.1.5.

(Reference 37) developed this recursive algorithm explicitly.[1] Kyner (Reference 38) and J. A. Morrison (Reference 39) have shown the Von Zeipel transformation method to be a special case of the generalized method of averaging, at least to second order, thus establishing a direct link to the methods used in developing analytical satellite theories.[2] F. Morrison (Reference 40) has presented a lucid discussion of the first-order application of the method. A discussion of the generalized method of averaging is also given by Nayfeh (Reference 41).

Although the discussion in this section is equally valid for many other dynamical systems, the primary objective of this report is the application of the method of averaging to the equations of motion for an artificial satellite. Consequently, the concepts of short and long period are developed in this context. Also, since the method of averaging can be applied to either the Gaussian (Equation (2-15)) or Lagrangian (Equation (2-31)) formulation of the VOP equations, the general expression

$$\frac{da_i}{dt} = \epsilon F_i(\vec{a}, \ell) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}1a)$$

$$\frac{d\ell}{dt} = n(a) + \epsilon F_6(\vec{a}, \ell) \quad (3\text{-}1b)$$

(where $\vec{a}$ consists of the five elements a_i) is used in the following discussion.

---

[1] This recursive algorithm is a general expression relating the jth-order term in the near-identity transformation to various combinations of the lower order terms in the transformation with lower order contributions to the mean element rates. This recursive algorithm is quite distinct from the recursively formulated first-order theory presented in Volume II of this report.

[2] An analytical satellite theory can be developed using successive applications of the method of averaging to remove first the short-period terms and then the long-period terms.

In this section, the generalized method of averaging is applied to the VOP equations of motion to obtain systematically the equations for the long-period and secular motion. A discussion of the criteria for the selection of short-period terms is presented in Section 3.1, and the averaged equations of motion for a single perturbing function are derived in Section 3.2. Section 3.3 extends the application of the method of averaging to cases with multiple perturbing functions. Next, in Section 3.4, the modifications required to extend the application of the method of averaging in the case of resonance phenomena is presented. Finally, the application of higher order averaging theories is discussed in Section 3.5.

3.1 CRITERIA FOR SELECTING SHORT-PERIOD TERMS

The criteria for distinguishing short-period terms are, in general, subjective. The shortest period of significance in the equations of motion effectively constrains the integration step size. For efficient computation, it is desirable to maximize this step size while retaining the essential character of the motion over an extended interval of time. This is the primary consideration in the selection of appropriate criteria for distinguishing short-period phenomena.

To illustrate this point, the following simple differential equation is considered:

$$\dot{a} = C_j \cos\left[j(\ell - \ell_0)\right]$$

In general, the minimum number of function evaluations required to integrate a function of this type over one period is four. The cosine function has three zeroes in the interval of one period. In view of the Fundamental Theorem of Algebra, any approximating polynomial which is valid over one complete period must be of at least third degree. Consequently, the function must be evaluated at four points to determine the coefficients of this third-degree approximating polynomial, or, equivalently, the function and its first three derivatives can be evaluated at a single point, requiring four function evaluations. This does not mean that four function evaluations per period provide the best representation of the element rate in the example, but only that this is the minimum number of function evaluations per period required to obtain the gross behavior of the real solution. The accurate integration of such a periodic function using arbitrary, equally spaced abscissae would probably require six, and more likely, eight function evaluations per period, requiring a corresponding number of integration steps.

A useful criterion for the selection of long-period terms is provided by careful examination of the frequencies in the artificial satellite problem.

3.1.1 Satellite-Dependent Frequencies

The perturbing functions $F_i(\vec{a}, \ell)$ in Equations (3-1) are assumed to be 2π periodic in the satellite fast variable, ℓ. Some of the slow variables are angular quantities (Keplerian elements) or functions of angular quantities (equinoctial elements) that produce fundamental periods in the motion of order $O(\epsilon^{-1})$. If P_i' denotes the fundamental period produced by one of the slowly varying angles and if the fundamental period produced by the fast variable ℓ is 2π, then the fundamental period, P_i', satisfies the relation

$$P_i' \geq \frac{2\pi}{\epsilon |F_i|_{max}}$$

If the quantity $\epsilon |F_i|_{max} \ll 1$, the period P_i' must be such that $P_i' \gg 2\pi$, i.e., it is much greater than the periods contributed by terms containing the fast variable ℓ. In addition, the VOP formulation implicitly assumes that the quantity $\epsilon |F_i|_{max}$ is not large. This discussion suggests that terms dependent on the satellite fast variable ℓ and all multiples of ℓ (i.e., $m\ell$, where m   1,2,3,...), which are of period 2π m, be considered to be short periodic as compared with terms containing the slowly varying angular quantities. Consequently, all terms with periods of the same order of magnitude as the satellite period and all smaller periods will be considered to be short period terms.

Other variables which can introduce short-period effects also appear in the perturbing function. More specifically, the effects on the satellite motion caused by the fast variable of the disturbing third body (i.e., Moon, Sun, etc.) or the Greenwich Hour Angle in the nonspherical gravitational potential model must be considered.

3.1.2 Third-Body Effects on the Motion

The presence of the disturbing third-body fast variable in the equations of motion will contribute terms with a fundamental period of approximately 28 days for the Moon and 1 year for the Sun. Either of these can certainly be considered to produce long-period effects (relative to the satellite period) in the motion of the vast majority of artificial Earth satellites. An infinity of multiples of the third-body fast variable also appear in the third-body model. Such terms will contribute the periodicities P_n^* to the motion of the satellite where

$$P_n^* = \frac{P^*}{n} \qquad (n = 1, 2, 3 \ldots)$$

and where P^* is the fundamental period produced by the fast variable of the disturbing body.

Clearly, as n increases, the periods P_n^* decrease; therefore, very high harmonics in the third-body perturbation model will contribute terms with periods similar to that of the Earth satellite, thus introducing third-body-dependent short-period terms. However, in the absence of resonance, the coefficients of these high-harmonic terms are very small in magnitude, rendering the contributions of these terms insignificant.[1] Consequently, the third-body motion (in the absence of resonance) contributes significant effects with periods of P^*/n, where n usually remains a small integer. Such periods are, in most cases, still very long compared with the periods of most Earth satellites.

However, certain classes of satellites (e.g., Interplanetary Monitoring Platform (IMP) satellites) have orbital periods comparable to the periods of the lower harmonic lunar terms cited above. For this class of satellites, the lunar effects on the motion cannot be considered to be long period. However, in the case of a strong resonance, a long-period component of the motion is introduced. The period of the resonant or critical term is significantly greater than the period of the satellite.

[1] In resonance, the commensurability between the mean motion of the satellite and the mean motion of the third body or the Earth's rotation rate causes the appearance of a small divisor in the coefficient of the critical term, resulting in a significantly increased magnitude for the coefficient and a corresponding increase in the contribution of the term.

3.1.3 Nonspherical Gravitational Effects on the Motion

In the case of perturbing effects caused by the nonsphericity of the Earth's gravitational field, the rotation of the Earth, represented as the Greenwich Hour Angle, θ, contributes terms with a fundamental period of 24 hours. These terms can be considered long period only for close-Earth satellites with periods of at most a few hours. These "long-period" contributions of 24 hours and fractional multiples thereof should not be grouped with the long-period contributions of several days or more caused by the third body and the slowly varying elements of the satellite. Consequently, these nonspherical gravitational contributions will be referred to as "medium-period" contributions.

Multiples of the Greenwich Hour Angle, $m\theta$ ($m = 1, 2, \ldots$), appear in the spherical harmonic expansion representing the Earth's gravitational model. The harmonics of moderately high degree (i.e., $m = 11, 12, 13$, etc.) will contribute terms with periods of a few hours or less. Even for close-Earth satellites, these terms obviously cannot be considered to be medium period and will be referred to as θ-dependent short-period terms. As in the third-body case, the coefficients of these high-degree harmonic terms are small, except in the case of resonance, and produce little effect on the motion. Consequently, to first order the harmonics of lower degree can be considered to produce relatively insignificant medium-period effects on close-Earth satellites, except in the case of resonance. For satellites with larger orbital periods, even the medium-period effects produced by the low-order tesseral harmonics must be considered as short period.

In summary, the key to the designation of short-period and long-period terms is, of course, the orbital period of the satellite. All periods introduced by the satellite fast variable are considered to be short period and are referred to as satellite-dependent or ℓ-dependent short-period terms. The other frequencies in the dynamical system, i.e., the frequencies introduced by the third body and by the rotation of the central body, must be considered in relation to the natural frequency of the satellite.

3.1.4 Implications for the Application of the Method of Averaging

The method of averaging is best suited to cases where quite distinct groups or families of frequencies are present. Each of these distinct families is introduced by its own source, and the distinction is found in the specific frequencies and amplitudes introduced. Occasionally, the higher frequencies in one family approach the primary frequency in another family and the separate contributions become more difficult to distinguish. Furthermore, elimination of one of these families of frequencies by a single application of the method of averaging does not eliminate the similar frequencies contributed by the other family.[1]

Additional applications of the averaging procedure are expensive in the numerical averaging approach or require multiple forms of the analytically averaged equations of motion necessary for all cases that might be encountered. Also, multiple applications of the averaging procedure are not always suitable as a technique for developing a reasonably accurate orbit generator. In contrast to a second averaging procedure, other means sometimes exist for eliminating unwanted high frequencies in the motion.

Proper restriction of the tesseral harmonic terms in the nonspherical gravitational model will eliminate the θ-dependent short-period terms they introduce into the motion. Such a restriction has no effect on the secular motion, at least to first order, since the tesseral harmonics produce no secular contributions to the motion to first order (Reference 2). In fact, for all nonresonant satellites, it is recommended that the contribution of all tesseral harmonic terms be deleted from the averaged equations of motion.

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[1] In the case of exact resonance, two of the families of frequencies are no longer distinct. The frequencies in one of the families appear to be integral multiples of the frequencies in the other family. Furthermore, a single application of the averaging procedure will remove all frequencies contributed by both sources up to a cut-off frequency specified in the averaging operation. This is discussed in more detail in Section 3.4.

The inclusion of these medium-period and θ-dependent short-period contributions in the evaluation of the mean element rates severely restricts the step size in the numerical integration procedure. The medium-period contributions have periods of 24 hours or less and they necessarily restrict the integration step size to at most 3 to 4 hours. Although the amplitudes of these terms are not negligible, they do not significantly affect the long-term motion as compared with the long-period and secular contributions of the zonal harmonics. Furthermore, these medium-period tesseral harmonic contributions can be evaluated analytically in the same manner, and at the same time if desired, as the short-period element variations discussed in Section 4.

If the medium-period effects contributed by the low-order zonal harmonics are retained in the equations of motion, the θ-dependent short-period terms should still be eliminated as described above, since it is inconsistent to eliminate the satellite-dependent short-period terms while retaining the θ-dependent terms with similar periods. This, in effect, defeats the whole purpose in the application of the method of averaging by imposing small step sizes in the numerical integration procedure. The arbitrary imposition of larger step sizes in this case causes these θ-dependent short-period terms to introduce spurious noise in the mean element rates and, consequently, in the numerically integrated solution. This is explained by the fact that the contributions of these short-period terms are propagated through the numerical integration as though it were part of the contribution of a term with a period approximately six to eight times the step-size interval.

The effects caused by the third body can be considered as exclusively long-period for the vast majority of artificial satellites. However, for very-long-period satellites (such as the IMP class with periods of several days), the third-body (lunar) contribution can in no way be considered to be long period and the application of the method of averaging must be reevaluated in this light.

The usual procedure in these cases has been to use Gauss' method of secular perturbations (Reference 32), which is also referred to as the method of double averaging. In this approach, the method of averaging is applied twice in succession, once to remove the satellite-dependent frequencies and then again to remove the third-body-dependent frequencies. While this method does isolate the secular motion of the satellite quite well, the periodic variations contributed by the third body to the motion of the satellite may have amplitudes of several thousands of kilometers. The elimination of such contributions is usually not suitable for generating a reasonably accurate satellite ephemeris. The alternative of using a high-precision technique to generate a satellite ephemeris should be strongly considered in this case, since such large step sizes are appropriate even for the frequencies in the high-precision case.

A strong resonance in the problem introduces a long-period contribution to the satellite motion of considerably larger period than either the satellite or lunar periods. In this instance, a single application of the method of averaging will isolate these contributions to the motion. However, due to the strong short-period variations in the problem contributed by the fast variables of the satellite and third body, a second or higher order averaging theory is probably required. This is also probably true for the double averaging approach discussed above.

Based on the above discussion, a single application of the method of averaging is used in the development of the semianalytical theory presented in this report. The θ-dependent short-period terms will be eliminated by appropriate restriction of the potential model. Although it is not recommended,[1] the theory for the medium-period contributions to the equations of motion will be developed. The third-body theory developed in this report is restricted to nonresonant cases only and to satellites with periods significantly shorter than the third-body orbital period.

[1]The analytical formulation of the medium-period contributions has not been implemented in the GTDS R&D version.

3.1.5 Mean Elements

Since the mean elements are defined operationally as the solution of the averaged equations of motion, the exact definition of a specific set of mean elements depends on the assumptions or constraints imposed in the development of the averaged equations of motion and on the interval over which the equations of motion are averaged. In this report, the averaged equations of motion are developed so that the mean elements obtained are, in principle, equivalent to the mean or average over the averaging interval of the osculating elements. This is demonstrated in Section 3.2.2. The averaging interval (in the absence of resonance) is selected to be the satellite period to ensure the elimination of all satellite-dependent short-period terms.

The dependence of the definition of a particular set of mean elements on the averaging interval has contributed to some confusion in the communication of results obtained by different investigators. For many, the term "mean elements" is immediately associated with the double-primed elements obtained by Brouwer (Reference 42). This element set reflects only the secular motion of the artificial satellite. The single-primed element set obtained by Brouwer in the same reference reflects both the long-period and secular motion of the satellite. The single-primed element set was obtained by the application of an averaging operation over an interval equal to the period of the satellite and, consequently, is the analog to the mean elements used in this report.

In an attempt to eliminate the confusion caused by terminology, several other names, including single averaged elements and long-period elements, have been suggested. However, these terms do not adequately define the elements. This is because the mean elements are defined wholly by the theory from which they are obtained and, therefore, no simple naming device can adequately describe them. To eliminate confusion when comparing separately obtained results, the corresponding theories must be understood. Therefore, the terminology "mean elements" will be used in this report, recognizing the inherent ambiguity in the phrase and also recognizing the lack of a satisfactory alternative.

3.2 THE AVERAGED EQUATIONS OF MOTION FOR A SINGLE PERTURBING FUNCTION

The following set of differential equations is considered:

$$\frac{da_i}{dt} = \epsilon F_i(\vec{a}, \ell) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}2a)$$

$$\frac{d\ell}{dt} = n(a_1) + \epsilon F_6(\vec{a}, \ell) \quad (3\text{-}2b)$$

where the vector $\vec{a}$ consists of the five slowly varying elements a_i. The near-identity transformation from $(\vec{a}, \ell)$ to $(\bar{\vec{a}}, \bar{\ell})$ is assumed to take the form

$$a_i = \bar{a}_i + \sum_{j=1}^{n} \epsilon^j \eta_{i,j}(\bar{\vec{a}}, \bar{\ell}) + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}3a)$$

$$\ell = \bar{\ell} + \sum_{j=1}^{n} \epsilon^j \eta_{6,j}(\bar{\vec{a}}, \bar{\ell}) + O(\epsilon^{n+1}) \quad (3\text{-}3b)$$

where the functions $\eta_{i,j}$ are 2π periodic in $\bar{\ell}$. The barred variables are referred to as mean elements. The quantity ϵ is assumed to be a small parameter, e.g., a coefficient in one of the terms of the spherical harmonic expansion of the geopotential model or the ratio of the semimajor axes of the satellite and third-body orbits in the series expansion of the third-body disturbing function. The presence of such a small parameter is basic to the method of averaging.

In the application of the method of averaging, the transform of the original system of equations (Equation (3-2)) (i.e., the equations of motion for the mean elements) is assumed to be of the form

$$\frac{d\bar{a}_i}{dt} = \sum_{j=1}^{n} \epsilon^j A_{i,j}(\bar{\bar{a}}) + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}4a)$$

$$\frac{d\bar{l}}{dt} = n(\bar{a}_1) + \sum_{j=1}^{n} \epsilon^j A_{6,j}(\bar{\bar{a}}) + O(\epsilon^{N+1}) \quad (3\text{-}4b)$$

so that the rate of change of the mean elements depends only on the slowly varying mean elements.

Basically, the procedure for obtaining the mean element equations of motion is to express both sides of Equation (3-2) in terms of the mean elements ($\bar{a}$, $\bar{l}$). Equations (3-3) and (3-4) are used to transform the left-hand side of Equation (3-2). The perturbing function on the right-hand side is expanded in a Taylor series about the mean elements and then rearranged as a power series in the small parameter ϵ. The resulting equations are averaged such that all dependence on the mean fast variable is eliminated. The final result yields order-by-order expressions for the mean element rates, $A_{i,j}(\bar{\bar{a}})$, in terms of suitably averaged functions of the perturbing function and its partial derivatives.

3.2.1 Formulation of the VOP Equations in Mean Elements

Equations (3-2) are expressed in terms of mean elements as follows. First, Equation (3-3) is differentiated, obtaining expressions for the osculating element rates which depend only on the corresponding mean elements and their rates, i.e.,

$$\frac{da_i}{dt} = \frac{d\bar{a}_i}{dt} + \sum_{j=1}^{N} \epsilon^j \sum_{k=1}^{6} \frac{\partial \eta_{i,j}}{\partial \bar{a}_k} \frac{d\bar{a}_k}{dt} + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad \text{(3-5a)}$$

$$\frac{d\ell}{dt} = \frac{d\bar{\ell}}{dt} + \sum_{j=1}^{N} \epsilon^j \sum_{k=1}^{6} \frac{\partial \eta_{6,j}}{\partial \bar{a}_k} \frac{d\bar{a}_k}{dt} + O(\epsilon^{N+1}) \quad \text{(3-5b)}$$

where $\bar{a}_6$ is understood to designate $\bar{\ell}$ under the summation. Substituting Equations (3-4) into Equations (3-5), thus introducing the functions $A_{i,j}$ into the equations of motion for the osculating elements, results in the expressions

$$\frac{da_i}{dt} = \sum_{j=1}^{N} \epsilon^j \left[A_{i,j}(\bar{a}) + n(\bar{a}_1) \frac{\partial \eta_{i,j}}{\partial \bar{\ell}} + \sum_{m=1}^{N-j} \epsilon^m \sum_{k=1}^{6} A_{k,m} \frac{\partial \eta_{i,j}}{\partial \bar{a}_k} \right] + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad \text{(3-6a)}$$

$$\frac{d\ell}{dt} = n(\bar{a}_1) + \sum_{j=1}^{N} \epsilon^j \left[A_{6,j}(\vec{\bar{a}}) + n(\bar{a}_1) \frac{\partial \eta_{6,j}}{\partial \bar{\ell}} + \sum_{m=1}^{N-j} \epsilon^m \sum_{k=1}^{6} A_{k,m}(\vec{\bar{a}}) \frac{\partial \eta_{6,j}}{\partial \bar{a}_k} \right] + O(\epsilon^{N+1}) \qquad (3\text{-}6b)$$

The form of these expressions can be rearranged to give

$$\frac{da_i}{dt} = \sum_{j=1}^{N} \epsilon^j \left[A_{i,j}(\vec{\bar{a}}) + n(\bar{a}_1) \frac{\partial \eta_{i,j}}{\partial \bar{\ell}} + \sum_{k=1}^{6} \sum_{p=1}^{j-1} A_{k,p}(\vec{\bar{a}}) \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right] + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}7a)$$

$$\frac{d\ell}{dt} = n(\bar{a}_1) + \sum_{j=1}^{N} \epsilon^j \left[A_{6,j}(\vec{\bar{a}}) + n(\bar{a}_1) \frac{\partial \eta_{6,j}}{\partial \bar{\ell}} + \sum_{k=1}^{6} \sum_{p=1}^{j-1} A_{k,p}(\vec{\bar{a}}) \frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} \right] + O(\epsilon^{N+1}) \qquad (3\text{-}7b)$$

The summation over p is not performed for $j = 1$ and thus does not contribute to the first-order terms.

