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A RECURSIVELY FORMULATED FIRST-ORDER SEMIANALYTIC ARTIFICIAL SATELLITE THEORY BASED ON THE GENERALIZED METHOD OF AVERAGING

VOLUME II

THE EXPLICIT DEVELOPMENT OF THE FIRST-ORDER AVERAGED EQUATIONS OF MOTION FOR THE NONSPHERICAL GRAVITATIONAL AND NONRESONANT THIRD-BODY PERTURBATIONS

(NASA-CR-156783) A RECURSIVELY FORMULATED FIRST-ORDER SEMIANALYTIC ARTIFICIAL SATELLITE THEORY BASED ON THE GENERALIZED METHOD OF AVERAGING. VOLUME 2: THE EXPLICIT DEVELOPMENT OF (Computer Sciences N78-28148 Unclas G3/15 27059

Prepared For

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Goddard Space Flight Center
Greenbelt, Maryland

CONTRACT NAS 5-24300
Task Assignment 895

MAY 1978

CSC

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Volume II. The Explicit Development of the First-Order Averaged Equations of Motion for the Nonspherical Gravitational and Nonresonant Third-Body Perturbations

Prepared by

COMPUTER SCIENCES CORPORATION

For

GODDARD SPACE FLIGHT CENTER

Under

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PREFACE

During the past several years, rapid orbit generation techniques, based on a first-order application of the generalized method of averaging, have been investigated for the National Aeronautics and Space Administration (NASA) Goddard Space Flight Center (GSFC). This investigation has culminated in the development of a hybrid averaged orbit generator which has been implemented in the Research and Development (R&D) version of the Goddard Trajectory Determination System (GTDS).

In order to satisfy the requirements of different audiences and because of the scope of this investigation, the results of the investigation have been documented in several parts. The primary documents are as follows:

> A Recursively Formulated First-Order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging. Volume I: The Generalized Method of Averaging Applied to the Artificial Satellite Problem. Computer Sciences Corporation Report No. CSC/TR-77/6010, Wayne D. McClain, November 1977.
>
> [This document presents a discussion of the application of the generalized method of averaging to the artificial satellite problem; the document is specifically directed to the analyst.]

> A Recursively Formulated First-Order Semianalytic Artificial Satellite Theory Based on the Generalized Method of Averaging. Volume II: The Explicit Development of the First-Order Averaged Equations of Motion for the Nonspherical Gravitational and Nonresonant Third-Body Perturbations. (The present document.)
>
> [This document presents the explicit development of the first-order averaged equations of motion for the nonspherical gravitational and nonresonant third-body perturbations. The document is directed to the analyst.]

> System Description and User's Guide for the GTDS R&D Averaged Orbit Generator. Computer Sciences Corporation Report No. CSC/SD-78/6020, Leo W. Early, June 1978.
>
> [This document presents an overview of the averaged orbit generator, a description of the software system, and instructions for program execution. The document is directed to a general audience consisting of analysts, programmers, and data technicians.]

The Numerical Evaluation of the GTDS R&D Averaged Orbit Generator. Computer Sciences Corporation Report No. CSC/TM-78/6138, W. D. McClain and L. W. Early, September 1978 (in preparation).

[This document is directed primarily to the analyst and user.]

Status Report on Numerical Averaging Methods. Computer Sciences Corporation Report No. CSC/TM-75/6039, Anne C. Long, September 1975

[This document presents a discussion of the numerical averaging capability in the GTDS R&D hybrid averaged orbit generator (parts of this document are superseded by the report CSC/SD-78/6020 described above). The document is directed primarily to the analyst.]

Development and Evaluation of Numerical Quadrature Procedures for Use in Numerically Averaged Variation-of-Parameters Orbit Generators. Computer Sciences Corporation Report No. CSC/TM-75/6038, Leo W. Early, July 1975.

[This document is also directed primarily to the analyst.]

Earlier documents reporting preliminary results for both analytical and numerical averaging techniques are referenced in the above documents.

ABSTRACT

This report presents, in two volumes, a recursively formulated, first-order, semianalytic artificial satellite theory, based on the generalized method of averaging. Volume I, which has been produced under a separate cover, discusses the theory of the generalized method of averaging applied to the artificial satellite problem.

The present volume, Volume II, presents a general first-order theory for the accurate computation of the long-period and secular motion of a satellite caused by the nonspherical gravitational field of the central body. Also, the development of the first-order averaged third-body disturbing function is presented, and the theory for the accurate computation of the long-period and secular motion for the special case of low-altitude nonresonant satellites is completed. Recursive algorithms are provided for efficient evaluation of the theory. In addition, several mathematical developments necessary for the construction of the first-order theory are presented. Also, sufficient information has been provided to construct the analytical formulation of the first-order short-period variations.

This theory has been implemented in the Research and Development (R&D) version of the Goddard Trajectory Determination System (GTDS), a large orbit determination system primarily devoted to research and development efforts supported by the Goddard Space Flight Center (GSFC).

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SECTION 1 - INTRODUCTION

The prediction and definitive determination of artificial satellite orbits is one of the more computationally expensive dynamical problems today. Maintaining accurate ephemerides for the ever-increasing number of artificial satellites (which include active scientific, defense, communication, and weather satellites as well as defunct satellites, launch vehicles, and other debris) requires a considerable expenditure in terms of computing time. Prelaunch mission analysis requires that several hundred satellite trajectories over periods of up to several years be generated for the purposes of lifetime and geometry constraint analysis.

Generally, these applications fall into two categories: those applications which require high accuracy, e.g., definitive orbit determination, and the low-to-moderate accuracy applications referred to under the broad category of mission analysis. The highest accuracy requirements are obtained through the extremely accurate high-precision orbit generation techniques which rely on the expensive process of numerical integration of Newton's equations of motion or some equivalent set of differential equations. Applications with less stringent accuracy requirements are often treated with analytical approximations. Mission analysis applications are often treated with analytical approximations and, in many cases, even two-body mechanics is used.

The analytical approach to the artificial satellite problem yields a set of analytical formulas for the coordinates or orbital elements which are usually obtained to first or second order in a small parameter. The approach is to separate the short-period, long-period, and secular components of the motion through a series of canonical transformations (Reference 1). The secular contributions to the motion are evaluated at a given time, and the canonical transformation used to remove the long-period component of motion is inverted to provide the long-period motion in terms of the secular elements. Finally, the transformation to remove the short-period terms is inverted and evaluated with the secular and long-period contributions to the elements, thus obtaining the short-period contributions to the motion.

Generally, existing analytical satellite theories[1] have severely restricted perturbation models. In addition, these analytical satellite theories are not very flexible with respect to the extension of the force model. This is partially because an analytical satellite theory has not been formulated for completely general central-body and third-body perturbation models.[2] More importantly, the fundamental problem of developing an accurate and flexible analytical drag theory remains unsolved. Consequently, as knowledge of the physical environment (e.g., atmospheric density, geopotential coefficients of high order and degree) and accuracy requirements increase, the current analytical theories cannot be expected to keep pace. If the more costly high-precision methods are to be avoided and the increased accuracy requirements are to be met, either more generally formulated analytical theories must be developed, including a much more satisfactory treatment of analytical drag, or an alternate approach must be found.

The method of averaging offers a very promising alternative approach for the artificial satellite problem. This approach shares similarities with both the numerical high-precision and the pure analytical methods and is classified as a semianalytical method. In essence, this method provides the long-period and secular motion of the satellite very efficiently through the numerical integration of the averaged equations of motion. In addition, the theory provides

[1] Y. Hagihara (Reference 2) gives an extensive list of references to the work in artificial satellite theory.

[2] Small (Reference 3) has developed a first-order analytical theory for an arbitrary number of zonal harmonic terms, and Mueller (Reference 4), using the Poincare-Von Zeipel technique, has developed a first-order analytical theory for the secular and long-period motion due to an arbitrary number of zonal harmonic terms. Recurrence relations are used in the evaluation of both theories.

for short-period variations in the osculating elements (Reference 5) which are required for high-accuracy applications.[1]

The method of averaging approach is particularly flexible, especially with respect to the atmospheric drag models. Not only are the complex drag models which are used in high-precision theories easily accommodated, but they are also easily interchanged without any impact on the theory (Reference 6). General models for the central body, nonspherical gravitational, and third-body perturbations can be developed in a straightforward manner, which is the subject of this volume.

A combination of numerical evaluation and theoretical considerations indicates that the method of averaging approach is generally two to three orders of magnitude more efficient than the high-precision techniques.[2] Specifically, it has been shown in References 6 and 7 that a first-order application of the method of averaging to the artificial satellite problem produces the long-period and secular motion very accurately and with the computational efficiency cited above.

Thus, the method of averaging provides a low-cost, long-term orbit prediction capability useful for the following applications:

- Mission analysis (lifetime and geometric constraints)
- Tracking station acquisition schedules
- Dynamic modeling required for differential correction (DC) procedures used to solve for dynamical parameters, e.g., high-order geopotential coefficients

[1]This is equivalent to inverting the first canoncial transformation in the analytical satellite theory to obtain first-order short-period variations in the osculating elements, which are then superimposed on the secular and long-period elements. See Section 4 of Reference 5 for more details.

[2]For very strongly drag perturbed satellites, the increase in efficiency may be reduced to between one and two orders of magnitude.

The effectiveness of representing the osculating elements by superimposing first-order short-period variations on the mean elements has been demonstrated by Lutzky and Uphoff (Reference 8). Also, it can be shown both from the discussion in Volume I of this report (Reference 5) and in the present volume and from the discussion in Reference 9 that the first-order short-period variations can be formulated analytically. It appears that the cost of the evaluation of these analytical formulas is quite feasible and would be roughly equivalent to a single evaluation of the mean element rates which are numerically integrated to obtain the long-period and secular motion. However, the cost of evaluating these short-period variations at several points in a single orbital revolution would be considerably less than the cost of evaluating the same number of mean element rates.[1] Consequently, it appears that this high-accuracy mode of the method of averaging could prove to be significantly more efficient than the usual high-precision techniques and, thus, may offer an efficient high-precision orbit prediction capability for definitive orbit determination procedures, particularly where extended data intervals with data gaps are encountered. In addition, the short-period variations can and should be used to develop an osculating-to-mean element conversion capability.

[1] Within the same revolution or "pass," the slowly varying mean elements are essentially constant. Only those functions which depend explicitly on the fast variable need to be reevaluated.

1.1 OVERVIEW OF THE METHOD OF AVERAGING

The efficiency of the method of averaging procedure arises from the fact that the maximum step size which can be used in the numerical integration of a set of differential equations is constrained by the highest significant frequency (i.e., shortest period) contained therein. The method of averaging is used to remove high-frequency components from the equations of motion. The resulting averaged equations of motion are integrated numerically but with a significantly greater step size than can be used with the high-precision equations. The long-period and secular components of the satellite motion are thus obtained. The short-period component of the motion can be computed either numerically (Reference 8) or from analytical formulas which can be constructed from the results contained in Reference 5 and Sections 3 and 4 of the present document. These formulas are also developed in Reference 9. In most cases, the computational savings achieved by the larger step size (which results in fewer force evaluations) far outweighs the possible additional cost of the derivative evaluation,[1] thereby effecting a significant decrease in the overall computational costs.

The technique of removing the high-frequency terms from the equations of motion was first used by Lagrange in his investigations of the planetary motion. In the particular formulation of the equations of motion developed by Lagrange, the high-frequency terms, in the case of conservative perturbing forces, could be isolated more or less by inspection. However, a rigorous mathematical foundation for this technique was not provided until the relatively recent work by Krylov and Bogoliubov (Reference 10) on asymptotic methods for nonlinear oscillations.

Two approaches are available for the application of the method of averaging. The high-frequency components of the equations of motion can be removed

[1] The exact cost of a derivative evaluation depends on the specific perturbations and the characteristics of the satellite orbit, which may permit significant truncation of the series expansions.

numerically by application of a quadrature around an appropriate formulation of the high-precision equations of motion. This procedure is known as the numerical averaging approach. If the perturbing forces are conservative, the equations of motion can be expressed using Lagrange's formulation (Reference 5), and the averaging quadrature can be performed analytically. Under certain assumptions,[1] this method produces the same result as that obtained by inspection. This semianalytical procedure of numerically integrating the analytically averaged equations of motion is referred to as the analytical averaging approach.

[1]The assumptions arise when either the Greenwich Hour Angle (i.e., the Earth's rotation) or the fast variable of the disturbing third body appear in the perturbation models. Specifically, these quantities are assumed to be completely independent of the satellite fast variable, both explicitly and implicitly through the time.

1.2 RECENT DEVELOPMENTS IN ANALYTICAL AVERAGING THEORY

Recently, several authors have investigated general, analytically averaged perturbation models for the third-body and nonspherical gravitational perturbations in terms of nonsingular element sets. Cefola and Broucke (Reference 11) developed recursively formulated models for the nonresonant third-body and zonal harmonic perturbations based on the nonsingular equinoctial elements. The development of the zonal harmonic model is similar to that of Cook (Reference 12), with the exception that the inclination function is developed in terms of associated Legendre polynomials and their derivatives and certain complex polynomials. Cefola's third-body model is developed in terms of the direction cosines of the disturbing third-body position vector, which proves computationally efficient but is limited to nonresonant cases. Cefola outlined an extension of his zonal harmonic model to include the nonresonant tesseral harmonic terms (Reference 13) and later completed and extended the model to include resonant phenomena (Reference 14). Giacaglia (Reference 15) reformulated Kaula's (References 16 and 17) perturbation models (using Allan's inclination function) in a nonsingular element set and provided a set of recursive algorithms for computational purposes. Finally, Nacozy and Dallas (Reference 18) also reformulated the Kaula geopotential model (using Allan's inclination function) in terms of a nonsingular element set. No recursive algorithms were provided.

1.3 SUMMARY

This report is the result of a series of task assignments with the objective of implementing in the Research and Development (R&D) version of the Goddard Trajectory Determination System (GTDS) a set of recursively evaluated, first-order analytically averaged equations of motion for an artificial satellite perturbed by nonresonant third-body and nonspherical gravitational perturbations. This analtyical averaging capability enhances the GTDS numerical averaging capability (Reference 6) and provides for optimal averaged perturbation models for each specific type of perturbation (with the exception of third-body resonance cases, which were not considered). Partial results obtained for some of the otpimal averaged perturbation models in GTDS have been presented in Reference 7.

For implementation, Cefola's averaged perturbation models (Reference 11) for the nonresonant third-body and zonal harmonic perturbations are adopted. The nonresonant tesseral harmonic model was developed as part of this task assignment using the approach outlined by Cefola in Reference 13. The resonant tesseral harmonics model was also developed as part of the task assignment from a completely general nonspherical gravitational theory designed to yield the zonal harmonics, nonresonant tesseral harmonics, and resonant tesseral harmonics models as special cases. In addition, all models were generalized to handle retrograde as well as direct equinoctial elements (see Appendix A of Reference 5).

The brute-force implementation of recursive algorithms can contribute to computational inefficiency and can possibly introduce artificial singularities (not in the equations of motion, but in the model evaluation). To insure against this possibility, careful consideration was given to the ordering of the terms in the models, such that the recurrence formulas proceed in the proper direction to avoid small divisors. Also, the amount of recomputation and storage requirements are minimized.

For implementation in the GTDS R&D system, it was felt that the resonant tesseral harmonic model should be very flexible with respect to the specific resonant harmonic terms used. The existence of a resonance dictates which terms in the potential expansion are significant to the long-period motion. Knowledge of the common characteristics of these terms and the proper use of the recursive algorithms could have provided a means for further optimization of this model. However, the procedure would have been automatic, with the program expecting a certain set of terms. Therefore, for the purposes of flexibility and at some additional computational costs, the contributions from each spherical harmonic term are computed entirely independently of all other terms.[1]

Due to the extensive new software for the analytical averaging capability, as well as to the extensive modifications required to the previously implemented averaging software (particularly the input processor and initialization procedures and the attendant added complexity of executing the GTDS R&D averaging capability), it was decided that a system description and user's guide for the GTDS R&D averaging capability would be issued under a separate cover (Reference 19). In addition, a document extending the numerical results beyond those presented in Reference 7 is also in preparation. This document (Reference 20) will discuss the computational costs in terms of machine processing time, the accuracy of the analytical averaging methods, and the procedure and algorithms used to develop an automatic truncation capability to further optimize the perturbation models for each particular case.

The present report consists of two volumes. Volume I (Reference 5) presents a comprehensive discussion of the application of the generalized method of averaging to the artificial satellite problem and the resulting formulation of the averaged equations of motion. Included in the discussion are the formulation of the Variation of Parameters (VOP) equations of motion and the application

[1]The capability to automatically select the resonant terms was implemented in the GTDS R&D version. However, no special relationship among them is assumed.

of the method of averaging to the VOP equations of motion. Other topics discussed include the criteria for the selection of short-period terms, the application of the method of averaging to the case of two or more perturbing functions, the application of the method of averaging to cases involving resonance phenomena, and a discussion of the first-order short-period variations in the osculating elements and their application to osculating-to-mean and mean-to-osculating element conversions.

Volume II (the present document) presents the mathematical formulation, in nonsingular equinoctial elements, of the nonspherical gravitational and nonresonant third-body disturbing functions required for the first-order averaged equations of motion. Section 2 of this document presents some mathematical developments required for the expansion of the disturbing functions. Specifically, Section 2.1 discusses the theory of the rotation of spherical harmonic functions. Next, Section 2.2 develops certain Fourier series expansions which are of importance in the development of the disturbing functions.

Section 3 presents the explicit theory for the nonspherical gravitational perturbation. The development of the nonspherical gravitational disturbing function is discussed in Section 3.1, and the disturbing function is expressed in equinoctial elements in Section 3.2. Also, a discussion relating Kaula's inclination function (References 16 and 17) to the inclination function developed in this report is presented. The nonspherical gravitational disturbing function is averaged in Section 3.3. The averaging operation and the concepts and implications of time-dependent and time-independent averaging are discussed. In addition, the averaged disturbing functions for the special cases of the zonal harmonics, combined zonal and nonresonant tesseral harmonics, and resonant tesseral harmonics are isolated and presented. In Section 3.4, the partial derivatives of the nonspherical gravitational disturbing function which are required for the averaged equations of motion are presented for each case, and the recurrence relations used for the evaluation of the constituent functions are given.

Section 4 of the present volume presents the explicit theory for the disturbing third-body perturbation. Section 4.1 discusses the development of the third-body disturbing function, and Section 4.2 gives the general expansion of the disturbing function. Due to time and other resource constraints, only an outline of the general development is presented. However, because of the similarities with the nonspherical gravitational theory presented in Section 3, the neglected details are straightforward. Section 4.3 presents another expansion of the third-body disturbing function which is well suited for cases of nonresonant (with the third body) near-Earth satellites. Section 4.4 presents the partial derivatives of the averaged disturbing function developed in Section 4.3 which are required for the averaged equations of motion in the special case. The necessary recurrence relations for evaluation of the theory are also provided.

The equations of motion for all models are given in what is considered to be an optimal form, taking into account the minimization of the combined computational and storage costs while avoiding computational singularities. It is this final form of the models that was implemented in the GTDS R&D system. These models reflect, to some extent, the computer environment in which they were implemented, i.e., the GSFC IBM S/360-95 computer. A variable can assume a moderately wide range of magnitudes in this environment. This, of course, is not true in all computer environments. Thus, for implementation on other computers, it may be necessary to introduce normalization factors and to otherwise redefine certain functions appearing in the models given in Sections 3 and 4 in order to minimize the magnitude differences between quantities.

1.4 CONCLUSIONS

Most satellite theories based on an averaging concept, e.g., Cefola's (Reference 14), Giacaglia's (Reference 15), and Kaula's (References 16 and 17), have been formulated using the classical assumptions of a time-independent perturbation model[1] or of exact resonance. While these classical assumptions guarantee the removal of the unwanted short-period contributions to the motion, they may, in some cases, produce a significant exaggeration of certain medium- and long-period contributions to the motion. This fact should be considered in the implementation of an averaging theory.

The zonal, nonresonant tesseral, and resonant tesseral harmonic models, which are presented in Section 3 of this volume and which have been implemented in the GTDS R&D system, comprise a completely general first-order theory for the contributions to the long-period and secular motion caused by the central-body nonspherical gravitational field. However, as discussed in Section 3.1 of Reference 5, it is recommended that the effects contributed by the nonresonant tesseral harmonics be excluded from the averaged equations of motion, since they unnecessarily restrict the integration step size and since they can be formulated analytically in the same way as the first-order short-period variations in the osculating elements. In addition, the analytical formulas for the first-order short-period variations in the osculating elements can, in essence, be developed from the information contained in both volumes of this report. In order to obtain the final formulas, it is only necessary to extract the appropriate formulas from Sections 2.2, 3, and 4 of the present volume and substitute them into Equations (4-14) of Section 4 in Volume I.

Numerical evaluation of the nonspherical gravitational theories for long-period and secular motion, performed as part of the investigation and documented in

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[1]See Section 3.3 for more details.

References 7 and 20, indicates that the theory is very efficient and accurate for all cases considered, except resonant effects on large-eccentricity orbits. The proper treatment of the eccentricity expansions (Hansen coefficients) in these cases still remains an open question. For the present, the numerical averaging method, with a properly reduced force model, provides an acceptable alternative for cases of deep resonance. However, the effects of the high eccentricities are observed through higher order quadrature algorithms.

Although the general third-body disturbing function is developed in Section 4.2 of the present volume, only the special case discussed in Section 4.3 was implemented in the GTDS R&D system. This implementation is restricted to those applications involving low to moderate altitude satellites that are not in resonance with the disturbing third body. For Earth satellites, these requirements translate into satellites with periods less than 3 to 4 days and which are not in resonance with the Moon.

The numerical evaluation of this capability (References 7 and 20) indicates that it produces the long-period and secular motion of nonresonant satellites of low to moderate altitude very accurately and efficiently. This theory is completely inadequate for Earth satellites with periods longer than 4 days. Also, the expense of numerically averaging such cases is completely unacceptable since it is at least as expensive, and usually more expensive, than high-precision techniques.

The application of averaging methods to satellites with longer periods requires a double-averaged (nonresonant cases) third-body theory or a single-averaged third-body resonance theory, depending on the orbital characteristics of the satellite. It also seems feasible for such theories that first-order short-period variations in the osculating elements can be analytically formulated to meet high-precision requirements.

SECTION 2 - MATHEMATICAL PRELIMINARIES

The explicit development in equinoctial elements of the nonspherical gravitational and third-body disturbing functions requires the mathematical formulation for the rotation of the spherical harmonic functions. In addition, Fourier series representations are required for functions of the form

$$\left(\frac{r}{a}\right)^n \left\{ \begin{matrix} \cos sL \\ \sin sL \end{matrix} \right\}$$

where r is the radial distance of the satellite, a is the semimajor axis of the osculating orbit, L is the true longitude of the satellite which describes the satellite position in the orbit relative to the origin of the longitudes, and s is an integer.

This section presents a general discussion of the rotation of the spherical harmonic functions and develops the Fourier expansions, in the true, eccentric, and mean longitudes, of the functions

$$\left(\frac{r}{a}\right)^n e^{jsL}$$

where

$$e^{jsL} = \exp(jsL) = \cos sL + j \sin sL$$

The results of this section are applied in Sections 3 and 4 to obtain the disturbing functions and averaged equations of motion for the nonspherical gravitational and third-body perturbations, respectively.

2.1 ROTATION OF THE SPHERICAL HARMONIC FUNCTIONS

A spherical harmonic function takes the form

$$\left\{ \begin{matrix} r^{\ell} \\ r^{-\ell-1} \end{matrix} \right\} P_{\ell,m}(\sin\phi) \left(C_{\ell,m} \cos m\lambda + S_{\ell,m} \sin m\lambda \right) \tag{2-1}$$

where (r, λ, ϕ) are the spherical coordinates of a given point, i.e., the satellite position, and $P_{\ell,m}(x)$ is the associated Legendre polynomial of degree ℓ and order m and is defined to be (see Section 3.1 for more details)

$$P_{\ell,m}(x) = (1 - x^2)^{m/2} \frac{1}{2^{\ell} \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^{\ell} \quad (-1 \leq x \leq 1) \tag{2-2}$$

The quantities $C_{\ell,m}$ and $S_{\ell,m}$ are referred to as the spherical harmonic coefficients and are usually empirically determined constants.

A complex variable representation of the spherical harmonic function is useful for the purposes of this discussion and Equation (2-1) is expressed in the form

$$\mathrm{Re}\left[\left\{ \begin{matrix} r^{\ell} \\ r^{-\ell-1} \end{matrix} \right\} \left(C_{\ell,m} - j\, S_{\ell,m} \right) P_{\ell,m}(\sin\phi) \exp(jm\lambda) \right] \tag{2-3}$$

where Re designates the real part and j is the imaginary unit, i.e.,

$$j = \sqrt{-1}$$

Often, it is necessary to transform the above expression to a different reference system in order to obtain an expression in terms of the transformed coordinates.

Most generally, such a transformation involves a translation and rotation of the coordinate reference system.[1] If, however, the two coordinate reference systems possess a common origin, the transformation reduces to only a rotation and the spherical harmonic function is to be expressed in the transformed latitude and longitude (ϕ', λ'). Since the radial distance, r, is invariant under a rotation and since the spherical harmonic coefficients are independent of the position, it is sufficient to rotate only the surface harmonic function

$$P_{\ell,m}(\sin\phi)\exp(jm\lambda) \tag{2-4}$$

In this section, a discussion of the mathematical formulation of a general rotation is presented. Specifically, the Euler angle and Euler parameter representations of a rotation are presented. Next, the rotation of a surface harmonic is discussed. Particular attention is devoted to the development of the generalized inclination function in terms of orthogonal polynomials.

2.1.1 Mathematical Representation of a General Rotation

Since a general rotation of a coordinate reference system leaves the origin invariant, only three independent parameters are required to describe the rotation. It follows that these three independent parameters specify the relative orientation of two coordinate reference systems with a common origin. The three parameters most frequently chosen are the Euler angles $0 \leq \Omega \leq 2\pi$, $0 \leq \omega \leq 2\pi$, and $0 \leq i \leq \pi$ shown in Figure 1.