Next, the perturbing functions on the right-hand side of Equations (3-2) are expanded via a Taylor series about the mean elements as follows:

$$F_i(\vec{a},\ell) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=1}^{6} \Delta a_k \frac{\partial}{\partial \bar{a}_k}\right)^n F_i(\vec{a},\ell)\Bigg|_{\substack{a=\bar{a}\\ \ell=\bar{\ell}}} \qquad (i = 1,2,\ldots,6) \quad (3\text{-}8)$$

where $\Delta a_k = a_k - \bar{a}_k$ are defined by Equation (3-3). The notation $\partial/(\partial \bar{a}_k)$ denotes the operation

$$\frac{\partial}{\partial a_k}\Bigg|_{a_k = \bar{a}_k}$$

and for the sake of conciseness will be used throughout this report. Rearranging Equation (3-8) as a power series in ϵ yields

$$F_i(\vec{a},\ell) = \sum_{j=0}^{N} \epsilon^j f_{i,j}(\vec{\bar{a}},\bar{\ell}) + O(\epsilon^{N+1}) \qquad (i = 1,2,\ldots,6) \quad (3\text{-}9)$$

where

$$f_{i,0}(\vec{\bar{a}},\bar{\ell}) = F_i(\vec{\bar{a}},\bar{\ell}) \qquad (3\text{-}10a)$$

$$f_{i,1}(\vec{\bar{a}},\bar{\ell}) = \sum_{k=1}^{6} \eta_{k,1} \frac{\partial F_i}{\partial \bar{a}_k} \qquad (3\text{-}10b)$$

$$f_{i,2}(\vec{\bar{a}},\bar{\ell}) = \sum_{k=1}^{6}\left(\eta_{k,2}\frac{\partial F_i}{\partial \bar{a}_k} + \frac{1}{2}\sum_{\ell=1}^{6} \eta_{k,1}\eta_{\ell,1}\frac{\partial^2 F_i}{\partial \bar{a}_k \partial \bar{a}_\ell}\right) \qquad (3\text{-}10c)$$

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$$f_{i,3}(\vec{\bar{a}},\bar{l}) = \sum_{k=1}^{6}\left\{\eta_{k,3}\frac{\partial F_i}{\partial \bar{a}_k} + \frac{1}{2}\sum_{l=1}^{6}\left[\left(\eta_{k,2}\eta_{l,1}+\eta_{k,1}\eta_{l,2}\right)\right.\right. \tag{3-10d}$$

$$\left.\left. \times \frac{\partial^2 F_i}{\partial \bar{a}_k \partial \bar{a}_l} + \frac{1}{3}\sum_{j=1}^{6}\eta_{k,1}\eta_{l,1}\eta_{j,1}\frac{\partial^3 F_i}{\partial \bar{a}_k \partial \bar{a}_l \partial \bar{a}_j}\right]\right\}$$

etc. The mean motion, $n(a_1)$, is also expanded in a Taylor series about the mean element, $\bar{a}_1$, i.e.,

$$n(a_1) = \sum_{k=0}^{\infty}\frac{(\Delta a_1)^k}{k!}\frac{d^k n}{d\bar{a}^k} \tag{3-11}$$

Rearranging Equation (3-11) as a power series in ϵ yields

$$n(a_1) = \sum_{k=0}^{N}\epsilon^k N_k(\vec{\bar{a}},\bar{l}) + O(\epsilon^{N+1}) \tag{3-12}$$

where

$$N_0(\vec{\bar{a}},\bar{l}) = n(\bar{a}_1) = \bar{n} \tag{3-13a}$$

$$N_1(\vec{\bar{a}},\bar{l}) = -\frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\eta_{1,1} \tag{3-13b}$$

$$N_2(\vec{\bar{a}},\bar{l}) = \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\eta_{1,1}^2 - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\eta_{1,2} \tag{3-13c}$$

$$N_3(\vec{\bar{a}},\bar{l}) = -\frac{35}{16}\frac{\bar{n}}{\bar{a}_1^3}\eta_{1,1}^3 + \frac{15}{8}\frac{\bar{n}}{a_1^2}\eta_{1,2}\eta_{1,1} - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\eta_{1,3} \tag{3-13d}$$

etc. Substituting Equations (3-7), (3-9), and (3-12) into Equation (3-2) completes the transformation. Equating terms with like powers of ϵ yields the expressions for the jth-order contribution to the osculating element rates, i.e.,

$$A_{i,j}(\bar{\vec{a}}) + \bar{n}\,\frac{\partial \eta_{i,j}}{\partial \bar{\ell}} + \sum_{k=1}^{6}\sum_{p=1}^{j-1} A_{k,p}(\bar{\vec{a}})\,\frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} = f_{i,j-1}(\bar{\vec{a}},\bar{\ell}) \tag{3-14a}$$

$$(i = 1, 2, \ldots, 5)$$

$$A_{6,j}(\bar{\vec{a}}) + \bar{n}\,\frac{\partial \eta_{6,j}}{\partial \bar{\ell}} + \sum_{k=1}^{6}\sum_{p=1}^{j-1} A_{k,p}(\bar{\vec{a}})\,\frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} = f_{6,j-1}(\bar{\vec{a}},\bar{\ell}) + N_j \tag{3-14b}$$

3.2.2 Elimination of the Fast Variable Dependence

In order to determine the averaged equations of motion (Equation (3-4)), the functions $A_{i,j}$, which depend only on the slowly varying mean elements, must be related to the perturbing function or its power series representation. At first glance, it appears that this is accomplished in Equations (3-14). However, the functions $\eta_{i,j}$ are as yet undetermined, except that they are constrained to be 2π periodic in the mean fast variable, $\bar{\ell}$. Fortunately, this condition permits the elimination of the mean fast variable dependence. Integrating both sides of Equation (3-14) over the mean fast variable, $\bar{\ell}$, on the interval $[0, 2\pi]$ eliminates the function $\partial\eta_{i,j}/\partial\bar{\ell}$. This procedure of definite integration is referred to as the averaging operation and is written as

$$\left\langle H(\bar{\vec{a}},\bar{\ell}) \right\rangle_{\bar{\ell}} = \frac{1}{2\pi}\int_0^{2\pi} H(\bar{\vec{a}},\bar{\ell})\, d\bar{\ell} \tag{3-15}$$

Some properties of the averaging operation derived from the above definition are as follows. If $X(\vec{\bar{a}}, \bar{\ell})$ and $Y(\vec{\bar{a}}, \bar{\ell})$ are two functions (appropriately continuous and differentiable) which are 2π periodic in $\bar{\ell}$, then

$$\left\langle X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} = C(\vec{\bar{a}}) \qquad (3\text{-}16a)$$

$$\left\langle X(\vec{\bar{a}}, \bar{\ell})\, Y(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \neq \left\langle X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \left\langle Y(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \qquad (3\text{-}16b)$$

$$\left\langle X(\vec{\bar{a}}, \bar{\ell}) + Y(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} = \left\langle X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} + \left\langle Y(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \qquad (3\text{-}16c)$$

$$\left\langle \rho\, X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} = \rho \left\langle X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \qquad (3\text{-}16d)$$

$$\frac{\partial}{\partial \bar{a}_k} \left\langle X(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} = \left\langle \frac{\partial X(\vec{\bar{a}}, \bar{\ell})}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (k = 1, 2, \ldots, 6) \qquad (3\text{-}16e)$$

where ρ is any function independent of $\bar{\ell}$. These properties will be used implicitly throughout the remainder of this section. Because $\eta_{i,j}$ is 2π periodic in $\bar{\ell}$ (a condition of the near-identity transformation),

$$\left\langle \bar{n} \frac{\partial \eta_{i,j}}{\partial \bar{\ell}} \right\rangle_{\bar{\ell}} = 0 \qquad (3\text{-}17)$$

In view of Equation (3-17), the averaging operation also yields

$$\left\langle \sum_{k=1}^{6} \sum_{p=1}^{j-1} A_{k,p}(\vec{\bar{a}}) \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} = \sum_{k=1}^{5} \sum_{p=1}^{j-1} A_{k,p}(\vec{\bar{a}}) \left\langle \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (3\text{-}18)$$

As a result, the averaged equations representing the jth-order contribution to the mean element rates are

$$A_{i,j}(\vec{\bar{a}}) = \left\langle f_{i,j-1}(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} - \sum_{p=1}^{j-1} \sum_{k=1}^{5} A_{k,p}(\vec{\bar{a}}) \left\langle \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (3\text{-}19a)$$

$$(i = 1, 2, \ldots, 5)$$

$$A_{6,j}(\vec{\bar{a}}) = \left\langle f_{6,j-1}(\vec{\bar{a}}, \bar{\ell}) + N_j \right\rangle_{\bar{\ell}} - \sum_{p=1}^{j-1} \sum_{k=1}^{5} A_{6,p}(\vec{\bar{a}}) \left\langle \frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (3\text{-}19b)$$

These equations can be simplified by requiring that

$$\left\langle \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} = 0 \qquad \begin{matrix} (i = 1, 2, \ldots, 6) \\ (k = 1, 2, \ldots, 5) \end{matrix} \qquad (3\text{-}20)$$

or, equivalently,

$$\left\langle \frac{da_i}{dt} \right\rangle_{\bar{\ell}} = \frac{d\bar{a}_i}{dt} \quad \text{and} \quad \left\langle \frac{d\ell}{dt} \right\rangle_{\bar{\ell}} = \frac{d\bar{\ell}}{dt} \qquad (3\text{-}21)$$

which follows from the application of the averaging operation to Equations (3-5). Consequently, the mean elements $(\bar{\vec{a}}, \bar{\ell})$ represent the long-period and secular contributions to the osculating elements $(\vec{a}, \ell)$ to within a constant, and

$$\left\langle \eta_{i,j}(\bar{\vec{a}},\bar{\ell}) \right\rangle_{\bar{\ell}} = C_{i,j} \tag{3-22}$$

where $C_{i,j}$ is a constant. Equation (3-22) follows from Equations (3-20) and (3-17). A logical extension of the constraint in Equation (3-20) is to require that these constants vanish identically, i.e.,

$$C_{i,j} \equiv 0 \tag{3-23}$$

such that

$$\left\langle a_i \right\rangle_{\bar{\ell}} = \bar{a} \quad \text{and} \quad \left\langle \ell \right\rangle_{\bar{\ell}} = \bar{\ell} \tag{3-24}$$

Initially, in the development of the averaged equations, the functions $\eta_{i,j}$ were quite arbitrary except for the condition of 2 periodicity in $\bar{\ell}$. Equation (3-20) restricts these functions to contain only short-period, mixed short-period,[1] and constant terms. Equation (3-23) further restricts these functions to pure and mixed short-period terms only. That such restricted functions can be determined is demonstrated below.

Applying the constraint expressed in Equation (3-20), Equations (3-19) reduce to

$$A_{i,j}(\bar{\vec{a}}) = \left\langle f_{i,j-1}(\bar{\vec{a}},\bar{\ell}) \right\rangle_{\bar{\ell}} \qquad (i = 1, 2, \ldots, 5) \tag{3-25a}$$

[1] A mixed short-period term is the product of a pure short-period term, $g(\bar{\ell})$, and a long-period term, $f(\bar{\vec{a}})$, i.e., $f(\bar{\vec{a}})\,g(\bar{\ell})$.

$$A_{6,j}(\bar{a}) = \left\langle f_{6,j-1}(\bar{a},\bar{l}) + N_j \right\rangle_{\bar{l}} \qquad (3\text{-}25b)$$

The averaged equations of motion are now completely specified in terms of the expansion of the perturbing function and, in the case of the variable $\bar{l}$, the expansion of the osculating mean motion. More explicitly, substituting Equations (3-25) into Equations (3-4) yields the following expressions for the averaged equations of motion:

$$\frac{d\bar{a}_i}{dt} = \sum_{j=1}^{N} \epsilon^j \left\langle f_{i,j-1}(\bar{a},\bar{l}) \right\rangle_{\bar{l}} + O(\epsilon^{N+1}) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}26a)$$

$$\frac{d\bar{l}}{dt} = n(\bar{a}_1) + \sum_{j=1}^{N} \epsilon^j \left\langle f_{6,j-1}(\bar{a},\bar{l}) + N_j \right\rangle_{\bar{l}} + O(\epsilon^{N+1}) \qquad (3\text{-}26b)$$

The functions $f_{i,k}$ and N_k for $k \geq 1$ are formulated in terms of the as yet undetermined short-periodic functions $\eta_{i,j}$. This dependence is shown explicitly in Equations (3-10) and (3-13). The averaging operation does not free the averaged equations of all contributions from the short-periodic terms. Such contributions are, in fact, the source of the higher order terms in the averaged equations of motion. The product of two short-period functions can yield a long-period term; for example, in the product

$$\left[h(\bar{a}) \sin \bar{l}\right]\left[g(\bar{a}) \sin \bar{l}\right] = \frac{1}{2}h(\bar{a})\, g(\bar{a}) - \frac{1}{2}h(\bar{a})\, g(\bar{a}) \cos 2\bar{l}$$

both factors are of short period, yet the product contains a long-period term (i.e., a term independent of $\bar{l}$).

Inspection of Equations (3-10) and (3-13) indicates that products of the partial derivatives of the osculating force function, F_i, with the functions $\eta_{i,j}$ appear, as do products involving two or more of the functions $\eta_{i,j}$. Such products can produce long-period terms, as described in the above example, that will always be of second or higher order in the small parameter.

3.2.3 Determination of the Short-Period Functions, $\eta_{i,j}$

The general formulation of the averaged equations of motion is completed by obtaining the functions $\eta_{i,j}$ from the information contained in the method of averaging. In the following discussion, these functions are determined without the constraints expressed by Equations (3-20) and (3-23). However, the justification for these constraints is demonstrated.

A partial differential equation for the functions $\eta_{i,j}$ is obtained by subtracting Equations (3-19) from Equation (3-14), yielding

$$\bar{n}\frac{\partial \eta_{i,j}}{\partial \bar{\ell}} + \sum_{p=1}^{j-1}\sum_{k=1}^{6} A_{k,p}\left(\frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} - \left\langle \frac{\partial \eta_{i,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}}\right) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}27a)$$

$$= f_{i,j-1} - \left\langle f_{i,j-1} \right\rangle_{\bar{\ell}}$$

$$\bar{n}\frac{\partial \eta_{6,j}}{\partial \bar{\ell}} + \sum_{p=1}^{j-1}\sum_{k=1}^{6} A_{k,p}\left(\frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} - \left\langle \frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}}\right) \qquad (3\text{-}27b)$$

$$= f_{6,j-1} + N_j - \left\langle f_{6,j-1} + N_j \right\rangle_{\bar{\ell}}$$

If the superscript S denotes the short-periodic part of a function such that

$$f^s_{i,j-1} = f_{i,j-1} - \left\langle f_{i,j-1} \right\rangle_{\bar{l}}$$

then the preceding equations can be expressed as

$$\bar{n}\,\frac{\partial \eta_{i,j}}{\partial \bar{l}} = f^s_{i,j-1} - \sum_{p=1}^{j-1}\sum_{k=1}^{6} A_{k,p}\,\frac{\partial \eta^s_{i,j-p}}{\partial \bar{a}_k} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}28a)$$

$$\bar{n}\,\frac{\partial \eta_{6,j}}{\partial \bar{l}} = f^s_{6,j-1} + N^s_j - \sum_{p=1}^{j-1}\sum_{k=1}^{6} A_{k,p}\,\frac{\partial \eta^s_{6,j-p}}{\partial \bar{a}_k} \qquad (3\text{-}28b)$$

Inspection of Equations (3-28) indicates that the functions $\eta_{i,j}$ $(i = 1,2,\ldots,5)$ depend only on quantities of lower order. In the case of the sixth variable $\bar{l}$, the function $\eta_{6,j}$ also depends on the jth-order function $\eta_{1,j}$ introduced through the term N_j. Hence, the function $\eta_{1,j}$ must be determined prior to the function $\eta_{6,j}$.

These functions are determined to within an arbitrary function of the slow variables, $\vec{a}$, by developing the right-hand side of Equations (3-28) into multiple Fourier series and integrating term by term. More explicitly,

$$\eta_{i,j} = \frac{1}{\bar{n}}\int \left[f^s_{i,j-1}(\bar{a},\bar{l}) - \sum_{k=1}^{6}\sum_{p=1}^{j-1} A_{k,p}\,\frac{\partial \eta^s_{i,j-p}}{\partial \bar{a}_k} \right] d\bar{l} \qquad (3\text{-}29a)$$

$$(i = 1,2,\ldots,5)$$

$$\eta_{6,j} = \frac{1}{\bar{n}}\int \left[\left\{ f^s_{6,j-1}(\bar{a},\bar{l}) + N^s_j \right\} - \sum_{k=1}^{6}\sum_{p=1}^{j-1} A_{k,p}\,\frac{\partial \eta_{6,j-p}}{\partial \bar{a}_k} \right] d\bar{l} \qquad (3\text{-}29b)$$

The functions $\eta_{i,j}$ therefore have the form

$$\eta_{i,j}(\vec{a},\bar{\ell}) = \alpha_{i,j}(\vec{a},\bar{\ell}) + C_{i,j}(\vec{a}) \tag{3-30}$$

where $\alpha_{i,j}$ is a 2π periodic function of $\bar{\ell}$ with zero mean, i.e.,

$$\left\langle \alpha_{i,j}(\vec{a},\bar{\ell}) \right\rangle_{\bar{\ell}} = 0 \tag{3-31}$$

and $C_{i,j}$ is an arbitrary function of integration depending only on the slowly varying mean elements.

It then follows from averaging Equation (3-30) that

$$\left\langle \eta_{i,j} \right\rangle_{\bar{\ell}} = C_{i,j}(\vec{a}) \tag{3-32}$$

This equation is a generalization of the constraints expressed in Equation (3-20) and Equation (3-23). Therefore, in order specify the functions $\eta_{i,j}$ most generally, a set of arbitrary functions of the slow variables is required. Because the function $C_{i,j}(\vec{a})$ is an arbitrary function of integration, it can be taken to be identically zero, i.e.,

$$C_{i,j}(\vec{a}) \equiv 0 \tag{3-33}$$

thereby reproducing the constraint used to obtain the form of the averaged equations of motion given in Equation (3-26). Consequently, the validity of the application of the constraint expressed in either Equation (3-20) or Equation (3-23) has been demonstrated. The use of the constraint given by Equation (3-33) requires that the $\eta_{i,j}$ functions be purely short periodic and or mixed short periodic, i.e.,

$$\eta_{i,j} = \eta^s_{i,j}$$

In summary, a set of functions $\eta_{i,j}$ containing only short-periodic terms can be obtained, and the near-identity transformation given by Equations (3-3) is completely specified by the expressions

$$a_i = \bar{a}_i + \frac{1}{\bar{n}} \sum_{j=1}^{N} \epsilon^j \int \left[f^s_{i,j-1} - \sum_{k=1}^{6} \sum_{p=1}^{j-1} A_{k,p} \frac{\partial \eta^s_{i,j-p}}{\partial \bar{a}_k} \right] d\bar{l} + O(\epsilon^{N+1}) \tag{3-34a}$$

$$(i = 1, 2, \ldots, 5)$$

$$l = \bar{l} + \frac{1}{\bar{n}} \sum_{j=1}^{N} \epsilon^j \int \left[f^s_{6,j-1} + N^s_j - \sum_{k=1}^{6} \sum_{p=1}^{j-1} A_{k,p} \frac{\partial \eta^s_{6,j-p}}{\partial \bar{a}_k} \right] d\bar{l} + O(\epsilon^{N+1}) \tag{3-34b}$$

3.2.4 Computational Procedure

The determination of the jth-order contribution to the mean element rates (Equations (3-26) and the functions $\eta_{i,j}$ are interdependent and must proceed serially on an order-by-order basis. To illustrate this procedure, the second-order equations are presented more explicitly. Expressing Equation (3-26) to second order yields

$$\frac{d\bar{a}_i}{dt} = \epsilon\left\langle f_{i,0}(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} + \epsilon^2\left\langle f_{i,1}(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} + O(\epsilon^3) \quad (i=1,2,\ldots,5) \tag{3-35a}$$

$$\frac{d\bar{l}}{dt} = \bar{n} + \epsilon\left\langle f_{6,0}(\bar{\vec{a}},\bar{l}) + N_1\right\rangle_{\bar{l}} + \epsilon^2\left\langle f_{6,1}(\bar{\vec{a}},\bar{l}) + N_2\right\rangle_{\bar{l}} + O(\epsilon^3) \tag{3-35b}$$

Using Equations (3-13) and the constraints given in Equations (3-22) and (3-23), the averaging operation yields the simplifications

$$\left\langle f_{i,j-1}(\bar{\vec{a}},\bar{l}) + N_j(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} = \left\langle f_{i,j-1}(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} + \left\langle N_j(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} \tag{3-36}$$

$$\left\langle N_1(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} = \left\langle -\frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\eta_{1,1}(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} = 0 \tag{3-37}$$

and

$$\begin{aligned}\left\langle N_2(\bar{\vec{a}},\bar{l})\right\rangle_{\bar{l}} &= \left\langle \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\eta_{1,1}^2 - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1^2}\eta_{1,2}\right\rangle_{\bar{l}} \\ &= \left\langle \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\eta_{1,1}^2\right\rangle_{\bar{l}} - \left\langle \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\eta_{1,2}\right\rangle_{\bar{l}} \\ &= \left\langle \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\eta_{1,1}^2\right\rangle_{\bar{l}}\end{aligned} \tag{3-38}$$

In view of Equations (3-10), (3-36), (3-37), and (3-38), Equations (3-35) simplify to

$$\frac{d\bar{a}_i}{dt} = \epsilon \left\langle F_i(\bar{a},\bar{l}) \right\rangle_{\bar{l}} + \epsilon^2 \left\langle \sum_{k=1}^{6} \eta_{k,1}(\bar{a},\bar{l}) \frac{\partial F_i(\bar{a},\bar{l})}{\partial \bar{a}_k} \right\rangle_{\bar{l}} + O(\epsilon^3) \qquad (3\text{-}39a)$$

$$(i = 1, 2, \ldots, 5)$$

$$\frac{d\bar{l}}{dt} = \bar{n} + \epsilon \left\langle F_6(\bar{a},\bar{l}) \right\rangle_{\bar{l}}$$

$$+ \epsilon^2 \left\langle \sum_{k=1}^{6} \eta_{k,1}(\bar{a},\bar{l}) \frac{\partial F_6(\bar{a},\bar{l})}{\partial \bar{a}_k} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \eta_{1,1}^2(\bar{a},\bar{l}) \right\rangle_{\bar{l}} \qquad (3\text{-}39b)$$

$$+ O(\epsilon^3)$$

Inspection of this equation indicates that the first-order contributions to the mean element rates, $A_{i,1}$, are independent of the functions $\eta_{i,j}$. However, the second-order contributions to the mean element rates, $A_{i,2}$, require knowledge of the functions $\eta_{i,j}$. Hence, the computation must proceed as follows:

$$A_{i,1} = \left\langle F_i(\bar{a},\bar{l}) \right\rangle_{\bar{l}} \qquad (i = 1,2,\ldots,6) \quad (3\text{-}40a)$$

$$\eta_{i,1} = \frac{1}{\bar{n}} \int F_i^s(\bar{a},\bar{l})\, d\bar{l} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}40b)$$

$$\eta_{6,1} = \frac{1}{\bar{n}} \int \left[F_6^s(\bar{a},\bar{l}) - \frac{3}{2} \frac{\bar{n}}{\bar{a}_1} \eta_{1,1} \right] d\bar{l} \qquad (3\text{-}40c)$$

$$A_{i,2} = \left\langle \sum_{k=1}^{6} \eta_{k,1} \frac{\partial F_i}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}40d)$$

$$A_{6,2} = \left\langle \sum_{k=1}^{6} \eta_{k,1} \frac{\partial F_6}{\partial \bar{a}_k} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \eta_{1,1}^2 \right\rangle_{\bar{\ell}} \qquad (3\text{-}40e)$$

This procedure is followed in extending to higher order the averaged equations of motion.

3.3 AVERAGED EQUATIONS OF MOTION FOR MULTIPLE PERTURBING FUNCTIONS

The preceding analysis can be extended in a straightforward manner to the case of multiple perturbations contributing to each element rate. Examples of such cases are: inclusion of more than one spherical harmonic from the nonspherical gravitational potential field, multiple third-body perturbations, and combinations of these effects with atmospheric drag and/or solar radiation pressure. To first order in the small parameters, this formulation is identical to summing the first-order averaged equations of motion (Equations (3-26)) for each perturbation. However, at higher orders, mixed (coupled) terms appear in the averaged equations of motion. To illustrate this phenomenon, the case of two perturbing functions is considered. The corresponding set of differential equations is given by

$$\frac{da_i}{dt} = \epsilon F_i(\vec{a}, \ell) + \nu G_i(\vec{a}, \ell) \qquad (i = 1, 2, \ldots, 5) \qquad \text{(3-41a)}$$

$$\frac{d\ell}{dt} = n + \epsilon F_6(\vec{a}, \ell) + \nu G_6(\vec{a}, \ell) \qquad \text{(3-41b)}$$

The near-identity transformation (Equation (3-3)) is generalized to

$$a_i = \bar{a}_i + \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k \psi_{i,j,k} + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \qquad \text{(3-42a)}$$

$$\ell = \bar{\ell} + \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k \psi_{6,j,k} + O(\epsilon^{N+1}) \qquad \text{(3-42b)}$$

where the functions $\psi_{i,j,k} = \psi_{i,j,k}(\bar{\bar{a}}, \bar{\ell})$ and are 2π periodic in the mean fast variable $\bar{\ell}$.

The transform of the original system (Equations (3-41)) is assumed to be of the form

$$\frac{d\bar{a}_i}{dt} = \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k B_{i,j,k} + O(\epsilon^{N+1}) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}43a)$$

$$\frac{d\bar{\ell}}{dt} = \bar{n} + \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k B_{6,j,k} + O(\epsilon^{N+1}) \qquad (3\text{-}43b)$$

where the functions $B_{i,j,k} = B_{i,j,k}(\bar{\bar{a}})$ depend only on the slowly varying elements. Equations (3-43) are a generalization of Equations (3-4) given previously.

The constraint $1 \le j+k$ is imposed on the lower limits of the double summation in Equations (3-42) by the assumption that the difference between the osculating and mean elements is, at most, of first order in one the the small parameters, i.e.,

$$|a_i - \bar{a}_i| \approx \max\left[O(\epsilon), O(\nu)\right] \qquad (3\text{-}44)$$

Similarly, the same constraint is imposed on the lower limits of the double summation in Equations (3-43) by the assumption that the magnitude of the mean eiement rates is, at most, of first order in one of the small parameters.

In Equations (3-42) and (3-43), the upper limit on the summation over j, N, is chosen such that all contributions through order $O(\epsilon^N)$ are retained. Terms with increasing powers of ν obviously require decreasing powers of ϵ in order to meet the criterion that only terms of order less than or equal to $O(\epsilon^N)$ be retained, i.e., $O(\epsilon^j \nu^k) \le O(\epsilon^N)$. Specifically, for a given value of j , the maximum value of k , M(j), is given by the integer part of the expression

$$M(j) = (N-j)\frac{\log\epsilon}{\log\nu} \tag{3-45}$$

and the range of M(j) is $0 \le M(j) \le I = \left[N\frac{\log\epsilon}{\log\nu}\right]$ for $N \le j \le 0$.

Differentiating Equations (3-42) and substituting Equations (3-43) for the mean element rates into the result yields

$$\frac{da_i}{dt} = \sum_{\substack{j=0 \\ (1\le j+k)}}^{N}\sum_{k=0}^{M(j)} \epsilon^j \nu^k \left(B_{i,j,k} + \bar{n}\frac{\partial\psi_{i,j,k}}{\partial\bar{l}} \right. \qquad (i = 1,2,\ldots,5) \tag{3-46a}$$

$$\left. + \sum_{q=1}^{6}\frac{\partial\psi_{i,j,k}}{\partial\bar{a}_q}\sum_{\substack{j'=0 \\ (1\le j'+k')}}^{N-j}\sum_{k'=0}^{M(j)-k}\epsilon^{j'}\nu^{k'}B_{q,j',k'}\right) + O(\epsilon^{N+1})$$

$$\frac{dl}{dt} = \bar{n} + \sum_{\substack{j=0 \\ (1\ \ j+k)}}^{N}\sum_{k=0}^{M(j)} \epsilon^j \nu^k \left(B_{6,j,k} + \bar{n}\frac{\partial\psi_{6,j,k}}{\partial\bar{l}} \right. \tag{3-46b}$$

$$\left. + \sum_{q=1}^{6}\frac{\partial\psi_{6,j,k}}{\partial\bar{a}_q}\sum_{\substack{j'=0 \\ (1\le j'+k')}}^{N-j}\sum_{k'=0}^{M(j)-k}\epsilon^{j'}\nu^{k'}B_{q,j',k'}\right) + O(\epsilon^{N+1})$$

which is a generalization of Equations (3-6). Rearranging Equations (3-46) yields the following generalization of Equations (3-7):

$$\frac{da_i}{dt} = \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k \left(B_{i,j,k} + \bar{n} \frac{\partial \psi_{i,j,k}}{\partial \bar{\ell}} + \sum_{\substack{q=1 \\ [1 \le (r+s) < (j+k)]}}^{6} \sum_{r=0}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi_{i,j-r,k-s}}{\partial \bar{a}_q} \right) + O(\epsilon^{N+1}) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}47a)$$

$$\frac{d\ell}{dt} = \bar{n} + \sum_{j=0}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k \left(B_{6,j,k} + \bar{n} \frac{\partial \psi_{6,j,k}}{\partial \bar{\ell}} + \sum_{\substack{q=1 \\ [1 \le (r+s) < (j+k)]}}^{6} \sum_{r=0}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi_{6,j-r,k-s}}{\partial \bar{a}_q} \right) + O(\epsilon^{N+1}) \qquad (3\text{-}47b)$$

It is advantageous to decompose the above double summation as follows:

$$\sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k \left(B_{i,j,k} + \bar{n} \frac{\partial \psi_{i,j,k}}{\partial \bar{\ell}} + \sum_{q=1}^{6} \sum_{\substack{r=0 \\ [1 \le (r+s) < (j+k)]}}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi_{i,j-r,k-s}}{\partial \bar{a}_q} \right)$$

$$= \sum_{j=1}^{N} \epsilon^j \left(B_{i,j,0} + \bar{n} \frac{\partial \psi_{i,j,0}}{\partial \bar{\ell}} + \sum_{q=1}^{6} \sum_{r=1}^{j-1} B_{q,r,0} \frac{\partial \psi_{i,j-r,0}}{\partial \bar{a}_q} \right)$$

$$+ \sum_{k=1}^{M(j=0)} \nu^k \left(B_{i,0,k} + \bar{n} \frac{\partial \psi_{i,0,k}}{\partial \bar{\ell}} + \sum_{q=1}^{6} \sum_{s=1}^{k-1} B_{q,0,s} \frac{\partial \psi_{i,0,k-s}}{\partial \bar{a}_q} \right) \qquad (3\text{-}48)$$

$$+ \sum_{j=1}^{N-1} \sum_{k=1}^{M(j)} \epsilon^j \nu^k \left(B_{i,j,k} + \bar{n} \frac{\partial \psi_{i,j,k}}{\partial \bar{\ell}} + \sum_{q=1}^{6} \sum_{\substack{r=0 \\ [1 \le (r+s) < (j+k)]}}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi_{i,j-r,k-s}}{\partial \bar{a}_q} \right)$$

for $i = 1, 2, \ldots, 6$.

Expanding the perturbing functions F_i and G_i in Equation (3-41) as a Taylor series and then arranging as a power series in the small parameter yields

$$
\begin{aligned}
\epsilon F_i(\bar{a}, l) + \nu G_i(\bar{a}, l) &= \sum_{j=0}^{N-1} \sum_{k=0}^{M(j+1)} \epsilon^j \nu^k \left(\epsilon f_{i,j,k} + \nu g_{i,j,k}\right) + O(\epsilon^{N+1}) \\
&= \sum_{j=1}^{N} \epsilon^j f_{i,j-1,0} + \sum_{k=1}^{M(j=0)} \nu^k g_{i,0,k-1} \\
&\quad + \sum_{j=1}^{N-1} \sum_{k=1}^{M(j)} \epsilon^j \nu^k \left(f_{i,j-1,k} + g_{i,j,k-1}\right) + O(\epsilon^{N+1})
\end{aligned}
\tag{3-49}
$$

where

$$f_{i,0,0} = F(\bar{a}, \bar{l}) \tag{3-50a}$$

$$f_{i,1,0} = \sum_{q=1}^{6} \psi_{q,1,0} \frac{\partial F_i}{\partial \bar{a}_q} \tag{3-50b}$$

$$f_{i,0,1} = \sum_{q=1}^{6} \psi_{q,0,1} \frac{\partial F_i}{\partial \bar{a}_q} \tag{3-50c}$$

$$f_{i,2,0} = \sum_{q=1}^{6} \left(\psi_{q,2,0} \frac{\partial F_i}{\partial \bar{a}_q} + \frac{1}{2} \sum_{t=1}^{6} \psi_{q,1,0} \psi_{t,1,0} \frac{\partial^2 F_i}{\partial \bar{a}_q \partial \bar{a}_t} \right) \tag{3-50d}$$

$$f_{i,0,2} = \sum_{q=1}^{6} \left(\psi_{q,0,2} \frac{\partial F_i}{\partial \bar{a}_q} + \frac{1}{2} \sum_{t=1}^{6} \psi_{q,0,1} \psi_{t,0,1} \frac{\partial^2 F_i}{\partial \bar{a}_q \partial \bar{a}_t} \right) \tag{3-50e}$$

$$f_{i,1,1} = \sum_{q=1}^{6} \left[\psi_{q,1,1} \frac{\partial F_i}{\partial \bar{a}_q} + \frac{1}{2} \sum_{t=1}^{6} \left(\psi_{q,1,0} \psi_{t,0,1} + \psi_{q,0,1} \psi_{t,1,0} \right) \frac{\partial^2 F_i}{\partial \bar{a}_q \partial \bar{a}_t} \right] \tag{3-50f}$$

etc. The functions $g_{i,j,k}$ are identical to the above, with the exception that the function $F_i(\vec{a}, \ell)$ and its partial derivatives are replaced by the function $G_i(\vec{a}, \ell)$ and its partial derivatives evaluated at $\vec{a} = \bar{\vec{a}}$, $\ell = \bar{\ell}$.

Expanding the mean motion, n, as a power series in the small parameter yields

$$\begin{aligned} n &= \bar{n} + \sum_{\substack{j=0 \\ (1 \le j+k)}}^{N} \sum_{k=0}^{M(j)} \epsilon^j \nu^k N_{j,k} + O(\epsilon^{N+1}) \\ &= \bar{n} + \sum_{j=1}^{N} \epsilon^j N_{j,0} + \sum_{k=1}^{M(j=0)} \nu^k N_{0,k} \\ &\quad + \sum_{j=1}^{N-1} \sum_{k=1}^{M(j)} \epsilon^j \nu^k N_{j,k} + O(\epsilon^{N+1}) \end{aligned} \tag{3-51}$$

where

$$N_{1,0} = -\frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\psi_{1,1,0} \tag{3-52a}$$

$$N_{0,1} = -\frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\psi_{1,0,1} \tag{3-52b}$$

$$N_{2,0} = \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\psi_{1,1,0}^2 - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\psi_{i,2,0} \tag{3-52c}$$

$$N_{0,2} = \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\psi_{1,0,1}^2 - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\psi_{i,0,2} \tag{3-52d}$$

$$N_{1,1} = \frac{15}{8}\frac{\bar{n}}{\bar{a}_1^2}\psi_{1,1,0}\psi_{1,0,1} - \frac{3}{2}\frac{\bar{n}}{\bar{a}_1}\psi_{1,1} \tag{3-52e}$$

and so forth.