2.1.1.1 The Euler Angle Representation

The primed coordinate system in Figure 1 can be obtained by performing three simple rotations of the unprimed system. Specifically, the first rotation is performed about the unprimed z axis through the angle Ω. The second rotation is

[1] Lee (Reference 21) discusses the effects on the spherical harmonic functions caused by this more general transformation. Also, see Aardoom (Reference 22).

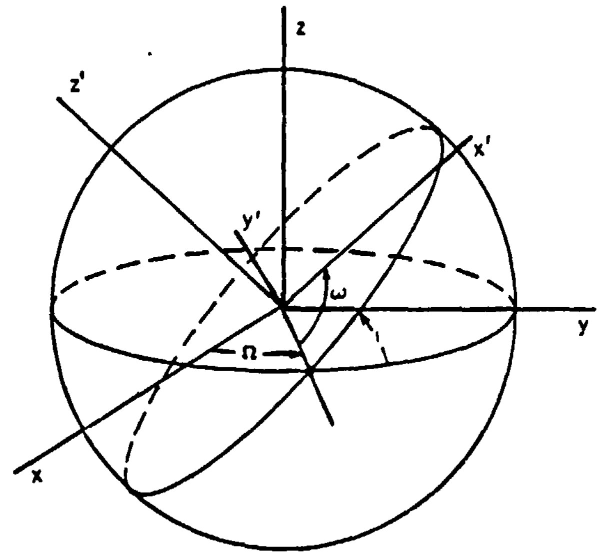

Figure 2-1. Euler Angles

performed about the new x axis through the angle i, thus rotating the xy plane into the x'y' plane. Finally, the third rotation is performed about the z' axis through the angle ω. The general rotation is then expressed as a composite of the three simple rotations, i.e.,

$$A = R_3(\omega)\, R_1(i)\, R_3(\Omega) \tag{2-5}$$

where

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{2-6}$$

and

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-7}$$

are the matrix representations of rotations about the x and z axes, respectively. The product of the three rotation matrices in Equation (2-5) yields the following general rotation matrix:

$$A = \begin{bmatrix} C_\omega C_\Omega - C_i S_\omega S_\Omega & C_\omega S_\Omega + C_i S_\omega C_\Omega & S_\omega S_i \\ -S_\omega C_\Omega - C_i C_\omega S_\Omega & -S_\omega S_\Omega + C_i C_\omega C_\Omega & C_\omega S_i \\ S_i S_\Omega & -S_i C_\Omega & C_i \end{bmatrix} \qquad (2\text{-}8)$$

where

$$C_x = \cos x$$

$$S_x = \sin x$$

This transformation matrix is used to transform the coordinates of the position vector in the unprimed reference system to the coordinates in the primed reference system through the equation

$$\vec{r}\,' = A\vec{r} \qquad (2\text{-}9)$$

The inverse transformation

$$\vec{r} = A^{-1}\vec{r}\,' \qquad (2\text{-}10)$$

is easily obtained since the transformation is orthogonal, i.e.,

$$\sum_{i=1}^{3} a_{ij}\, a_{ik} = \delta_{jk} \qquad \begin{matrix}(j = 1, 2, 3)\\(k = 1, 2, 3)\end{matrix} \qquad (2\text{-}11)$$

where the quantities $a_{n,m}$ are the elements of the transformation matrix A, i.e.,

$$A = \{a_{n,m}\} \qquad (2\text{-}12)$$

It follows that

$$A^{-1} = A^{T} \tag{2-13}$$

where

$$A^{T} = \left| a_{m,n} \right| \tag{2-14}$$

is the transpose of the A matrix obtained by interchanging the rows and columns of the matrix (Reference 23).

2.1.1.2 The Euler Parameter Representation

Other representations of the transformation matrix exist. For example, by using elementary trigonometric identities, it is easily shown that matrix A can be expressed as

$$A = \begin{bmatrix} S_{i/2}^2 C_{\Omega-\omega} + C_{i/2}^2 C_{\Omega+\omega} & S_{i/2}^2 S_{\Omega-\omega} + C_{i/2}^2 S_{\Omega+\omega} & S_\omega S_i \\ -C_{i/2}^2 C_{\Omega-\omega} + S_{i/2}^2 C_{\Omega+\omega} & -S_{i/2}^2 C_{\Omega-\omega} + C_{i/2}^2 C_{\Omega+\omega} & C_\omega S_i \\ S_i S_\Omega & -S_i C_\Omega & C_i \end{bmatrix} \tag{2-15}$$

Making the substitutions

$$\sigma = \frac{\Omega - \omega}{2} \tag{2-16a}$$

$$\rho = \frac{\Omega + \omega}{2} \tag{2-16b}$$

$$\tau = \frac{i}{2} \tag{2-16c}$$

and using the identities

$$C_{2x} = C_x^2 - S_x^2 \tag{2-17a}$$

$$S_{2x} = 2 S_x C_x \tag{2-17b}$$

yields the matrix expression

$$A = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{2-18}$$

where

$$q_1 = S_\tau C_\sigma = \sin\frac{i}{2} \cos\frac{\Omega - \omega}{2} \tag{2-19a}$$

$$q_2 = S_\tau S_\sigma = \sin\frac{i}{2} \sin\frac{\Omega - \omega}{2} \tag{2-19b}$$

$$q_3 = C_\tau S_\rho = \cos\frac{i}{2} \sin\frac{\Omega + \omega}{2} \tag{2-19c}$$

$$q_4 = C_\tau C_\rho = \cos\frac{i}{2} \cos\frac{\Omega + \omega}{2} \tag{2-19d}$$

are the well-known Euler parameters (Reference 24).

The Euler parameters are frequently used in the quaternion representation for the transformed coordinates, i.e.,

$$\vec{r}\,' = q\,\vec{r}\,q^{-1} \tag{2-20}$$

where the quaternion q is a hypercomplex variable defined to be

$$q = q_4 + q_1 i + q_2 j + q_3 k$$

and where

$$\vec{r} = xi + yj + zk$$

$$i^2 = j^2 = k^2 = -1$$

$$q^{-1} = -q$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

Equation (2-20) should be taken only as a convenient algorithm and does not imply that the quaternion is the rotational operator (Reference 24).

2.1.2 Rotation of a Surface Harmonic Function

Essentially, there are two approaches to the rotation of the surface harmonic function

$$P_{\ell,m}(\sin\phi)\, e^{jm\lambda} \tag{2-21}$$

One approach relies on the brute-force substitution of the transformed coordinates using the expressions[1]

$$\sin\phi = \frac{1}{2} S_i \left[e^{j(\omega-\pi/2)} e^{j\lambda'} + e^{-j(\omega-\pi/2)} e^{-j\lambda'} \right] \cos\phi' + C_i \sin\phi' \tag{2-22}$$

$$\cos\phi\, e^{j\lambda} = \cos\phi' \left[C^2_{i/2}\, e^{j(\Omega+\omega)} e^{j\lambda'} + S^2_{i/2}\, e^{j(\Omega-\omega)} e^{-j\lambda'} \right] + S_i\, e^{j(\Omega-\pi/2)} \sin\phi' \tag{2-23}$$

into Equations (2-21). Equations (2-22) and (2-23) are obtained from Equations (2-10), (2-13), and (2-8) or (2-15). This approach can yield complicated expressions which may obscure the real nature of the transformation.

The second approach, based on the work of G. Herglotz and W. Magnus (Reference 25) is to rotate the entire expression for the surface harmonic instead of direct substitution of the rotated position components. Since the derivation of the theory is quite lengthy and since it is provided in Reference 25 and also by Lee (Reference 21), it will not be presented here.

---

[1] In real variables, Equation (2-22) takes on the more familiar form

$$\sin\phi = \sin i\, \sin(\lambda'+\omega) \cos\phi' + \cos i\, \sin\phi'$$

The real and imaginary parts of Equation (2-23) also assume a familiar form.

The rotation of a surface harmonic function of (λ, ϕ) to a reference system with the corresponding coordinates (λ', ϕ') takes the form

$$P_{\ell,m}(\sin\phi)\, e^{jm\lambda} = \sum_{s=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell-m)!}\, S_{2\ell}^{m,s}(\rho,\sigma,\tau)\, P_{\ell,s}(\sin\phi')\, e^{js\lambda'} \qquad (2\text{-}24)$$

where the associated Legendre polynomial of negative order is defined in terms of the corresponding polynomial of positivie order by the relation

$$P_{\ell,-n}(x) = (-1)^n \frac{(\ell-n)!}{(\ell+n)!} P_{\ell,n}(x) \qquad (n \geq 0) \quad (2\text{-}25)$$

The function $S_{2\ell}^{m,s}(\rho, \sigma, \tau)$ is discussed next.

2.1.2.1 Development of the Function $S_{2\ell}^{m,s}(\rho, \sigma, \tau)$.

The function $S_{2\ell}^{m,s}(\rho, \sigma, \tau)$ is defined to be

$$S_{2\ell}^{m,s}(\rho,\sigma,\tau) = \exp\left\{ j\left[(m-s)\left(\sigma - \frac{\pi}{2}\right) - (m+s)\rho\right]\right\} U_{2\ell}^{m,s}(\tau) \qquad (2\text{-}26)$$

where

$$U_{2\ell}^{m,s}(\tau) = \begin{cases} (-1)^{\ell+m}\binom{\ell-m}{\ell+s} C_\tau^{-m-s} S_\tau^{m-s} F\left(-\ell-s,\ \ell-s+1,\ 1-m-s;\ C_\tau^2\right) & (\text{for } m+s \leq 0) \quad (2\text{-}27\text{a}) \\ (-1)^{\ell+s}\binom{\ell+m}{\ell-s} C_\tau^{m+s} S_\tau^{s-m} F\left(s-\ell,\ \ell+s+1,\ 1+m+s;\ C_\tau^2\right) & (\text{for } m+s \geq 0) \quad (2\text{-}27\text{b}) \end{cases}$$

and corresponds, in part, to the inclination function in References 16 and 17.

The parameters ρ, σ, τ describe the orientation of the unprimed or original coordinate reference system relative to the primed or transformed reference system. The corresponding transformation from the primed reference system to the unprimed reference system is given by Equation (2-9) if the symbols $\vec{r}$ and $\vec{r}'$ are interchanged. The parameters ρ, σ, and τ are defined in terms of the Euler angles through Equation (2-16).[1] Otherwise, the transformation given in Equation (2-10) is used and the Euler angles of the inverse transformation (Ω', i', ω') are required to determine ρ, σ, and τ through Equations (2-16).

The notation $F(a, b, c, x)$ in Equations (2-27) designates the well-known hypergeometric series defined by (Reference 26)

$$F(a,b,c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{2-28}$$

where Pochhammer's symbol, $(a)_n$, is defined by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \tag{2-29}$$

Clearly, if a is a nonnegative integer, then Equation (2-29) can be expressed using the well-known Gamma Function as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2-30}$$

---

[1] The resulting definition of σ differs from that found in Courant and Hilbert (Reference 25) by $\pi/2$, i.e.,

$$\sigma_{C\&H} = \sigma - \frac{\pi}{2}$$

In addition, Equation (2-26) is a modification of the corresponding equation in Reference 25 in order to account for the change in the definition of σ.

For negative integers a, Equation (2-29) can be expressed as

$$(a)_n = (-1)^n \frac{(-a)!}{(-a-n)!} \tag{2-31}$$

Ultimately, a set of recursive algorithms is desired for evaluating the function $S_{2\ell}^{m,s}(\rho,\sigma,\tau)$. Gauss' contiguous relations for the hypergeometric series (Reference 26) could be used as a basis for these recurrence relations; however, they are not well suited for the purposes of this investigation. This is discussed in Section 2.2.1.4.

Inspection of Equations (2-28) and (2-29) indicates that if either of the first two arguments is a negative integer, the hypergeometric series terminates in a polynomial. These polynomials are the orthogonal polynomials named after Jacobi, and they possess some very useful recurrence relations (as will be shown in Section 3).

2.1.2.2 Jacobi Polynomial Representation of the Function $S_{2\ell}^{m,s}(\rho,\sigma,\tau)$

Inspection of Equations (2-27) indicates that the hypergeometric series terminates to yield a polynomial of degree $\ell - |s|$. The relationship between the hypergeometric series and the Jacobi polynomial taken from Reference 26 is

$$F(-n, a+b+n+1, a+1; x) = \frac{n!\,a!}{(a+n)!} P_n^{a,b}(1-2x) \tag{2-32}$$

where the Jacobi polynomial takes the form

$$P_n^{a,b}(x) = 2^{-n} \sum_{m=0}^{n} \binom{n+a}{m}\binom{n+b}{n-m}(x-1)^{n-m}(x+1)^m \tag{2-33}$$

or, equivalently,

$$P_n^{a,b}(x) = \frac{\Gamma(a+n+1)}{n!\,\Gamma(a+b+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(a+b+n+m+1)}{2^m\,\Gamma(a+m+1)} (x-1)^m \qquad (2\text{-}34)$$

In principle, the indexes a, b can assume any value with the exception of negative integers, i.e.,

$$a \neq -n \; ; \quad b \neq -m \qquad (2\text{-}35)$$

However, for this investigation, only nonnegative integer values are considered. Applying Equation (2-32) to the hypergeometric series in Equations (2-27) and noting that

$$1 - 2C_\tau^2 = -C_{2\tau}$$

yields

$$F(-\ell-s,\ \ell-s+1,\ 1-m-s;\ C_\tau^2) = \frac{(\ell+s)!\,|m+s|!}{(\ell-m)!} P_{\ell+s}^{-m-s,\,m-s}(-C_{2\tau}) \qquad (2\text{-}36)$$

for $m + s \leq 0$ and

$$F(s-\ell,\ \ell+s+1,\ 1+m+s;\ C_\tau^2) = \frac{(\ell-s)!\,(m+s)!}{(\ell+m)!} P_{\ell-s}^{m+s,\,s-m}(-C_{2\tau}) \qquad (2\text{-}37)$$

for $m + s \geq 0$.

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Care must be taken to insure that the Jacobi polynomial is valid over the entire range of validity for the hypergeometric series. Since only integer values of m and s are of concern, Equations (2-35) can be expressed as

$$a \geq 0; \quad b \geq 0$$

For Equation (2-36), the constraints on the Jacobi polynomial are

$$-m-s \geq 0; \quad m-s \geq 0$$

which are simultaneously satisfied only by those values of s where $s \leq -m$. Hence, the Jacobi polynomial and hypergeometric series in Equation (2-36) are valid over the same range, i.e., $m+s \quad 0$. The constraints on the Jacobi polynomial in Equation (2-37) are

$$m+s \geq 0; \quad s-m \geq 0$$

which are simultaneously satisfied only for $s \geq m$. Thus, while the hypergeometric series is valid for $m+s \geq 0$ or $s \geq -m$, the Jacobi polynomial is valid only for $s \geq m$.

A valid Jacobi polynomial representation for Equation (2-37) over the range $-m \leq s \leq m$ is obtained through a linear transformation of the hypergeometric series (Reference 26), i.e.,

$$F(s-\ell, \ell+s+1, 1+m+s; C_\tau^2) = S_\tau^{2(m-s)} F(\ell+m+1, m-\ell, 1+m+s; C_\tau^2) \qquad (2\text{-}38)$$

Both of these hypergeometric series are valid when $m+s \geq 0$ is satisfied. It follows from the definition of the hypergeometric series that the first two arguments can be interchanged as follows:

$$F(\ell+m+1, m-\ell, 1+m+s; C_\tau^2) = F(m-\ell, \ell+m+1, 1+m+s; C_\tau^2) \qquad (2\text{-}39)$$

In view of Equation (2-32),

$$F(m-\ell, \ell+m+1, 1+m+s; C_\tau^2) = \frac{(\ell-m)!\,(m+s)!}{(\ell+s)!} P_{\ell-m}^{m+s,\, m-s}(-C_{2\tau}) \quad (2\text{-}40)$$

The constraints on the Jacobi polynomial are $m + s \geq 0$ and $m - s \geq 0$, which are satisfied simultaneously by $-m \leq s \leq m$. In summary, the hypergeometric series in Equation (2-27) can be expressed as

$$F(s-\ell, \ell+s+1, 1+m+s; C_\tau^2) = \begin{cases} \dfrac{(\ell-m)!\,(m+s)!}{(\ell+s)!} S_{2\tau}^{2(m-s)} P_{\ell-m}(-C_{2\tau}) & (-m \leq s \leq m) \quad (2\text{-}41a) \\ \dfrac{(\ell-s)!\,(m+s)!}{(\ell+m)!} P_{\ell-s}^{m+s}(-C_{2\tau}) & (m \leq s \leq \ell) \quad (2\text{-}41b) \end{cases}$$

Substituting Equations (2-36) and (2-41) into Equations (2-27) and using the relation

$$P_n^{a,b}(-x) = (-1)^n P_n^{b,a}(x) \quad (2\text{-}42)$$

yields

$$U_{2\ell}^{m,s}(\tau) = \begin{cases} (-1)^{m+s} C_\tau^{-m-s} S_\tau^{m-s} P_{\ell+s}^{m-s,\,-m-s}(C_{2\tau}) & (-\ell \leq s \leq -m) \quad (2\text{-}43a) \\ (-1)^{s-m} \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!} C_\tau^{m+s} S_\tau^{m-s} P_{\ell-m}^{m-s,\,m+s}(C_{2\tau}) & (-m \leq s \leq m) \quad (2\text{-}43b) \\ C_\tau^{m+s} S_\tau^{s-m} P_{\ell-s}^{s-m,\,m+s}(C_{2\tau}) & (m \leq s \leq \ell) \quad (2\text{-}43c) \end{cases}$$

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Substitution of this expression into Equation (2-26) yields the complete expression for the function $S_{2\ell}^{m,s}(\rho,\sigma,\tau)$ in terms of the Jacobi polynomials, i.e.,

$$S_{2\ell}^{m,s} = \exp\left\{ j\left[(m-s)(\sigma-\pi/2)-(m+s)\rho\right]\right\}$$

$$\times \begin{cases} (-1)^{m+s}\, C_\tau^{-m-s}\, S_\tau^{m-s}\, P_{\ell+s}^{m-s,-m-s}(C_{2\tau}) & (-\ell\le s\le -m) \quad (2\text{-}44a)\\ (-1)^{s-m}\dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!}\, C_\tau^{m+s}\, S_\tau^{m-s}\, P_{\ell-m}^{m-s,m+s}(C_{2\tau}) & (-m\le s\le m) \quad (2\text{-}44b)\\ C_\tau^{m+s}\, S_\tau^{s-m}\, P_{\ell-s}^{s-m,m+s}(C_{2\tau}) & (m\le s\le \ell) \quad (2\text{-}44c)\end{cases}$$

2.1.2.3 The Function $S_{2\ell}^{m,s}$ in Terms of the Euler Angles

Expressing Equations (2-44) explicitly in the Euler angles (Ω,ω,i), which describe the rotation from the primed reference system to the unprimed reference system, through Equations (2-16) yields

$$S_{2\ell}^{m,s} = e^{-j(m-s)\frac{\pi}{2}}\; e^{-j(s\Omega+m\omega)}$$

$$\times \begin{cases} (-1)^{m+s}\, C_{i/2}^{-m-s}\, S_{i/2}^{m-s}\, P_{\ell+s}^{m-s,-m-s}(C_i) & (-\ell\le s\le -m) \quad (2\text{-}45a)\\ (-1)^{s-m}\dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!}\, C_{i/2}^{m+s}\, S_{i/2}^{m-s}\, P_{\ell-m}^{m-s,m+s}(C_i) & (-m\le s\le m) \quad (2\text{-}45b)\\ C_{i/2}^{m+s}\, S_{i/2}^{s-m}\, P_{\ell-s}^{s-m,m+s}(C_i) & (m\le s\le \ell) \quad (2\text{-}45c)\end{cases}$$

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2.1.2.4 Symmetry of the Function $S_{2l}^{m,s}$

In view of the symmetry in Equation (2-45), it is possible to formally collapse the definition of the $S_{2l}^{m,s}$ function. Defining $\epsilon = 1$ for $s \leq 0$ ($\epsilon = -1$ for $s \geq 0$) and using the relation given in Equation (2-42) yields the following expressions for Equations (2-45a) and (2-45c):

$$e^{-j(m-\epsilon s)\frac{\pi}{2}} \, e^{-j(m\omega+\epsilon s\Omega)} \, \epsilon^{l+m} \, C_{i/2}^{s+\epsilon m} \, S_{i/2}^{s-\epsilon m} \, P_{l-s}^{s-m,m+s}(\epsilon C_i) \tag{2-46}$$

for $\epsilon = \pm 1$ and $s \geq m$.

Similarly, Equation (2-45b) is symmetric about $s = 0$ and can be expressed as

$$e^{-j(m-\epsilon s)\frac{\pi}{2}} \, e^{-j(m\omega+\epsilon s\Omega)} \, (-1)^{s-m} \, \epsilon^{l+m} \, \frac{(l+m)!\,(l-m)!}{(l+s)!\,(l-s)!} \, C_{i/2}^{m+\epsilon s} \, S_{i/2}^{m-\epsilon s} \, P_{l-m}^{m-s,m+s}(\epsilon C_i) \tag{2-47}$$

for $\epsilon = \pm 1$ and $0 \leq s \leq m$.

The rotation of a surface harmonic function given by Equation (2-24) can then be expressed as

$$P_{l,m}(\sin\phi)\, e^{jm\lambda} = \sum_{s=0}^{l} \delta_s \frac{(l-\epsilon s)!}{(l-m)!} \, S_{2l}^{m,s,\epsilon} \, P_{l,\epsilon s}(\sin\phi')\, e^{j\epsilon s\lambda'} \tag{2-48}$$

where ϵ takes on the values ± 1,

$$\delta_s = \begin{cases} 1/2 & s = 0 \\ 1 & s > 0 \end{cases} \tag{2-49}$$

and

$$S^{*\,m,s,\epsilon}_{2\ell} = e^{-j(m-\epsilon s)\frac{\pi}{2}}\, e^{-j(m\omega+\epsilon s\Omega)}$$

$$\times \begin{cases} \epsilon^{\ell+m}(-1)^{s-m}\dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!}\, C_{i/2}^{m+\epsilon s}\, S_{i/2}^{m-\epsilon s}\, P_{\ell-m}^{m-s,\,m+s}(\epsilon C_i) & (0 \le s \le m) \quad (2\text{-}50a) \\ \epsilon^{\ell+m}\, C_{i/2}^{s+\epsilon m}\, S_{i/2}^{s-\epsilon m}\, P_{\ell-s}^{s-m,\,m+s}(\epsilon C_i) & (m \le s \le \ell) \quad (2\text{-}50b) \end{cases}$$

In view of Equation (2-25)

$$P_{\ell,\epsilon s}(x) = \epsilon^{s}\, \frac{(\ell+\epsilon s)!}{(\ell+s)!}\, P_{\ell,s}(x) \qquad (s \ge 0) \qquad (2\text{-}51)$$

Therefore, Equation (2-48) can be simplified to read

$$P_{\ell,m}(\sin\phi)\, e^{jm\lambda} = \sum_{s=0}^{\ell} \delta_s\, \epsilon^{s}\, \frac{(\ell-s)!}{(\ell-m)!}\, S^{*\,m,s,\epsilon}_{2\ell}\, P_{\ell,s}(0)(\sin\phi')\, e^{j\epsilon s\lambda'} \qquad (2\text{-}52)$$

for $\epsilon = \pm 1$, since

$$\frac{(\ell+\epsilon s)!\,(\ell-\epsilon s)!}{(\ell+s)!} = (\ell-s)! \qquad (2\text{-}53)$$

Equations (2-50) and (2-52) are the recommended formulation for those cases where the symmetry is easily taken advantage of. Otherwise, the formulation should be developed using Equations (2-24) and (2-45).

2.2 EXPANSIONS OF THE PRODUCT $(r/a)^n e^{jsL}$

Expansions of the form

$$\left(\frac{r}{a}\right)^n e^{jsL} = \sum_t A_t^{n,s} e^{jtX} \tag{2-54}$$

(where r, a, and L are the radial distance, semimajor axis, and true longitude, respectively, and X is the true (L), eccentric (F), or mean (λ) longitude) play a major role in the development in equinoctial elements of the disturbing functions for the nonspherical gravitational and third-body perturbations. Consequently, they are also important in the development of the analytically averaged equations of motion and in the analytical development of the short-period variations in the osculating elements.

For certain cases, each longitude possesses a particular advantage as the expansion variable, X. Specifically, for $n \leq 0$, the above expansion is finite in terms of the true longitude, and, for $n \geq 0$, the expansion is finite in terms of the eccentric longitude. While the expansion in the mean longitude is always infinite, it is of considerable importance because of the simple relationship between the time and mean longitude.

Similar expansions of the form

$$\left(\frac{r}{a}\right)^n e^{jsF} = \sum_t a_t^{n,s} e^{jtx} \tag{2-55}$$

(where f designates the true anomaly and x is the true (f), the eccentric (u), or the mean (ℓ) anomaly) played an important role in the development of the classical disturbing function of planetary theory. These expansions were investigated extensively by Hansen (Reference 27) and good, if less exhaustive, discussion can be found in Brown and Shook (Reference 28). A somewhat more theoretical discussion, based on Cauchy's first and second theorems, is given by Hagihara (Reference 29).