Equations (3-47), (3-49), and (3-51) are substituted into Equations (3-41) and terms with like powers of the small parameters are set equal, thus obtaining

$$B_{i,j,0} + \bar{n}\frac{\partial \psi_{i,j,0}}{\partial \bar{\ell}} + \sum_{q=1}^{6}\sum_{r=1}^{j-1} B_{q,r,0}\frac{\partial \psi_{i,j-r,0}}{\partial \bar{a}_q} = f_{i,j-1,0} \qquad (i = 1, 2, \ldots, 5) \tag{3-53a}$$

$$B_{6,j,0} + \bar{n}\frac{\partial \psi_{6,j,0}}{\partial \bar{\ell}} + \sum_{q=1}^{6}\sum_{r=1}^{j-1} B_{q,r,0}\frac{\partial \psi_{6,j-r,0}}{\partial \bar{a}_q} = f_{6,j-1,0} + N_{j,0} \tag{3-53b}$$

where $1 \le j \le N$.

$$B_{i,0,k} + \bar{n}\frac{\partial\psi_{i,0,k}}{\partial\bar{\ell}} + \sum_{q=1}^{6}\sum_{s=1}^{k-1} B_{q,0,s}\frac{\partial\psi_{i,0,k-s}}{\partial\bar{a}_q} = g_{i,0,k-1} \quad (i = 1,2,\ldots,5) \tag{3-54a}$$

$$B_{6,0,k} + \bar{n}\frac{\partial\psi_{6,0,k}}{\partial\bar{\ell}} + \sum_{q=1}^{6}\sum_{s=1}^{k-1} B_{q,0,s}\frac{\partial\psi_{6,0,k-s}}{\partial\bar{a}_q} = g_{6,0,k-1} + N_{0,k} \tag{3-54b}$$

where $1 \le k \le M(j=0)$

$$B_{i,j,k} + \bar{n}\frac{\partial\psi_{i,j,k}}{\partial\bar{\ell}} + \sum_{q=1}^{6}\sum_{r=0}^{j}\sum_{s=0}^{k} B_{q,r,s}\frac{\partial\psi_{i,j-r,k-s}}{\partial\bar{a}_q}$$
$$[1 \le (r+s) < (j+k)]$$
$$= f_{i,j-1,k} + g_{i,j,k-1} \quad (i = 1,2,\ldots,5) \tag{3-55a}$$

$$B_{6,j,k} + \bar{n}\frac{\partial\psi_{6,j,k}}{\partial\bar{\ell}} + \sum_{q=1}^{6}\sum_{r=0}^{j}\sum_{s=0}^{k} B_{q,r,s}\frac{\partial\psi_{6,j-r,k-s}}{\partial\bar{a}_q}$$
$$[1 \le (r+s) < (j+k)]$$
$$= f_{6,j-1,k} + g_{6,j,k-1} + N_{j,k} \tag{3-55b}$$

where $1 \le j \le (N-1)$ and $1 \le k \le M(j)$.

Equations (3-53) and (3-54) are essentially identical in form to the corresponding equations for a single perturbing function given by Equations (3-14). Consequently, to obtain the complete higher order contributions for the case of two perturbing forces, Equations (3-55) representing the coupled terms must be added to the equations for each perturbation given by Equations (3-14). Equations (3-53), (3-54), and (3-55) are then averaged (essentially as before), yielding the averaged equations of motion; the remainder of the solution then proceeds as before. The final results are as follows:

$$B_{i,j,0} = \left\langle f_{i,j-1,0} \right\rangle_{\bar{l}} \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}56a)$$

$$B_{6,j,0} = \left\langle f_{6,j-1,0} + N_{j,0} \right\rangle_{\bar{l}} \quad (3\text{-}56b)$$

where $1 \le j \le N$

$$B_{i,0,k} = \left\langle g_{i,0,k-1} \right\rangle_{\bar{l}} \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}56c)$$

$$B_{6,0,k} = \left\langle g_{6,0,k-1} + N_{0,k} \right\rangle_{\bar{l}} \quad (3\text{-}56d)$$

where $1 \le k \le M (j=0)$

$$B_{i,j,k} = \left\langle f_{i,j-1,k} + g_{i,j,k-1} \right\rangle_{\bar{l}} \quad (3\text{-}56e)$$

$$B_{6,j,k} = \left\langle f_{6,j-1,k} + g_{6,j,k-1} + N_{j,k} \right\rangle_{\bar{l}} \tag{3-56f}$$

where $1 \leq j \leq N-1$ and $1 \leq k \leq M(j)$

$$\psi_{i,j,0} = \frac{1}{\bar{\bar{n}}} \int \left[f^{S}_{i,j-1,0} - \sum_{q=1}^{6} \sum_{r=1}^{j-1} B_{q,r,0} \frac{\partial \psi^{S}_{i,j-r,0}}{\partial \bar{a}_q} \right] d\bar{l} \tag{3-57a}$$

$(i = 1, 2, \ldots, 5;\ 1 \leq j \leq N)$

$$\psi_{6,j,0} = \frac{1}{\bar{\bar{n}}} \int \left[f^{S}_{6,j-1,0} + N^{S}_{j,0} - \sum_{q=1}^{6} \sum_{r=1}^{j-1} B_{q,r,0} \frac{\partial \psi^{S}_{6,j-r,0}}{\partial \bar{a}_q} \right] d\bar{l} \tag{3-57b}$$

$(1 \leq j \leq N)$

$$\psi_{i,0,k} = \frac{1}{\bar{\bar{n}}} \int \left[g^{S}_{i,0,k-1} - \sum_{q=1}^{6} \sum_{s=1}^{k-1} B_{q,0,s} \frac{\partial \psi^{S}_{i,0,k-s}}{\partial \bar{a}_q} \right] d\bar{l} \tag{3-57c}$$

$(i = 1, 2, \ldots, 5;\ 1 \leq k \leq M(j=0))$

$$\psi_{6,0,k} = \frac{1}{\bar{\bar{n}}} \int \left[g^{S}_{6,0,k-1} + N^{S}_{0,k} - \sum_{q=1}^{6} \sum_{s=1}^{k-1} B_{q,0,s} \frac{\partial \psi^{S}_{6,0,k-s}}{\partial \bar{a}_q} \right] d\bar{l} \tag{3-57d}$$

$(1 \leq k \leq M(j=0))$

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$$\psi_{i,j,k} = \frac{1}{\bar{n}} \int \Bigg[f^{S}_{i,j-1,k} + g^{S}_{i,j,k-1} - \sum_{q=1}^{6} \sum_{r=0}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi^{S}_{i,j-r,k-s}}{\partial \bar{a}_q} \Bigg] d\bar{\ell}$$

$$[1 \le (r+s) < (j+k)]$$

$(i = 1, 2, \ldots, 5;\ 1 \le j \le N-1;\ 1 \le k \le M(j))$

$$\psi_{6,j,k} = \frac{1}{\bar{n}} \int \Bigg[f^{S}_{6,j-1,k} + g^{S}_{6,j,k-1} + N^{S}_{j,k} - \sum_{q=1}^{6} \sum_{r=0}^{j} \sum_{s=0}^{k} B_{q,r,s} \frac{\partial \psi^{S}_{6,j-r,k-s}}{\partial \bar{a}_q} \Bigg] d\bar{\ell} \tag{3-57f}$$

$(1 \le j \le N-1;\ 1 \le k \le M(j))$

where the superscript S again denotes the short-period part of the function. The assumptions used to obtain Equations (3-56) and (3-57) require that

$$\psi_{i,j,k} = \psi^{S}_{i,j,k}$$

as in the case for the single perturbing function.

The explicit averaged equations of motion to second order in both small parameters reduce to

$$\frac{d\bar{a}_i}{dt} = \epsilon \left\langle F_i(\vec{\bar{a}},\bar{l})\right\rangle_{\bar{l}} + \nu \left\langle G_i(\vec{\bar{a}},\bar{l})\right\rangle_{\bar{l}}$$

$$+ \sum_{q=1}^{6} \left[\epsilon^2 \left\langle \psi_{q,1,0} \frac{\partial F_i}{\partial \bar{a}_q} \right\rangle_{\bar{l}} + \epsilon\nu \left\langle \psi_{q,0,1} \frac{\partial F_i}{\partial \bar{a}_q} + \psi_{q,1,0} \frac{\partial G_i}{\partial \bar{a}_q} \right\rangle_{\bar{l}} \right. \tag{3-58a}$$

$$\left. + \nu^2 \left\langle \psi_{q,0,1} \frac{\partial G_i}{\partial \bar{a}_q} \right\rangle_{\bar{l}} \right]$$

$$(i = 1, 2, \ldots, 5)$$

$$\frac{d\bar{l}}{dt} = \bar{n} + \epsilon \left\langle F_6(\vec{\bar{a}},\bar{l})\right\rangle_{\bar{l}} + \nu \left\langle G_6(\vec{\bar{a}},\bar{l})\right\rangle_{\bar{l}}$$

$$+ \sum_{q=1}^{6} \left[\epsilon^2 \left\langle \psi_{q,1,0} \frac{\partial F_6}{\partial \bar{a}_q} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \psi_{1,1,0}^2 \right\rangle_{\bar{l}} \right. \tag{3-58b}$$

$$+ \epsilon\nu \left\langle \psi_{q,0,1} \frac{\partial F_6}{\partial \bar{a}_q} + \psi_{q,1,0} \frac{\partial G_6}{\partial \bar{a}_q} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \psi_{1,1,0} \psi_{1,0,1} \right\rangle_{\bar{l}}$$

$$\left. + \nu^2 \left\langle \psi_{q,0,1} \frac{\partial G_i}{\partial \bar{a}_q} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \psi_{1,0,1}^2 \right\rangle_{\bar{l}} \right]$$

and the order of computation follows as in the single perturbing function case.

Extension of the discussion to an arbitrary number of perturbing functions is a straightforward but tedious exercise. The differential equations take the form

$$\frac{da_i}{dt} = \sum_{k=1}^{K} \gamma_k H_{i,k}(\vec{a},\ell) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}59a)$$

$$\frac{d\ell}{dt} = n + \sum_{k=1}^{K} \gamma_k H_{6,k}(\vec{a},\ell) \qquad (3\text{-}59b)$$

where γ_k (k = 1.2,. . ,K) are K distinct small parameters (i.e., $\gamma_1 \equiv \epsilon$, $\gamma_2 \equiv \nu$, etc.) and $H_{i,k}$ is the kth perturbing function acting on the ith element (i.e., $H_{i,1} = F_i(\bar{\vec{a}}, \ell)$, $H_{i,2} = G_i(\bar{\vec{a}}, \bar{\ell})$, etc.). The near-identity transformation and the transform of the above differential equations become

$$a_i = \bar{a}_i + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_K} \gamma_1^{j_1} \gamma_2^{j_2} \cdots \gamma_K^{j_K} \psi_{i,j_1 j_2,\ldots,j_K}(\vec{\bar{a}},\bar{\ell}) \qquad (3\text{-}60)$$

$$(i = 1,2,\ldots,6)$$

and

$$\frac{d\bar{a}_i}{dt} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_K} \gamma_1^{j_1} \gamma_2^{j_2} \cdots \gamma_K^{j_K} B_{i,j_1,j_2,\ldots,j_K}(\bar{\vec{a}}) \qquad (3\text{-}61a)$$

$$(i = 1,2,\ldots,5)$$

$$\frac{d\bar{\ell}}{dt} = \bar{n} + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_K} \gamma_1^{j_1} \gamma_2^{j_2} \cdots \gamma_K^{j_K} B_{6,j_1,j_2,\ldots,j_K}(\bar{\vec{a}}) \qquad (3\text{-}61b)$$

respectively.

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The perturbing functions are expanded in the form

$$H_{i,k}(\vec{a},\ell) = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_K} \gamma^{j_1} \gamma^{j_2} \cdots \gamma^{j_K} h_{i,k,j_1,j_2,\cdots,j_K}(\vec{a},\ell) \qquad (3\text{-}62)$$

etc. Further pursuit of this procedure adds little additional insight except for an appreciation of the cumbersome expressions obtained for the final results.

A less involved approach is presented next, based on the fact that the practical application of such theories is almost always limited to at most second order in the small parameters (see Section 3.5). As previously shown in the case of two perturbing functions, the second-order averaged equations of motion for a single perturbation are summed and the coupled term is then added. For K perturbing functions, the same procedure holds to second order, i.e., K equations of the same form as in the single perturbing function case are summed. The coupled terms are then evaluated to complete the second-order contributions. The number of coupled terms is simply the distinct number of pairs obtained from K objects taken two at a time, i.e.,

$$\frac{K!}{(K-2)!\,2!} = \frac{K(K-1)}{2}$$

This procedure provides for all contributions from the K perturbing functions to the averaged equations of motion through second order in all K small parameters, γ_K.

3.4 MODIFICATION OF THE AVERAGING OPERATION FOR RESONANT PHENOMENA

A commensurability of two mean motions appearing in the dynamical system, e.g., the satellite and third-body mean motions or the satellite mean motion and the central body rotation rate, can contribute significantly to the long-period motion of the satellite. The generalized method of averaging presented in Sections 3.2 and 3.3 is directly applicable to cases involving such resonance phenomena.

The basic objective in applying the method of averaging to the orbital equations of motion is the removal of short-period terms. The averaging procedure defined by Equation (3-15) removes the high-frequency components of the motion for the majority of problems but is not suitable for the treatment of all resonance phenomena. In those cases for which resonance phenomena are significant, the averaging operation given in Equation (3-15) may have to be modified. The necessity of this modification depends on the criteria used for selecting short-period terms and the characteristics of the perturbing functions.

3.4.1 Frequency Characteristics Specific to Resonant Phenomena

The existence of a resonance condition, i.e., a commensurability in the mean motions of the fast variables of the perturbed and perturbing bodies, dictates that these fast variables cannot be considered mutually independent. An arbitrary term in the Fourier series expansion for the perturbing function takes the general form

$$A_{j,k} \cos(j\ell - k\ell' + \theta_1) + B_{j,k} \sin(j\ell - k\ell' + \theta_2) \tag{3-63}$$

where ℓ and ℓ' are the fast variables of the perturbed and perturbing bodies and θ_1 and θ_2 are linear combinations of slowly varying angles.

The fast variables ℓ and ℓ' are assumed to have the mean motions n and n' , respectively. If the ratio of the mean motions is approximately equal to the ratio of two integers, i.e.,

$$\frac{n}{n'} \approx \frac{N}{N'} \tag{3-64}$$

then

$$N'n - Nn' \approx 0 \tag{3-65}$$

The fast variables thus obey the relationship

$$N'\ell - N\ell' = \mu \tag{3-66}$$

where the function $\mu = \mu(t)$ is a slowly varying angle which produces only long-period effects.

One of the fast variables can be eliminated from the perturbing function using Equation (3-66), resulting in a formulation dependent on only one fast variable and an additional slow variable $\mu(t)$. Eliminating the fast variable ℓ' from terms of the form given in Equation (3-63) yields arguments of the form

$$\left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \left\{ (jN - kN') \frac{\ell}{N} + \theta_i' \right\} \tag{3-67}$$

where

$$\theta_i' = \theta_i - \frac{k}{N}\mu \qquad (i = 1, 2)$$

Elimination of the fast variable ℓ in favor of ℓ' yields trigonometric arguments of the same form. More specifically, the quantities N and N' are interchanged, ℓ is replaced by ℓ', and θ_i' is defined by the sum rather than the difference.

In general, arguments of the form given in Equation (3-67) produce fractional as well as integral multiples of the fast variable ℓ. This is specifically the case when kN' is not a multiple of N. An arbitrary decision to consider only integral multiples of the fast variable as short period is not practical in this case, particularly in view of the desire to maximize the integration step size. For example, the case of a close-Earth satellite in a 12:1 resonance with the Earth's rotation is considered. From Equations (3-64), $N = 12$ and $N' = 1$, and the argument in Equation (3-67) can be expressed as

$$j\ell + \frac{k}{12}\ell + \theta_i'$$

This argument will contribute terms containing the fractional arguments

$1/12\,\ell$, $1/6\,\ell$, $1/4\,\ell$, $1/3\,\ell$, $5/12\,\ell$, $1/2\,\ell$, $7/12\,\ell$, $2/3\,\ell$, $3/4\,\ell$, and $11/12\,\ell$

for those values of k which are not multiples of 12. The averaging operation defined by Equation (3-15) will not remove terms with these arguments. Defining terms containing the arguments 2ℓ and ℓ as short period and terms containing $1/2\,\ell$, $11/12\,\ell$, etc., as long period would restrict the integration step size to approximately one-eighth of the satellite revolution period. To maximize the integration step size (hopefully to the order of several orbital periods), while retaining the basic long-period behavior of the dynamical system, all dependence on the fast variable should be eliminated. This requirement is identical in philosophy to that imposed in the selection of the averaging operation for nonresonant phenomena (Equation (3-15)).

3.4.2 The Averaging Operation for Resonant Phenomena

When resonant phenomena are included in the equations of motion, the selection of an optimal averaging operation is dependent on the form of the perturbing function. The resonant contribution is embedded in this function and is isolated by the application of the averaging operation to the function. For this discussion, the resonant perturbing functions are separated into two categories: embedded resonant terms and quasi-isolated resonant terms. These categories are distinguished according to whether or not the perturbing functions contribute terms with fractional multiples of the fast variable.

An embedded resonant term contributes fractional multiples of the fast variable. Such formulations of the perturbing function are frequently encountered in numerical averaging applications where the perturbing function is formulated in terms of the complete perturbing acceleration (Equation (2-15)).

The second category of perturbing functions (i.e., quasi-isolated resonant terms) contributes only integral multiples of the fast variable. The resonant contribution has been partly isolated from the complete perturbing function such that only integral multiples of the fast variable appear. More specifically, the perturbing function is restricted such that the integer k in Equation (3-67) takes on only values which are multiples of N, i.e.,

$$k = pN \qquad (p = 1, 2, \ldots)$$

It is important to note that no restriction has been placed on the integer j in Equation (3-67). Since only particular values of j produce the resonant contribution, the quasi-isolated resonant term contributes both short-period (integral multiples of the fast variable only) and resonant contributions to the motion. If the values of j are restricted appropriately, the resonant term is completely isolated from the perturbation function.

As an example of a quasi-isolated resonant term, the 12:1 resonance example cited previously is again considered. If k is restricted to multiples of N, i.e., multiples of 12, then all fractional multiples of the fast variable are eliminated. These terms correspond to the tesseral harmonic terms in the geopotential of order 12. In this case, any geopotential term of order 12 would be a quasi-isolated resonant term. The specific resonant term, which will be isolated by the application of the averaging operation, corresponds in this case to the value of j where $j = 1$.