2.2.1 Reduction to the Expansion of the Product $(r/a)^n e^{jsf}$

Hansen's results are directly applicable to the expansions in the longitude (Equation (2-54)). This is easily demonstrated using the relation between the equinoctial longitudes and the corresponding classical anomalies

$$X = x + \omega + I\Omega \tag{2-56}$$

In view of Equation (2-56), the left-hand side of Equation (2-54) can be expressed as

$$\left(\frac{r}{a}\right)^n e^{jsL} = \left(\frac{r}{a}\right)^n e^{js(\omega + I\Omega)} e^{jsf} \tag{2-57}$$

Substituting the expansion in Equation (2-55) yields

$$\left(\frac{r}{a}\right)^n e^{jsL} = e^{js(\omega + I\Omega)} \sum_t a_t^{n,s} e^{jtx} \tag{2-58}$$

Inverting Equation (2-56) and substituting the result into the expansion in Equation (2-58) yields

$$\begin{aligned} \left(\frac{r}{a}\right)^n e^{jsL} &= e^{js(\omega + I\Omega)} \sum_t a_t^{n,s} e^{jt(X - \omega - I\Omega)} \\ &= \sum_t a_t^{n,s} e^{j(s-t)(\omega + I\Omega)} e^{jtX} \end{aligned} \tag{2-59}$$

A comparison of Equations (2-54) and (2-59) indicates that

$$A_t^{n,s} = e^{j(s-t)(\omega + I\Omega)} a_t^{n,s} \tag{2-60}$$

In view of the definition of the equinoctial elements h, k (Reference 5, Appendix A),

$$e^{j(\omega + I\Omega)} = e^{-1}(k + jh) \tag{2-61a}$$

and

$$e^{-j(\omega + I\Omega)} = e^{-1}(k - jh) \tag{2-61b}$$

Therefore,

$$A_t^{n,s} = e^{t-s}(k + jh)^{s-t} a_t^{n,s} \tag{2-62a}$$

or

$$A_t^{n,s} = e^{s-t}(k - jh)^{t-s} a_t^{n,s} \tag{2-62b}$$

and the applicability of Hansen's results is demonstrated.

The explicit development of the Fourier series expansions of the form given in Equation (2-55) is presented next. Much of the discussion follows the approach of Hansen but is of much more limited scope. In addition, some results obtained by Hill and Newcomb, for expansions in the mean anomaly, will be presented.

2.2.1.1 Expansion in the True Anomaly

The Fourier expansion of the form

$$\left(\frac{r}{a}\right)^n e^{jsf} = \sum_t a_t^{n,s} e^{jtf} \tag{2-63}$$

is desired. More correctly, the coefficients in the expansion have only two indexes since the Fourier expansion of the imaginary exponential function is not required. If

$$\left(\frac{r}{a}\right)^n = \sum_t V_t^n e^{jtf} \tag{2-64}$$

then

$$\left(\frac{r}{a}\right)^n e^{jsf} = \sum_t V_t^n e^{j(t+s)f} \tag{2-65}$$

and it is sufficient to develop the expansion for $(r/a)^n$ (Equation (2-64)).

It follows from Fourier's Theorem (Reference 30) that

$$V_k^n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n e^{-jkf}\,df \tag{2-66}$$

This expression is obtained by multiplying e^{-jkf} through Equation (2-64) and integrating both sides appropriately. The resulting series collapses to a single term, specifically, the term when $t = k$, in view of the orthogonality conditions for Fourier series or, equivalently, the 2π periodicity of the imaginary exponential function.

Since r/a is a real function and since Equation (2-64) is also satisfied by the complex conjugates of each side, it follows that

$$\left(\frac{r}{a}\right)^n = \sum_t V_t^n e^{-jtf} \tag{2-67}$$

A comparison of Equations (2-64) and (2-67) shows that

$$V_t^n = V_{-t}^n \tag{2-68}$$

An explicit representation of the coefficient V_t^n can be obtained by a brute-force expansion of $(r/a)^n$ using the well-known relation

$$\frac{r}{a} = \frac{1 - e^2}{1 + e \cos f} \tag{2-69}$$

Using Hansen's approach, the following definitions are made:

$$\sin \phi = e \tag{2-70}$$

$$x = e^{jf} \tag{2-71}$$

Then,

$$\cos \phi = \sqrt{1 - e^2} \tag{2-72}$$

$$\beta = \tan(\phi/2) = \frac{e}{1 + \sqrt{1 - e^2}} \tag{2-73}$$

and Equation (2-69) can be expressed as

$$\frac{r}{a} = (1 - \beta^2) \cos \phi \, (1 + \beta X)^{-1} \left(1 + \frac{\beta}{X}\right)^{-1} \tag{2-74}$$

Therefore,

$$\left(\frac{r}{a}\right)^{n} = (1-\beta^2)^{n} \cos^{n}\phi \, (1+\beta X)^{-n}\left(1+\frac{\beta}{X}\right)^{-n} \tag{2-75}$$

$$\left(\frac{r}{a}\right)^{-n} = (1-\beta^2)^{-n} \cos^{-n}\phi \, (1+\beta X)^{n}\left(1+\frac{\beta}{X}\right)^{n} \tag{2-76}$$

for n, a nonnegative integer. Hence, the expansions for $(r/a)^{\pm n}$ reduces to the expansions of the product

$$(1+\beta X)^{\mp n}\left(1+\frac{\beta}{X}\right)^{\mp n} \tag{2-77}$$

2.2.1.1.1 Expansions of the product $(1+\beta x)^{-n}\,[1+(\beta/x)]^{-n}$

Using the Binomial Theorem yields the following result:

$$\begin{aligned}(1+\beta X)^{-n}\left(1+\frac{\beta}{X}\right)^{-n} &= \sum_{k=0}^{\infty} \frac{(-1)^k (n+k-1)!}{(n-1)!\,k!}\beta^k X^k \sum_{m=0}^{\infty} \frac{(-1)^m (n+m-1)!}{(n-1)!\,m!}\beta^m X^{-m} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} (-1)^{k+m} \frac{(n+k-1)!\,(n+m-1)!}{(n-1)!\,k!\,(n-1)!\,m!}\beta^{k+m} X^{k-m}\end{aligned} \tag{2-78}$$

If the following definitions are made:

$$t = k-m \tag{2-79a}$$

$$p = k+m \tag{2-79b}$$

it follows that $-\infty < t < \infty$, since $0 \leq k < \infty$ and $0 \leq m < \infty$. To determine the range of p, it is necessary to invert Equations (2-79) to yield

$$m = \frac{p-t}{2} \geq 0 \tag{2-80a}$$

$$k = \frac{p+t}{2} \geq 0 \tag{2-80b}$$

Clearly, $p \geq |t|$ and $p \pm t$ must be even; consequently, p is defined as

$$p = 2i + |t| \tag{2-81}$$

Then, if $|t| \leq p < \infty$, it follows that $0 \leq i < \infty$ and

$$m = 2i + |t| - t \tag{2-82a}$$

$$k = 2i + |t| + t \tag{2-82b}$$

It also follows that Equation (2-78) can be expressed as

$$(1+\beta X)^{-n}\left(1+\frac{\beta}{X}\right)^{-n} = \sum_{t=-\infty}^{\infty}\sum_{i=0}^{\infty} (-1)^{|t|} \frac{(n+|t|+i-1)!\,(n+i-1)!}{(n-1)!\,(n-1)!\,(|t|+i)!\,i!}\,\beta^{2i+|t|} X^{t} \tag{2-83}$$

Since

$$\begin{aligned} \frac{(n+|t|+i-1)!\,(n+i-1)!}{(n-1)!\,(n-1)!\,(|t|+i)!} &= \frac{(n+|t|-1)!}{(n-1)!\,|t|!}\,\frac{(n+|t|+i-1)!}{(n+|t|-1)!}\,\frac{(n+i-1)!}{(n-1)!}\,\frac{|t|!}{(|t|+i)!} \\ &= \binom{n+|t|-1}{|t|}\frac{(n+|t|)_i\,(n)_i}{(|t|+1)_i} \end{aligned} \tag{2-84}$$

and

$$\sum_{i=0}^{\infty} \frac{(n+|t|)_i\,(n)_i}{(|t|+1)_i}\,\frac{\beta^{2i}}{i!} = F\left(n+|t|,\, n,\, |t|+1;\, \beta^2\right) \qquad (2\text{-}85)$$

then

$$\left(1+\beta X\right)^{-n}\left(1+\frac{\beta}{X}\right)^{-n} = \sum_{t=-\infty}^{\infty} (-1)^{|t|}\,\beta^{|t|}\binom{n+|t|-1}{|t|} F\left(n+|t|,\, n,\, |t|+1, \beta^2\right) X^t \qquad (2\text{-}86)$$

2.2.1.1.2 Expansion of the Product $(1+\beta x)^n[1+(\beta/x)]^n$

A straightforward application of the Binomial Theorem yields

$$\left(1+\beta X\right)^{n}\left(1+\frac{\beta}{X}\right)^{n} = \sum_{k=0}^{n}\sum_{m=0}^{n}\binom{n}{k}\binom{n}{m}\beta^{k+m}X^{k-m} \qquad (2\text{-}87)$$

The quantities t, p, and i are defined as above, with the exception that the ranges in this case are $-n \leq t \leq n$, $|t| \leq p \leq 2n$, and $0 \leq i \leq [n-(|t|/2)]$ (where $[\ ]$ denotes the integer part). Equation (2-87) can then be expressed as

$$\left(1+\beta X\right)^{n}\left(1+\frac{\beta}{X}\right)^{n} = \sum_{t=-n}^{n}\ \sum_{i=0}^{[n-|t|/2]}\binom{n}{i+|t|}\binom{n}{i}\beta^{2i+|t|}X^t \qquad (2\text{-}88)$$

The definition of the binomial coefficient $\binom{n}{i+|t|}$ is

$$\binom{n}{i+|t|} = \frac{n!}{(n-|t|-i)!\,(i+|t|)!} \qquad (2\text{-}89)$$

Also,

$$\frac{n!}{(n-|t|-i)!} = n(n-1)(n-2)\cdots(n-|t|)(n-|t|-1)\cdots(n-|t|-i+1)$$

$$= \frac{n!}{(n-|t|)!}(-1)^i(|t|-n)(|t|-n+1)\cdots(|t|-n-1+i) \tag{2-90}$$

which can be expressed using Pochhamer's symbol as

$$\frac{n!}{(n-|t|-i)!} = \frac{n!}{(n-|t|)!}(-1)^i(|t|-n)_i \tag{2-91}$$

and

$$\frac{1}{(i+|t|)!} = \frac{1}{|t|!}\,\frac{|t|!}{(i+|t|)!} = \frac{1}{|t|!\,(|t|+1)_i} \tag{2-92}$$

Therefore,

$$\binom{n}{i+|t|} = \frac{n!}{(n-|t|)!\,|t|!}(-1)^i\frac{(|t|-n)_i}{(|t|+1)_i} \tag{2-93}$$

Similarly,

$$\binom{n}{i} = \frac{n!}{(n-i)!\,i!} = \frac{n(n-1)\cdots(n-i+1)}{i!} = \frac{(-1)^i(-n)_i}{i!} \tag{2-94}$$

Finally,

$$\binom{n}{i+|t|}\binom{n}{i} = \binom{n}{|t|}\frac{(|t|-n)_i\,(-n)_i}{(|t|+1)_i\,i!} \tag{2-95}$$

Substituting this result into Equation (2-88) yields the result

$$\left(1+\beta X\right)^n\left(1+\frac{\beta}{X}\right)^n = \sum_{t=-n}^{n}\binom{n}{|t|}\beta^{|t|}\sum_{i=0}^{[n-|t|/2]}\frac{(|t|-n)_i\,(-n)_i}{(|t|+1)_i}\,\frac{\beta^{2i}}{i!}\,X^t \qquad (2\text{-}96)$$

which can be expressed in terms of the hypergeometric series as

$$\left(1+\beta X\right)^n\left(1+\frac{\beta}{X}\right) = \sum_{t=-n}^{n}\binom{n}{|t|}\beta^{|t|}F\left(|t|-n,-n,|t|+1;\beta^2\right)X^t \qquad (2\text{-}97)$$

The expansions for $(r/a)^{\pm n}$ in the true anomaly are obtained by substituting Equations (2-86) and (2-97) into Equations (2-75) and (2-76), respectively, which yields

$$\left(\frac{r}{a}\right)^n = \left(1-\beta^2\right)^n\cos^n\phi\sum_{t=-\infty}^{\infty}(-1)^{|t|}\beta^{|t|}\binom{n+|t|-1}{|t|}F\left(n+|t|,n,|t|+1;\beta^2\right)X^t \qquad (2\text{-}98)$$

$$\left(\frac{r}{a}\right)^{-n} = \left(1-\beta^2\right)^{-n}\cos^{-n}\phi\sum_{t=-n}^{n}\binom{n}{|t|}\beta^{|t|}F\left(|t|-n,-n,|t|+1;\beta^2\right)X^t \qquad (2\text{-}99)$$

Expressing these expansions explicitly in the true anomaly yields

$$\left(\frac{r}{a}\right)^n = \sum_{t=-\infty}^{\infty}V_t^n\,e^{jtf} \qquad (2\text{-}100a)$$

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where[1]

$$V_t^n = (1-\beta^2)^n \cos^n\phi \, (-1)^{|t|} \beta^{|t|} \binom{n+|t|-1}{|t|} F(n+|t|, n, |t|+1; \beta^2) \quad \text{(2-100b)}$$

and

$$\left(\frac{r}{a}\right)^{-n} = \sum_{t=-n}^{n} V_t^{-n} e^{jtf} \quad \text{(2-101a)}$$

where

$$V_t^{-n} = (1-\beta^2)^{-n} \cos^{-n}\phi \binom{n}{|t|} \beta^{|t|} F(|t|-n, -n, |t|+1; \beta^2) \quad \text{(2-101b)}$$

2.2.1.2 Expansion in the Eccentric Anomaly

It is apparent that if $(r/a)^{\pm n}$ and e^{jsf} are expandible in a Fourier series in the eccentric anomaly, u, i.e.,

$$\left(\frac{r}{a}\right)^n = \sum_k p_k^n e^{jku} \quad \text{(2-102)}$$

$$e^{jsf} = \sum_m q_m^s e^{jmu} \quad \text{(2-103)}$$

---

[1] Although this expression is not of closed form, it is easily transformed to a closed-form expression using the linear transformation (Reference 26)

$$F(a, b, c, x^2) = (1-x^2)^{c-a-b} F(c-a, c-b, c, x^2)$$

which yields the result

$$V_t^n = (1-\beta^2)^{-(n-1)} \cos^n\phi \, (-\beta)^{|t|} F(1-n, |t|-n+1, |t|+1; \beta^2)$$

Then

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_k \sum_m p_k^{\pm n} q_m^s \, e^{j(k+m)u} \tag{2-104}$$

or, equivalently,

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_t W_t^{\pm n,s} e^{jtu} \tag{2-106}$$

where

$$W_t^{\pm n,s} = \sum_m p_{t-m}^{\pm n} \, q_m^s \qquad \text{(2-10[illegible])}$$

Hence, the coefficients $W_t^{\pm n,s}$ can be determined from the coefficients of the simpler expansions in Equations (2-102) and (2-103).

Another expression for the coefficients $W_t^{\pm n,s}$ is provided by multiplying Equation (2-105) by e^{-jku} and integrating over $-\pi \leq u \leq \pi$, which yields

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^{\pm n} e^{j(sf-ku)}\,du &= \sum_t W_t^{\pm n,s}\,\frac{1}{2\pi}\int e^{j(t-k)u}\,du \\ &= \sum_t W_t^{\pm n,s}\,\delta_{t,k} = W_k^{\pm n,s} \end{aligned} \tag{2-107}$$

In addition, the symmetry condition

$$W_k^{\pm n,s} = W_{-k}^{\pm n,-s}$$

follows, as before, by substituting the complex conjugates in Equation (2-107) and comparing the results with those given in Equation (2-107).

An explicit representation of the coefficients is developed next. This representation is obtained directly by expanding the left-hand side of Equation (2-105) (rather than by using the expansions given by Equations (2-102) and (2-103) and then constructing the coefficients through Equation (2-106)). The expansions in Equations (2-102) and (2-103) are also obtained, since they are special cases of Equation (2-105).

Again, following Hansen's method, the definitions are made

$$x = e^{jf} \tag{2-108}$$

$$y = e^{ju} \tag{2-109}$$

Since

$$\frac{r}{a} = 1 - e\cos u \tag{2-110}$$

it is easily verified that

$$\frac{r}{a} = (1+\beta^2)^{-1}\left[1+\beta^2-\beta(y+y^{-1})\right] = (1+\beta^2)^{-1}(1-\beta y)\left(1-\frac{\beta}{y}\right) \tag{2-111}$$

and

$$x = y\,\frac{(1-\beta y^{-1})}{(1-\beta y)} \tag{2-112}$$

It follows from Equations (2-111) and (2-112) that

$$\left(\frac{r}{a}\right)^n e^{jsf} = \left(\frac{r}{a}\right)^n x^s = (1+\beta^2)^{-n} y^s (1-\beta y)^{n-s} (1-\beta y^{-1})^{n+s} \tag{2-113}$$

$$\left(\frac{r}{a}\right)^{-n} e^{jsf} = \left(\frac{r}{a}\right)^{-n} x^s = (1+\beta^2)^{n} y^s (1-\beta y)^{-n-s} (1-\beta y^{-1})^{s-n} \tag{2-114}$$

where n is a nonnegative integer. For the purposes of this investigation, the relation $n \geq |s|$ is always satisfied; hence, $n-s \geq 0$ and $n+s \geq 0$, and $-n-s \leq 0$ and $s-n \leq 0$.

2.2.1.2.1 Expansion of the Product $(1-\beta y)^{n-s} (1-\beta y^{-1})^{n+s}$

Since $n \geq |s|$, the Binomial Theorem yields the expansion

$$(1-\beta y)^{n-s}(1-\beta y^{-1})^{n+s}$$

$$= \sum_{k=0}^{n-s} \sum_{m=0}^{n+s} \binom{n-s}{k} \binom{n+s}{m} (-\beta)^{k+m} y^{k-m}$$

$$= \sum_{t=-s-n}^{-s+n} \sum_{\substack{p=|t| \\ (p \pm t \text{ even})}}^{2n} \binom{n-s}{\frac{p+t}{2}} \binom{n+s}{\frac{p-t}{2}} (-\beta)^p y^t \tag{2-115}$$

$$= \sum_{t=-n}^{n} \sum_{i=0}^{n-|t-s|/2} (-1)^{|t-s|} \binom{n-s}{i+(|t-s|+t-s)/2} \binom{n+s}{i+[|t-s|-(t-s)]/2} \beta^{2i+|t-s|} y^{t-s}$$

If

$$\alpha = \frac{|t-s|+t-s}{2} \tag{2-116a}$$

and

$$\rho = \frac{|t-s| - (t-s)}{2} \tag{2-116b}$$

then it follows from Equation (2-93) that

$$\binom{n-s}{i+\alpha} = \binom{n-s}{\alpha} (-1)^i \frac{(\alpha-n+s)_i}{(\alpha+1)_i} \tag{2-117}$$

and

$$\binom{n+s}{i+\rho} = \binom{n+s}{\rho} (-1)^i \frac{(\rho-n-s)_i}{(\rho+1)_i} \tag{2-118}$$

Therefore,

$$\binom{n-s}{i+\alpha}\binom{n+s}{i+\rho} = \binom{n-s}{\alpha}\binom{n+s}{\rho} \frac{(\alpha-n+s)_i\,(\rho-n-s)_i}{(|t|+1)_i\; i!} \tag{2-119}$$

since

$$(\alpha+1)_i\,(\rho+1)_i = (|t|+1)_i\,(1)_i \tag{2-120}$$

and

$$(1)_i = i! \tag{2-121}$$

Substituting Equation (2-119) into Equation (2-115) and expressing the result in terms of the hypergeometric series yields

$$(1-\beta y)^{n-s}\left(1-\frac{\beta}{y}\right)^{n+s} = \sum_{t=-n}^{n} (-1)^{|t-s|}\binom{n-s}{\frac{|t-s|+t-s}{2}}\binom{n+s}{\frac{|t-s|-(t-s)}{2}}\beta^{|t-s|} \tag{2-122}$$

$$\times F\left(\frac{|t-s|+t-s}{2}-n+s,\ \frac{|t-s|-(t-s)}{2}-n-s,\ |t-s|+1;\beta^2\right)y^{t-s}$$

2.2.1.2.2 Expansion of the Product $(1-\beta y)^{-n-s}(1-\beta y^{-1})^{s-n}$

Since $n \geq |s|$, it follows that

$$(1-\beta y)^{-n-s}(1-\beta y^{-1})^{s-n} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+s+k-1)!}{(n+s-1)!\,k!}\ \frac{(n-s+m-1)!}{(n-s-1)!\,m!}\ \beta^{k+m} y^{k-m} \tag{2-123}$$

which can be expressed as

$$(1-\beta y)^{-n-s}(1-\beta y^{-1})^{s-n} = \sum_{t=-\infty}^{\infty}\sum_{i=0}^{\infty} \frac{(n+s+i+\alpha-1)!}{(n+s-1)!\,(i+\alpha)!}\ \frac{(n-s+i+\rho-1)!}{(n-s-1)!\,(i+\rho)!}\ \beta^{2i+|t|} y^{t-s} \tag{2-124}$$

where α and ρ retain the definitions given in Equations (2-116). Also,

$$\frac{(n+s+i+\alpha-1)!}{(n+s-1)!} = \frac{(n+s+\alpha-1)!}{(n+s-1)!\,\alpha!}\,\alpha!\,\frac{(n+s+\alpha+i-1)!}{(n+s+\alpha-1)!} \tag{2-125}$$

$$= \binom{n+s+\alpha-1}{\alpha}\alpha!\,(n+s+\alpha)_i$$

and, similarly

$$\frac{(n-s+i+\rho-1)!}{(n-s-1)!} = \binom{n-s+\rho-1}{\rho}!\,(n-s+\rho)_i \tag{2-126}$$

Substituting Equations (2-125) and (2-126) into (2-124) yields

$$(1-\beta y)^{-n-s}(1-\beta y^{-1})^{s-n} = \sum_{t=-\infty}^{\infty} \binom{n+s+\alpha-1}{\alpha}\binom{n-s+\rho-1}{\rho} \times \sum_{i=0}^{\infty} \frac{(n+s+\alpha)_i\,(n-s+\rho)_i}{(\alpha+1)_i\,(\rho+1)_i}\,\beta^{2i+|t-s|}\,y^{t-s} \tag{2-127}$$

Again, it follows from the definitions of α and ρ that

$$(\alpha+1)_i\,(\rho+1)_i = (|t-s|+1)_i\,i! \tag{2-128}$$

Thus, Equation (2-127) admits the hypergeometric series representation

$$(1-\beta y)^{-n-s}(1-\beta y^{-1})^{s-n} =$$

$$\sum_{t=-\infty}^{\infty} \binom{n+s+\frac{|t-s|+t-s}{2}-1}{\frac{|t-s|+t-s}{2}}\binom{n-s+\frac{|t-s|-(t-s)}{2}-1}{\frac{|t-s|-(t-s)}{2}}\beta^{|t-s|} \times F\left(n+s+\frac{|t-s|+(t-s)}{2},\ n-s+\frac{|t-s|-(t-s)}{2},\ |t-s|+1;\ \beta^2\right) y^{t-s} \tag{2-129}$$

Substituting Equations (2-122) and (2-129) into Equations (2-113) and (2-114), respectively, yields the desired expansions

$$\left(\frac{r}{a}\right)^n e^{jsf} = \sum_{t=-n}^{n} W_t^{n,s}\, e^{jtu} \tag{2-130a}$$

where

$$W_t^{n,s} = (1+\beta^2)^{-n} (-1)^{|t-s|} \binom{n-s}{\frac{|t-s|+t-s}{2}} \binom{n+s}{\frac{|t-s|-(t-s)}{2}} \beta^{|t-s|}$$

$$\times F\left(\frac{|t-s|+t-s}{2} - n+s,\ \frac{|t-s|-(t-s)}{2} - n-s,\ |t-s|+1;\ \beta^2\right) \tag{2-130b}$$

and

$$\left(\frac{r}{a}\right)^{-n} e^{jsf} = \sum_{t=-\infty}^{\infty} W_t^{-n,s} e^{jtu} \tag{2-131a}$$

where[1]

$$W_t^{-n,s} = (1+\beta^2)^{n} \binom{n+s+\frac{|t-s|+t-s}{2}-1}{\frac{|t-s|+t-s}{2}} \binom{n-s+\frac{|t-s|-(t-s)}{2}-1}{\frac{|t-s|-(t-s)}{2}} \beta^{|t-s|}$$

$$\times F\left(n+s+\frac{|t-s|+t-s}{2},\ n-s+\frac{|t-s|-(t-s)}{2},\ |t-s|+1;\ \beta^2\right) \tag{2-131b}$$

---

[1] A closed-form expression can be obtained by a linear transformation (see Equation (2-101b) and the accompanying footnote) which yields the expression

$$W_t^{n,s} = (1-\beta^2)^{-(n-1)} \cos^{-n}\phi \binom{n+s+\frac{|t-s|+(t-s)}{2}-1}{\frac{|t-s|+(t-s)}{2}} \binom{n-s+\frac{|t-s|-(t-s)}{2}-1}{\frac{|t-s|-(t-s)}{2}} \beta^{|t-s|}$$

$$\times F\left(1-n-s+\frac{|t-s|-(t-s)}{2},\ 1-n+s+\frac{|t-s|-(t-s)}{2},\ |t-s|+1;\ \beta^2\right)$$

2.2.1.3 Expansion in the Mean Anomaly

The Fourier expansion in the mean anomaly takes the form

$$\left(\frac{r}{a}\right)^n e^{jsf} = \sum_t X_t^{n,s} e^{jt\ell} \tag{2-132}$$

It is clear from the discussion in Sections 2.2.1.1 and 2.2.1.2 that the coefficients $X_t^{n,s}$, referred to as Hansen's coefficients, can be expressed as

$$X_k^{n,s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n e^{j(sf-k\ell)} d\ell \tag{2-133}$$

and also that the symmetry relation

$$X_k^{n,s} = X_{-k}^{n,-s} \tag{2-134}$$

holds.

The above expansion has played a central role in the development of the classical planetary theories because of the desire for explicit time-dependent theories and because of the simple relationship between the mean anomaly and the time. Accordingly, this expansion and similar ones have been studied by a host of investigators.[1]

There are several approaches which can be taken to develop an explicit representation for the Hansen coefficients. For example, the Hansen coefficients can be expressed in terms of either of the previously derived coefficients, $V_t^{\pm n}$ or

[1] Leverrier (Reference 31) and Cayley (Reference 32) developed extensive tables for the expansions through the seventh power of the eccentricity.

$W_t^{\pm n}$. The $V_t^{\pm n}$ representation is obtained from the Fourier expansion (Equation (2-65))

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_q V_{q-s}^{\pm n} e^{jqf} \tag{2-135}$$

and a variation of the equation of the center (Reference 1)

$$e^{jq(f-\ell)} = \sum_{t=-\infty}^{\infty} C_{t+q}^{q} e^{jt\ell} \tag{2-136}$$

Substituting Equation (2-136) into Equation (2-135) and rearranging the summation yields

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_{t=-\infty}^{\infty} \sum_q V_{q-s}^{\pm n} C_t^q e^{jt\ell} \tag{2-137}$$

Comparison of this result with Equations (2-132) yields the relation

$$X_t^{\pm n,s} = \sum_q V_{q-s}^{n} C_t^q \tag{2-138}$$

Similarly, the Hansen coefficients can also be expressed in terms of the coefficient $W_t^{\pm n,s}$. The resulting expression of the Hansen coefficients has sometimes been referred to as Hill's formulation of the Hansen coefficients (Reference 33). However, this expression was given by Hansen (Reference 27) some 20 years before Hill.

In addition to Hill's formulation of the Hansen coefficients, some of the representations obtained by Hansen and by Newcomb and Poincare will be presented.

2.2.1.3.1 Hill's Representation for $X_t^{n,s}$

Hill (Reference 33) developed a representation of the form

$$X_t^{n,s} = \sum_p J_p(te)\, X_{t,p}^{n,s} \tag{2-139}$$

where the function $X_{t,p}^{n,s}$ is, to within a factor, a hypergeometric series, and where

$$J_s(x) = \sum_{i=0}^{\infty} \frac{(-1)^i (x/2)^{2i+s}}{(s+i)!\, i!} \tag{2-140a}$$

$$J_{-s} = (-1)^s J_s(x) \tag{2-140b}$$

is the Bessel function of the first kind (Reference 30).

This same form can be easily constructed using Equations (2-130) and (2-131), i.e.,

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_r W_r^{\pm n,s} e^{jru} \tag{2-141}$$

and the expansion[1]

$$e^{jru} = \sum_t B_t^r e^{jt\ell} \tag{2-142}$$

[1] This expansion is of major importance in the expansion of the classical disturbing function (Reference 1).

where

$$B_0^r = \begin{cases} 1 & \text{for } r = 0 \\ -e/2 & \text{for } |r| = 1 \\ 0 & \text{for } |r| > 1 \end{cases} \tag{2-143}$$

and where

$$B_t^r = \frac{r}{t} J_{t-r}(te) \qquad (t \neq 0) \tag{2-144}$$

Substituting Equation (2-142) into Equation (2-141) yields

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsf} = \sum_t \sum_r W_r^{\pm n,s} B_t^r e^{jt\ell} \tag{2-145}$$

and a comparison of Equations (2-132) and (2-145) gives the relation

$$X_t^{\pm n,s} = \sum_r W_r^{\pm n,s} B_t^r \tag{2-146}$$

The range of r is $-n \le r \le n$ for $n \ge 0$, $-\infty \le r \le \infty$ for $n < 0$ and $-1 \le r \le 1$ for $t = 0$.

The $W_t^{\pm n,s}$ coefficients were shown in Section 2.2.1.2 to be proportional to the hypergeometric series; consequently, it is clear that Equations (2-139) and (2-146) are of the same general form.

2.2.1.3.2 Hansen's Representation for $X_t^{n,s}$

Another approach to the explicit development of Hansen's coefficients is to expand the integrand in Equation (2-133). Clearly, only the constant term in the expansion will remain after the definite integral is evaluated. Defining

$$x = e^{jf} \qquad (2\text{-}147a)$$

$$y = e^{ju} \qquad (2\text{-}147b)$$

$$z = e^{j\ell} \qquad (2\text{-}147c)$$

and substituting the relation

$$d\ell = \frac{dz}{jz} \qquad (2\text{-}148)$$

into Equation (2-133) yields the contour integral

$$X_k^{n,s} = \frac{1}{2\pi j} \int_c \left(\frac{r}{a}\right)^n x^s z^{-k-1} dz \qquad (2\text{-}149)$$

where the contour c is the unit circle $|z| = 1$.

Expansion of the integrand in Equation (2-149) in powers of z will yield the results presented earlier in this section. However, the definite integral formulation is quite flexible in that the integration variable can be transformed to either x or y via the relations

$$dz = \frac{r^2}{a^2 \cos\phi} \frac{z}{x} dx \qquad (2\text{-}150)$$

$$dz = \frac{r}{a} \frac{z}{y} dy \qquad (2\text{-}151)$$

which follow from the well-known relations

$$d\ell = \frac{r^2\, df}{a^2 \cos\phi} \tag{2-152}$$

$$d\ell = \frac{r}{a}\, du \tag{2-153}$$

Substituting Equation (2-150) into Equation (2-149) yields the contour integral

$$X_k^{n,s} = \frac{1}{2\pi j \cos\phi} \int_c \left(\frac{r}{a}\right)^{n+2} x^{s-1} z^{-k}\, dx \tag{2-154}$$

where the contour c is defined by $|x| = 1$.

Making the substitutions

$$\frac{r}{a} = \frac{(1-\beta^2)^2}{1+\beta^2} (1+\beta x)^{-1} \left(1+\frac{\beta}{x}\right)^{-1} \tag{2-155}$$

$$z = y\, e^{-(e/2)(y-y^{-1})} \tag{2-156}$$

$$y = x\, \frac{1+\beta x^{-1}}{1+\beta x} \tag{2-157}$$

$$\cos\phi = \frac{1-\beta^2}{1+\beta^2} \tag{2-158}$$

yields

$$X_k^{n,s} = \frac{1}{2\pi j} \frac{(1-\beta^2)^{2n+3}}{(1+\beta^2)^{n+1}} \int_c x^{s-k-1} (1+\beta x)^{-n+k-2} (1+\beta x^{-1})^{-n-k-2} \tag{2-159}$$

$$\times \exp\left\{\mu\beta\left(\frac{x}{1+\beta x} - \frac{x^{-1}}{1+\beta x^{-1}}\right)\right\} dx$$

where

$$\mu = k \cos \phi \tag{2-160}$$

Grouping factors in the integrand, developing expansions for each group, and then multiplying all constituent expansions together yields the final expansion for the integrand in Equation (2-149). The form of the final expansion depends on how the factors are grouped. Clearly, the product

$$(1+\beta x)^{-n+k-2}\left(1+\frac{\beta}{x}\right)^{-n-k-2}$$

will yield a hypergeometric series representation as shown in Section 2.2.1.2. Hanson considers the factors

$$x^{s-k-1}; \quad (1+\beta x)^{-n+k-2}\exp\left\{\mu\,\frac{\beta x}{1+\beta x}\right\}; \quad \left(1+\frac{\beta}{x}\right)^{-n-k-2}\exp\left\{\frac{-\mu\beta x^{-1}}{1+\beta x^{-1}}\right\} \tag{2-161}$$

The expansion for the exponential function yield

$$\exp\left\{\frac{\mu\beta x}{1+\beta x}\right\} = \sum_{m=0}^{\infty}\frac{(\mu\beta x)^m}{m!}(1+\beta x)^{-m} \tag{2-162}$$

and, therefore,

$$(1+\beta x)^{-n+k-2}\exp\left\{\frac{\mu\beta x}{1+\beta x}\right\} = \sum_{m=0}^{\infty}\frac{(\mu\beta x)^m}{m!}(1+\beta x)^{-n+k-m-2} \tag{2-163}$$

It follows from the Binomial Theorem that

$$(1+\beta x)^{-n+k-m-2} = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(n-k+m+2+\ell-1)!}{(n-k+m+1)!\ \ell!} (\beta x)^{\ell} \tag{2-164}$$

Substituting Equation (2-164) into Equation (2-163) yields the result

$$(1+\beta x)^{-n+k-2} \exp\left\{ \frac{\mu\beta x}{1+\beta x} \right\} = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\mu^m}{m!} (\beta x)^{m+\ell} \frac{(n-k+m+1+\ell)!}{(n-k+m+1)!\ \ell!} \tag{2-165}$$

If the following definition is made,

$$p = m + \ell$$

then the left-hand side of Equation (2-165) becomes

$$(1+\beta x)^{-n+k-2} \exp\left\{ \frac{\mu\beta x}{1+\beta x} \right\} = \sum_{p=0}^{\infty} (-1)^{p} \sum_{m=0}^{p} (-1)^{m} \frac{(n-k+p+1)!}{(n-k+m+1)!\,(p-m)!} \frac{\mu^m}{m!} (\beta x)^{p} \tag{2-166}$$

Thus,

$$(1+\beta x)^{-n+k-2} \exp\left\{ \frac{\mu\beta x}{1+\beta x} \right\} = \sum_{p=0}^{\infty} (-1)^{p} M_p (\beta x)^{p} \tag{2-167a}$$

where

$$M_p = \sum_{m=0}^{p} (-1)^{m} \frac{(n-k+p+1)!}{(n-k+m+1)!\,(p-m)!} \frac{\mu^m}{m!} \tag{2-167b}$$

The expansion of the product

$$(1+\beta x)^{-n-k-2} \exp\left\{\frac{-\mu\beta x^{-1}}{1+\beta x^{-1}}\right\}$$

is obtained by substituting $-k$ for k, $-\mu$ for μ, and x^{-1} for x in Equation (2-166). The result is

$$(1+\beta x^{-1})^{-n-k-2} \exp\left\{\frac{-\mu\beta x^{-1}}{1+\beta x^{-1}}\right\} = \sum_{q=0}^{\infty} (-1)^q N_q \beta^q x^{-q} \tag{2-168a}$$

where

$$N_q = \sum_{m=0}^{q} \frac{(n+k+q+1)!}{(n+k+m+1)!\,(q-m)!} \frac{\mu^m}{m!} \tag{2-168b}$$

The product of Equations (2-167a) and (2-167b) yields the result

$$\begin{aligned}(1+\beta x)^{-n+k-2}(1+\beta x^{-1})^{-n-k-2} \exp\left\{\mu\beta\left(\frac{x}{1+\beta x} - \frac{x^{-1}}{1+\beta x}\right)\right\} \\ = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{p+q} M_p N_q \beta^{p+q} x^{p-q}\end{aligned} \tag{2-169}$$

If the following definitions are made

$$t = p-q$$

$$r = p+q$$

then $-\infty \leq t \leq \infty$.

Since

$$p = \frac{t+r}{2}$$

$$q = \frac{r-t}{2}$$

it follows that $r \pm t$ must be even and $r \geq |t|$. Hence, r is defined as follows:

$$r = 2i + |t|$$

the right-hand side of Equation (2-169) takes the form

$$\sum_{t=-\infty}^{\infty} (-\beta)^{|t|} \sum_{i=0}^{\infty} M_{i+(|t|+t)/2} \, N_{i+(|t|-t)/2} \, \beta^{2i} x^{t} \tag{2-170}$$

Multiplying this result by x^{s-k-1} yields the expansion of the integrand in Equation (2-159). Equation (2-159) then takes the form

$$X_{k}^{n,s} = \frac{1}{2\pi j} \frac{(1-\beta^{2})^{2n+3}}{(1+\beta^{2})^{n+1}} \int_{c} \sum_{t=-\infty}^{\infty} (-\beta)^{|t|} \times \sum_{i=0}^{\infty} M_{i+(|t|+t)/2} \, N_{i+(|t|-t)/2} \, \beta^{2i} x^{t+s-k-1} \, dx \tag{2-171}$$

Evaluating this integral reduces to evaluating the integral

$$\int_{c} x^{t+s-k-1} \, dx \tag{2-172}$$

Clearly,

$$\int_C x^{t+s-k-1}\, dx = \begin{cases} 2\pi j & \text{for } t+s-k-1=-1 \\ 0 & \text{for } t+s-k-1 \neq -1 \end{cases} \tag{2-173}$$

from the theory of integration in complex variables or, equivalently, from Cauchy's Residue Theorem (Reference 30).

In view of Equation (2-173), the expression for the Hansen coefficient in Equation (2-171) reduces to

$$X_k^{n,s} = \frac{(1-\beta^2)^{2n+3}}{(1+\beta^2)^{n+1}} (-\beta)^{|k-s|} \sum_{i=0}^{\infty} M_{i+(|k-s|+k-s)/2}\, N_{i+[|k-s|-(k-s)]/2}\, \beta^{2i} \tag{2-174}$$

where the coefficients M and N are defined by Equations (2-167b) and (2-168b), respectively.

Hansen obtained another expression for the Hansen coefficients by substituting Equation (2-151) into Equation (2-149) and using the relations

$$\frac{r}{a} = (1-\beta^2)^{-1}(1-\beta y)\left(1-\frac{\beta}{y}\right) \tag{2-175}$$

$$x = y\,\frac{(1-\beta y^{-1})}{(1-\beta y)} \tag{2-176}$$

and Equation (2-156) to obtain the expression

$$X_k^{n,s} = \frac{1}{2\pi j}(1+\beta^2)^{-n-1}\int_C y^{s-k-1}(1-\beta y)^{n-s+1}\left(1-\frac{\beta}{y}\right)^{n+s+1} \exp\left\{\frac{ke}{2}(y-y^{-1})\right\} dy \tag{2-177}$$

Using the decomposition

$$\exp\left\{\frac{ke}{2}(y-y^{-1})\right\} = \exp\left\{-\nu(1-\beta y)\right\} \exp\left\{\nu(1-\beta y^{-1})\right\} \tag{2-178}$$

where

$$\nu = \frac{k}{1+\beta^2} \tag{2-179}$$

Hansen develops expansions for the factors

$$(1-\beta y)^{n-s+1} e^{-\nu(1-\beta y)}$$

and

$$(1-\beta y^{-1})^{n+s-1} e^{\nu(1-\beta y^{-1})}$$

Substituting the expansions for these factors into Equation (2-177) and evaluating the result yields the expression

$$X_k^{n,s} = (1+\beta^2)^{-n-1} (-\beta)^{|k-s|} \sum_{i=0}^{\infty} G_{i+(|k-s|+k-s)/2} H_{i+[|k-s|-(k-s)]/2} \beta^{2i} \tag{2-180}$$

where

$$G_p = \sum_{\ell=0}^{\infty} (-1)^{\ell} \binom{n-s+\ell+1}{p} \frac{\nu^{\ell}}{\ell!} \tag{2-181a}$$

$$H_q = \sum_{\ell=0}^{\infty} \binom{n+s+\ell+1}{q} \frac{\nu^{\ell}}{\ell!} \tag{2-181b}$$

It is interesting to note that

$$G_0 = e^{-\nu} \tag{2-182a}$$

$$H_0 = e^{\nu} \tag{2-182b}$$

Hansen provided other representation for the Hansen coefficients which can be found in Reference 27.

2.2.1.3.3 The Newcomb-Poincare Formulation of $X_t^{n,s}$

Newcomb (Reference 34) applied an operator approach to the problem of the expansion of the classical disturbing function. This operator development relies on certain differential operators to produce an expansion in the eccentricity. The resulting development is analagous to that obtained by using analytical expressions for the Hansen coefficients and, consequently, provides another representation for the Hansen coefficients.

In essence, the Newcomb operator method produces a power series in the square of the eccentricity, where the Newcomb operators are the coefficients. Evaluation of the Newcomb operators yields pure rational numbers. In addition, these coefficients can be evaluated recursively using the rec uce relations that exist for the Newcomb operators.

A complete discussion of the Newcomb operators would require an in-depth discussion of the operator approach to the expansion of the classical disturbing function; this discussion is beyond the scope of this section. However, in addition to Newcomb's original work, the method is discussed and simplified by Poincare (Reference 35). Other treatments of the subject can be found in References 1, 2, 29, and 36.

For the purpose of describing the Hansen coefficients in terms of the Newcomb operators, the following result given by Poincare (Reference 35) is considered:

$$\left(\frac{r}{a}\right)^n e^{js(f-\ell)} = \sum_{q=-\infty}^{\infty} \sum_{m=0}^{\infty} \Pi_q^{2m+|q|}(n|s) e^{2m+|q|} e^{jq\ell} \tag{2-183}$$

where the Newcomb operator $\Pi_q^{2m+|q|}(n|s)$ is a polynomial in n and s. Comparison of Poincare's result to a variation of the series of Hansen in Equation (2-132), i.e.,

$$\left(\frac{r}{a}\right)^n e^{js(f-\ell)} = \sum_{q=-\infty}^{\infty} X_{q+s}^{n,s} e^{jq\ell} \tag{2-184}$$

yields the relation

$$X_{q+s}^{n,s} = \sum_{m=0}^{\infty} \Pi_q^{2m+|q|}(n|s) e^{2m+|q|} \tag{2-185}$$

thus relating the Hansen coefficients to the Newcomb operators.

The remainder of this discussion follows closely that of Iszak, et al. (Reference 36). Inspection of Equation (2-183) indicates that it can be expressed as

$$\left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^s = \sum_{\rho=0}^{\infty} \sum_{\sigma=0}^{\infty} X_{\rho,\sigma}^{n,s} e^{\rho+\sigma} z^{\rho-\sigma} \tag{2-186}$$

where

$$\rho = m + \frac{|q|+q}{2} \tag{2-187}$$

$$\sigma = m + \frac{|q| - q}{2} \tag{2-188}$$

$$X_{\rho,\sigma}^{n,s} = \Pi_q^{2m+|q|}(n|s) \tag{2-189}$$

and x and z are defined by Equations (2-147).

It follows from the definitions of ρ and σ that

$$X_{\rho,\sigma}^{n,s} = \Pi_{\rho-\sigma}^{\rho+\sigma}(n|s) \tag{2-190}$$

and

$$\Pi_q^{2m+|q|}(n|s) = X_{m+(|q|+q)/2,\, m+(|q|-q)/2}^{n,s} \tag{2-191}$$

According to Iszak, the change in the indexes from q, m to ρ, σ simplifies the development of Von Zeipel's recurrence relations (Reference 37) for the Newcomb operators. These recurrence relations follow from the partial differential equation of Von Zeipel which is derived in Appendix A and is given by

$$\begin{aligned}(1-e^2)\, e\, \frac{\partial X^{n,k}}{\partial e} + (1-e^2)^{3/2}\, z\, \frac{\partial X^{n,k}}{\partial z} \\ = \left\{ k\left[1-(1-e^2)^{3/2}\right] + (k-n)\frac{e^2}{2} + (2k-n)\, e x + (k-n)\frac{e^2}{2} x^2 \right\}\end{aligned} \tag{2-192}$$

where

$$X^{n,k} = \left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^k \tag{2-193}$$

Eliminating the explicit appearance of x in Equation (2-192) through the substitution

$$x = z\left(\frac{x}{z}\right)$$

and developing the resulting equation into a power series in e yields the partial differential equation

$$2\left(e\frac{\partial}{\partial e} + z\frac{\partial}{\partial z}\right)X^{n,k} = 2(2k-n)ezX^{n,k+1}$$

$$+ (k-n)e^2 z^2 X^{n,k+2} + e^2\left[(4k-n) + 2e\frac{\partial}{\partial e} + 3z\frac{\partial}{\partial z}\right]X^{n,k} \qquad (2\text{-}195)$$

$$- 2\sum_{\tau \geq 2}\binom{3/2}{\tau}(-e^2)^{\tau}\left(k + z\frac{\partial}{\partial z}\right)X^{n,k}$$

Substituting Equation (2-186) into Equation (2-195) and comparing similar terms yields the recurrence relation

$$4\rho X^{n,k}_{\rho,\sigma} = 2(2k-n)X^{n,k+1}_{\rho-1,\sigma} + (k-n)X^{n,k+2}_{\rho-2,\sigma}$$

$$+ (5\rho-\sigma-4+4k-n)X^{n,k}_{\rho-1,\sigma-1} \qquad (2\text{-}196)$$

$$- 2(\rho-\sigma+k)\sum_{\tau \geq 2}(-1)^{\tau}\binom{3/2}{\tau}X^{n,k}_{\rho-\tau,\sigma-\tau}$$

The subscripts in this relation are restricted to nonnegative integers.

Another recurrence relation can be obtained from Equation (2-196) by interchanging the subscripts, changing k to -k, and using the symmetry relation

$$X_{\rho,\sigma}^{n,-k} = X_{\sigma,\rho}^{n,+k} \tag{2-197}$$

which follows from the symmetry relation for the Hansen coefficients (Equation (2-134)). The result is

$$\begin{aligned} 4\sigma X_{\rho,\sigma}^{n,k} = & -2(2k+n) X_{\rho,\sigma-1}^{n,k-1} - (k+n) X_{\rho,\sigma-2}^{n,k-2} \\ & - (\rho - 5\sigma + 4 + 4k + n) X_{\rho-1,\sigma-1}^{n,k} \\ & + 2(\rho - \sigma + k) \sum_{\tau \geq 2} (-1)^{\tau} \binom{3/2}{\tau} X_{\rho-\tau,\sigma-\tau}^{n,k} \end{aligned} \tag{2-198}$$

Finally, a third recurrence relation can be obtained by summing Equations (2-196) and (2-198) to yield

$$\begin{aligned} 4(\rho+\sigma) X_{\rho,\sigma}^{n,k} = & \ 2(2k-n) X_{\rho-1,\sigma}^{n,k+1} - 2(2k+n) X_{\rho,\sigma-1}^{n,k-1} \\ & + (k-n) X_{\rho-2,\sigma}^{n,k+2} - (k+n) X_{\rho,\sigma-2}^{n,k-2} \\ & + 2(2\rho + 2\sigma - 4 - n) X_{\rho-1,\sigma-1}^{n,k} \end{aligned} \tag{2-199}$$

thus eliminating the summation over τ.

Initialization for the recurrence relation is provided by

$$X_{0,0}^{n,k} = 1 \tag{2-200}$$

and the fact that quantities with negative subscripts are treated as identically zero. Consequently,

$$X_{1,0}^{n,k} = k - \frac{n}{2} \tag{2-201}$$

$$X_{0,1}^{n,k} = -k - \frac{n}{2} \tag{2-202}$$

etc., are easily obtained.

Although the Newcomb operators are rational numbers, the problem of generating them can be reduced to integer arithmetic by using the polynomials (Reference 36)

$$J_{\rho,\sigma}^{h,k} = 2^{\rho+\sigma} \rho! \, \sigma! \, X_{\rho,\sigma}^{n,k} \tag{2-203}$$

and the corresponding recurrence relations

$$J_{\rho,0}^{n,k} = (2k-n) J_{\rho-1,0}^{n,k+1} + (\rho-1)(k-n) J_{\rho-2,0}^{n,k+2} \tag{2-204}$$

$$\begin{aligned} J_{\rho,\sigma}^{n,k} = &- (2k+n) J_{\rho,\sigma-1}^{n,k-1} - (\sigma-1)(k+n) J_{\rho,\sigma-2}^{n,k-2} \\ &- \rho(\rho - 5\sigma + 4 + 4k + n) J_{\rho-1,\sigma-1}^{n,k} \\ &+ \rho(\rho-\sigma+k) \sum_{\tau \geq 2} C_{\rho,\sigma,\tau} J_{\rho-\tau,\sigma-\tau}^{n,k} \end{aligned} \tag{2-205}$$

where

$$C_{\rho,\sigma,\tau} = \frac{(\rho-1)!}{(\rho-\tau)!}\,\frac{(\sigma-1)!}{(\sigma-\tau)!}\,C_\tau \tag{2-206}$$

and where

$$C_\tau = (-1)^\tau \binom{3/2}{\tau} 2^{2\tau-1} \tag{2-207}$$

2.2.1.3.4 The Hansen Coefficient $X_0^{n,s}$ - A Special Case

The Hansen coefficient $X_0^{n,s}$ is of major importance in the development of the averaged equations of motion (in the absence of resonance phenomena), which are presented in Sections 3 and 4 of this document. Because of this importance and because it possesses a characteristic distinct from all other Hansen coefficients $X_k^{n,s}$ $(k \neq 0)$, it is singled out for a special discussion.

This particular Hansen coefficient is the constant term in the Fourier expansion, in the mean anomaly, of the product

$$\left(\frac{r}{a}\right)^n e^{jsf}$$

It possesses a finite hypergeometric series representation in either of the arguments β^2 or e^2, contrary to the general Hansen coefficients which have infinite series representations. These finite representations can be obtained through a brute-force expansion of the integrand of the expression

$$X_0^{n,s} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n e^{jsf}\,d\ell \tag{2-208}$$

the special case of Equation (2-133) where $k = 0$. However, this development is unnecessary since almost all of the work has already been performed in

Sections 2.2.1.1 and 2.2.1.2. More specifically, since

$$d\ell = \frac{r^2 df}{a^2 \cos\phi}$$

and

$$d\ell = \frac{r}{a} du$$

it then follows that changing the integration variable in Equation (2-208) yields

$$X_0^{n,s} = \frac{1}{2\pi \cos\phi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^{n+2} e^{jsf} df \tag{2-209}$$

and

$$X_0^{n,s} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^{n+1} e^{jsf} du \tag{2-210}$$

which, in view of Equations (2-66) and (2-107), yields the relations

$$X_0^{n,s} = \frac{V_s^{n+2}}{\cos\phi} \tag{2-211}$$

and

$$X_0^{n,s} = W_0^{n+1,s} \tag{2-212}$$

A Finite Hypergeometric Series Representation in β^2

It has already been shown that the coefficients $V_s^{\pm n}$, $W_t^{\pm n,s}$ admit finite representations in β^2 and, therefore, they yield finite representation of the Hansen coefficients $X_0^{n,s}$ through Equations (2-211) and (2-212). Replacing n by $-(n+1)$ in Equation (2-211) yields

$$X_0^{-(n+1),s} = \frac{V_s^{-(n-1)}}{\cos\phi} \tag{2-213}$$

Substituting n-1 for n in Equation (2-101b) and dividing by $\cos\phi$ yields the expression

$$X_0^{-(n+1),s} = (1-\beta^2)^{-(n-1)} \cos^{-n}\phi \binom{n-1}{|s|} \beta^{|s|} F(|s|-n+1, 1-n, |s|+1; \beta^2) \tag{2-214}$$

where $|s| \leq n-1$.