3.4.2.1 The averaging Operation for Embedded Resonant Terms

In the case of embedded resonant terms, fractional multiples of the fast variable appear in the perturbing function. In view of the form of the argument given in Equation (3-67), all dependence on the fast variable ℓ can be removed by defining the averaging operation to be the definite integral over the angle $\sigma = \ell/N$ on the interval $0 \leq \sigma \leq 2\pi$. Expressing a function of two fast variables denoted by H in terms of the fast variable σ and the slow variable μ yields

$$H(\vec{a},\ell,\ell') = H'(\vec{a},\ell,\mu) = H^*(\vec{a},\sigma,\mu)$$

The average of the function $H^*(\vec{a}, \sigma, \mu)$ is defined as

$$\left\langle H^*(\vec{a},\sigma,\mu)\right\rangle_\sigma = \frac{1}{2\pi}\int_0^{2\pi} H^*(\quad,\sigma,\mu)\, d\sigma \qquad (3\text{-}68)$$

The averaging definition can be expressed explicitly in terms of the fast variable ℓ. If $0 \le \sigma \le 2\pi$, then $0 \le \ell \le 2\pi N$ and

$$\frac{1}{2\pi}\int_0^{2\pi} H^*(\vec{a},\sigma,\mu)\,d\sigma = \frac{1}{2\pi N}\int_0^{2\pi N} H'(\vec{a},\ell,\mu)\,d\ell$$

$$= \frac{1}{2\pi N}\int_0^{2\pi N} H(\vec{a},\ell,\ell')\,d\ell \tag{3-69}$$

Therefore, in the case of an embedded r sonant term, the definition of the averaging operation should be specified as

$$\left\langle H(\vec{a},\ell,\ell')\right\rangle_\sigma = \frac{1}{2\pi N}\int_0^{2\pi N} H(\vec{a},\ell,\ell')\,d\ell \tag{3-70}$$

This definition has been used by Schubart (Reference 43) for performing a numerical investigation of the Hilda group of minor planets which exhibit a 3:2 commensurability with Jupiter. Also, Benson and Williams (Reference 44) used the same definition in their numerical investigation of resonances in the Neptune-Pluto system.

It should be noted that the above averaging operation removes only those terms with periods of $2\pi N$ or less. It does not remove any contributions to the motion caused by the resonance, since the fundamental period in the motion caused by the resonance is contributed by the angular variable μ and is given by

$$\frac{2\pi}{N'n - Nn'}$$

Clearly, if Equation (3-65 holds,

$$\frac{2\pi}{N'n - Nn'} \gg 2\pi N$$

3.4.2.2 The Averaging Operation for Quasi-Isolated Resonant Terms

Since only integral multiples of the fast variable, ℓ , appear in the case of quasi-isolated resonant terms, the averaging operation given in Equation (3-15) is applicable. It is repeated here for convenience:

$$\left\langle H^*(\vec{a},\ell,\ell') \right\rangle_\ell = \frac{1}{2\pi}\int_0^{2\pi} H^*(\vec{a},\ell,\ell')\, d\ell \qquad (3\text{-}71)$$

where H* denotes a quasi-isolated resonant term.

The distinction in the averaging operations given in Equations (3-70) and (3-71) has an important implication for numerical averaging theories where the averaging is performed using a numerical quadrature. The perturbation model must be evaluated at each abscissa in the quadrature interval (usually between 12 and 96 points per interval). Numerically averaging an embedded resonant term requires N times as many force evaluations as the numerical averaging of a quasi-isolated resonant term for a total of between 12N and 96N force evaluations. Therefore, in the application of the numerical averaging methods, the perturbation models should be restricted to the quasi-isolated resonant terms whenever possible.

The spherical harmonic expansion representing the nonspherical gravitational potential is well suited for obtaining by inspection the quasi-isolated resonant terms. The commensurability is directly related to the order of those terms which contribute to the resonance. Such is not the case for the closed-form, third-body perturbing acceleration or even for the standard expansion in

Legendre polynomials for the third-body disturbing function. The resonance contributions remain embedded in these particular forms. However, the third-body disturbing function can be expanded in spherical harmonics using the associated Legendre polynomials (Reference 18). The quasi-isolated resonant terms are then immediately obvious as in the case of the nonspherical gravitational potential.

3.5 THE APPLICATION OF HIGHER ORDER AVERAGING THEORIES

The implementation of a jth-order theory requires the explicit determination of the near-identity transformation through order j-1. Consequently, higher order averaging theories are significantly more complex than the first-order theory. Examination of the second-order averaged equations of motion indicates that a first-order theory should suffice for all cases where the amplitude of the first-order short-period variations in the osculating elements are small, either absolutely or relative to the amplitude of the long-period variations in the mean elements.

In cases where a second-order theory is needed, it should be applied selectively to those terms producing the largest short-period perturbations, e.g., the oblateness (J_2) term in the zonal harmonic expansion or the first few terms in the expansion of the third-body disturbing function. Such restrictions are usually justified on physical grounds and by the practical considerations of implementing a higher order theory. For those cases where such restrictions cannot be justified on physical grounds, an alternate formulation of a problem, e.g., a restricted three-body problem, should be considered.

3.5.1 The Significance of Second-Order Terms

Two questions are of particular interest concerning the possible significance of second-order terms in the averaged equations of motion:

- How do the solutions of the first-order equations and second-order equations differ with time?
- What are sufficient conditions such that second-order terms can be neglected over the time interval $0 \leq t \leq T$?

A precise answer to the first question is impossible without generating the actual solutions; however, a qualitative estimate of this behavior is possible. The answer to the second question is provided by inspection of the second-order averaged equations of motion.

3.5.1.1 A Qualitative Comparison of the First- and Second-Order Theories[1]

The quantity $[\vec{\bar{a}}'(t), \bar{\ell}'(t)]$ is defined to be the solution of the following system of second-order averaged equations:

$$\frac{d\bar{a}_i'}{dt} = \epsilon A_{i,1}(\vec{\bar{a}}') + \epsilon^2 A_{i,2}(\vec{\bar{a}}') \qquad (i = 1,2,\ldots,5) \quad (3\text{-}72a)$$

$$\frac{d\bar{\ell}'}{dt} = n(\bar{a}_1') + \epsilon A_6(\vec{\bar{a}}') + \epsilon^2 A_{6,2}(\vec{\bar{a}}') \qquad (3\text{-}72b)$$

Similarly, $[\vec{\bar{a}}^*(t), \bar{\ell}^*(t)]$ designates the solution of the system of first-order averaged equations

$$\frac{d\bar{a}_i^*}{dt} = \epsilon A_{i,1}(\vec{\bar{a}}^*) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}73a)$$

$$\frac{d\bar{\ell}^*}{dt} = n(\bar{a}_1^*) + \epsilon A_{6,1}(\vec{\bar{a}}^*) \qquad (3\text{-}73b)$$

The difference of the solutions is designated as

$$r_i(t) = \bar{a}_i'(t) - \bar{a}_i^*(t) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}74a)$$

$$r_6(t) = \bar{\ell}'(t) - \bar{\ell}^*(t) \qquad (3\text{-}74b)$$

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[1] This discussion follows closely that given by W. T. Kyner in a series of lectures on the topic of nonlinear resonance (see Reference 8).

A set of differential equations representing these differences is given by

$$\frac{dr_i}{dt} = \epsilon\left[A_{i,1}(\vec{\bar{a}}') - A_{i,1}(\vec{\bar{a}}^*)\right] + \epsilon^2 A_{i,2}(\vec{\bar{a}}') \qquad (i = 1,2,\ldots,5) \quad (3\text{-}75a)$$

$$\frac{dr_6}{dt} = n(\bar{a}_1') - n(\bar{a}_1^*) + \epsilon\left[A_{6,1}(\vec{\bar{a}}') - A_{6,2}(\vec{\bar{a}}^*)\right] + \epsilon^2 A_{6,2}(\vec{\bar{a}}') \quad (3\text{-}75b)$$

It follows that

$$\begin{aligned} |r_i(t)| &= \epsilon\left|\int_0^t \left[A_{i,1}(\vec{\bar{a}}') - A_{i,2}(\vec{\bar{a}}^*)\right] dt' + \epsilon^2 \int_0^t A_{i,2}(\vec{\bar{a}}')\, dt'\right| \\ &\le \epsilon \int_0^t \left|A_{i,1}(\vec{\bar{a}}') - A_{i,1}(\vec{\bar{a}}^*)\right| dt' + \epsilon^2 \int_0^t \left|A_{i,2}(\vec{\bar{a}}')\right| dt' \end{aligned} \quad (3\text{-}76a)$$

and, similarly,

$$\begin{aligned} |r_6(t)| \le & \int_0^t \left|n(\bar{a}_1') - n(\bar{a}_1^*)\right| dt' \\ & + \epsilon \int_0^t \left|A_{6,1}(\vec{\bar{a}}') - A_{6,1}(\vec{\bar{a}}^*)\right| dt' + \epsilon^2 \int_0^t \left|A_{6,2}(\vec{\bar{a}}')\right| dt' \end{aligned} \quad (3\text{-}76b)$$

The functions $A_{i,1}$ and n are assumed to satisfy the Lipschitz condition

$$\left|A_{i,1}(\vec{\bar{a}}') - A_{i,1}(\vec{\bar{a}}^*)\right| \le L_i \left|\vec{\bar{a}}' - \vec{\bar{a}}^*\right| \equiv L_i \left|\vec{r}(t)\right| \qquad (i = 1,2,\ldots,6) \quad (3\text{-}77a)$$

$$\left| n(\bar{a}'_1) - n(\bar{a}^*_1) \right| \le L' \left| \bar{a}'_1 - \bar{a}^*_1 \right| \le L' \left| \vec{a}' - \vec{a}^* \right| \equiv L' \left| \vec{r}(t) \right| \quad (3\text{-}77b)$$

on the interval $0 \le t \le T$, where L_i and L' are positive constants and where the vector $\vec{r}$ consists of the components r_i (where $i = 1, 2, \ldots, 5$). It is sufficient that the partial derivatives of the functions $A_{i,1}$ and n exist and are bounded on the interval $0 \le t \le T$ for Equations (3-77) to be satisfied. It is also assumed that the absolute value of the second-order function $A_{i,2}$ is bounded from above on the interval $0 \le t \le T$, i.e.,

$$|A_{i,2}| \le M_i \qquad \text{for } 0 \le t \le T$$

Substituting Equations (3-77) into Equations (3-76) yields the inequalities

$$|r_i(t)| \le \epsilon L_i \int_0^t |\vec{r}(t')| \, dt' + \epsilon^2 M_i t \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}78a)$$

$$|r_6(t)| \le \epsilon L_6 \int_0^t |\vec{r}(t')| \, dt' + L' \int_0^t |\vec{r}(t')| \, dt' + \epsilon^2 M_6 t \quad (3\text{-}78b)$$

To simplify the discussion, the positive constant L is chosen such that

$$L \ge \sum_{i=1}^{6} L_i$$

$$L \ge L'$$

and

$$L \ge \sum_{i=1}^{6} M_i$$

Then, summing Equations (3-78a) over i yields the inequality

$$|\vec{r}(t)| \le \sum_{i=1}^{5} |r_i(t)| \le \epsilon L \int_0^t |\vec{r}(t')|\, dt' + \epsilon^2 L t \tag{3-79a}$$

It also follows that

$$|r_6(t)| \le (1+\epsilon) L \int_0^t |\vec{r}(t')|\, dt' + \epsilon^2 L t \tag{3-79b}$$

Using the generalized Gronwall inequality,[1] it is easily shown that

$$\begin{aligned} |\vec{r}(t)| &\le \int_0^t \exp\left(\epsilon L \int_\tau^t d\theta\right) \epsilon^2 L\, d\tau \\ &= \int_0^t \epsilon^2 L \exp\left[\epsilon L (t-\tau)\right] d\tau \\ &= \epsilon \left[\exp(\epsilon L t) - 1\right] \le \epsilon^2 L t \exp(\epsilon L t) \end{aligned} \tag{3-80}$$

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[1] The Generalized Gronwall Inequality (Reference 45)

If the following four conditions are met:

(1) $\lambda(t)$, $\phi(t)$, and $u(t)$ are defined on the interval $t_0 \le t \le T$
(2) $\lambda(t)$ is greater than or equal to zero and is summable
(3) $\phi(t)$ and $u(t)$ are absolutely continuous
(4) the following inequality is satisfied

$$u(t) \le \int_{t_0}^{t} \lambda(\tau)\, u(\tau)\, d\tau + \phi(t) \qquad (t_0 \le t \le t_1)$$

then

$$u(t) \le \phi(t_0) \exp\left(\int_{t_0}^{t} \lambda(\tau)\, d\tau\right) + \int_{t_0}^{t} \exp\left(\int_{\tau}^{t} \lambda(\theta)\, d\theta\right) \frac{d\phi}{d\tau}\, d\tau$$

C-2

Substituting the minimum of the upper bounds for $|\vec{r}(t)|$, i.e.,

$$|\vec{r}(t)| \leq \epsilon\left[\exp(\epsilon L t) - 1\right] \tag{3-81}$$

into the inequality for $|r_6(t)|$ yields

$$\begin{aligned} |r_6(t)| &\leq (1+\epsilon)\,\epsilon L \int_0^t \left[\exp(\epsilon L t') - 1\right] dt' + \epsilon^2 L t \\ &= (1+\epsilon)\left[\exp(\epsilon L t) - 1 - \epsilon L t\right] + \epsilon^2 L t \\ &\leq \epsilon L t\,(\epsilon + \epsilon L t)\exp(\epsilon L t) \end{aligned} \tag{3-82}$$

It is noted that this last result is not in agreement with that obtained by Kyner, i.e.,

$$|r_6(t)| \leq \epsilon L t\left[2\exp(\epsilon L t) + \epsilon - t\right] \tag{3-83}$$

In summary, the difference of the first- and second-order solutions is bounded by the functions

$$|\bar{\bar{a}}' - \bar{\bar{a}}^*| \leq \epsilon^2 L t \exp(\epsilon L t) \tag{3-84a}$$

$$|\bar{l}' - \bar{l}^*| \leq \left(\epsilon^2 L t + \epsilon^2 L^2 t^2\right)\exp(\epsilon L t) \tag{3-84b}$$

If the time t is restricted such that $\epsilon L t \ll 1$ (i.e., $0 \le t \le T \ll (\epsilon L)^{-1}$), then generally the order of magnitude estimate of the divergence between the first- and second-order theories is given by

$$|\vec{\bar{a}}' - \vec{\bar{a}}^*| \sim O(\epsilon^2 t) \qquad (\text{for } 0 \le t \le T \ll (\epsilon L)^{-1}) \tag{3-85a}$$

and

$$|\bar{\ell}' - \bar{\ell}^*| \sim O(\epsilon^2 t) \qquad (\text{for } 0 \le t < L^{-1}) \tag{3-85b}$$

$$|\bar{\ell}' - \bar{\ell}^*| \sim O(\epsilon^2 t^2) \qquad (\text{for } L^{-1} < t \le T \ll (\epsilon L)^{-1}) \tag{3-85c}$$

The above error estimates can be mapped back into the osculating elements using the near-identity transformation. Only first-order terms are assumed, since only first-order terms are required for the second-order averaged equations of motion. Evaluating the near-identity transformation

$$\vec{a} = \vec{\bar{a}} + \epsilon\vec{\eta}(\vec{\bar{a}},\bar{\ell}) + O(\epsilon^2)$$

$$\ell = \bar{\ell} + \epsilon\eta_6(\vec{\bar{a}},\bar{\ell}) + O(\epsilon^2)$$

with the elements obtained from the first- and second-order solutions and taking the absolute value of the difference yields the inequalities

$$|\vec{a}' - \vec{a}^*| \le |\vec{\bar{a}}' - \vec{\bar{a}}^*| + \epsilon|\vec{\eta}(\vec{\bar{a}}', \bar{\ell}') - \vec{\eta}(\vec{\bar{a}}^*, \bar{\ell}^*)| \tag{3-86a}$$

$$|\ell' - \ell^*| \le |\bar{\ell}' - \bar{\ell}^*| + \epsilon|\eta_6(\vec{\bar{a}}', \bar{\ell}') - \eta_6(\vec{\bar{a}}^*, \bar{\ell}^*)| \tag{3-86b}$$

where $\vec{\eta}$ is a vector with the components $\eta_{i,1}$ $(i = 1, 2, \ldots, 5)$.

Since the constant L can be chosen to satisfy the Lipschitz conditions

$$|\vec{\eta}(\vec{a}', l') - \vec{\eta}(\vec{a}^*, l^*)| \leq L|\vec{r}(t)| + L|r_6(t)| \quad \text{(3-87a)}$$

$$|\eta_{6,1}(\vec{a}', l') - \eta_{6,1}(\vec{a}^*, l^*)| \leq L|\vec{r}(t)| + L|r_6(t)| \quad \text{(3-87b)}$$

Equations (3-86) can be simplified to give

$$|\vec{a}' - \vec{a}^*| \leq (1 + \epsilon L)|\vec{r}(t)| + \epsilon L|r_6(t)| \quad \text{(3-88a)}$$

$$|l' - l^*| \leq \epsilon L|\vec{r}(t)| + (1 + \epsilon L)|r_6(t)| \quad \text{(3-88b)}$$

Substituting the upper bounds for $|r(t)|$ and $|r_6(t)|$ into Equations (3-88) yields the inequalities

$$|\vec{a}' - \vec{a}^*| \leq \epsilon^2 Lt(1 + 2\epsilon L + \epsilon L^2 t) \exp(\epsilon Lt) \quad \text{(3-89a)}$$

$$|l' - l^*| \leq \epsilon^2 Lt(1 + 2\epsilon L + \epsilon L^2 t + Lt) \exp(\epsilon Lt) \quad \text{(3-89b)}$$

which yields the following qualitative estimates for the osculating elements:

$$|\vec{a}' - \vec{a}^*| \sim O(\epsilon^2 t) \qquad (\text{for } 0 \leq t \leq T \ll (\epsilon L)^{-1}) \quad \text{(3-90a)}$$

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$$|\ell' - \ell^*| \sim O(\epsilon^2 t) \qquad (\text{for } 0 \le t \le L^{-1}) \tag{3-90b}$$

$$|\ell' - \ell^*| \sim O(\epsilon^2 t^2) \qquad (\text{for } 0 \le t \le T \ll (\epsilon L^{-1}) \tag{3-90c}$$

when the restriction $\epsilon L t \ll 1$ is imposed. Thus, the qualitative behavior of the osculating elements is, in general, the same as that of the mean elements (Equations (3-85)).

In summary, for arbitrarily small ϵ, the difference in the first- and second-order theories is arbitrarily small. For a given ϵ, the difference in the theories will be sufficiently small for some interval of time $0 \le t \le T$, where $T \sim O(\epsilon^{-1})$.