The expression for the coefficient $X_0^{n,s}$ is obtained by replacing n by $n+1$ in Equation (2-130b) and restricting the value of the subscript t to be identically zero. The result obtained is

$$X_0^{n,s} = (1+\beta^2)^{-(n+1)} (-\beta)^{|s|} \binom{n-s+1}{\frac{|s|-s}{2}} \binom{n+s+1}{\frac{|s|+s}{2}} \times F\left(\frac{|s|+s}{2}-n-1, \frac{|s|-s}{2}-n-1, |s|+1; \beta^2\right) \tag{2-215}$$

This expression can be simplified in view of the fact that interchanging the first two arguments in the hypergeometric series has no effect on its value. Since the only effect of a change in the sign of s on the hypergeometric series in Equation (2-215) is a permutation of the first two arguments, it follows that

$$F\left(\frac{|s|+s}{2}-n-1, \frac{|s|-s}{2}-n-1, |s|+1; \beta^2\right) = F(|s|-n-1, -n-1, |s|+1; \beta^2) \tag{2-216}$$

Similarly, a change in the sign of s also causes a permutation in the binomial coefficients in Equation (2-215). Clearly,

$$\binom{n-s+1}{\frac{|s|-s}{2}}\binom{n+s+1}{\frac{|s|+s}{2}} = \binom{n+|s|+1}{|s|} \qquad (2\text{-}217)$$

In view of Equations (2-216) and (2-217), Equation (2-215) simplifies to[1]

$$X_0^{n,s} = (1+\beta^2)^{-(n+1)}(-\beta)^{|s|}\binom{n+|s|+1}{|s|} F\left(|s|-n-1, -n-1, |s|+1; \beta^2\right) \qquad (2\text{-}218)$$

A Finite Hypergeometric Series Representation in e^2

A hypergeometric series representation in e^2 can be obtained through a quadratic transformation of the hypergeometric series in Equations (2-214) and (2-218). Inspection of the transformed hypergeometric series in e^2 indicates that they terminate after a finite number of terms.

It is well known that any hypergeometric series $F(a, b, c; x)$ admits a quadratic transformation if and only if the quantities

$$\pm(1-c), \quad \pm(a-b), \quad \pm(a+b-c)$$

[1] This expression is more easily obtained using the relation in Equation (2-211) and the closed-form expression for the coefficients $V_s^{\pm n}$ (see Equation (2-101b) and the accompanying footnote). However, it is also of value to pursue the simplifications which arise for the coefficients $W_0^{\pm n}$.

are such that any two of them are equal or one of them is equal to 1/2 (Reference 26). Inspection of the hypergeometric series in Equations (2-214) and (2-218) show that they satisfy the condition

$$a - b = \pm(1 - c) \tag{2-219}$$

Consequently, they are of the form

$$F(a, b, a-b+1; \beta^2) \tag{2-220}$$

The quadratic transformation which yields the e^2 representation is[1] (Reference 26)

$$F(a, b, a-b+1; \beta^2) = (1+\beta^2)^{-a} F\left(\frac{a}{2}, \frac{a+1}{2}, a-b+1; \frac{4\beta^2}{(1+\beta^2)^2}\right) \tag{2-221}$$

[1]There are two other quadratic transformations which can be applied in this case. They yield representations in terms of the arguments

$$\pm \frac{2e}{1 \pm e} \quad \text{and} \quad \frac{-e^2}{1-e^2}$$

with the simplification

$$\frac{4\beta^2}{(1+\beta^2)^2} = e^2 \tag{2-222}$$

which is easily verified from the definition of β. Applying Equation (2-221) to Equations (2-214) and (2-218) yields the expressions

$$X_0^{-(n+1),s} = \cos^{-(2n-1)}\phi \left(\frac{e}{2}\right)^{|s|} \binom{n-1}{|s|} F\left(\frac{|s|-n+1}{2}, \frac{|s|-n+2}{2}, |s|+1; e^2\right) \tag{2-223}$$

$$X_0^{n,s} = \left(-\frac{e}{2}\right)^{|s|} \binom{n+|s|+1}{|s|} F\left(\frac{|s|-n-1}{2}, \frac{|s|-n}{2}, |s|+1; e^2\right) \tag{2-224}$$

since

$$\frac{1-\beta^2}{1+\beta^2} = \cos\phi \tag{2-225}$$

and

$$\frac{\beta}{1+\beta^2} = \frac{e}{2} \tag{2-226}$$

An Associated Legendre Polynomial Representation

Any hypergeometric series which admits a quadratic transformation can be expressed as an associated Legendre Polynomial of the first kind (Reference 38). The hypergeometric series in Equations (2-214) and (2-218), which are of the form

$$F(a, b, a-b+1;\ z)$$

admit the associated Legendre polynomial representation[1] (Reference 26)

$$F(a,b,a-b+1;\beta^2) = \Gamma(a-b+1)\,\beta^{b-a}(1-\beta^2)^{-b}\,P_{-b}^{b-a}\left(\frac{1+\beta^2}{1-\beta^2}\right) \tag{2-227}$$

Application of Equation (2-227) to Equations (2-214) and (2-218) the following relation for the integer values of n and m (Reference 26)

$$P_n^{-m}(x) = \frac{(n-m)!}{(n+m)!}\,P_n^m(x) \tag{2-228}$$

yield the expressions

$$X_0^{-(n+1),s} = \frac{(n-1)!}{(n+|s|-1)!}\,x^n\,P_{n-1}^{|s|}(x) \tag{2-229}$$

$$X_0^{n,s} = (-1)^s\,\frac{(n-|s|+1)!}{(n+1)!}\,x^{-n-1}\,P_{n+1}^{|s|}(x) \tag{2-230}$$

where

$$x = \frac{1+\beta^2}{1-\beta^2} = \frac{1}{\cos\phi} \tag{2-231}$$

---

[1] Hobson's definition of the associated Legendre polynomial for arguments in the range $1 < x < \infty$ is (Reference 30)

$$P_n^m(x) = \frac{1}{2^n n!}\,(x^2-1)^{m/2}\,\frac{d^{n+m}}{dx^{n+m}}\,(x^2-1)^n$$

These relations are particularly useful in view of the many recurrence relations available for the associated Legendre polynomials. The above expressions were also obtained by Cook (Reference 12), with the exception that a discrepancy of $(-1)^s$ appears between Equation (2-230) and the corresponding result of Cook. A review of Cook's work and the results he cites from Whitaker and Watson (Reference 30) indicate that the missing factor does, in fact, belong in Cook's expression if Hobson's definition of the associated Legendre polynomials is used.

Evaluation of Equations (2-229) and (2-230) for a few values of n and s yields the following Hansen coefficients:

$$X_0^{-2,0} = (1-e^2)^{-1/2}$$

$$X_0^{-3,0} = (1-e^2)^{-3/2} \qquad X_0^{-3,1} = \frac{e}{2}(1-e^2)^{-3/2}$$

$$X_0^{-4,0} = \frac{3}{2}(1-e^2)^{-5/2} - \frac{1}{2}(1-e^2)^{-3/2} = \left(1+\frac{e^2}{2}\right)(1-e^2)^{-5/2}$$

$$X_0^{-4,1} = \frac{e}{2}\,2(1-e^2)^{-5/2}$$

$$X_0^{-4,2} = \frac{1}{4}\left[(1-e^2)^{-5/2} - (1-e^2)^{-3/2}\right] = \frac{e^2}{4}(1-e^2)^{-5/2}$$

and

$$X_0^{0,0} = 1$$

$$X_0^{1,0} = 1+\frac{e^2}{2} \qquad X_0^{1,1} = -\frac{3}{2}e$$

$$X_0^{2,0} = 1+\frac{3}{2}e^2 \qquad X_0^{2,1} = -2e-\frac{e^3}{2} \qquad X_0^{2,2} = \frac{5}{2}e^2$$

$$X_0^{3,0} = 1+3e^2+\frac{3}{8}e^4 \qquad X_0^{3,1} = -\frac{5}{2}e-\frac{15}{8}e^3 \qquad X_0^{3,2} = \frac{15}{4}e^2+\frac{5}{8}e^4 \qquad X_0^{3,3} = -\frac{35}{8}e^3$$

2.2.1.4 Recurrence Relations for the Fourier Series Coefficients

Recurrence relations prove very useful for the efficient evaluation of the coefficients of the Fourier series expansions developed in the previous sections, provided the recurrence relations are sufficiently stable.[1] Hansen gives several recurrence relations for the coefficients of the Fourier series expansions in the true, eccentric, and mean anomalies. More recently, Vinh (Reference 39) has discussed recurrence relations for the coefficients $W_t^{n,s}$ and $X_t^{n,s}$. Cook (Reference 12) and Cefola (Reference 11) have discussed recurrence relations for functions related to the special case of the Hansen coefficients $X_0^{n,s}$. Giacaglia (References 15 and 40) also developed recurrence formulas for the Hansen coefficients, and Cefola (Reference 41) has developed a new recurrence relation for the coefficients $W_t^{n,s}$.

Previously, the coefficients $V_s^{\pm n}$, $W_t^{\pm n,s}$, and $X_0^{\pm n,s}$ were shown to be expressed simply in terms of the hypergeometric series. It follows that Gauss' contiguous relations (Reference 26) for the hypergeometric series could serve as the basis for constructing recurrence relations for these functions. However, recurrence relations that permit only one varying index are often the most desirable. Since the parameters of the hypergeometric series (a, b, c) (Equation (2-28) are linear combinations of the indexes n, s, and sometimes t, the form of Gauss' contiguous relations is not well suited for developing recurrence relations with a single varying index.

Orthogonal polynomials are suited for single-varying-parameter recurrence relations. Furthermore, it was shown in the previous section that the coefficients $X_0^{\pm n,s}$ are simply related to the associated Legendre polynomials. Consequently,

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[1] Since truncation and rounding errors are introduced into the evaluation of the recurrence relations, it is important to know how these errors are propagated through the recurrence process. If the errors do not grow relative to the magnitude of the function being evaluated, the recurrence relation is said to be stable; otherwise, the recurrence relation is unstable.

the recurrence relations for this set of orthog· al polynomials provide the foundation for the recurrence relations for the coefficients $X_0^{\pm n, s}$ and for the coefficients $V_s^{\pm n}$ and $W_0^{\pm n, s}$, in view of Equations (2-211) and (2-212). Recurrence relations for the coefficients $W_t^{\pm n, s}$ and $X_t^{\pm n, s}$ are obtained through a more classical approach used by Hansen.

2.2.1.4.1 Recurrence Relations for the Coefficients $X_0^{\pm n, s}$, $V_s^{\pm n}$, $W_s^{\pm n, s}$

Recurrence relations for these coefficients are obtained from the following recurrence relations for the associated Legendre polynomials obtained from References 26 and 42. The fixed-order recurrence relation is given by

$$P_{n+1}^{m}(x) = \frac{1}{n-m+1}\left[(2n+1)\, x\, P_n^{m}(x) - (n+m)\, P_{n-1}^{m}(x)\right] \tag{2-232}$$

the fixed-degree recurrence relation is

$$P_n^{m+1}(x) = \frac{-2mx}{\sqrt{x^2-1}}\, P_n^{m}(x) + (n-m+1)(n+m)\, P_n^{m-1}(x) \tag{2-233}$$

and the varying-order-and-degree relation is

$$P_{n+1}^{m}(x) = P_{n-1}^{m}(x) + (2n+1)(x^2-1)^{1/2}\, P_n^{m-1}(x) \tag{2-234}$$

Equations (2-232) and (2-235) can be combined to yield the recurrence relation

$$P_n^{m}(x) = \frac{1}{x}\left[P_{n-1}^{m}(x) + (n-m+1)\sqrt{x^2-1}\; P_n^{m-1}(x)\right] \tag{2-235}$$

It is easily demonstrated from the results in Reference 26 that

$$P_n^n(x) = (2n-1)!!\,(x^2-1)^{n/2} \tag{2-236}$$

and it follows from the principle of induction that

$$P_{n+1}^{n+1}(x) = (2n+1)\sqrt{x^2-1}\;P_n^n(x) \tag{2-237}$$

In addition, it is easily shown from the definition of the associated Legendre polynomial that

$$P_{n+1}^n(x) = (2n+1)\,x\,P_n^n(x) \tag{2-238}$$

Combining Equations (2-237) and (2-238) yields

$$P_{n+1}^{n+1}(x) = \frac{\sqrt{x^2-1}}{x}\;P_{n+1}^n(x) \tag{2-239}$$

and it follows from Equations (2-238) and (2-239) that

$$P_{n+1}^n(x) = (2n+1)\sqrt{x^2-1}\;P_n^{n-1}(x) \tag{2-240}$$

Inverting Equation (2-229) and substituting the result into Equations (2-232) through (2-235) yields the following recurrence relations:

$$X_0^{-(n+1),s} = \frac{(n-1)(1-e^2)^{-1}}{(n+s-1)(n-s-1)}\left[(2n-3)X_0^{-n,s} - (n-2)X_0^{-(n-1),s}\right] \tag{2-241}$$

$$X_0^{-(n+1),s+1} = \frac{1}{n+s}\left[-\frac{2s}{e}X_0^{-(n+1),s} + (n-s)X_0^{-(n+1),s-1}\right] \tag{2-242}$$

$$X_0^{-(n+1),s} = \frac{(n-1)(1-e^2)^{-1}}{(n+s-1)(n+s-2)}\left[(2n+1)e\,X_0^{-n,s-1} + (n-2)X_0^{-(n-1),s}\right] \tag{2-243}$$

$$X_0^{-(n+1),s} = \frac{1}{(n+s-1)}\left[(n-1)X_0^{-n,s} + (n-s)e\,X_0^{-(n+1),s-1}\right] \tag{2-244}$$

In the above equations, the superscript s is restricted to nonnegative values, i.e., $s \geq 0$. This restriction is quite satisfactory in view of the symmetry relation (Equation (2-134))

$$X_k^{n,s} = X_{-k}^{n,-s}$$

In view of Equations (2-229) and (2-236), it follows that

$$X_0^{-(n+1),n-1} = \sqrt{1-e^2}\left(\frac{e/2}{1-e^2}\right)^{n-1} \tag{2-245}$$

and it follows from Equations (2-229) and (2-239) that

$$X_0^{-(n+1),n-1} = \frac{1}{n-1}\,\frac{e}{2}\,X_0^{-(n+1),n-2} \tag{2-246}$$

Also, Equations (2-237) and (2-229) can be combined to yield

$$X_0^{-(n+1),n-1} = \frac{e}{2(1-e^2)}\,X_0^{-n,n-2} \tag{2-247}$$

In addition, inspection of the definition given by Equation (2-229) indicates that

$$X_0^{-(n+1),n} \equiv 0 \tag{2-248}$$

and, more generally,

$$X_0^{-(n+1),s} \equiv 0 \tag{2-249}$$

for all s such that $|s| \geq n$.

The recurrence relations for $X_0^{n,s}$ are obtained by substituting Equations (2-230) into Equations (2-232), which yields

$$X_0^{n,s} = \frac{1}{n+1}\left[(2n+1)X_0^{n-1,s} - \frac{(n+s)(n-s)(1-e^2)}{n} X_0^{n-2,s}\right] \tag{2-250}$$

$$X_0^{n,s+1} = \frac{1}{n-s+1}\left[\frac{2s}{e} X_0^{n,s} + (n+s+1) X_0^{n,s-1}\right] \tag{2-251}$$

$$X_0^{n,s} = \frac{1}{n+1}\left[\frac{(n-s+1)(n-s)(1-e^2)}{n} X_0^{n-2,s} - (2n+1)\, e\, X_0^{n-1,s-1}\right] \tag{2-252}$$

$$X_0^{n,s} = \frac{(n-s+1)}{n+1}(1-e^2)\, X_0^{n-1,s} - e X_0^{n,s-1} \tag{2-253}$$

The superscript s is again restricted to nonnegative values.

Applying Equations (2-236) and (2-238) to Equation (2-230) yields the result for the special case

$$X_0^{n,n} = \frac{(2n+1)!!}{(n+1)!} (-e)^n \tag{2-254}$$

Also, inverting Equation (2-230) and substituting the result into Equation (2-240) yields the recurrence relation

$$X_0^{n,n} = -\frac{2n+1}{n+1} e X_0^{n-1,n-1} \tag{2-255}$$

and two successive applications of Equation (2-240) with Equation (2-230) yield

$$X_0^{n,n} = \frac{(2n+1)(2n-1)}{n(n+1)} e^2 X_0^{n-2,n-2} \tag{2-256}$$

Clearly,

$$X_0^{n,n+1} \equiv 0 \tag{2-257a}$$

follows from Equation (2-230) and, generally,

$$X_0^{n,s} \equiv 0 \tag{2-257b}$$

for s such that $|s| \geq n+1$.

Inspection of the recurrence relations in Equations (2-242) and (2-251) indicate the appearance of e as a divisor. Consequently, these recurrence relations appear to be of little value for cases with small eccentricities. This difficulty is easily avoided by using the expressions

$$X_0^{n,s} = e^s A_n^s$$
$$X_0^{-(n+1),s} = e^s B_{n+1}^s$$

where the definitions of the functions A_n^s and B_{n+1}^s follow from Equations (2-224) and (2-223), respectively. The resulting recurrence relations for the functions A_n^s and B_{n+1}^s are free of the eccentricity divisor.

The recurrence relations for the coefficients $V_s^{\pm n}$ and $W_0^{\pm n,s}$ are easily obtained by substituting Equations (2-211), (2-212), and (2-213) into the recurrence relations for the special case of the Hansen coefficients $X_0^{n,s}$ and $X_0^{-(n+1),s}$.

2.2.1.4.2 Recurrence Relations for the General Hansen Coefficients

Since the general Hansen coefficients $X_k^{n,s}$ do not apparently admit a simple orthogonal polynomial representation, the classical approach of Hansen will be used to develop the recurrence relations for these coefficients.

For convenience, the following definitions are made:

$$\rho = \frac{r}{a}$$
$$x = e^{jf}$$
$$z = e^{j\ell}$$

Then,

$$z \frac{d}{dz} \rho^n x^s = n\rho^{n-1} x^s z \frac{d\rho}{dz} + s\rho^n x^{s-1} z \frac{dx}{dz} \qquad \text{(2-258)}$$

Using the relations

$$z \frac{d}{dz} = z \frac{dx}{dz} \frac{d}{dx} \qquad \text{(2-259a)}$$

$$\frac{dx}{dz} = \frac{x}{z}\frac{a^2}{r^2}\cos\phi \tag{2-259b}$$

$$x\frac{d\rho}{dx} = \frac{1}{2}\frac{r^2}{a^2}\frac{\sin\phi}{\cos^2\phi}\left(\frac{1}{x} - x\right) \tag{2-259c}$$

Equation (2-257) can be simplified to yield

$$z\frac{d}{dz}\rho^n x^s = \frac{n\sin\phi}{2\cos\phi}\rho^{n-1}(x^{s-1} - x^{s+1}) + s\cos\phi\,\rho^{n-2}x^s \tag{2-260}$$

Since[1]

$$\rho - \frac{\cos^2\phi}{1 + \frac{\sin\phi}{2}(x + x^{-1})} \equiv 0 \tag{2-261}$$

or, equivalently,

$$\sin\phi\,(x + x^{-1}) - 2\cos^2\phi\,\rho^{-1} + 2 = 0 \tag{2-262}$$

then the products of Equation (2-262) with the factors

$$\pm\frac{n\rho^{n-1}x^s}{2\cos\phi} \quad \text{and} \quad \frac{s\rho^{n-1}x^s}{2\cos\phi}$$

---

[1] In classical elements, this equation takes on the more familiar form

$$\frac{r}{a} - \frac{(1 - e^2)}{1 + e\cos f} \equiv 0$$

are identically zero and can therefore be added to the right-hand side of Equation (2-260) to yield

$$z\frac{d}{dz}\rho^n x^s = n\frac{\sin\phi}{\cos\phi}\rho^{n-1}x^{s-1} + (s-n)\cos\phi\,\rho^{n-2}x^s + \frac{n}{\cos\phi}\rho^{n-1}x^s \qquad (2\text{-}263)$$

$$z\frac{d}{dz}\rho^n x^s = -\frac{n\sin\phi}{\cos\phi}\rho^{n-1}x^{s+1} + (s+n)\cos\phi\,\rho^{n-2}x^s - \frac{n}{\cos\phi}\rho^{n-1}x^s \qquad (2\text{-}264)$$

and

$$z\frac{d}{dz}\rho^n x^s = \frac{(n+s)\sin\phi}{2\cos\phi}\rho^{n-1}x^{s-1} - \frac{(n-s)\sin\phi}{2\cos\phi}\rho^{n-1}x^{s+1} + \frac{s}{\cos\phi}\rho^{n-1}x^s \qquad (2\text{-}265)$$

Also, since[1]

$$\frac{\sin\phi}{2\cos^2\phi}\rho\,(x^{-1}+x) + \frac{\rho}{\cos^2\phi} = 1 \qquad (2\text{-}266)$$

the product of Equation (2-266) with the right-hand side of Equation (2-265) yields

$$\begin{aligned} z\frac{d}{dz}\rho^n x^s = {} & \frac{(n+s)\sin^2\phi}{4\cos^3\phi}\rho^n x^{s-2} + \frac{(n+2s)\sin\phi}{2\cos^3\phi}\rho^n x^{s-1} \\ & + \frac{2s+s\sin^2\phi}{2\cos^3\phi}\rho^n x^s - \frac{(n-2s)\sin\phi}{2\cos^3\phi}\rho^n x^{s+1} \qquad (2\text{-}267) \\ & - \frac{(n-s)\sin^2\phi}{4\cos^3\phi}\rho^n x^{s+2} \end{aligned}$$

---

[1] A less familiar form of the expression $1 + e\cos f = \dfrac{r}{a(1-e^2)}$

Hansen also obtains the result

$$z^2 \frac{d^2}{dz^2} \rho^n x^s + z \frac{d}{dz} \rho^n x^s = \left[n(n-2)+s^2\right] \cos^2\phi \, \rho^{n-4} x^s$$
$$- n(2n-3) \rho^{n-3} x^s + n(n-1) \rho^{n-2} x^s \qquad (2\text{-}268)$$
$$+ s(n-1) \sin\phi \, \rho^{n-3} \left(x^{s-1} - x^{s+1}\right)$$

by differentiating Equations (2-263) and (2-264) with the operator $z \frac{d}{dz}$ and substituting those equations back into the result. Equations (2-262) are also helpful in obtaining the final form. In addition, multiplying Equation (2-262) by the factor

$$s(n-1) \rho^{n-3} x^s$$

and alternately adding and subtracting the result from Equation (2-268) yields the expressions

$$z^2 \frac{d^2}{dz^2} \rho^n x^s + z \frac{d}{dz} \rho^n x^s = \left[n(n-2) + s^2 - 2s(n-1)\right] \cos^2\phi \, \rho^{n-4} x^s$$
$$- \left[n(2n-3) - 2s(n-1)\right] \rho^{n-3} x^s \qquad (2\text{-}269)$$
$$+ n(n-1) \rho^{n-2} x^s + 2s(n-1) \sin\phi \, \rho^{n-3} x^{s-1}$$

$$z^2 \frac{d^2}{dz^2} \rho^n x^s + z \frac{d}{dz} \rho^n x^s = \left[n(n-2) + s^2 + 2s(n-1)\right] \cos^2\phi \, \rho^{n-4} x^s$$
$$- \left[n(2n-3) + 2s(n-1)\right] \rho^{n-3} x^s \qquad (2\text{-}270)$$
$$+ n(n-1) \rho^{n-2} x^s - 2s(n-1) \sin\phi \, \rho^{n-3} x^{s+1}$$

The recurrence relations are obtained by substituting the series representation for $\rho^n x^s$ and the first and second derivatives, i.e.,

$$\rho^{\pm n} x^s = \sum_t X_t^{\pm n, s} z^t \qquad (2\text{-}271a)$$

$$z \frac{d}{dz} \rho^{\pm n} x^s = \sum_t t X_t^{\pm n, s} z^t \qquad (2\text{-}271b)$$

$$z \frac{d^2}{dz^2} \rho^{\pm n} x^s = \sum_t t(t-1) X_t^{\pm n, s} z^t \qquad (2\text{-}271c)$$

into the above differential equations to yield

$$n \sin\phi\, X_t^{n-1, s-1} - n \sin\phi\, X_t^{n-1, s+1} + 2s \cos^2\phi\, X_t^{n-2, s} - 2t \cos\phi\, X_t^{n, s} = 0 \qquad (2\text{-}272)$$

$$n \sin\phi\, X_t^{n-1, s-1} + (s-n) \cos^2\phi\, X_t^{n-2, s} + n X_t^{n-1, s} - t \cos\phi\, X_t^{n, s} = 0 \qquad (2\text{-}273)$$

$$-n \sin\phi\, X_t^{n-1, s+1} + (s+n) \cos^2\phi\, X_t^{n-2, s} - n X_t^{n-1, s} - t \cos\phi\, X_t^{n, s} = 0 \qquad (2\text{-}274)$$

$$(n+s) \sin\phi\, X_t^{n-1, s-1} - (n-s) \sin\phi\, X_t^{n-1, s+1} + 2s X_t^{n-1, s} - 2t \cos\phi\, X_t^{n, s} = 0 \qquad (2\text{-}275)$$

$$\begin{aligned} &(n+s) \sin^2\phi\, X_t^{n, s-2} + 2(n+2s) \sin\phi\, X_t^{n, s-1} + (4s + 2s \sin^2\phi - 4t \cos^3\phi) X_t^{n, s} \\ &- 2(n-2s) \sin\phi\, X_t^{n, s+1} - (n-s) \sin^2\phi\, X_t^{n, s+2} = 0 \end{aligned} \qquad (2\text{-}276)$$

$$\left[n(n-2)+s^2\right]\cos^2\phi\, X_t^{n-4,s} - n(2n-3)\,X_t^{n-3,s} + n(n-1)\,X_t^{n-2,s}$$
$$+\ s(n-1)\sin\phi\, X_t^{n-3,s-1} - s(n-1)\sin\phi\, X_t^{n-3,s+1} - t^2 X_t^{n,s} = 0 \qquad (2\text{-}277)$$

$$\left[n(n-2)+s^2-2s(n-1)\right]\cos^2\phi\, X_t^{n-4,s} - \left[n(2n-3)-2s(n-1)\right]X_t^{n-3,s}$$
$$+\ n(n-1)\,X_t^{n-2,s} + 2s(n-1)\sin\phi\, X_t^{n-3,s-1} - t^2 X_t^{n,s} = 0 \qquad (2\text{-}278)$$

$$\left[n(n-2)+s^2+2s(n-1)\right]\cos^2\phi\, X_t^{n-4,s} - \left[n(2n-3)+2s(n-1)\right]X_t^{n-3,s}$$
$$+\ n(n-1)\,X_t^{n-2,s} - 2s(n-1)\sin\phi\, X_t^{n-3,s+1} - t^2 X_t^{n,s} = 0 \qquad (2\text{-}279)$$

Setting Equations (2-260) and (2-263) equal and substituting Equation (2-271a) into the result yields the recurrence relation

$$\sin\phi\, X_t^{n-1,s-1} + \sin\phi\, X_t^{n-1,s+1} + 2X_t^{n-1,s} - 2\cos^2\phi\, X_t^{n-2,s} \qquad (2\text{-}280)$$

Also, replacing n with n-2 in Equation (2-272) and setting the result equal to Equation (2-277) yields the recurrence relation

$$n\left[(n-2)^2-s^2\right]\cos^2\phi\, X_t^{n-4,s} - n(n-2)(2n-3)\,X_t^{n-3,s}$$
$$+\ (n-1)\left[n(n-2)+2ts\cos\phi\right]X_t^{n-2,s} - t^2(n-2)\,X_t^{n,s} = 0 \qquad (2\text{-}281)$$

This recurrence relation is particularly attractive since the superscript s is fixed and since the eccentricity $e = \sin\phi$ does not appear.