3.5.1.2 Sufficient Conditions for Neglecting Second-Order Terms

The second-order averaged equations of motion are given by

$$\frac{d\bar{a}_i}{dt} = \epsilon A_{i,1}(\vec{\bar{a}}) + \epsilon^2 A_{i,2}(\vec{\bar{a}}) + O(\epsilon^3) \qquad (i = 1, 2, \ldots, 5) \tag{3-91a}$$

$$\frac{d\bar{\ell}}{dt} = n(\bar{a}_1) + \epsilon A_{6,1}(\vec{\bar{a}}) + \epsilon^2 A_{6,2}(\vec{\bar{a}}) + O(\epsilon^3) \tag{3-91b}$$

where

$$A_{i,1} = \left\langle F_i(\vec{\bar{a}}, \bar{\ell}) \right\rangle_{\bar{\ell}} \qquad (i = 1, 2, \ldots, 6) \tag{3-92a}$$

$$A_{i,2} = \sum_{k=1}^{6} \left\langle \eta_{k,1} \frac{\partial F_i}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \qquad (i = 1, 2, \ldots, 5) \tag{3-92b}$$

$$A_{6,2} = \sum_{k=1}^{6} \left\langle \eta_{k,1} \frac{\partial F_6}{\partial \bar{a}_k} \right\rangle_{\bar{l}} + \frac{15}{8} \frac{\bar{n}}{\bar{a}_1^2} \left\langle \eta_{1,1}^2 \right\rangle_{\bar{l}} \qquad (3\text{-}92c)$$

Inspection of the second-order averaged equations of motion indicates that, for the limiting case in which the first-order short-period variations of the osculating elements are identically equal to zero, the second-order contributions to the mean element rates vanish identically. Similarly, if the amplitudes of the first-order short-period variations are small in magnitude, the second-order contribution to the mean element rates will be small, provided that the short-periodic part of the function $\partial F_i / \partial \vec{a}_k$ is not large. Finally, inspection of the second-order equations indicates that the effect of nonzero second-order terms will be most significant when the first-order contribution to the mean element rates is very small or zero. Consequently, the inadequacy of a first-order theory will be most apparent when the element history approaches a local maximum or minimum value.

Before further discussion, the following relation will be demonstrated:

$$\frac{\partial F_i(\vec{\bar{a}}, \bar{l})}{\partial \bar{a}_k} = \frac{\partial A_{i,1}(\vec{\bar{a}})}{\partial \bar{a}_k} - \frac{\partial \dot{\eta}_{i,1}(\vec{\bar{a}}, \bar{l})}{\partial \bar{a}_k} \qquad (3\text{-}93)$$

where $(\dot{\ })$ indicates $d(\ )/dt$. (Since extension of the following discussion to the case $i = 6$ is straightforward, it is not presented.)

Substituting the relation

$$F_i(\vec{a}, l) = F_i(\vec{\bar{a}}, \bar{l}) + O(\epsilon) \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}94)$$

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into the high-precision equation (from Equation (3-2))

$$\frac{da_i}{dt} = \epsilon F_i(\vec{a}, \ell) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}95)$$

yields the result

$$\frac{da_i}{dt} = \epsilon F_i(\vec{\bar{a}}, \bar{\ell}) + O(\epsilon^2) \qquad (i = 1,2,\ldots,5) \quad (3\text{-}96)$$

Differentiating with respect to time the near-identity transformation from Equation (3-3)

$$\frac{da_i}{dt} = \frac{d\bar{a}_i}{dt} + \epsilon \frac{d\eta_{i,1}}{dt}(\vec{\bar{a}}, \bar{\ell}) + O(\epsilon^2) \quad (3\text{-}97)$$

and substituting into the result the expansion of the mean element rate from Equation (3-4), i.e.,

$$\frac{d\bar{a}_i}{dt} = \epsilon A_{i,1}(\vec{\bar{a}}) + O(\epsilon^2) \quad (3\text{-}98)$$

yields the relation

$$\frac{da_i}{dt} = \epsilon \left[A_{i,1}(\vec{\bar{a}}) + \frac{d}{dt} \eta_{i,1}(\vec{\bar{a}}, \bar{\ell}) \right] + O(\epsilon^2) \quad (3\text{-}99)$$

Comparison of Equations (3-96) and (3-99) yields the result

$$F_i(\vec{\bar{a}},\bar{\ell}) = A_{i,1}(\vec{\bar{a}}) + \frac{d\eta_{i,1}(\vec{\bar{a}},\bar{\ell})}{dt} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}100)$$

and Equation (3-93) follows immediately. As a result, the second-order contribution to the mean element rates reduces to

$$\begin{aligned} A_{i,2}(\vec{\bar{a}}) &= \sum_{k=1}^{6} \left\langle \eta_{k,1} \left(\frac{\partial A_{i,1}}{\partial \bar{a}_k} + \frac{\partial \dot{\eta}_{i,1}}{\partial \bar{a}_k} \right) \right\rangle_{\bar{\ell}} \\ &= \sum_{k=1}^{6} \left\langle \eta_{k,1} \frac{\partial \dot{\eta}_{i,1}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} \end{aligned} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}101)$$

since, by Equations (3-22) and (3-23),

$$\left\langle \eta_{k,1} \frac{\partial A_{i,1}}{\partial \bar{a}_k} \right\rangle_{\bar{\ell}} = \frac{\partial A_{i,1}}{\partial \bar{a}_k} \left\langle \eta_{k,1} \right\rangle_{\bar{\ell}} \equiv 0 \qquad (3\text{-}102)$$

The requirement that the magnitude of the short-period part of the function $\partial F_i / \partial \bar{a}_k$ is not large then reduces to the requirement that the magnitude of the function $\partial \dot{\eta}_{i,1} / \partial \bar{a}_k$ is not large. It seems reasonable to expect that, if the function $\dot{\eta}_{i,1}$ has a small absolute variation and there are few local extrema over the interval corresponding to one satellite period, then the first time derivative of the function should not be large. This assumption should also extend to the partial derivatives.

A somewhat more formal criterion for neglecting second-order terms requires simply that the integrated effect of the second-order term over the interval $0 \le t \le T$ be less than some specified tolerance δ, i.e.,

$$\epsilon^2 \int_0^t A_{i,2}(\bar{a})\,dt < \delta \qquad (i = 1,2,\ldots,6) \quad (3\text{-}103)$$

or, more specifically (in view of Equation (3-101)),

$$\epsilon^2 \int_0^t \sum_{k=1}^{6} \left\langle \eta_{k,1} \frac{\partial \dot{\eta}_{i,1}}{\partial \bar{a}_k} \right\rangle_{\bar{l}} dt < \delta \qquad (3\text{-}104)$$

Clearly, the integral of the second-order contribution can be bounded as follows:

$$\epsilon^2 \int_0^t A_{i,2}(\bar{a})\,dt \le \epsilon^2 \int_0^t \left| A_{i,2}(\bar{a}) \right| dt \qquad (3\text{-}105)$$

and it follows from Equation (3-101) that

$$\begin{aligned} \left| A_{i,2}(\bar{a}) \right| &\le \sum_{k=1}^{6} \left| \left\langle \eta_{k,1} \frac{\partial \dot{\eta}_{i,1}}{\partial \bar{a}_k} \right\rangle_{\bar{l}} \right| \\ &\le \sum_{k=1}^{6} \rho_k \gamma_{i,k} = M_i \end{aligned} \qquad (i = 1,2,\ldots,5) \quad (3\text{-}106)$$

where ρ_k and $\gamma_{i,k}$ designate the maximum variations of the functions $\eta_{k,1}$ and $\partial\eta_{i,k}/\partial\bar{a}_k$, respectively, i.e.,

$$\left|\eta_{k,1}\right| \le \rho_k \tag{3-107a}$$

$$\left|\frac{\partial\dot{\eta}_{i,1}}{\partial\bar{a}_k}\right| \le \gamma_{i,k} \qquad (i = 1,2,\ldots,5) \tag{3-107b}$$

For the case of the fast variable (i.e., where i = 6),

$$A_{6,2} = \sum_{k=1}^{6}\left\langle \eta_{k,1}\frac{\partial\dot{\eta}_{6,1}}{\partial\bar{a}_k}\right\rangle_{\bar{\ell}} + \frac{15}{8}\,\frac{\bar{n}}{\bar{a}_1^2}\left\langle \eta_{1,1}^2\right\rangle_{\bar{\ell}} \tag{3-108}$$

and, therefore,

$$\left|A_{6,2}\right| \le \sum_{k=1}^{6}\rho_k\,\gamma_{6,k} + K\rho_1^2 = M_6 \tag{3-109}$$

where

$$\frac{15}{8}\left|\frac{\bar{n}}{\bar{a}_1^2}\right| \le K$$

It follows that

$$\epsilon^2\int_0^t A_{i,2}(\vec{\bar{a}})\,dt \le \epsilon^2\int_0^t M_i\,dt = \epsilon^2 M_i t \qquad (i = 1,2,\ldots,6) \tag{3-110}$$

Thus, the second-order term may be neglected when $\epsilon^2 M_i t < \delta$ $(i = 1, 2, \ldots, 6)$, that is, over the interval $0 \le t \le T$, where $T = \delta/(\epsilon^2 M_i)$. Therefore, the time interval over which a first-order theory is valid depends inversely on the magnitude of the first-order short-period variations in the osculating elements.

A relative criterion for neglecting the second-order terms provides a little more insight in practical applications. Essentially, it is required that the integrated second-order contribution be negligible when compared with the integrated first-order contribution over the interval $0 \le t \le T$. Specifically, the condition

$$\left| \epsilon^2 \int_0^t A_{i,2}\, dt \right| \ll \max \left| \epsilon \int_0^t A_{i,1}\, dt \right| \quad (0 \le t \le T) \tag{3-111}$$

is to be satisfied.[1] As before, the integral of the second-order term is easily bounded by

$$\left| \epsilon^2 \int_0^t A_{i,2}\, dt \right| \le \epsilon^2 M_i t \tag{3-112}$$

Also, if the following definition is made

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$$\Delta \bar{a}_i = \max \left| \epsilon \int_0^t A_{i,1}\, dt \right| \quad (i = 1, 2, \ldots, 6) \tag{3-113}$$

[1] The coupling between the second-order and first-order contributions is assumed to be negligible. This argument is valid only for the bounded periodic elements or very slowly growing secular elements, since the rapid first-order secular growth of the fast variable would satisfy the inequality even for large second-order contributions. This criterion is really useful as a negative criterion specifying when second-order terms are definitely necessary rather than when they can be neglected.

then the inequality in Equation (3-111) will be satisfied when

$$\epsilon^2 M_i t \ll \Delta \bar{a}_i \qquad (i = 1, 2, \ldots, 5) \quad (3\text{-}114)$$

or

$$\epsilon^2 \frac{M_i}{\Delta \bar{a}_i} t \ll 1 \qquad (3\text{-}115)$$

If $\gamma_{i,k}$ (Equation (3-107b)) is replaced by the order of magnitude estimate

$$\gamma_{i,k} \approx \frac{\Delta \dot{\eta}_{i,1}}{\Delta \bar{a}_k} \approx \frac{\rho_i'}{\Delta \bar{a}_k} \qquad (3\text{-}116)$$

where Δ denotes the maximum variation of the element $\bar{a}_k$ over the interval $0 \le t' \le t$ and ρ_i' is defined to be an upper bound of the time derivative of the short-period variation $\eta_{i,1}$, i.e.,

$$|\dot{\eta}_{i,1}| \le \rho_i'$$

Then it follows that

$$\frac{M_i}{\Delta \bar{a}_i} = \sum_{k=1}^{6} \frac{\rho_k}{\Delta \bar{a}_i} \gamma_{i,k} \approx \sum_{k=1}^{6} \frac{\rho_k}{\Delta \bar{a}_k} \frac{\rho_i'}{\Delta \bar{a}_i} \qquad (3\text{-}117)$$

and the second-order terms can be neglected when

$$\epsilon^2 t \sum_{k=1}^{6} \frac{\rho_k}{\Delta \bar{a}_k} \frac{\rho_i'}{\Delta \bar{a}_i} \ll 1 \qquad (3\text{-}118)$$

Thus, the second-order terms can be neglected over the interval $0 < t \leq T$, where

$$T \ll \frac{1}{\epsilon^2 \sum_{k=1}^{6} \frac{\rho_k \rho_i'}{\bar{a}_k \bar{a}_i}} \tag{3-120}$$

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The smaller the ratio of the short-period variation to the first-order long-period variation, the greater the interval over which a first-order theory is valid. How small these ratios must be depends on the time interval over which the first-order theory is to be valid. The answer to this question can be provided only by a thorough investigation for each dynamical system. However, an upper bound of at most a few percent would be a likely guess for retaining a period of validity of a few years.

On the other hand, a first-order theory is clearly inadequate when the amplitude of the short-period variations is 20 to 30 percent of the long-period variations. The author has investigated the case of a near-circular satellite (IMP-J) in 2:1 resonance with the Moon. The amplitude of the short-period variations was approximately 30 percent of the magnitude of the long-period variation caused by the resonance. The first-order averaging theory produced poor results in the neighborhood of a local extremum of the semimajor axis history, an indication of significant second-order contributions to the motion.

3.5.2 Application of a Restricted Second-Order Theory of Averaging

The application of a second-order averaging theory to all perturbations would compromise the advantage of the low computational cost, which is characteristic of the first-order theory. However, the application of a second-order averaging theory, restricted to selected perturbations, may yield more accurate results where the application of a first-order theory is marginal, or it may extend the time interval over which the first-order theory is valid with a minimal increase in cost.

3.5.2.1 Nonspherical Gravitational Perturbation

The spherical harmonic expansion representing the potential of the nonspherical gravitational field of the central body (Earth, Moon) contains the small parameters

$$C_{n,m}\left(\frac{a_e}{a}\right)^n \quad ; \quad S_{n,m}\left(\frac{a_e}{a}\right)^n$$

where a_e designates the equatorial radius of the central body, the quantity a designates the semimajor axis of the satellite orbit, and the coefficients $C_{n,m}$ and $S_{n,m}$ are observed quantities.

The zonal harmonic coefficients J_n are defined by

$$J_n = -C_{n,0}$$

These small parameters are obviously bounded above by the numerical coefficients $C_{n,m}$ and $S_{n,m}$. Since $J_2 \approx O(10^{-3})$ and since all other coefficients are of the order of J_2^n for $n \geq 2$, the oblateness term, J_2, in the geopotential might seem to be a logical candidate for the application of a second-order averaging procedure. In fact, a consistency argument is often made that second-order oblateness contributions should be included if any other terms in the spherical harmonic expansion for the geopotential model are also included. According to the previous discussion, this is not necessarily the case since the second-order contributions depend on the first-order short-period variations of the osculating elements and their time derivatives and not on the first-order contribution to the long-period motion. However, it is reasonable to expect that if second-order terms are necessary, the J_2 contribution would strongly dominate over the other harmonics. Consequently, any second-order theory for the nonspherical gravitational field could be limited, in most cases, to the J_2 oblateness contribution.

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3.5.2.2 Third-Body Perturbation

The case for the third-body perturbation is not generally as simple. The relevant small parameters are the nth power of the parallax factor, i.e.,

$$\epsilon_n = \left(\frac{a}{a'}\right)^n \qquad (n = 2, 3, \ldots)$$

where a and a' are the semimajor axes of the satellite orbit and third-body orbit, respectively. (It is tacitly assumed that the disturbing third body is an exterior perturbation, i.e., $a < a'$. If, however, $a > a'$, the expansion proceeds in powers of the inverse of the above parameter.)

The upper bound of this set of small parameters is unity in contrast to the upper bound for the small parameters in the nonspherical gravitational model which is of the order $O(10^{-3})$. Clearly, for high-altitude satellites, the small parameters are not really very small except for the large values of n. Physically, as the parallax factor grows towrad unity, the third body produces stronger disturbances (both short- and long-period) in the satellite motion. These larger disturbances require a more complex model which is manifested by a greater number of terms in the disturbing function expansion.

The recursive formulation of the disturbing function presented in Volume II of this report can, in principle, be used to produce expansions to any arbitrary order. However, high-order expansions can produce increased computational cost, unavoidable numerical round-off and truncation errors, and, possibly, errors due to unstable recursion formulas. Also, the first-order averaged equations of motion can be formulated in terms of the perturbing acceleration using the Gaussian formulation (Equation (2-15)) to avoid entirely the problem of slow convergence of the disturbing function expansion.

The slow convergence of the disturbing function is of far greater significance to the application of the method of averaging itself. The strong short-period disturbances can no longer be neglected in formulating the averaged equations of motion: a higher-order theory becomes necessary. The stronger the short-period disturbances, the greater the number of terms in the disturbing function expansion which must be developed to second or higher order in the application of the method of averaging.

An additional complexity is presented because the small parameters are not entirely independent, specifically,

$$(\epsilon_n)^m = \epsilon_{nm}$$

$$\epsilon_{n+m} = \epsilon_n \epsilon_m$$

Under these constraints, an order of magnitude argument would indicate that if the term containing the small parameter ϵ_6 produces significant short-period contributions to the motion, then the terms containing the powers ϵ_2^2, ϵ_2^3, ϵ_3^2, and the products $\epsilon_2 \cdot \epsilon_3$ and $\epsilon_2 \cdot \epsilon_4$ might contribute significant short-period variations and, therefore, could be required in the averaged equations of motion.

On the other hand, the numerical coefficients in the higher order terms of the averaged equations of motion may render their contributions less significant than the order of magnitude argument would indicate. If this were generally true, the slow convergence of the disturbing function would have a less severe impact on the order required in the application of the method of averaging, and development of the first few terms of the disturbing function expansion to second order might considerably extend the range of the parallax factor where the averaged equations of motion are valid. At the very least, it would extend the application of the method of averaging to those cases where a first-order application was only marginal and/or extend the interval of validity to tens of years[1] in those cases where a first-order theory already provides adequate results over a few years.

[1] Such very long predictions may be necessary to meet future mission analysis requirements, e.g., permanent space station missions, solar power satellites, etc. High-precision techniques are not well suited for such investigations.

Furthermore, even though rough qualitative estimates of the behavior of the averaged equations is known (see Section 3.5.1), very little is known quantitatively of the time intervals over which a first-order application of the method of averaging is valid. The basic difficulty in ascertaining an accurate estimate of this time interval is obtaining a suitable reference solution with which to gauge the averaging theory. The standard approach has been to compare the results obtained from the averaging theory with a high-precision reference solution directly or with mean elements obtained in some manner from the high-precision reference. Beyond the time intervals over which high-precision techniques are valid, the only reference is that obtained by direct observation.[1]

It is assumed that these mean elements continue to provide an accurate picture of the long-period and secular motion of the dynamical system for several years or more. Of course, the exact length of this time interval depends on the magnitude of the short-period variations in the osculating elements as shown in Section 3.5.1.2 of this document. Eventually, the prediction from the first-order theory will gradually diverge from the real solution. Without some comparison, this divergence will probably become apparent only after it has reached extreme proportions. A comparison with a complete first-order averaging theory augmented to include the dominant second-order contributions from the oblateness and third-body perturbations would provide some insight into the period of validity of the first-order theory in addition to extending it. This approach, in essence, computes the value of the integral in Equation 3-103).

In summary, a first-order application of the method of averaging is adequate for several years in those cases where the short-period variations in the osculating elements are either absolutely small or small relative to the long-period variations of the first-order mean elements. The implementation of a complete second

[1] This situation is not peculiar to the averaged orbit generator but also affects the high-precision generator, since the only reference is provided by observations.

or higher order averaging theory would reduce the computational advantages characteristic of the first-order implementation. However, a second-order implementation, restricted to only the dominant perturbations, would extend the application of the method of averaging to a wider class of problems while minimizing the additional computational cost, and it would also provide some estimate of the time interval over which first-order averaging is adequate.