Several more recurrence relations can be obtained by combining the above recurrence relations or, equivalently, the differential equations. All of the above recurrence relations are valid for positive and negative values of n.

2.2.1.4.3 Recurrence Relations for the Coefficients $W_t^{n,s}$

The procedure for generating the recurrence relations for this case is identical to the procedure used above. Since

$$\frac{d\rho}{dy} = \beta \cos^2 \frac{\phi}{2}\left(\frac{1}{y^2} - 1\right) \tag{2-282}$$

and

$$\frac{dx}{dy} = \frac{1-\beta^2}{(1-\beta y)^2} \tag{2-283}$$

where

$$y = e^{ju}$$

it follows that

$$\frac{d\rho^n x^s}{dy} = n\beta \cos^2 \frac{\phi}{2} \rho^{n-1} x^s \left(\frac{1}{y^2} - 1\right) + s(1-\beta^2)\frac{\rho^n x^{s-1}}{(1-\beta y^2)} \tag{2-284}$$

which can be expressed as

$$\left[y(1+\beta^2) - \beta - \beta y^2\right]\frac{d\rho^n x^s}{dy} = n\beta\rho^n x^s\left(\frac{1}{y} - y\right) + s(1-\beta^2)\rho^n x^s \tag{2-285}$$

Substituting the series representations

$$\rho^n x^s = \sum_t W_t^{n,s} y^t \tag{2-286a}$$

$$\frac{d\rho^n x^s}{dy} = \sum_t t W_t^{n,s} y^{t-1} \tag{2-286b}$$

yields the recurrence relation

$$(t+1+n)\beta W_{t+1}^{n,s} - \left[t-s+(t+s)\beta^2\right]W_t^{n,s} + (t-1-n)\beta W_{t-1}^{n,s} = 0 \tag{2-287}$$

Another recurrence relation can be obtained from the expression

$$\rho^{n+1} x^{s-1} = \cos^2\frac{\phi}{2}\left(\frac{1}{y} - 2\beta + \beta^2 y\right)\rho^n x^s \tag{2-288}$$

which is easily verified. Substituting Equation (2-286a) into Equation (2-288) yields the recurrence relation

$$W_t^{n+1,s-1} = \cos^2\frac{\phi}{2}\left[W_{t+1}^{n,s} - 2\beta W_t^{n,s} + \beta^2 W_{t-1}^{n,s}\right] \tag{2-289}$$

In addition, Hansen also gives the recurrence relation

$$W_t^{n+1,s+1} = \cos^2\frac{\phi}{2}\left[W_{t-1}^{n,s} - 2\beta W_t^{n,s} + \beta^2 W_{t+1}^{n,s}\right] \tag{2-290}$$

Recently, Cefola (Reference 41), using the technique of Hansen, obtained the recurrence relation in which the superscript s is fixed, i.e.,

$$\begin{aligned}(n-1)(t^2-n^2)W_t^{n,s} = {} & (2n-1)\left[ts\cos\phi - n(n-1)\right]W_t^{n-1,s} \\ & + n(n-s-1)(n+s-1)\cos^2\phi\, W_t^{n-2,s}\end{aligned} \tag{2-291}$$

This recurrence relation is a generalization of the recurrence formula given by Vinh in Equation (3.8) of Reference 39.

Cefola's recurrence relation is reminiscent of recurrence relations for orthogonal polynomials and thus leads to the conjecture that the coefficients $W_t^{n,s}$ may have an orthogonal polynomial representation. This conjecture is shown to be true in Appendix B of this document.

2.2.2 Fourier Expansions in the Longitudes

The results obtained in Section 2.2.1 can now be used to construct the Fourier series expansions in the true, eccentric, and mean longitudes of the product (Equation (2-54)),

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_t A_t^{n,s} e^{jtX} \tag{2-292}$$

where X is any of the three longitudes. It was previously shown that the coefficients $A_t^{n,s}$ of the expansion in a particular longitude are related to the coefficients $a_t^{n,s}$ of the expansion in the corresponding anomaly by the expression

$$A_t^{n,s} = e^{-|t-s|} (k - j\eta h)^{|s-t|} a_t^{n,s} \tag{2-293}$$

where $\eta = \text{sign}(t-s)$. Equation (2-293) is equivalent to Equations (2-62a) and (2-62b). The form of Equation (2-293) is used to avoid reciprocals of the complex polynomial $(k+jh)^m$ by replacing it with its conjugate polynomial, i.e.,

$$\left[\frac{k+jh}{e}\right]^{-m} = \left[\frac{k-jh}{e}\right]^{+m} \tag{2-294}$$

It appears that the form of Equation (2-293) admits a "computational singularity" for vanishing eccentricity; however, the coefficients $a_t^{n,s}$ contain a factor that absorbs the eccentricity divisor, as will be demonstrated.

2.2.2.1 Expansion in the True Longitude

Since only the expansion of the parallax factor, r/a, is required, it is considered first. In view of Equations (2-100), (2-292), and (2-293), it follows that

$$\left(\frac{r}{a}\right)^{\pm n} = \sum_{t} e^{-|t|} (k - j\eta h)^{|-t|} V_t^{\pm n} e^{jtL} \tag{2-295}$$

where $-\infty < t < \infty$ for $+n$ and $-n \le t \le n$ for $-n$, and where η = sign t.

Inspection of the expressions for the coefficients $V_t^{\pm n}$ (Equations (2-100b) and (2-101b)) show that they are proportional to $\beta^{|t|}$ or, equivalently, $e^{|t|}$ (Equation (2-73)). Hence, if the coefficient $\nu_t^{\pm n}$ is defined as

$$V_t^{\pm n} = e^{|t|} \nu_t^{\pm n} \tag{2-296}$$

then Equations (2-295) take the form

$$\left(\frac{r}{a}\right)^{\pm n} = \sum_{t} (k - j\eta h)^{|-t|} \nu_t^{\pm n} e^{jtL} \tag{2-297}$$

Multiplying both sides of Equation (2-295) by e^{jsL} yields the final result

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_{t} (k - j\eta h)^{|-t|} \nu_t^{\pm n} e^{j(t+s)L} \tag{2-298}$$

where $-\infty < t < \infty$ for $+n$ and $-n \le t \le n$ for $-n$.

2.2.2.2 Expansion in the Eccentric Longitude

In view of Equations (2-130), (2-131), and (2-293), it follows that

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_{t} e^{-|t-s|} (k - j\eta h)^{|s-t|} W_t^{\pm n, s} e^{jtF} \tag{2-299}$$

where $\eta = \text{sign}(t-s)$, $-n \le t \le n$ for $+n$ and $-\infty < t < \infty$ for $-n$. Inspection of Equations (2-130b) and (2-131b) show that the coefficients $W_t^{\pm n, s}$ are proportional to $\beta^{|t-s|}$ or $e^{|t-s|}$; hence, if $w_t^{\pm n, s}$ is defined by

$$W_t^{\pm n,s} = e^{|t-s|}\, w_t^{\pm n,s} \tag{2-300}$$

then Equation (2-299) takes the form

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_t (k - j\eta h)^{|s-t|}\, w_t^{\pm n,s}\, e^{jtF} \tag{2-301}$$

where the definition of η and the ranges of t are the same as in Equation (2-299).

2.2.2.3 Expansion in the Mean Longitude

The expansion in the mean longitude follows from Equations (2-132) and (2-293) to yield

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_{t=-\infty}^{\infty} e^{-|t-s|} (k - j\eta h)^{|s-t|}\, X_t^{\pm n,s}\, e^{jtL} \tag{2-302}$$

Also, inspection of Equation (2-174), (2-180), or (2-185) shows that the Hansen coefficient $X_t^{\pm n, s}$ is proportional to $e^{|t-s|}$; hence, if the coefficient $K_t^{\pm n, s}$ is defined by

$$X_t^{\pm n,s} = e^{|t-s|}\, K_t^{\pm n,s} \tag{2-303}$$

then Equation (2-242) takes the form

$$\left(\frac{r}{a}\right)^{\pm n} e^{jsL} = \sum_{t=-\infty}^{\infty} (k - j\eta h)^{|s-t|}\, K_t^{\pm n,s}\, e^{jt\lambda} \tag{2-304}$$

where $\eta = \text{sign}(t-s)$.

2.2.2.4 Recurrence Relations

Recurrence relations for the coefficients of the expansions in the longitudes follow immediately from the recurrence relations for the coefficients of the expansions in the anomalies presented in Section 2.2.1.4. For example, substituting Equation (2-303) into the recurrence relations for the general Hansen coefficients yields the recurrence relations for the functions $K_t^{\pm n, s}$. Combining these recurrence relations with the simple recurrence properties of the complex polynomial $(k - j\eta h)^{|s-t|}$ yields the recurrence relation of the product

$$Y_t^{\pm n,s} = (k - j\eta h)^{|s-t|} K_t^{\pm n,s} \tag{2-305}$$

As an example, substituting Equation (2-303) into Equation (2-281) yields the result

$$\begin{aligned} n\left[(n-2)^2 - s^2\right]\cos^2\phi\, Y_t^{n-4,s} - n(n-2)(2n-3)\,Y_t^{n-3,s} \\ + (n-1)\left[n(n-2) + 2st\cos\phi\right] Y_t^{n-2,s} - t^2(n-2)\,Y_t^{n,s} = 0 \end{aligned} \tag{2-306}$$

SECTION 3 - EXPLICIT THEORY FOR THE NONSPHERICAL GRAVITATIONAL PERTURBATION

This section presents the explicit development of the first-order averaged equations of motion for the nonspherical gravitational perturbation. Section 3.1 presents a general discussion of the development in spherical coordinates of the nonspherical gravitational disturbing function. In Section 3.2, the disturbing function is developed explicitly in terms of the equinoctial elements. Specifically, Section 3.2.1 presents the rotation of the disturbing to the equinoctial frame and Section 3.2.2 introduces the necessary Fourier series expansions.

In Section 3.3, the averaged disturbing function is obtained. The concepts of time-dependent and time-independent averaging are introduced and compared. It is shown that time-dependent averaging does not always remove all short-period terms in the disturbing functions and that time-independent averaging may exaggerate the amplitudes of the remaining medium- and long-period terms in the averaged disturbing function for the cases of nonresonant and resonant tesseral harmonic terms, respectively. In addition, the zonal harmonic, combined zonal and nonresonant tesseral harmonic, and resonant tesseral harmonic disturbing functions are isolated.

Section 3.4 develops the partial derivatives necessary for the averaged equations of motion. The equations of motion are developed separately for the zonal harmonic, the combined zonal and nonresonant tesseral harmonic, and the resonant tesseral harmonic models.

3.1 THE NONSPHERICAL GRAVITATIONAL DISTURBING FUNCTION

It is well known that the gravitational force, $\vec{F}$, can be represented as the gradient of a potential function V, i.e.,

$$\vec{F} = -\nabla V(x, y, z) \tag{3-1}$$

where the gradient operator ∇ is defined in Cartesian coordinates as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \tag{3-2}$$

where $(\hat{i}, \hat{j}, \hat{k})$ is the orthogonal triad of the Cartesian reference system. The particular form of the potential function associated with the gravitational force exerted by the attracting body depends on the mass distribution of that body. The potential function must satisfy Poisson's equation

$$\nabla^2 V(x, y, z) = 4\pi G \rho(x, y, z) \tag{3-3}$$

where the divergence operator is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2}\hat{i} + \frac{\partial^2}{\partial y^2}\hat{j} + \frac{\partial^2}{\partial z^2}\hat{k} \tag{3-4}$$

and where G is the universal gravitational constant and $\rho(x, y, z)$ is the density per unit volume at the point (x, y, z). At all points where the density vanishes, i.e., outside the attracting body, Poisson's equation reduces to Laplace's equation, i.e.,

$$\nabla^2 V(x, y, z) = 0 \tag{3-5}$$

The general solution of Laplace's equation yields the potential function for the gravitational force exerted by a body of arbitrary mass configuration on an exterior particle located at the position (x, y, z). The potential function for a given mass configuration is specified by the appropriate boundary conditions in addition to the general solution of Laplace's equation. In general, these boundary conditions are unknown, and, in practice, the potential function is ultimately determined by a semiempirical method. This method assumes knowledge of the form of the general solution of Laplace's equation which is obtained below.

The method of solution for Laplace's equations is the standard separation of variables technique and is usually developed in spherical coordinates (r, ψ, ϕ) where $r \geq 0$, $0 \leq \psi \leq 2\pi$, and $-\pi/2 \leq \phi \leq \pi/2$. In spherical coordinates, Laplace's equation takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\cos\phi} \frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial V}{\partial \phi}\right) + \frac{1}{\cos^2\phi} \frac{\partial^2 V}{\partial \psi^2} = 0 \qquad (3\text{-}6)$$

The solution is assumed to be of the form

$$V = \Lambda(r)\, \Psi(\psi)\, \Phi(\phi) \qquad (3\text{-}7)$$

Substituting Equation (3-7) into Equation (3-6) and dividing the result by Equation (3-7) yields the differential equation

$$\frac{1}{\Lambda(r)} \frac{d}{dr}\left(r^2 \frac{d\Lambda(r)}{dr}\right) + \frac{1}{\Phi \cos\phi} \frac{d}{d\phi}\left(\cos\phi \frac{d\Phi(\phi)}{d\phi}\right)$$
$$+ \frac{1}{\cos^2\phi\, \Psi(\psi)} \frac{d^2\Psi(\psi)}{d\psi^2} = 0 \qquad (3\text{-}8)$$

Since only the first term is dependent on r, it must be (most generally) a constant to satisfy the above differential equation. It is convenient to choose the constant to be of the form $\ell(\ell+1)$, hence

$$\frac{1}{\Lambda(r)} \frac{d}{dr}\left(r^2 \frac{d\Lambda(r)}{dr}\right) = \ell(\ell+1) \tag{3-9}$$

which can be expressed in the form

$$r^2 \frac{d^2\Lambda(r)}{dr^2} + 2r \frac{d\Lambda(r)}{dr} - \ell(\ell+1)\Lambda(r) = 0 \tag{3-10}$$

The agreement between the power of r and the order of the derivative in each term of this equation suggests a solution of the form

$$\Lambda(r) = r^n \tag{3-11}$$

Substituting this form into Equation (3-10) yields the equation

$$n(n-1) + 2n - \ell(\ell+1) = 0 \tag{3-12}$$

which admits the solutions

$$n = \begin{cases} \ell & \text{(3-13a)} \\ -(\ell+1) & \text{(3-13b)} \end{cases}$$

Therefore, the general solution of Equations (3-10) is of the form

$$\Lambda(r) = c_1 r^{\ell} + c_2 r^{-\ell-1} \tag{3-14}$$

where c_1 and c_2 are arbitrary constants. For the gravitational potential,

$$c_1 \equiv 0 \tag{3-15}$$

since the gravitational potential is assumed to vanish at infinity, i.e.,

$$\lim_{r \to \infty} V(r, \theta, \phi) = 0 \tag{3-16}$$

The remainder of the potential function is then determined. Substituting Equation (3-9) into Equation (3-8) and multiplying the result by $\cos^2 \phi$ yields the differential equation

$$\ell(\ell+1) \cos^2 \phi + \frac{\cos \phi}{\Phi} \frac{d}{d\phi} \left(\cos \phi \frac{d\Phi}{d\phi} \right) + \frac{1}{\Psi} \frac{d^2}{d\psi^2} \Psi = 0 \tag{3-17}$$

The last term is clearly constant since it alone depends on ψ. If this constant is denoted by $-m^2$, then

$$\frac{d^2 \Psi}{d\psi^2} + m^2 \Psi(\psi) = 0 \tag{3-18}$$

which is the equation for a harmonic oscillator and which admits the solution

$$\Psi = C \cos m\psi + S \sin m\psi \tag{3-19}$$

where C and S designate arbitrary constants of integration.

The final function $\Phi(\phi)$ is determined by substituting the constant $-m^2$ into Equation (3-17) and multiplying the result by $\Phi/\cos^2\phi$ which yields the differential equation

$$\frac{1}{\cos\phi}\frac{d}{d\phi}\left(\cos\phi\frac{d\Phi}{d\phi}\right)+\left[\ell(\ell+1)-\frac{m^2}{\cos^2\phi}\right]\Phi=0 \qquad (3\text{-}20)$$

Transforming the independent variable in Equation (3-20) through the relation

$$x=\sin\phi \qquad (3\text{-}21)$$

yields the resulting differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\Phi\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]\Phi=0 \qquad (3\text{-}22)$$

For $m=0$, this equation reduces to the classical equation named after Legendre (Reference 30) and which admits as solutions the Legendre polynomials $P_\ell(x)$, i.e.,

$$\Phi(\phi)=c_3P_\ell(x)=C_3P_\ell(\sin\phi) \qquad \text{(for } m=0) \qquad (3\text{-}23)$$

where c_3 is an arbitrary constant. The Legendre polynomial is defined on the interval $-1\leq x\leq 0$ by Rodrigues' formula (Reference 30)

$$P_\ell(x)=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(1-x^2)^\ell \qquad (3\text{-}24)$$

The solution to the more general differential equation given in Equation (3-22) is denoted by

$$\Phi = c_3 P_{\ell,m}(x) = c_3 P_{\ell,m}(\sin\phi) \tag{3-25}$$

and is called the associated Legendre polynomial of degree ℓ and order m. The associated Legendre polynomial is defined on the interval $-1 \le x \le 1$ by Ferrer (Reference 30) to be[1]

$$P_{\ell,m}(x) = (1-x^2)^{m/2} \frac{d^m P_\ell(x)}{dx^m} \tag{3-26}$$

A complete discussion of the general differential equation is given in Reference 30. However, one further point is of interest for this discussion. The general differential equation given in Equation (3-22) can be obtained from Legendre's differential equation for which $m = 0$ by differentiating the latter m times. However, since Legendre's differential equation admits a polynomial solution of degree ℓ, m must be restricted to the range $0 \le m \le \ell$ in order to obtain a nontrivial result.

In view of Equation (3-14), (3-15), (3-19), and (3-25), the potential V depending on the constants ℓ and m and denoted by $V_{\ell,m}$ can be expressed as

$$V^*_{\ell,m} = r^{-\ell-1} P_{\ell,m}(\sin\phi)\left(C^*_{\ell,m}\cos m\psi + S^*_{\ell,m}\sin m\psi\right) \tag{3-27}$$

---

[1] Some authors include the factor $(-1)^m$ in the definition of the associated Legendre polynomial given by Equation (3-26) (Reference 42). These differing definitions have contributed to a certain amount of unnecessary confusion. The effects of this definition difference will be observed in the sign of the spherical harmonic coefficients of odd order m. Reference 30 adopts the notation of $P_{\ell,m}(x)$ for Ferrer's definition and the notation

$$P_\ell^m(x) = (-1)^m P_{\ell,m}(x)$$

for the alternate definition. This convention is observed in this report.

where

$$C^*_{\ell,m} = c_1 c_3 C \tag{3-28a}$$

$$S^*_{\ell,m} = c_1 c_3 S \tag{3-28b}$$

are dimensional spherical harmonic coefficients. Since the dimensions of a gravitational potential function are (length/time)^2, a set of dimensionless spherical harmonic coefficients ($C_{\ell,m}$, $S_{\ell,m}$) are obtained through the definitions

$$C^*_{\ell,m} = \mu a_e^{\ell} C_{\ell,m} \tag{3-29a}$$

$$S^*_{\ell,m} = \mu a_e^{\ell} S_{\ell,m} \tag{3-29b}$$

where μ is the gravitational parameter of the attracting body, defined in terms of the mass of the attracting body, M, and the universal gravitational constant, G, by[1]

$$\mu = GM \tag{3-30}$$

and a_e designates the mean equatorial radius of the central body.

A complete account of the method is given by Fitzpatrick (Reference 43) and a more physical approach is provided by Battin (Reference 44).

The complete general solution is obtained by summing over all admissible values of ℓ and m, which obtains the following final result:

$$V = -\frac{\mu}{r} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} V_{\ell,m} \tag{3-31}$$

[1] Values for the gravitational parameter are better obtained through observation of the satellite mean motion, n, and the semimajor axis of the satellite orbit and through the use of Kepler's Third Law expressed as $n^2 a^3 = \mu$.

where

$$V_{\ell,m} = \left(\frac{a_e}{r}\right)^{\ell} P_{\ell,m}(\sin\phi)\left(C_{\ell,m}\cos m\psi + S_{\ell,m}\sin m\psi\right) \tag{3-32}$$

and where μ = the gravitational parameter = GM

r = the distance of the satellite from the origin of the coordinate system reference frame

a_e = the mean equatorial radius of the attracting body

ϕ = the latitude of the satellite

ψ = the body-fixed longitude of the satellite

$C_{\ell,m}$, $S_{\ell,m}$ = the spherical harmonic coefficients which are determined empirically for a given body

$P_{\ell,m}(x)$ = the associated Legendre polynomial of degree ℓ, order m, and argument x

The term of zero degree and zero order, i.e.,

$$V_{0,0} = -\frac{\mu}{r} \tag{3-33}$$

corresponds to the potential function of the point mass of classical two-body theory. Inspection of the expression in Equation (3-32) indicates that the values for the spherical harmonic coefficients $S_{\ell,0}$ are completely arbitrary and can be taken to be identically zero, i.e.,

$$S_{\ell,0} \equiv 0 \tag{3-34}$$

Furthermore, it can be shown that the spherical harmonic coefficients $C_{1,0}$, $C_{1,1}$, and $S_{1,1}$ vanish identically if the origin of the coordinate system is placed at the center of mass of the attracting body (Reference 43). Consequently, under this condition,

$$V_{10} \equiv 0 \tag{3-35a}$$

$$V_{1,1} \equiv 0 \tag{3-35b}$$

and the nonspherical gravitational potential function takes the form

$$V = -\frac{\mu}{r}\left[1 + \sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a_e}{r}\right)^l P_{l,m}(\sin\phi)\left(C_{l,m}\cos m\psi + S_{l,m}\sin m\psi\right)\right] \tag{3-36}$$

The disturbing function R is obtained by taking the negative of the potential function and deleting the point-mass term to yield

$$R = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a_e}{r}\right)^l P_{l,m}(\sin\phi)\left(C_{l,m}\cos m\psi + S_{l,m}\sin m\psi\right) \tag{3-37}$$

The partial derivatives with respect to the equinoctial elements of the disturbing function are required by the equations of motion. It is desirable to express the disturbing function directly in terms of the equinoctial elements, rather than relying on the application of the chain rule. A complex variable representation of the disturbing function will facilitate the transformation to equinoctial elements. The disturbing function is given by the real part of the expression[1]

$$R^* = \frac{\mu}{r}\sum_{l=2}^{\infty}\sum_{m=0}^{l}\left(\frac{a_e}{r}\right)^l\left(C_{l,m} - jS_{l,m}\right)P_{l,m}(\sin\phi)\,e^{jm(g-\theta)} \tag{3-38}$$

[1] Reference to the real part of the complex variable representation will be dropped until the final form of the potential and its partial derivatives are obtained.

where j is the imaginary unit, $\sqrt{-1}$, and the body-fixed longitude, ψ, has been expressed as the difference of the right ascension of the satellite, α, and the Greenwich Hour Angle or some equivalent angle for a central body other than the Earth.

3.2 TRANSFORMATION TO THE EQUINOCTIAL ELEMENTS

All quantities in the disturbing function which are dependent on the satellite position must be expressed in terms of the equinoctial elements. This transformatio .nplicitly or explicitly requires a rotation of the coordinate reference frame associated with the coordinates (ψ, ϕ) to one of the equinoctial reference frames (direct or retrograde). A general discussion of rotations is presented in Section 2.1.1 and the specific rotation required is discussed in Appendix A of Reference 5.

Inspection of Equation (3-38) indicates that the only quantities dependent on the satellite position are the spherical harmonic functions

$$r^{-\ell-1} P_{\ell,m}(\sin\phi)\, e^{jmg}$$

The general theory of the rotation of the spherical harmonic functions is discussed in Section 2.1.2.

In addition, the radial distance of the satellite, which is invariant under a rotation, must be expressed in terms of the equinoctial elements. This is accomplished through a Fourier series representation in the true, eccentric, or mean longitude. Since sines and cosines of multiples of the true longitude, L, are introduced by the rotation of the coordinate reference system, it is really necessary to develop Fourier series expansions of functions of the form

$$\left(\frac{r}{a}\right)^n \cos sL \; ; \quad \left(\frac{r}{a}\right)^n \sin sL$$

The development of these Fourier series expansions is discussed in Section 2.2.