SECTION 4 - FIRST-ORDER SHORT-PERIOD CONTRIBUTIONS TO THE OSCULATING ELEMENTS

For many plications, the solution to Equations (3-2) (the true instantaneous or osculating elements) is desired. Several techniques have been developed for the solution of these equations (e.g., Cowell, Encke, etc.--see Reference 29); however, these high-precision techniques share the characteristic of high computational cost. To reduce this cost, the averaged equations of motion were developed, which provide mean elements for the dynamical system.

In addition to the mean trajectory, the method of averaging provides (in principle) a way to compute a jth-order approximation to the osculating elements from the mean elements. First-order, and possibly second-order, approximations to the osculating elements are sufficiently accurate for most applications. The computational complexity of these approximations increases tremendously with the order of the small parameter.

The effectiveness of representing osculating elements by applying a first-order short-period variation to mean elements has been demonstrated by Lutsky and Uphoff (Reference 5). It might appear that such a procedure would vitiate the computational advantages associated with the method of averaging, and it has already been demonstrated that mean elements are sufficiently accurate for many applications (References 4 and 9). However, for some applications, e.g., definitive orbit determination procedures, the additional accuracy provided by the first-order short-period variations might be necessary.

Based on the following discussion, it appears that the cost of evaluating the first-order short-period variations using an analytical formulation[1] would be no more costly than a single averaged derivative evaluation. This estimate is based on the assumption that the evaluation of first-order short-period variations is performed independently of the derivative evaluation.

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[1] The cost of evaluating the first-order short-period variations by a numerical technique can be estimated by reviewing the method presented in Reference 5.

As will be shown in Volume II, the mathematical formalism for the mean element rates is also common to the first-order short-period variations. Consequently, it is estimated that, if proper advantage is taken of this commonality, the cost of evaluating the analytical formulation of the first-order short-period variations could be reduced to possibly 20 percent (or even less) of the cost of a derivative evaluation. For those environments where the utmost computational efficiency is required, these variations should be applied only at judiciously selected points along and/or at the end of the trajectory for applications with high-accuracy requirements.

An equally important application for such approximations to the osculating elements is the conversion of osculating elements to mean elements. An osculating-to-mean element conversion can be developed by inverting the equations which specify the mean-to-osculating element transformation.

The mean elements describing the long-period variations in the trajectory are only as accurate as the initial mean elements and, hence, only as accurate as the osculating-to-mean conversion. Existing conversion procedures are strictly numerical (except for the Brouwer theory, which is limited to the low-order zonal perturbations) and are based on quadratures or costly differential correction procedures which require a high-precision orbit generator. Therefore, either the initial conditions must be predetermined or the software system must have access to a high-precision orbit generator as well as to the averaged orbit generator.[1] In addition, implementation of the short-period corrections appears to require no additional theory beyond that necessary for the averaged equations of motion.

This section presents a discussion of the first-order short-period variations of the osculating elements and their application to both osculating-to-mean and mean-to-osculating element conversions. This discussion is developed in the context of

[1] If nonconservative perturbing forces, e.g., drag, etc., are present, there is no recourse (at present) to the numerical osculating-to-mean conversions.

the Lagrange Planetary Equations (Equations (2-31)). A discussion of the first-order short-period variations in the context of the Gaussian Variation of Parameters (VOP) equations and the numerical averaging approach can be found in Reference 5.

4.1 MEAN-TO-OSCULATING ELEMENT CONVERSION

The near-identity transformation (Equation (3-3)) establishes the relation between mean elements and osculating elements. A general expression for the jth-order term in this transformation is given in Equation (3-29). Evaluation of this expression for higher orders is quite complicated if not prohibitive. However, evaluation of the first-order term is manageable. (This term also appears in the formulation of the second-order averaged equations of motion (Equations (3-39).)

Expressing the near-identity transformation to first order in the small parameter yields

$$a_i = \bar{a}_i + \epsilon\eta_{i,1}(\vec{\bar{a}},\bar{\ell}) \qquad (i = 1,2,\ldots,5) \quad (4\text{-}1a)$$

$$\ell = \bar{\ell} + \epsilon\eta_{6,1}(\vec{\bar{a}},\bar{\ell}) \qquad (4\text{-}1b)$$

where

$$\epsilon\eta_{i,1}(\vec{\bar{a}},\bar{\ell}) = \frac{1}{\bar{n}}\int \epsilon F_i^S(\vec{\bar{a}},\bar{\ell})\,d\bar{\ell} \qquad (i = 1,2,\ldots,5) \quad (4\text{-}2a)$$

$$\epsilon\eta_{6,1}(\vec{\bar{a}},\bar{\ell}) = \frac{1}{\bar{n}}\int \epsilon\left[F_6^S(\vec{\bar{a}},\bar{\ell}) - \frac{3}{2}\frac{\bar{n}}{\bar{a}}\eta_{1,1}(\vec{\bar{a}},\bar{\ell})\right]d\bar{\ell} \qquad (4\text{-}2b)$$

and F_i^S denotes the short-periodic part of the perturbing function, i.e.,

$$F_i^S = F_i - \langle F_i \rangle_{\bar{\ell}} \qquad (4\text{-}3)$$

If the functions $\eta_{i,1}$ (where i = 1 through 6) can be evaluated, then Equations (4-1) provide a first-order mean-to-osculating element conversion.

Using the Lagrange Planetary Equations (Equation (2-31)), it follows from Equation (4-3) that

$$\epsilon F_i^{S}(\vec{\bar{a}},\bar{\ell}) = -\sum_{j=1}^{6} \left[(\bar{a}_i,\bar{a}_j) \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} - \left\langle (\bar{a}_i,\bar{a}_j) \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} \right\rangle_{\bar{\ell}} \right] \qquad (4\text{-}4)$$

Because the nonzero Poisson Brackets are independent of the fast variable (see Appendix A),

$$\left\langle (\bar{a}_i,\bar{a}_j) \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} \right\rangle_{\bar{\ell}} = (\bar{a}_i,\bar{a}_j) \left\langle \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} \right\rangle_{\bar{\ell}} \qquad (4\text{-}5)$$

Consequently, Equation (4-4) can be expressed as

$$\epsilon F_i^{S}(\vec{\bar{a}},\bar{\ell}) = -\sum_{j=1}^{6} (\bar{a}_i,\bar{a}_j) \left[\frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} - \left\langle \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} \right\rangle_{\bar{\ell}} \right] \qquad (4\text{-}6)$$

Since

$$\left\langle \frac{\partial R(\vec{\bar{a}},\bar{\ell})}{\partial \bar{a}_j} \right\rangle_{\bar{\ell}} = \frac{\partial \langle R \rangle_{\bar{\ell}}}{\partial \bar{a}_j} \qquad (j = 1, 2, \ldots, 6) \quad (4\text{-}7)$$

Equation (4-6) can be simplified to read

$$\epsilon F_i^S(\vec{a},\bar{l}) = -\sum_{j=1}^{6}(\bar{a}_i,\bar{a}_j)\,\frac{\partial R^S}{\partial \bar{a}_j} \tag{4-8}$$

where

$$R^S = R - \langle R \rangle_{\bar{l}} \tag{4-9}$$

Substituting Equations (4-8) into Equations (4-2) and simplifying yields

$$\epsilon \eta_{i,1}(\vec{a},\bar{l}) = -\frac{1}{\bar{n}}\sum_{j=1}^{6}(\bar{a}_i,\bar{a}_j)\int \frac{\partial R^S(\vec{a},\bar{l})}{\partial \bar{a}_j}\,d\bar{l} \tag{4-10a}$$

and

$$\epsilon \eta_{6,1}(\vec{a},\bar{l}) = -\frac{1}{\bar{n}}\left[\sum_{j=1}^{6}(\bar{l},\bar{a}_j)\int \frac{\partial R^S(\vec{a},\bar{l})}{\partial \bar{a}_j}\,d\bar{l} + \frac{3}{2}\,\frac{\bar{n}\epsilon}{\bar{a}}\int \eta_{1,1}(\vec{a},\bar{l})\,d\bar{l}\right] \tag{4-10b}$$

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Since the disturbing function R is assumed to be appropriately continuous and differentiable,

$$\int \frac{\partial R^S(\bar{\vec{a}},\bar{l})}{\partial \bar{a}_j}\, d\bar{l} = \frac{\partial}{\partial \bar{a}_j} \int R^S(\bar{\vec{a}},\bar{l})\, d\bar{l} \tag{4-11}$$

If the short-periodic function $S(\bar{\vec{a}},\bar{l})$ is defined as

$$S(\bar{\vec{a}},\bar{l}) = \int R^S(\bar{\vec{a}},\bar{l})\, d\bar{l} \tag{4-12}$$

then Equations (4-10) take the form

$$\epsilon\eta_{i,1} = -\frac{1}{\bar{n}} \sum_{j=1}^{6} (a_i,\bar{a}_j) \frac{\partial S}{\partial \bar{a}_j} \qquad (i = 1,2,\ldots,5) \tag{4-13a}$$

$$\epsilon\eta_{6,1} = -\frac{1}{\bar{n}} \left[\sum_{j=1}^{6} (\bar{l},\bar{a}_j) \frac{\partial S}{\partial \bar{a}_j} + \frac{3}{2}\frac{\bar{n}\epsilon}{\bar{a}} \int \eta_{1,1}(\bar{\vec{a}},\bar{l})\, d\bar{l} \right] \tag{4-13b}$$

Equations (4-13) are almost identical in form to the general form of the Lagrange Planetary Equations (Equations (2-28)), with the exception of the reciprocal average mean motion factor and the second term in the equation for the short-period variation of the fast variable, $\epsilon\eta_{6,1}$. Expressing Equations (4-13) explicitly in equinoctial elements (a, h, k, p, q, λ) results in the following:

$$\epsilon\eta_{1,1} = \Delta a^S = \frac{2\bar{a}}{\bar{n}A}\frac{\partial S}{\partial\bar{\lambda}} \tag{4-14a}$$

$$\epsilon\eta_{2,1} = \Delta h^S = \frac{B}{\bar{n}A}\left(\frac{\partial S}{\partial\bar{k}} - \frac{\bar{h}}{1+B}\frac{\partial S}{\partial\bar{\lambda}}\right) + \frac{\bar{k}}{2\bar{n}AB}\left(\bar{p}\frac{\partial S}{\partial\bar{p}} + \bar{q}\frac{\partial S}{\partial\bar{q}}\right) \tag{4-14b}$$

$$\epsilon\eta_{3,1} = \Delta k^S = -\frac{B}{\bar{n}A}\left(\frac{\partial S}{\partial\bar{h}} + \frac{\bar{k}}{1+B}\frac{\partial S}{\partial\bar{\lambda}}\right) - \frac{\bar{h}C}{2\bar{n}AB}\left(\bar{p}\frac{\partial S}{\partial\bar{p}} + \bar{q}\frac{\partial S}{\partial\bar{q}}\right) \tag{4-14c}$$

$$\epsilon\eta_{4,1} = \Delta p^S = -\frac{\bar{p}C}{2\bar{n}AB}\left(\bar{k}\frac{\partial S}{\partial\bar{h}} - \bar{h}\frac{\partial S}{\partial\bar{k}} + \frac{\partial S}{\partial\bar{\lambda}}\right) + \frac{IC^2}{4\bar{n}AB}\frac{\partial S}{\partial\bar{q}} \tag{4-14d}$$

$$\epsilon \eta_{5,1} = \Delta q^s = -\frac{\bar{q} C}{2\bar{n}AB}\left(\bar{k}\frac{\partial S}{\partial \bar{h}} - \bar{h}\frac{\partial S}{\partial \bar{k}} + \frac{\partial S}{\partial \bar{\lambda}}\right) - \frac{IC^2}{4\bar{n}AB}\frac{\partial S}{\partial \bar{p}} \tag{4-14e}$$

$$\epsilon \eta_{6,1} = \Delta \lambda^s = -\frac{2\bar{a}}{\bar{n}A}\frac{\partial S}{\partial \bar{a}} + \frac{B}{\bar{n}A(1+B)}\left(\bar{h}\frac{\partial S}{\partial \bar{h}} + \bar{k}\frac{\partial S}{\partial \bar{k}}\right) + \frac{C}{2\bar{n}AB}\left(\bar{p}\frac{\partial S}{\partial \bar{p}} + \bar{q}\frac{\partial S}{\partial \bar{q}}\right) - \frac{3}{\bar{n}A} S \tag{4-14f}$$

where

$$S = S(\vec{\bar{a}}, \bar{\lambda})$$

$$A = \bar{n}\bar{a}^2$$

$$B = \sqrt{1 - \bar{h}^2 - \bar{k}^2}$$

$$C = 1 + \bar{p}^2 + \bar{q}^2$$

I = the retrograde factor

The indefinite integral of Equation (4-14a) yields the last term in Equation (4-14f). These equations can also be expressed in terms of the direction cosines $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ through Equations (2-35) and (2-36).

Explicit computation of S for the nonspherical gravitational perturbing function and for the third-body perturbing function is discussed in Volume II of this report.

4.2 OSCULATING-TO-MEAN ELEMENT CONVERSION

An osculating-to-mean element conversion is immediately obtained by inverting Equations (4-1). These equations are identical in form to Kepler's equation and can be numerically inverted, i.e., solved for the mean elements, by the same techniques. These techniques require an iterative scheme, since these Kepler-type equations are transcendental.

Expressions for the mean elements are obtained by writing Equations (4-1) in the form

$$\bar{a}_i = a_i - \epsilon\eta_{i,1}(\bar{\vec{a}},\bar{\ell}) \tag{4-15a}$$

and

$$\bar{\ell} = \ell - \epsilon\eta_{6,1}(\bar{\vec{a}},\bar{\ell}) \tag{4-15b}$$

An a priori estimate of the mean elements will permit evaluation of the right-hand sides of Equations (4-15). This, in turn, permits a computed approximation to the mean elements. These approximate mean elements are used to reevaluate the right-hand sides of the equations and to compute a new approximation to the mean elements. The kth approximation to the mean elements is expressed simply as

$$\bar{a}_{i,k} = a_i - \epsilon\eta_{i,1}(\bar{a}_{k-1},\bar{\ell}_{k-1}) \quad (i = 1, 2, \ldots, 5) \tag{4-16a}$$

$$\bar{\ell}_k = \ell - \epsilon\eta_{6,1}(\bar{a}_{k-1},\bar{\ell}_{k-1}) \tag{4-16b}$$

Such a procedure should converge within two or three iterations, provided a good a priori estimate is used.

A good estimate of the initial mean elements is provided by the osculating elements. The osculating elements differ from the true mean elements by order ϵ and, hence, introduce an error of only second order (i.e., $O(\epsilon^2)$) when used to evaluate the right-hand sides of Equations (4-16).

It should be noted that Equations (4-15) are of the same form as those given by Brouwer (Reference 44) and for transforming from the Brouwer primed element set (containing the long-period and secular motion) to the Brouwer unprimed element set (a first-order approximation to the osculating elements). Equations (4-15) take on the more familiar form of Brouwer's formulas when the expression for the functions $\eta_{i,j}$, given in Equations (4-14), are introduced.

APPENDIX A - THE EQUINOCTIAL ELEMENT SET AND REFERENCE SYSTEM

A.1 DEFINITION OF THE EQUINOCTIAL ELEMENT SET

The equinoctial elements defined in terms of the Keplerian or classical elements are given by

$$
\begin{aligned}
a &= a \\
h &= e \sin(\omega + I\Omega) \\
k &= e \cos(\omega + I\Omega) \\
p &= \tan^{I}(i/2) \sin\Omega \\
q &= \tan^{I}(i/2) \cos\Omega \\
\lambda &= \ell + \omega + I\Omega
\end{aligned}
\tag{A-1}
$$

where I is the retrograde factor and assumes the values

$$
\begin{aligned}
I &= 1 \quad \text{for } 0 \leq i \leq \pi/2 \\
I &= -1 \quad \text{for } \pi/2 < i \leq \pi
\end{aligned}
$$

If $I = 1$, the resulting element set is referred to as the direct equinoctial elements and for $I = -1$ the retrograde equinoctial elements are obtained. The direct equinoctial element set produces a singularity in the Variation of Parameters (VOP) equations for the inclination value $i = \pi$ and the retrograde element set produces a singularity for the inclination value $i = 0$. Hence, both element sets are required if the possibility of a singularity in the VOP equations is to be avoided. Since the inclination value $i = \pi$ is seldom encountered, the direct elements will suffice for the vast majority of applications.

Defining the value of the retrograde factor based on the cut-off value $i = \pi/2$ is quite arbitrary, and there is no compelling reason to change from direct to retrograde elements (or vice versa) in the middle of a numerical integration simply because the value of the inclination passed through this arbitrary cut-off value. On the contrary, this cut-off value is intended only as a guideline for choosing, at the initiation of the integration procedure, the element set to be used.

In Equations (A-1), the elements h and k are the components in the appropriate (direct or retrograde) orbital frame of the eccentric vector, with magnitude e, directed toward the periapse. The elements p and q can be considered as the components of a vector with magnitude tan(i/2) directed toward the ascending node. The element λ is the mean longitude.

Equations (A-1) are easily inverted to provide the transformation from the equinoctial to the classical elements, i.e.,

$$
\begin{aligned}
a &= a \\
e &= \sqrt{h^2 + k^2} \\
i &= \arccos\left[\frac{(1-p^2-q^2)\,I}{1+p^2+q^2}\right] \\
\omega &= \arctan\left(\frac{h}{k}\right) - I \arctan\left(\frac{p}{q}\right) \\
\Omega &= \arctan\left(\frac{p}{q}\right) \\
\ell &= \lambda - \omega - I\Omega
\end{aligned}
\tag{A-2}
$$

A.2 THE EQUINOCTIAL REFERENCE SYSTEMS

The equinoctial reference frames (direct and retrograde), designated by the orthogonal triad $(\hat{f}, \hat{g}, \hat{w})$, are right-hand systems and use the satellite orbit plane as the fundamental plane of reference. The unit vector $\hat{f}$ is directed toward a point in the satellite orbit displaced from the ascending node through the angle $-\Omega$ for the direct system and through the angle Ω for the retrograde system. The unit vector $\hat{w}$ points toward the north equinoctial pole and is identically the unit angular momentum vector. The vector $\hat{g}$ is directed toward a point in the orbital plane 90 degrees in advance of the unit vector $\hat{f}$ and can be expressed as

$$\hat{g} = \hat{w} \times \hat{f}$$

The relationship between the equinoctial reference systems and an arbitrary right-hand reference system, e.g., the equatorial system, is shown in Figures A-1 and A-2. Clearly, in both the direct and retrograde cases, a series of three rotations is required to make the arbitrary reference system coincide with each of the equinoctial reference systems. More specifically, a positive rotation about the z axis through the angle Ω points the x axis toward the ascending node. A positive rotation about this new x axis through the inclination angle, i, rotates the x,y plane into the f,g plane. Finally, for the direct case, a rotation about the current z axis (coincident with the $\hat{w}$ vector) through the angle $-\Omega$ points the x axis along the $\hat{f}$ vector. For the retrograde case, this last roatation about the z axis is performed through the angle Ω to align the x axis with the $\hat{f}$ vector of the retrograde system. This series of rotations provides the transformation of the coordinates of any point (e.g., satellite position) in the arbitrary system to the appropriate coordinates in the equinoctial system.