3.2.1 Rotation of the Spherical Harmonic Functions

The rotation to the equinoctial reference frame can be expressed by the product of three simple rotation matrices (see Appendix A of Reference 5)

$$T = R_3(-I\Omega)\, R_1(i)\, R_3(\Omega) \tag{3-39}$$

where Ω and i are the right ascension of the ascending node and the inclination of the satellite orbit, respectively, and I is the retrograde factor. The rotation matrices R_1 and R_3 are defined by Equations (2-6) and (2-7). The rotation transforms the spherical coordinates (r, g, ϕ) to the spherical coordinates $(r, L, \phi' \equiv 0)$. The satellite latitude is identically zero in the equinoctial reference system since the fundamental plane is the orbital plane.

There are two approaches to the transformation of the spherical harmonic functions. Kaula (Reference 16) used a brute-force substitution of the Keplerian elements for the right ascension and latitude of the satellite.[1] Expressions were obtained of the form (in complex variables)

$$P_{\ell,m}(\sin\phi)\, e^{jmg} = \sum_{p=0}^{\ell} j^{\ell-m} F_{\ell,m,p}(i)\, e^{j[(\ell-2p)(f+\omega)+m\Omega]} \tag{3-40}$$

where f is the true anomaly and the inclination function, $F_{\ell,m,p}(i)$ is defined to be

$$F_{\ell,m,p}(i) = \sum_{t} \frac{(2\ell-2t)!\, \sin^{\ell-m-2t}(i)}{t!\,(\ell-t)!\,(\ell-m-2t)!\, 2^{2\ell-2t}} \sum_{s=0}^{m} \binom{m}{s} \cos^{s} i \times \sum_{c} \binom{\ell-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \tag{3-41}$$

---

[1] This substitution can be obtained from Equations (2-22) and (2-23) by designating λ as the right ascension and λ' as the true anomaly, i.e., $\lambda' = f$.

where k is the integer part of $(\ell - m)/2$, the summation index, t, varies through the range $0 \le t \le \min(p, k)$, and c is summed over all values for which the binomial coefficients are defined.

Iszak (Reference 45) and Allan (Reference 46) simplified the expression for the inclination function by using the inclination half angle to obtain[1]

$$F_{\ell,m,p}(i/2) = \frac{(\ell+m)!}{2^{\ell}\, p!\,(\ell-p)!} \sum_{k=k_1}^{k_2} \left[(-1)^k \binom{2\ell-2p}{k} \binom{2p}{\ell-m-k} \times \cos^{3\ell-m-2p-2k}(i/2)\, \sin^{m-\ell+2p+2k}(i/2) \right] \tag{3-42}$$

where $k_1 = \max(0, \ell - m - 2p)$
$k_2 = \min(2\ell - 2p, \ell - m)$
$j = \sqrt{-1}$

To facilitate the development of a simple recursive scheme for evaluating the inclination functions, Cefola (Reference 13) suggested the alternate approach based on the theory of the rotation of the spherical harmonic functions presented in Section 2. The rotation of the spherical harmonic functions to the equinoctial reference frame takes the form (Reference 25)

$$P_{\ell,m}(\sin\phi)\, e^{jmg} = \sum_{s=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, S_{2\ell}^{m,s}(\rho,\sigma,\tau)\, e^{jsL} \tag{3-43}$$

[1] This particular expression was obtained by Iszak; Allan's expression incorporates the factor $j^{\ell-m}$ in the above definition.

since the latitude of the satellite in the equinoctial reference frame, ϕ', is identically zero. The parameters ρ, σ, and τ for this case will be defined shortly.

It should be noted that the function $P_{\ell,s}(0)$ is a constant depending only on the indexes ℓ and s.

It follows from Equation (8.6.1) of Reference 26 and from the footnote on page 3-7 that

$$P_{\ell,s}(0) = \frac{(-2)^s}{\sqrt{\pi}} \cos\left[(\ell+s)\frac{\pi}{2}\right] \frac{\Gamma\left[\frac{\ell+s+1}{2}\right]}{\Gamma\left[\frac{\ell-s}{2}+1\right]} \tag{3-44}$$

(This expression is valid for all integers s.)

It is obvious that for odd values of $\ell + s$,

$$P_{\ell,s}(0) \equiv 0 \tag{3-45}$$

Using the relations

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\Gamma\left(\frac{1}{2}\right) \tag{3-46a}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{3-46b}$$

it is easily shown that

$$P_{\ell,s}(0) = (-1)^{(\ell-s)/2}\frac{(\ell+s-1)!!}{(\ell-s)!!} \tag{3-47}$$

provided the definition

$$(-1)!! = 1$$

is made. An alternate definition which avoids the double factorial notation is

$$P_{\ell,s}(0) = (-1)^{(\ell-s)/2} \frac{(\ell+s)!}{2^{\ell}\left(\frac{\ell+s}{2}\right)!\left(\frac{\ell-s}{2}\right)!} \tag{3-48}$$

Since only terms which satisfy the condition $\ell \pm s = 2p$ contribute to the summation in Equation (3-43), the range $-\ell \leq s \leq \ell$ can be replaced by the range $0 \leq p \leq \ell$ to yield

$$\sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{\ell} \frac{(\ell-s)!}{(\ell-m)!} S_{2\ell}^{m,s}(\rho,\sigma,\tau) P_{\ell,s}(0) e^{jsL} = \sum_{p=0}^{\ell} \frac{(2p)!}{(\ell-m)!} S_{2\ell}^{m,\ell-2p}(\rho,\sigma,\tau) P_{\ell,\ell-2p}(0) e^{j(\ell-2p)L} \tag{3-49}$$

This modification is of no significance for machine processing and will not be adopted in this report. However, it does demonstrate the relationship between the present formulation and the standard Kaula approach given by Equation (3-40).

3.2.1.1 Determination of the Function $S_{2\ell}^{m,s}(\rho, \sigma, \tau)$

To obtain the appropriate form of the function $S_{2\ell}^{m,s}(\rho,\sigma,\tau)$, it is necessary to determine the parameters ρ, σ, and τ in terms of the Euler angles (Ω', i', ω'), which describe the orientation of the equatorial reference system relative to the equinoctial reference systems. The rotation from the equatorial to the equinoctial reference systems, which was derived in Appendix A of Reference 5, is given by Equation (3-13).

For the purpose of this discussion, the inverse transformation is required, i.e.,

$$T^{-1} = R_3^{-1}(\Omega)\, R_1^{-1}(i)\, R_3^{-1}(-I\Omega) \qquad (3\text{-}50)$$

Since the constituent rotation matrices, $R_1(\theta)$ and $R_3(\theta)$, are orthogonal, it follows that

$$R^{-1}(\theta) = R^T(\theta) = R(-\theta) \qquad (3\text{-}51)$$

Consequently,

$$T^{-1} = R_3(-\Omega)\, R_1(-i)\, R_3(I\Omega) \qquad (3\text{-}52)$$

Comparison of this result with Equation (2-5) indicates that, for this case, the Euler angles are

$$\omega' = -\Omega \qquad (3\text{-}53a)$$

$$i' = -i \qquad (3\text{-}53b)$$

$$\Omega' = I\Omega \qquad (3\text{-}53c)$$

The primed symbols denote the general Euler angles and the unprimed symbols denote the familiar Keplerian orbital elements. The parameters ρ, σ, and τ are thus obtained from Equations (2-16), i.e.,

$$\sigma = \frac{\Omega' - \omega'}{2} = \frac{(I+1)\Omega}{2} \qquad (3\text{-}54a)$$

$$\rho = \frac{\Omega' + \omega'}{2} = \frac{(I-1)\Omega}{2} \tag{3-54b}$$

$$\tau = \frac{i'}{2} = -\frac{i}{2} \tag{3-54c}$$

These expressions can be substituted into Equations (2-44) to obtain the function $S_{2\ell}^{m,s}$; however, it is just as easy to substitute the above Euler angles directly into Equation (2-45), which yields the result

$$S_{2\ell}^{m,s} = \exp\left[j(s-m)\frac{\pi}{2}\right] \exp\left[j(m-Is)\Omega\right]$$

$$\times \begin{cases} C_{i/2}^{-m-s} S_{i/2}^{m-s} P_{\ell+s}^{m-s,-m-s}(C_i) & (-\ell \leq s \leq -m) \quad (3\text{-}55a) \\ \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!} C_{i/2}^{m+s} S_{i/2}^{m-s} P_{\ell-m}^{m-s,-m+s}(C_i) & (-m \leq s \leq m) \quad (3\text{-}55b) \\ (-1)^{s-m} C_{i/2}^{m+s} S_{i/2}^{s-m} P_{\ell-s}^{s-m,m+s}(C_i) & (m \leq s \leq \ell) \quad (3\text{-}55c) \end{cases}$$

The substitutions

$$C_{-x} = C_x$$

$$S_{-x} = -S_x$$

were used to obtain the above results.

The collapsed form of the inclination function is obtained by substituting the Euler angles (Equations (3-54)) into Equation (2-50) to yield

$$S^{*\,m,s}_{2\ell} = \exp\left[j(\epsilon s - m)\frac{\pi}{2}\right] \exp\left[j(m - \epsilon I s)\Omega\right] \epsilon^{\ell+m}$$

$$\times \begin{cases} \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!} C_{i/2}^{m+\epsilon s} S_{i/2}^{m-\epsilon s} P_{\ell-m}^{m-s,\,m+s}(\epsilon C_i) & (0 \le s \le m) \quad (3\text{-}56a) \\[2ex] (-1)^{s-m} C_{i/2}^{s+\epsilon m} S_{i/2}^{s-\epsilon m} P_{\ell-s}^{s-m,\,m+s}(\epsilon C_i) & (m \le s \le \ell) \quad (3\text{-}56b) \end{cases}$$

where ϵ assumes the values $\epsilon = \pm 1$.

The inclination functions can be expressed in terms of either the equinoctial elements p and q or the direction cosines, with respect to the equinoctial reference system, of the z axis of the equatorial system. The relationship between the direction cosines (α, β, γ), the elements p and q, and the Keplerian elements Ω and i are

$$\alpha = \hat{f} \cdot \hat{z} = \frac{-2pI}{1+p^2+q^2} = -I S_i S_\Omega \qquad (3\text{-}57a)$$

$$\beta = \hat{g} \cdot \hat{z} = \frac{2q}{1+p^2+q^2} = S_i C_\Omega \qquad (3\text{-}57b)$$

$$\gamma = \hat{w} \cdot \hat{z} = \frac{(1-p^2-q^2)I}{1+p^2+q^2} = C_i \qquad (3\text{-}57c)$$

where, clearly,

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

These expressions are easily obtained from the transformation matrix in Appendix A of Reference 5.

3.2.1.1.1 The Function $S_{2\ell}^{m,s}$ in Terms of the Direction Cosines α, β, γ

It follows from the definition of the direction cosines that

$$e^{j\Omega} = \frac{\beta - jI\alpha}{\sqrt{1-\gamma^2}} \tag{3-58}$$

By direct substitution and some algebraic manipulation, it can be shown that

$$\exp\left[j(s-m)\frac{\pi}{2}\right] \exp\left[j(m-Is)\Omega\right] = I^m (-1)^{m-Is} \left(\frac{\alpha + jI\beta}{\sqrt{1-\gamma^2}}\right)^{m-Is} \tag{3-59}$$

Using the relations

$$C_{i/2} = \sqrt{\frac{1+\gamma}{2}} \tag{3-60a}$$

$$S_{i/2} = \sqrt{\frac{1-\gamma}{2}} \tag{3-60b}$$

$$\sqrt{\frac{1-\gamma}{1+\gamma}} = \frac{\sqrt{1-\gamma^2}}{1+\gamma} \tag{3-60c}$$

it can also be shown that

$$C_{i/2}^{-m-s}\, S_{i/2}^{m-s} = 2^{s}\,(1+\gamma)^{-m}\left(\sqrt{1-\gamma^{2}}\right)^{m-s} \tag{3-61a}$$

$$C_{i/2}^{m+s}\, S_{i/2}^{m-s} = 2^{-m}\,(1+\gamma)^{s}\left(\sqrt{1-\gamma^{2}}\right)^{m-s} \tag{3-61b}$$

$$C_{i/2}^{m+s}\, S_{i/2}^{s-m} = 2^{-s}\,(1+\gamma)^{m}\left(\sqrt{1-\gamma^{2}}\right)^{s-m} \tag{3-61c}$$

Substituting Equations (3-59) and (3-61) into Equations (3-55), using the relation

$$\left(\frac{\alpha+j\beta}{\sqrt{1-\gamma^{2}}}\right)^{-1} = \left(\frac{\alpha-j\beta}{\sqrt{1-\gamma^{2}}}\right) \tag{3-62}$$

and examining the direct and retrograde cases yields (after some algebraic manipulation) the following final expression:

$$S_{2\ell}^{m,s} = \begin{cases} I^{m}(-1)^{m-s}\,2^{s}\,(\alpha+j\beta)^{Im-s}\,(1+I\gamma)^{-Im}\,P_{\ell+s}^{m-s,-m-s}(\gamma) & (-\ell \le s \le -m) \quad (3\text{-}63a) \\ I^{m}(-1)^{m-s}\,2^{-m}\,\dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!}\,(\alpha+jI\beta)^{m-Is}\,(1+I\gamma)^{Is}\,P_{\ell-m}^{m-s,m+s}(\gamma) & (-m \le s \le m) \quad (3\text{-}63b) \\ I^{m}\,2^{-s}\,(\alpha-j\beta)^{s-Im}\,(1+I\gamma)^{Im}\,P_{\ell-s}^{s-m,s+m}(\gamma) & (m \le s \le \ell) \quad (3\text{-}63c) \end{cases}$$

Similarly, the collapsed form of the function $S_{2\ell}^{m,s}$ takes the form

$$S_{2\ell}^{*m,s,\epsilon} = I^{m} \epsilon^{\ell+m}$$

$$\times \begin{cases} (-1)^{m-s} \, 2^{-m} \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!} (\alpha + jI\beta)^{m-\epsilon Is} (1+I\gamma)^{\epsilon Is} \, P_{\ell-m}^{m-s,\,m+s}(\epsilon\gamma) & (0 \le s \le m) \quad (3\text{-}64a) \\ 2^{-s} (\alpha - j\epsilon\beta)^{s-\epsilon Im} (1+I\gamma)^{\epsilon Im} \, P_{\ell-s}^{s-m,\,s+m}(\epsilon\gamma) & (m \le s \le \ell) \quad (3\text{-}64b) \end{cases}$$

3.2.1.1.2 The Function $S_{2\ell}^{m,s}$ in Terms of the Equinoctial Elements p and q

It follows from Equations (3-57) that

$$\alpha \pm j\beta = \frac{-2I}{1+p^2+q^2} (p \mp jIq) \tag{3-65a}$$

$$\alpha + jI\beta = \frac{-2I}{1+p^2+q^2} (p - jq) \tag{3-65b}$$

$$1 + I\gamma = \frac{2}{1+p^2+q^2} \tag{3-65c}$$

Substituting these expressions into Equations (3-63) and (3-64), respectively, and simplifying yields the expressions

$$S_{2\ell}^{m,s} = \begin{cases} I^{s} (p - jIq)^{Im-s} (1+p^2+q^2)^{s} \, P_{\ell+s}^{m-s,\,-m-s}(\gamma) & (-\ell \le s \le -m) \quad (3\text{-}66a) \\ I^{s} \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!} (p - jq)^{m-Is} (1+p^2+q^2)^{-m} \, P_{\ell-m}^{m-s,\,m+s}(\gamma) & (-m \le s \le m) \quad (3\text{-}66b) \\ I^{s} (-1)^{m-s} (p + jIq)^{s-Im} (1+p^2+q^2)^{-s} \, P_{\ell-s}^{s-m,\,s+m}(\gamma) & (m \le s \le \ell) \quad (3\text{-}66c) \end{cases}$$

$$
S_{2\ell}^{*m,s,\epsilon} = \begin{cases} I^{s}\dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+s)!\,(\ell-s)!}\, a^{\ell+m}\,(1+p^2+q^2)^{-m}\,(p-jq)^{m-\epsilon I s}\, P_{\ell-m}^{m-s,\,m+s}(\epsilon\gamma) & (0 \le s \le m) \quad \text{(3-67a)} \\[2ex] I^{s}\, a^{\ell+m}\,(-1)^{s-m}\,(1+p^2+q^2)^{-s}\,(p+j\epsilon I q)^{s-\epsilon I m}\, P_{\ell-s}^{s-m,\,s+m}(\gamma) & (m \le s \le \ell) \quad \text{(3-67b)} \end{cases}
$$

This completes the discussion on the development of the function $S_{2\ell}^{m,s}$.

The disturbing function in terms of the rotated spherical harmonic functions is obtained by substituting Equation (3-43) into Equation (3-38), which yields

$$
R^{*} = \frac{\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{\ell} \left(\frac{a_e}{r}\right)^{\ell} \left(C_{\ell,m} - j\,S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!}\, P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, e^{j(sL-m\theta)} \qquad \text{(3-68)}
$$

The collapsed form of the disturbing function is obtained by substituting Equation (2-52) (with $\lambda = g - \theta$ and $\phi' = 0$) into Equation (3-38) to yield

$$
R^{*} = \frac{\mu}{r} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\substack{s=0 \\ [\ell \pm s \text{ even}]}}^{\ell} \delta_s\, \epsilon^{s} \left(\frac{a_e}{r}\right)^{\ell} \left(C_{\ell,m} - j\,S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!}\, P_{\ell}^{s}(0)\, S_{2\ell}^{*m,s,\epsilon}\, e^{j(\epsilon sL-m\theta)} \qquad \text{(3-69)}
$$

where ϵ assumes the values ± 1 and δ_s is defined in Equation (2-49).

3.2.2 Expansion of the Function $[(1/r^{\ell+1})(e^{jsL})]$

The representation of the disturbing function in equinoctial elements is completed by using a Fourier series expansion for the factors

$$
\frac{1}{r^{\ell+1}}\, e^{jsL} = \frac{1}{a^{\ell+1}} \left(\frac{a}{r}\right)^{\ell+1} e^{jsL} \qquad \text{(3-70)}
$$

3.2.2.1 Choice of the Expansion Variable

Any one of the mean, eccentric, and true longitudes can be chosen as the expansion variable. In most applications, the mean longitude is chosen since it is a simple linear function of the time which is the natural independent variable. The mean longitude is therefore well suited for generating ephemerides or any application where the time history of the elements is closely monitored.

Use of either the true or the eccentric longitude in the disturbing function with time as the independent variable in the equations of motion requires that more complicated expressions relating the chosen longitude to the time be evaluated at every step. If the eccentric longitude is chosen, then Kepler's equation

$$\lambda = F - k \sin F + h \cos F \tag{3-71}$$

must be evaluated. If the true longitude is chosen, then the expression

$$\tan\left[\frac{L - \tan^{-1}\left(\frac{h}{k}\right)}{2}\right] = \sqrt{\frac{1+e}{1-e}}\,\tan\left[\frac{F - \tan^{-1}\left(\frac{h}{k}\right)}{2}\right] \tag{3-72}$$

must be evaluated in addition to the evaluation of Kepler's equation.

Transforming the independent variable in the equations of motion to the desired longitude will remove the necessity of evaluating these expressions, but the relation to time is no longer apparent. If the explicit time dependence is required, one or both of the above expressions must be inverted, depending on the longitude. The iversions of Kepler's equation requires an iterative procedure, e.g., the Newton-Raphson method, and is considerably more expensive than simply evaluating the expression. Consequently, the eccentric and true longitudes as independent variables are best suited for those applications where knowledge of the time is not required or, at least, is required infrequently.

In addition to the above considerations, the particular characteristics of the Fourier expansion must be considered. In Section 2.2, Fourier expansions for the function

$$\left(\frac{r}{a}\right)^n e^{jsL}$$

were developed in terms of the mean, eccentric, and true longitudes. For the nonspherical gravitational disturbing function, n is a negative integer, i.e.,

$$n = -(\ell+1)$$

For negative values of n, the expansion in the mean longitude introduces the infinite series

$$\left(\frac{r}{a}\right)^{-(\ell+1)} e^{jsL} = \sum_{q=-\infty}^{\infty} Y_q^{-\ell-1,s} e^{jq\lambda} \tag{3-73}$$

where the modified Hansen coefficients, $Y_q^{-\ell-1,s}$, (defined in Section 2 in Equation (2-305)) are infinite power series in either β^2 or e^2, both of which are functions of h and k. The expansion in the eccentric longitude also introduces a similar infinite series.

In contrast, for negative n, the expansion in the true longitude introduces the finite expression

$$\left(\frac{r}{a}\right)^{-(\ell+1)} e^{jsL} = \sum_{q=-(\ell+1)}^{\ell+1} (k-j\eta h)^{|q|} V_q^{-(\ell+1)} e^{j(q+s)L} \tag{3-74}$$

Consequently, when choosing the expansion variable, the simple time-mean longitude relation and the infinite Fourier series representation must be weighed against the more complicated time-true longitude relation and a closed-form Fourier series representation. Expansion of the nonspherical gravitational disturbing function in the eccentric longitude offers the worst of both of the other alternatives, thus introducing Kepler's equation as well as an infinite Fourier series in the development. If the complete nonspherical gravitational disturbing function is required, the true longitude appears to be the better choice for moderate to large eccentricity satellites and the mean longitude is the better choice for small eccentricity satellites.

However, in the averaged disturbing function, all short-period contributions are to be eliminated, regardless of the particular formulation used. In the absence of resonance, only the constant term of the Fourier series should survive the averaging process.[1] Hence, any advantage in one expansion over the others depends ultimately on whether the constant term in one of the expansions has a computational advantage over the constant terms in either of the other two expansions. This discussion in Section 2.2 gives simple relations between the constant terms in the three Fourier expansions (see Equations (2-211) and (2-212)), i.e.,

$$X_0^{2n,s} = W_0^{2n-1,s} = \frac{V_s^{2n-2}}{\sqrt{1-e^2}} \tag{3-75}$$

and, therefore, no one constant term possesses a significant computational advantage over the other two. Thus, in the absence of resonance, insofar as the averaged nonspherical gravitational disturbing function is concerned, it makes no difference whether the true or mean longitude is chosen for the expansion.[2]

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[1]This, however, requires certain assumptions about the perturbation model. This is discussed in more detail in Section 3.3.

[2]However, for the formulation of the first-order short-period variations in the osculating elements (Reference 5, Section 4), which requires the short-periodic part of the disturbing function, the finite formulation afforded by the true longitude expansion is very important in all but the very-near circular cases.

If the possiblity of resonance phenomena is considered, the choice of the expansion variable is no longer arbitrary. The mean longitude (anomaly) must be selected for the Fourier series expansion if the resonant contributions are to be isolated. This is because a resonance occurs when the ratio of the mean motion of the satellite, n, to the central body rotation rate, ω, is very nearly the ratio of two small integers, i.e.,

$$\frac{n}{\omega} \approx \frac{N}{N'} \tag{3-76}$$

This causes the term

$$N'\lambda - N\theta = \mu(t) \tag{3-77}$$

referred to as the critical term, to vary quite slowly, thus introducing a very long-period, λ-dependent component to the motion instead of the usual short-period contribution produced in the absence of resonance. There is apparently no corresponding formulation of the critical argument in the true or eccentric longitudes (anomalies).

3.2.2.2 Introduction of the Fourier Expansion in the Mean Longitude

Since the averaged equations of motion are to be developed for the resonant tesseral harmonics in addition to the zonal and nonresonant tesseral harmonics, the Fourier expansion in the mean longitude is required. Substituting Equation (3-73) into Equation (3-68) yields the following complete expansion of the disturbing function:

$$R^* = \frac{\mu}{a} \sum_{\substack{\ell=2}}^{\infty} \sum_{\substack{m=0}}^{\ell} \sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{\ell} \sum_{q=-\infty}^{\infty} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!}$$
$$\times P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, Y_q^{-\ell-1,s}\, e^{j(q\lambda - m\theta)} \tag{3-78}$$

The corresponding collapsed expression for the disturbing function, where $0 \leq s \leq \ell$ and $0 \leq q < \infty$, follows immediately by replacing s by ϵs and q by νq in the above expression, where the parameters ϵ and ν are mutually independent and assume the values

$$\epsilon = \begin{cases} -1 & (s < 0) \\ +1 & (s \geq 0) \end{cases} \tag{3-79a}$$

$$\nu = \begin{cases} -1 & (q < 0) \\ +1 & (q \geq 0) \end{cases} \tag{3-79b}$$

and the quantities s and q are restricted to be nonnegative integers. In view of Equation (2-52), the resulting expression for the collapsed disturbing function is

$$R^* = \frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{\substack{m=0 \\ [\ell \pm s \text{ even}]}}^{\ell} \sum_{s=0}^{\ell} \sum_{q=0}^{\infty} \sum_{\epsilon=\pm 1} \sum_{\nu=\pm 1} \epsilon^{s} \delta_s \delta_q \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - j S_{\ell,m}) \times \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) S_{2\ell}^{*ms\epsilon} Y_{\nu q}^{-\ell-1,\epsilon s} e^{j(\nu q \lambda - m\theta)} \tag{3-80}$$

where δ_s and δ_q are defined by

$$\delta_k = \begin{cases} 1/2 & (k = 0) \\ 1 & (k > 0) \end{cases} \tag{3-81}$$

The above expression is simplified by substituting the values of ϵ and ν in the factors

$$\epsilon^{s} S_{2\ell}^{*m,s,\epsilon} Y_{\nu q}^{-\ell-1,\epsilon s} e^{j(\nu q \lambda - m\theta)} \tag{3-82}$$

and using the definition (Equation (2-305))

$$Y_{\nu q}^{-\ell-1,\,\epsilon s} = (k - j\eta h)^{|\epsilon s - \nu q|}\, K_{\nu q}^{-\ell-1,\,\epsilon s} \tag{3-83}$$

where $\eta = \mathrm{sgn}(\nu q - \epsilon s)$ and

$$K_{-q}^{-\ell-1,-s} = K_q^{-\ell-1,\,s} \tag{3-84}$$

which follows from Equations (2-134) and (2-303). The final result can be expressed in terms of ϵ and ν as

$$\begin{aligned}\epsilon^s S_{2\ell}^{*m,s,\epsilon}\, Y_{\nu q}^{-\ell-1,\,\epsilon s}\, e^{j(\nu q\lambda - m\theta)} \\ = \epsilon^s S_{2\ell}^{*m,s,\epsilon} (k - j\eta h)^{|\epsilon(s-\nu q)|}\, K_q^{-\ell-1,\,\nu s}\, e^{j(\epsilon\nu q\lambda - m\theta)}\end{aligned} \tag{3-85}$$

where $\eta = \mathrm{sgn}[\epsilon(\nu q - s)]$.