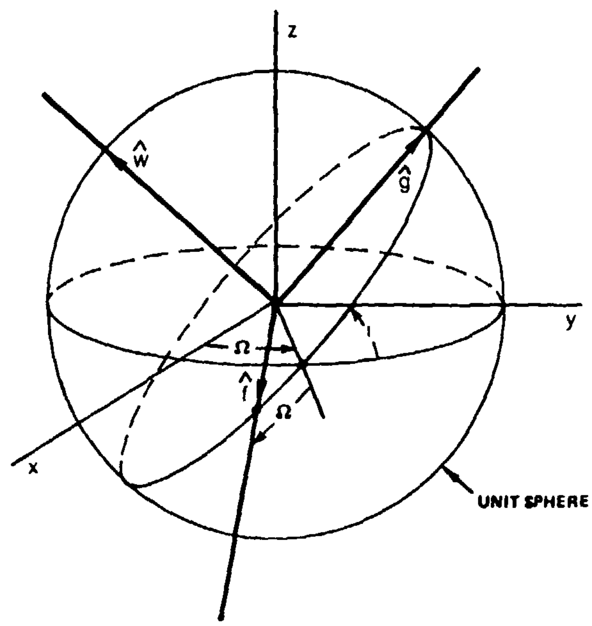

Figure A-1. Direct Equinoctial Coordinate Frame

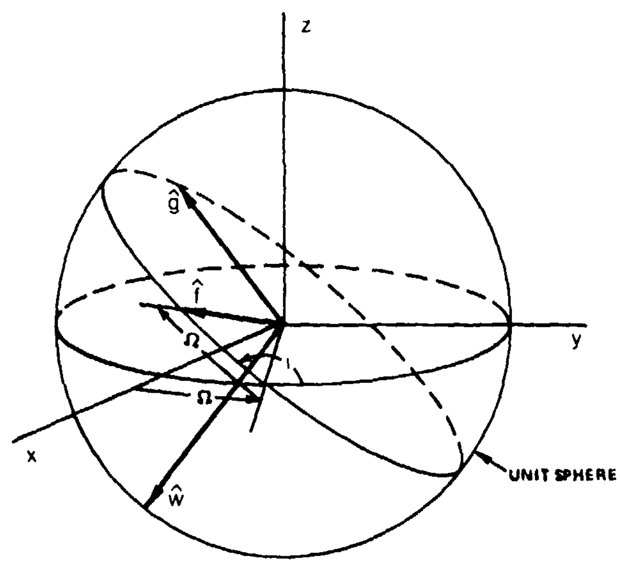

Figure A-2. Retrograde Equinoctial Coordinate Frame

The transformation from the arbitrary system to the equinoctial system is expressed as

$$\vec{r}_e = T\,\vec{r}_a \tag{A-3}$$

where

$$T = R_z(-\Omega)\, R_x(i)\, R_z(\Omega) \tag{A-4}$$

and $\vec{r}_a$ designates the position vector referred to the arbitrary reference system, $\vec{r}_e$ designates the same position vector referred to either the direct or retrograde equinoctial system (depending on the value of the retrograde factor), and where

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-5}$$

and

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{A-6}$$

are the matrix representations of the rotation through the angle θ about the z and x axes, respectively.

(In the discussion that follows, the definitions

$$C_\theta \equiv \cos\theta$$

$$S_\theta \equiv \sin\theta$$

are made, and it follows that

$$C_{-I\Omega} = C_\Omega$$

$$S_{-I\Omega} = -IS_\Omega$$

Multiplication of the three rotation matrices in Equation (A-4) yields the transformation matrix

$$T = \begin{bmatrix} C_\Omega^2 + IS_\Omega^2 C_i & C_\Omega S_\Omega (1 - IC_i) & -IS_\Omega S_i \\ IC_\Omega S_\Omega (1 - IC_i) & I(S_\Omega^2 + IC_\Omega^2 C_i) & C_\Omega S_i \\ S_\Omega S_i & -C_\Omega S_i & C_i \end{bmatrix} \tag{A-7}$$

It can be easily verified from Equations (A-1) that

$$S_\Omega = \frac{p}{\sqrt{p^2 + q^2}} \tag{A-8a}$$

$$C_\Omega = \frac{q}{\sqrt{p^2 + q^2}} \tag{A-8b}$$

$$S_i = \frac{2\sqrt{p^2+q^2}}{1+p^2+q^2} \tag{A-8c}$$

$$C_i = \frac{(1-p^2-q^2)I}{1+p^2+q^2} \tag{A-8d}$$

and the transformation matrix is expressed in the equinoctial elements[1] p and q as

$$T = \frac{1}{1+p^2+q^2}\begin{bmatrix} 1-p^2+q^2 & 2pq & -2pI \\ 2pqI & (1+p^2-q^2)I & 2q \\ 2p & -2q & (1-p^2-q^2)I \end{bmatrix} \tag{A-9}$$

The rows of this transformation matrix are the components (direction cosines) of the $\hat{f}$, $\hat{g}$, $\hat{w}$ vectors, respectively, in the arbitrary reference system, i.e.,

$$\hat{f} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 1-p^2+q^2 \\ 2pq \\ -2pI \end{bmatrix} \tag{A-10a}$$

---

[1]The definition of the elements p and q must, of course, be consistent with the value of the retrograde factor.

$$\hat{g} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 2pqI \\ (1+p^2-q^2)I \\ 2q \end{bmatrix} \tag{A-10b}$$

$$\hat{w} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 2p \\ -2q \\ (1-p^2-q^2)I \end{bmatrix} \tag{A-10c}$$

This is easily demonstrated since

$$\vec{r}_e = x'\hat{f} + y'\hat{g} + z'\hat{w}$$

and

$$x' = \vec{r}_a \cdot \hat{f}$$

$$y' = \vec{r}_a \cdot \hat{g}$$

$$z' = \vec{r}_a \cdot \hat{w}$$

A.3 TRANSFORMATION FROM EQUINOCTIAL ELEMENTS TO POSITION AND VELOCITY

The key to this transformation is the transformation from equinoctial elements to position and velocity in the equinoctial reference system. The position and velocity in any right-hand orthogonal reference system is then obtained by inverting the transformation matrix given in Equation (A-7), i.e.,

$$\vec{r}_a = T^{-1} \vec{r}_e \tag{A-11}$$

$$\dot{\vec{r}}_a = T^{-1} \dot{\vec{r}}_e \tag{A-12}$$

The transformation from equinoctial elements to the position and velocity in the equinoctial reference system makes use of the mean, eccentric, and true longitudes, respectively, which are defined by

$$\lambda = \ell + \omega + I\Omega \tag{A-13}$$

$$F = u + \omega + I\Omega \tag{A-14}$$

$$L = f + \omega + I\Omega \tag{A-15}$$

where ℓ, u, and f are the mean, eccentric, and true anomalies.

The position and velocity vectors can be expressed as

$$\vec{r}_e = X\hat{f} + Y\hat{g} \tag{A-16}$$

and

$$\dot{\vec{r}}_e = \dot{X}\hat{f} + \dot{Y}\hat{g} \tag{A-17}$$

since there is no motion out of the orbital plane.

Expressions for the coordinates of the position (X, Y) in terms of the true longitude follow directly from analytical geometry and are given by

$$X = r\cos L \tag{A-18}$$

$$Y = r\sin L \tag{A-19}$$

where

$$r = \frac{a(1-h^2-k^2)}{1+k\cos L+h\sin L} \tag{A-20}$$

The coordinates of the velocity vector are easily obtained by differentiating the expressions for the position coordinates and substituting the following two-body relation into the result:

$$\dot{L} = \frac{a^2\sqrt{1-h^2-k^2}\,\dot{\lambda}}{r^2} = \frac{na^2\sqrt{1-h^2-k^2}}{r^2} \tag{A-21}$$

The final results are

$$\dot{X} = \frac{-na(h+\sin L)}{\sqrt{1-h^2-k^2}} \tag{A-22}$$

and

$$\dot{Y} = \frac{na(k+\cos L)}{\sqrt{1-h^2-k^2}} \tag{A-23}$$

The position coordinates can be expressed in terms of the eccentric longitude, F, using the two-body relations

$$r\cos(L-\phi) = a\cos(F-\phi) - ae \tag{A-24}$$

and

$$r\sin(L-\phi) = a\frac{(1-\beta)}{\beta}\sin(F-\phi) \tag{A-25}$$

where

$$\beta = \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \tag{A-26a}$$

and

$$\phi = \omega + I\Omega \tag{A-26b}$$

The final results are

$$X = a\left[(1 - h^2\beta)\cos F + hk\beta\sin F - k\right] \tag{A-27}$$

and

$$Y = a\left[(1 - k^2\beta)\sin F + hk\beta\cos F - h\right] \tag{A-28}$$

The velocity coordinates follow by differentiating Equations (A-27) and (A-28) and substituting the two-body relation

$$\dot{F} = \frac{a\dot{\lambda}}{r} = \frac{na}{r} \tag{A-29}$$

yielding

$$\dot{X} = \frac{na^2}{r}\left[hk\beta\cos F - (1 - h^2\beta)\sin F\right] \tag{A-30}$$

and

$$\dot{Y} = \frac{na^2}{r}\left[\left(1 - k^2\beta\right)\cos F - hk\beta \sin F\right] \tag{A-31}$$

where the radial distance is expressed as

$$r = a\left(1 - k\cos F - h\sin F\right)$$

Equations (A-30) and (A-31) can also be obtained by combining Equations (A-18) and (A-19) with Equations (A-27) and (A-28) to yield expressions for $\cos L$ and $\sin L$. These expressions are substituted into Equations (A-22) and (A-23) to yield the final result.

A.4 TRANSFORMATION FROM POSITION AND VELOCITY TO EQUINOCTIAL ELEMENTS

The transformation from position and velocity in an arbitrary reference system to the equinoctial elements could be obtained by inverting the proper equations in Section A.3. However, appealing directly to the classical two-body problem permits a more concise derivation. The semimajor axis is immediately obtained by inverting the well known energy integral for the two-body problem which yields

$$a = \left(\frac{2}{|\vec{r}|} - \frac{|\dot{\vec{r}}|^2}{\mu} \right)^{-1} \tag{A-32}$$

where $\vec{r}$ is the position vector of the satellite in the $(\hat{x}, \hat{y}, \hat{z})$ reference system. The eccentricity vector is given by

$$\vec{e} = -\frac{\vec{r}}{|\vec{r}|} - \frac{(\vec{r} \times \dot{\vec{r}}) \times \dot{\vec{r}}}{\mu} \tag{A-33}$$

and the unit vector normal to the orbital plane is the normalized angular momentum vector given by

$$\hat{w} = \frac{\vec{r} \times \dot{\vec{r}}}{|\vec{r} \times \dot{\vec{r}}|} \tag{A-34}$$

In view of Equation (A-10c) relating the elements p and q to the vector $\hat{w}$, it follows that

$$p = \frac{w_x}{1 + \hat{w}_z I} \qquad \text{(A-35)}$$

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and

$$q = \frac{-\hat{w}_y}{1 + \hat{w}_z I} \qquad \text{(A-36)}$$

The elements p and q determined from Equations (A-35) and (A-36) are consistent with the value of the retrograde factor I .

The unit vectors f and g may now be computed using Equations (A-10a) and (A-10b). The equinoctial orbital elements h and k are computed using the formulas

$$h = \vec{e} \cdot \hat{g} \qquad \text{(A-37)}$$

and

$$k = \vec{e} \cdot \hat{f} \qquad \text{(A-38)}$$

The elements h and k are consistent with the vectors $\hat{f}$ and $\hat{g}$ with regard to the direct and retrograde definitions.

The remaining element to be computed is the mean longitude, λ. First, the position coordinates X and Y of the satellite relative to the orbital frame $\hat{f}$, $\hat{g}$, and $\hat{w}$ are computed from the expressions

$$X = r\cos L = \vec{r}\cdot\hat{f} \tag{A-39}$$

$$Y = r\sin L = \vec{r}\cdot\hat{g} \tag{A-40}$$

Inverting Equations (A-27) and (A-28) yields the expressions

$$\cos F = k + \frac{(1+k^2\beta)X - hk\beta Y}{a\sqrt{1-h^2-k^2}} \tag{A-41}$$

$$\sin F = h + \frac{(1-h^2\beta)Y - hk\beta X}{a\sqrt{1-h^2-k^2}} \tag{A-42}$$

which, when substituted into Kepler's equation

$$\lambda = F - k\sin F + h\cos F \tag{A-43}$$

yields the desired result.

A.5 POISSON BRACKETS

In the present application, the Poisson Brackets must be given in terms of the equinoctial elements. The results are obtained by direct substitution into the previously obtained results of Broucke and Cefola (Reference 33) and are listed in Table A-1.

Table A-1. Poisson Brackets of Equinoctial Elements[1,2]

$(a, \lambda_0) = -2as_1$		$(h, k) = -s_1 s_3$	
$(\lambda_0, h) = -hs_4$		$(h, p) = -kps_5$	
$(\lambda_0, k) = -ks_4$		$(h, q) = -kqs_5$	
$(\lambda_0, p) = -ps_5$		$(k, p) = hps_5$	
$(\lambda_0, q) = -qs_5$		$(k, q) = hqs_5$	
		$(p, q) = -(1/2)s_2 s_5 I$	

[1] Auxiliary Variables:

$$s_1 = 1/na^2$$

$$s_2 = 1 + p^2 + q^2$$

$$s_3 = \sqrt{1 - h^2 - k^2}$$

$$s_4 = s_1 s_3/(1 + s_3)$$

$$s_5 = s_1 s_2/(2s_3)$$

[2] These expressions are valid for both the direct and retrograde element sets.

A.6 PARTIAL DERIVATIVES OF THE EQUINOCTIAL ELEMENTS WITH RESPECT TO VELOCITY

The partial derivatives $\partial a/\partial\dot{\vec{r}}$, $\partial p/\partial\dot{\vec{r}}$, and $\partial q/\partial\dot{\vec{r}}$ are obtained directly as functions of the equinoctial elements by using the results of Broucke and Cefola (Reference 33). However, the expressions for $\partial h/\partial\dot{\vec{r}}$, $\partial k/\partial\dot{\vec{r}}$, and $\partial\lambda_0/\partial\dot{\vec{r}}$ in terms of the classical orbital elements are not as easily translated into the equinoctial elements. To compute these quantities, the following relationship (obtained by Broucke (Reference 30)) is used:

$$\frac{\partial a_\alpha}{\partial\dot{\vec{r}}} = -\sum_{\beta=1}^{6}(a_\alpha, a_\beta)\frac{\partial\vec{r}}{\partial a_\beta} \qquad \text{(A-44)}$$

which requires the Poisson Brackets from Table A-1 and the partial derivatives of the position vector. To obtain $\partial\vec{r}/\partial h$ and $\partial\vec{r}/\partial k$, the following partial derivatives of X and Y are needed:

$$\frac{\partial X}{\partial h} = -\frac{k\beta\dot{X}}{n} + \frac{a}{G}Y\dot{Y} \qquad \text{(A-45a)}$$

$$\frac{\partial X}{\partial k} = \frac{h\beta\dot{X}}{n} + \frac{a}{G}(\dot{X}Y - G) \qquad \text{(A-45b)}$$

$$\frac{\partial Y}{\partial h} = -\frac{k\beta\dot{Y}}{n} - \frac{a}{G}(X\dot{Y} + G) \qquad \text{(A-45c)}$$

$$\frac{\partial Y}{\partial k} = -\frac{a}{G}X\dot{X} + \frac{h\beta\dot{Y}}{n} \qquad \text{(A-45d)}$$

With these results, the position partial derivatives can be specified, as shown in Table A-2. Substitution of the results of Tables A-1 and A-2 into Equation (A-44) gives the desired results, which are listed in Table A-3.

Table A-2. Partial Derivatives of Position

$$\frac{\partial \vec{r}}{\partial a} = \frac{1}{a}\left(\vec{r} - \frac{3}{2}\dot{\vec{r}}\,t\right)$$

$$\frac{\partial \vec{r}}{\partial h} = \frac{\partial X}{\partial h}\,\hat{f} + \frac{\partial Y}{\partial h}\,\hat{g}$$

$$\frac{\partial \vec{r}}{\partial k} = \frac{\partial X}{\partial k}\,\hat{f} + \frac{\partial Y}{\partial k}\,\hat{g}$$

$$\frac{\partial \vec{r}}{\partial \lambda_0} = \frac{\dot{\vec{r}}}{n}$$

$$\frac{\partial \vec{r}}{\partial p} = \frac{2}{1+p^2+q^2}\left[q(Y\,\hat{f} - X\,\hat{g})I - X\,\hat{w}\right]$$

$$\frac{\partial \vec{r}}{\partial q} = \frac{2}{1+p^2+q^2}\left[p(X\,\hat{g} - Y\,\hat{f}) + Y\,\hat{w}\right]I$$

Table A-3. Partial Derivatives of the Equinoctial Elements With Respect to Velocity

$$\frac{\partial a}{\partial \dot{\vec{r}}} = \frac{2\dot{\vec{r}}}{n^2 a}$$

$$\frac{\partial h}{\partial \dot{\vec{r}}} = -\frac{1}{\mu}\left[G\hat{f} + r\dot{X}\,\hat{y}\right] + \frac{k}{G}(qYI - pX)\hat{w}$$

$$\frac{\partial k}{\partial \dot{\vec{r}}} = \frac{1}{\mu}\left[G\hat{g} + r\dot{Y}\,\hat{y}\right] - \frac{h}{G}(qYI - pX)\hat{w}$$

$$\frac{\partial p}{\partial \dot{\vec{r}}} = \frac{(1+p^2+q^2)Y\,\hat{w}}{2G}$$

$$\frac{\partial q}{\partial \dot{\vec{r}}} \quad \frac{(1+p^2+q^2)X\,\hat{w}}{2G}$$

$$\frac{\partial \lambda}{\partial \dot{\vec{r}}} = \frac{-2}{na^2}\vec{r} + \beta\left(k\frac{\partial h}{\partial \dot{\vec{r}}} - h\frac{\partial k}{\partial \dot{\vec{r}}}\right) + \frac{1}{na^2}(qIY - pX)\hat{w}$$

$$\hat{y} = \frac{\hat{w} \times \vec{r}}{r}$$

$$G = na^2\sqrt{1-h^2-k^2}$$

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