Substituting the right-hand side of Equation (3-85) into Equation (3-80) yields the final expression for the collapsed nonspherical gravitational disturbing function, i.e.,

$$\begin{aligned}R^* = \frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{q=0}^{\infty} \sum_{\epsilon=\pm 1} \sum_{\nu=\pm 1} \epsilon^s \delta_s \delta_q \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - j S_{\ell,m}) \\ \times \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, S_{2\ell}^{*m,s,\epsilon} (k - j\eta h)^{|\epsilon(s-\nu q)|}\, K_q^{-\ell-1,\,\nu s}\, e^{j(\epsilon\nu q\lambda - m\theta)}\end{aligned} \tag{3-86}$$

where $\eta = \epsilon\,\mathrm{sgn}(\nu q - s)$.

3.3 THE AVERAGED DISTURBING FUNCTION

This section discusses the application of the averaging operation to the nonspherical gravitational disturbing function. The concepts of time-independent and time-dependent averaging are discussed and related to the classical assumptions of a stationary central body and of exact resonance. The discussion of resonance is extended to include the phenomenon of near resonance.

Also presented are the time independently averaged disturbing functions for the zonal harmonic, the combined zonal and tesseral harmonic, and the reduced resonant tesseral harmonic fields.

3.3.1 Application of the Averaging Operation

The first-order averaged equations of motion require that the disturbing function, expressed in terms of the mean elements $(\bar{\vec{a}}, \bar{\lambda})$, be averaged over an appropriate interval to remove all short-period components. The development of these equations was discussed in Section 3.2 of Reference 4. In the absence of resonance phenomena, the appropriate form of the averaging operation is

$$\left\langle R(\bar{\vec{a}},\bar{\lambda})\right\rangle_{\bar{\lambda}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} R(\bar{\vec{a}},\bar{\lambda})\, d\bar{\lambda} \tag{3-87}$$

However, in the case of resonance, this averaging operation is strictly valid only for properly reduced force models consisting of quasi-isolated resonant terms in the disturbing function.[1] If the force model is not properly reduced, the averaging operation should be defined as

$$\left\langle R(\bar{\vec{a}},\bar{\lambda})\right\rangle_{\bar{\lambda}} = \frac{1}{2\pi N}\int_{-N\pi}^{N\pi} R(\bar{\vec{a}},\bar{\lambda})\, d\bar{\lambda} \tag{3-88}$$

where N is the number of satellite revolutions performed during the smallest period common to both fast variables (in this case, the satellite mean-mean

[1] See Section 3.4 of Reference 4 for a detailed discussion.

longitude and the Greenwich Hour Angle or its equivalent describing the central body rotation). For example, for the case of 2:1 resonance (i.e., $n/\omega = 2/1$), $N = 2$ and for the case of 7:4 resonance, $N = 7$.

In practice, the averaging interval will not generally be centered around the origin of longitudes, i.e., $\bar{\lambda} = 0$, as implied by Equations (3-87) and (3-88), but will, in fact, be centered about the value of the mean-mean longitude associated with the time of the numerical integration step, $\bar{\lambda}_0$. Consequently, Equation (3-87) is more correctly expressed as

$$\left\langle R(\vec{a},\bar{\lambda})\right\rangle_{\bar{\lambda}} = \frac{1}{2\pi}\int_{\lambda_0-\pi}^{\lambda_0+\pi} R(\vec{a},\bar{\lambda})\, d\lambda \tag{3-89}$$

The same modification is also appropriate for Equation (3-88).

This dependence on the value of the mean-mean longitude at the integration step time has important implications, particularly for the numerical averaging method. This is discussed in Section 3.3.1.2.

One form of the averaged disturbing function is obtained by applying Equation (3-89) to Equation (3-78), which yields

$$\left\langle R(\vec{a},\bar{\lambda})\right\rangle_{\bar{\lambda}} = \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} \frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{\ell} \sum_{q=-\infty}^{\infty} \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - j S_{\ell,m}) \tag{3-90}$$

$$\times \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, Y_q^{-\ell-1,s}\, e^{j(q\bar{\lambda}-m\theta)}$$

Clearly, only the imaginary exponential function is dependent on the mean-mean longitude, $\bar{\lambda}$, and, therefore,

$$\left\langle R(\bar{\vec{a}},\bar{\lambda})\right\rangle_{\bar{\lambda}} = \frac{\mu}{\bar{a}} \sum_{\substack{\ell=2}}^{\infty} \sum_{\substack{m=0}}^{\ell} \sum_{\substack{s=-\ell \\ [\ell-s \text{ even}]}}^{\ell} \sum_{q=-\infty}^{\infty} \left(\frac{a_e}{\bar{a}}\right)^{\ell} \left(C_{\ell,m} - j\, S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!} \tag{3-91}$$

$$\times\, P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, Y_q^{-\ell-1,s} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j(q\bar{\lambda}-m\theta)}\, d\bar{\lambda}$$

3.3.1.1 Evaluation of the Averaging Integral

The evaluation of the above definite integral is straightforward. The classical approach has been to assume that either θ is completely independent of λ, which tacitly requires a constant Greenwich Hour Angle for Earth satellites, or that the ratio of the central body rotation rate to the mean-mean motion of the satellite is in the ratio of two integers, i.e., exact resonance.

The first approach of holding θ constant over the averaging interval during the averaging operation will be referred to as time-independent averaging. However, since the value of θ is to be evaluated at each integration step, the effects of the rotation of the central body are introduced in the long-term satellite motion.

The second classical assumption of exact resonance is a special case of time-dependent averaging. Specifically, the Greenwich Hour Angle, θ, (or its equivalent for some central body other than the Earth) is permitted to vary during the averaging operation according to the constraint

$$\frac{\dot{\theta}}{\bar{n}} = \frac{N'}{N} \tag{3-92}$$

or, equivalently,

$$j\lambda - m\theta = 0 \tag{3-93}$$

where the integers j and m are multiples of the integers N' and N, respectively, i.e.,

$$j = kN' \tag{3-94a}$$

$$m = kN \tag{3-94b}$$

In practice, neither of the classical assumptions may be strictly valid for a particular satellite. In the following discussion, the more general time-dependent averaging approach is used in evaluating the definite integral in Equation (3-91). Each of the special cases corresponding to the classical assumptions is then deduced from the general result. Viewing the special cases against the background for the general result provides additional insight into the application of the method of averaging.

The method of averaging requires the disturbing function to be averaged over some time interval, in this application, the mean revolution period of the satellite. Because of the simple relation between the mean-mean longitude, $\bar{\lambda}$, and the time, this requirement is easily translated to averaging over $\bar{\lambda}$ on the interval $[0, 2\pi]$. However, the time dependence of other parameters, i.e., the Greenwich Hour Angle (or its equivalent) should not be discounted simply because the averaging operation is defined in terms of $\bar{\lambda}$ instead of time.

The Greenwich Hour Angle (or its equivalent) is easily expressed in terms of $\bar{\lambda}$ to accommodate the $\bar{\lambda}$-form of the averaging operation. First, θ can be expressed as a function of time through the relation

$$\theta = \omega t + \theta_0 \tag{3-95}$$

where ω is the rotation rate of the central body, considered to be constant over the averaging interval, and θ_0 is the apparent Greenwich Hour Angle at the integration step time. Equation (3-95) ignores the effects of precession and nutation on the value of θ, over the averaging interval, except at the center of the interval.

For the purposes of the following discussion, it is assumed that the ratio $\omega/\bar{n}$ is constant with respect to the time. This assumption is not inconsistent with the basic assumption of the method of averaging, i.e., that the slowly varying elements remain constant over the averaging interval.

The mean-mean longitude of the satellite is expressed explicitly in the time by

$$\bar{\lambda} = \bar{n}t + \bar{\lambda}_0 \tag{3-96}$$

where $\bar{n}$ is the mean-mean motion of the satellite defined by

$$\bar{n} = \sqrt{\frac{\mu}{\bar{a}^3}} \tag{3-97}$$

The quantity μ is the gravitational parameter of the central body and $\bar{a}$ is the mean semimajor axis. Hence, the mean motion, $\bar{n}$, is either constant or slowly varying, depending on the exact perturbations.

Eliminating the time between Equation (3-95) and (3-96), i.e.,

$$\frac{\theta - \theta_0}{\omega} = t = \frac{\bar{\lambda} - \bar{\lambda}_0}{\bar{n}} \tag{3-98}$$

yields the relation

$$\theta = \frac{\omega}{\bar{n}}(\lambda - \lambda_0) + \theta_0 \tag{3-99}$$

The special cases of a constant Greenwich Hour Angle and exact resonance correspond, respectively, to the conditions

$$\frac{\omega}{\bar{n}} = 0 \tag{3-100a}$$

and

$$\frac{\omega}{\bar{n}} = \frac{N'}{N} \tag{3-100b}$$

where N and N' are integers.

Substituting Equation (3-99) into the definite integral in Equation (3-91) yields the expression

$$\frac{1}{2\pi}\int_{\lambda_0-\pi}^{\lambda_0+\pi} e^{j(q\bar{\lambda}-m\theta)}\,d\bar{\lambda} = \frac{1}{2\pi}\, e^{-jm\left(\theta_0-\frac{\omega}{\bar{n}}\lambda_0\right)} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j\left(q-m\frac{\omega}{\bar{n}}\right)\bar{\lambda}}\,d\bar{\lambda} \tag{3-101}$$

Evaluation of the right-hand side of Equation (3-101) yields the general result

$$\frac{1}{2\pi}\int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j(q\bar{\lambda}-m\theta)}\,d\bar{\lambda} = \begin{cases} 1 & \text{(for } q = m = 0 \text{ or } q = m\omega/\bar{n}) \quad \text{(3-102a)} \\[2ex] \dfrac{\sin\left[\left(q-m\frac{\omega}{\bar{n}}\right)\pi\right]}{\left(q-m\frac{\omega}{\bar{n}}\right)\pi}\, e^{j(q\lambda_0-m\theta_0)} & \text{(otherwise)} \quad \text{(3-102b)} \end{cases}$$

Inspection of this result indicates that the time-dependent averaging operation does not generally remove all satellite-dependent ($\bar{\lambda}$-dependent) short-period terms in the disturbing function. Nor does it remove the θ-dependent medium- and short-period contributions except in the case of resonance. (This last observation is, or course, to be expected.) The residual short-period terms shown in Equation (3-102b) possess the periods

$$\frac{\tau}{q\bar{n} - m\omega} \tag{3-103}$$

where τ is the mean period of the satellite, i.e.,

$$\tau = \frac{2\pi}{\bar{n}} \tag{3-104}$$

Their contributions at the integration step time, i.e., $\bar{\lambda} = \bar{\lambda}_0$, may survive the averaging operation. The sine factor in Equation (3-102b), referred to as the averaging factor, determines whether in fact the residual short-period effects really persist after the averaging operation and, if so, their degree of significance.

3.3.1.2 The Averaging Factor

It is apparent that the averaging factor acts to suppress both the amplitudes of the residual satellite-dependent short-period terms and the amplitudes of the θ-dependent medium- and short-period terms, since

$$\left| \frac{\sin\left[\left(q - m\frac{\omega}{\bar{n}}\right)\pi\right]}{\left(q - m\frac{\omega}{\bar{n}}\right)\pi} \right| \leq 1 \tag{3-105}$$

is always satisfied. The degree to which the averaging factor actually suppresses the residual periodic terms depends, among other things, on the period of the term.

Inspection of the right-hand sides of Equations (3-101) and (3-102) shows that frequency of a given term is proportional to the denominator of the averaging factor. Hence, the averaging factor is more effective in suppressing shorter period terms in general. The effect of the averaging factor is discussed separately for the zonal and tesseral harmonic terms.

3.3.1.2.1 The Averaging Factor for the Zonal Harmonic Terms

The zonal harmonic terms in the disturbing function are those for which $m = 0$. Consequently, the averaging factor in Equation (3-102b) reduces to

$$1 \qquad \text{(for } q = 0) \qquad (3\text{-}106a)$$

$$\frac{\sin(q\pi)}{q\pi} \equiv 0 \qquad \text{(for } q \neq 0) \qquad (3\text{-}106b)$$

The case where $q = 0$ corresponds to those terms which contribute only to the long-period and secular motion of the satellite. The case $q \neq 0$ corresponds to the λ-dependent short-period terms which are completely suppressed by the averaging factor.

3.3.1.2.2 The Averaging Factor for the Tesseral Harmonic Terms

The tesseral harmonic terms (including the sectoral terms) in the disturbing function are those for which $m \neq 0$. The discussion for the tesseral harmonic terms is presented separately for the m-daily nonresonant terms ($q = 0$), the general nonresonant terms, and the resonant terms for which

$$\frac{q'}{m'} = \frac{\omega}{\bar{n}}$$

is satisfied.

3.3.1.2.2.1 The Averaging Factor for the m-Daily Terms

The m-daily terms in the tesseral harmonic field are those terms for which $q = 0$. The periods of the m-daily terms are given by

$$\frac{\tau'}{m} \tag{3-107}$$

where τ' is the rotation period of the central body. Hence, the m-daily terms produce effects in the satellite motion with a frequency of m cycles per day (rotation period).

Substituting $q = 0$ in the expression for the averaging factor in Equation (3-102b) yields the averaging factor for the m-daily terms, which is

$$\frac{\sin\left[\frac{\omega}{\bar{n}} m\pi\right]}{\frac{\omega}{\bar{n}} m\pi} \tag{3-108}$$

Clearly,

$$\left| \frac{\sin\left[\frac{\omega}{\bar{n}} m\pi\right]}{\frac{\omega}{\bar{n}} m\pi} \right| \begin{cases} = 1 & \left(\text{for } \frac{\omega}{\bar{n}} = 0\right) \quad (3\text{-}109\text{a}) \\ \leq 1 & \left(\text{for } \frac{\omega}{\bar{n}} \neq 0\right) \quad (3\text{-}109\text{b}) \end{cases}$$

Equation (3-109a) is easily obtained by using the theory of limits on Equation (3-108) or substituting the values $q = 0$ and $\omega/\bar{n} = 0$ directly into the right-hand side of Equation (3-101) before evaluating the definite integral. Equation (3-109b) follows from the properties of the sine function.

Equation (3-109a) corresponds to the classical assumption of a stationary central body. In view of Equations (3-109b), the significance of the m-daily terms in the averaged disturbing function is inversely proportional to the rotation rate of the central body and directly proportional to the mean-mean motion of the satellite. It is not surprising that the m-daily effects become less significant as the central body rotation period approaches the satellite mean revolution period. Furthermore, for supersynchronous satellites, the m-daily effects become much less significant than the $\bar{\lambda}$-dependent short-period terms.

In addition, for a fixed ratio $\omega/\bar{n}$, inspection of Equation (3-109) indicates that the m-daily terms become less significant for increasing order, i.e., increasing m. This is expected, since the higher order m-daily terms are of shorter periods.

For close-Earth satellites, the difference in the two averaging factors (Equation 3-109) is negligible for the low-order m-daily effects, i.e., m = 1, 2, 3. However, the discrepancy grows dramatically, percentage-wise, as the order m increases. Fortunately, the amplitudes of these high-order m-daily terms decrease rapidly. Thus, although the large percentage errors contribute much smaller absolute errors, the effects of each high-order m-daily term is significantly corrupted by using the averaging factor for the time-independent averaging theory.

More importantly, there exists a cutoff value of m where the two averaging factors produce a discrepancy in the sign of the term, thus introducing a phase error of π radians. This is easily demonstrated as follows. If it is assumed that the ratio $\omega/\bar{n}$ is bounded by the reciprocals of the integers k and k+1, i.e.,

$$\frac{1}{k+1} < \frac{\omega}{\bar{n}} < \frac{1}{k}$$

then

$$\operatorname{sgn}\left[\sin\left(\frac{\omega}{\bar{n}} m\pi\right)\right] = \begin{cases} 1 & \text{for } m \leq k \\ -1 & \text{for } m > k \end{cases}$$

3.3.1.2.2.2 Nonresonant Tesseral Harmonic Terms Excluding the m-Daily Terms

Excluding the m-daily terms, the remaining nonresonant tesseral harmonic terms are those for which $q \neq 0$ and $m \neq 0$. The corresponding averaging factor takes the general form given in Equation (3-102b). This can also be expressed as

$$\frac{\sin\left[\left(\frac{q}{m} - \frac{\omega}{\bar{n}}\right) m\pi\right]}{\left(\frac{q}{m} - \frac{\omega}{\bar{n}}\right) m\pi} \tag{3-110}$$

For the classical assumption $\omega/\bar{n} = 0$, the averaging factor reduces to

$$\frac{\sin q\pi}{q\pi} \equiv 0 \tag{3-111}$$

Thus, it suppresses all residual nonresonant, λ-dependent, short-period terms in the averaged disturbing function analogous to the case of the zonal harmonic terms. In the general case for which $\omega/\bar{n} \neq 0$, the averaging factor in Equation (3-110) is appropriate and can be expressed in the form

$$\frac{\sin\left[\left(\frac{q}{m} - \frac{\omega}{\bar{n}}\right) m\pi\right]}{\left(\frac{q}{m} - \frac{\omega}{\bar{n}}\right) m\pi} = (-1)^{q+1} \frac{\sin\left(\frac{\omega}{\bar{n}} m\pi\right)}{\left(\frac{q}{m} - \frac{\omega}{\bar{n}}\right) m\pi} \tag{3-112}$$

3.3.1.2.2.3 Resonant Tesseral Harmonic Terms

Exact Resonance

For the case of exact resonance, i.e., where

$$\frac{\omega}{\bar{n}} = \frac{N'}{N}$$

the averaging factor given in Equation (3-110) takes the form

$$\frac{\sin\left(\frac{q}{m} - \frac{N'}{N}\right)m\pi}{\left(\frac{q}{m} - \frac{N'}{N}\right)m\pi} \tag{3-113}$$

Those terms in the disturbing function for which

$$q = kN' \tag{3-114a}$$

$$m = kN \tag{3-114b}$$

produce the resonance phenomena. The averaging factor for this limiting case is unity.

The discussion for the averaging factor in the case of exact resonance is broadened to include not only the pure resonant terms, but also the quasi-isolated and embedded resonant terms discussed in Section 3.4 of Reference 5.

The entire nonspherical gravitational disturbing function can be considered to be an embedded resonant term in a general sense. The expansion of the disturbing function in the orbital elements results in a proliferation of terms as shown in Section 3.2. More particularly, the imaginary exponential function which contains all the periodic information depends generally on two indexes q and m. The index q is introduced by the expansion in the mean longitude, and the index m is the order of a given spherical harmonic term in the nonspherical gravitational disturbing function. These indexes can assume any value in the range $-\infty < q < \infty$ and $0 \leq m \leq \ell$, where ℓ is the degree of the given spherical harmonic term.

A quasi-isolated resonant term is any term or group of terms in the disturbing function for which Equation (3-114b) alone is satisfied. The range of the indexes in the imaginary exponential function is then $-\infty < q < \infty$ and $m = kN$ (where N is called the order of the resonance). Quasi-isolated resonant terms are a subset of the embedded resonant term. The pure resonant terms are those quasi-isolated resonant terms for which Equation (3-114a) is also satisfied and are therefore a subset of the quasi-isolated resonant terms.

For quasi-isolated resonant terms, the averaging factor assumes the form and values

$$\frac{\sin(q - kN')\pi}{(q - kN')\pi} = \begin{cases} 1 & \text{for } q = kN' \\ 0 & \text{for } q \neq kN' \end{cases} \qquad (3\text{-}115)$$

This result is obtained by substituting Equation (3-114b) into Equation (3-113). Consequently, the pure resonant terms for which $q = kN'$ are completely transparent to the averaging operation and the averaging operation completely suppresses all other quasi-isolated resonant terms for which $q \neq kN'$.

For the remaining terms in the embedded resonant term, i.e., the disturbing function, the averaging factor retains the form given in Equation (3-113). It will tend to reduce the effects of the corresponding residual short-period terms but will never completely suppress these contributions.

However, if the averaging operation defined in Equation (3-88) and centered at $\overline{\lambda}_0$ is used, it is easily verified that the corresponding averaging factor takes the form

$$\frac{\sin\left[\left(\frac{q}{m} - \frac{N'}{N}\right) m N \pi\right]}{\left(\frac{q}{m} - \frac{N'}{N}\right) m N \pi} \qquad (3\text{-}116)$$

Clearly, this averaging factor vanishes for all cases where the argument of the sine function assumes a nonzero integral multiple of π, i.e., if

$$qN - mN' = k \qquad \text{(for } k \neq 0\text{)} \qquad (3\text{-}117)$$

This condition is satisfied by all possible values for the indexes q and m, with the exception of those that yield k = 0, which specify the pure resonant terms. Consequently, the averaging operation based on Equation (3-88) completely suppresses all nonresonant terms in a general embedded resonant "term."

This fact is of little significance for analytical theories in which the pure resonance terms are easily isolated. However, the significance cannot be overstated for numerical averaging applications. Failure to properly reduce the force model to only the quasi-isolated terms requires an increase by a factor of N in the cost of the averaging operation, if all residual short-period terms are to be completely suppressed.

Near Resonance

Exact resonance is a limiting case which is seldom, if ever, encountered. However, near resonance, defined by the constraint

$$\frac{\omega}{\bar{n}} = \frac{N'}{N} + \epsilon \qquad (3\text{-}118)$$

is of considerable practical importance. Substituting the condition of near resonance into the expression for the averaging factor in Equation (3-113) yields the expression

$$\frac{\sin\left[\left(\frac{q}{m} - \frac{N'}{N} - \epsilon\right) m\pi\right]}{\left(\frac{q}{m} - \frac{N'}{N} - \epsilon\right) m\pi} \qquad (3\text{-}119)$$

The pure near-resonant terms are still specified by Equations (3-114) and the averaging factor reduces to

$$\frac{\sin \epsilon k N \pi}{\epsilon k N \pi} \tag{3-120}$$

The periods of these near-resonant terms are given by the expression

$$\frac{\tau}{\epsilon k N \pi} \tag{3-121}$$

Clearly, as ϵ approaches zero, i.e., as the resonance becomes deeper, the averaging factor approaches unity and the periods of the near-resonant terms become longer. Conversely, as ϵ grows larger, the resonance becomes shallower; the periods of the near-resonant terms grow shorter; and the averaging factor decreases in magnitude, resulting in the greater suppression of the amplitudes of the near-resonant terms.

The quasi-isolated near-resonant terms are restricted to those terms which satisfy the constraint given by Equation (3-114b) as in the case for exact resonance. Substitution of this constraint into Equation (3-119) yields the averaging factor for the quasi-isolated near-resonant terms, which simplifies to

$$\frac{\sin (q - kN - \epsilon kN)\pi}{(q - kN - \epsilon kN)\pi} = (-1)^{q-kN} \frac{\sin \epsilon kN\pi}{[(1+\epsilon)kN - q]\pi} \tag{3-122}$$

For $q = kN$, this expression reduces to the expression given in Equation (3-120) for the pure near-resonant terms.

Generally, Equation (3-122) indicates that the residual short-period contributions from the averaged quasi-isolated near-resonant term will be quite small since the numerator is quite small for small ϵ and small values of kN. For larger values of kN, the denominator grows larger, which helps to suppress the residual short-period contributions.

In summary, the analysis of the averaging operation and, particular , the averaging factor shows that time-dependent averaging methods can fail to eliminate all short-period contributions in the averaged resonant and nonresonant tesseral harmonic disturbing functions. In contrast, the classical assumptions either of a stationary central body during the averaging interval (time-independent averaging) or of an exact resonance do permit the complete elimination of all short-period terms. However, these assumptions can also produce an overestimate of the actual amplitudes of the long-period (resonance cases) and medium-period m-daily terms which survive the averaging operation.

This circumstance presents something of a dilemma. For the analytical averaging method, the residual short-period effects can be suppressed by simply deleting them from the force model. However, this is, in essence, equivalent to inserting the averaging factor for the general time-dependent averaging operation into the time independently averaged tesseral harmonics models (resonant and nonresonant). The method of numerical averaging is, however, not amenable to such an accommodation, since the averaging factor differs for each term in the fully expanded disturbing function and since numerical averaging methods in general avoid this fully expanded disturbing function representation.

Generally, the choice for both analytical and numerical averaging methods seems quite clear. The nonresonant tesseral harmonics terms should be deleted from the perturbation model, both to avoid the present dilemma and, more importantly, to maximize the numerical integration step size as discussed in Section 3.4 of Reference 5 or alternatively, to minimize the averaging interval. These contributions can be evaluated from analytical formulas (Reference 49) and superimposed on the integrated mean elements when necessary.

If, however, these averaged nonresonant tesseral harmonic terms are retained in the perturbation model in numerical averaging applications, the error (usually small) in the amplitudes of the m-daily effects introduced by the time-independent averaging operation is definitely preferable to the noise generated by the residual short-period terms[1] introduced by the time-dependent averaging operation or to the small step size required to accurately numerically integrate these residual short-period contributions.

For the case of tesseral resonance, time-dependent averaging is necessary. The residual short-period contributions are easily eliminated by restricting the force model to quasi-isolated resonant terms alone or by averaging full force models over the appropriate number of satellite revolutions.

3.3.2 The Averaged Zonal Harmonic Disturbing Function

The averaged zonal harmonic disturbing function consists of those terms in the averaged nonspherical gravitational disturbing function for which $m = 0$. Restricting the value of m accordingly in Equation (3-91) yields one form for the zonal harmonic disturbing function. If $J_\ell = -C_{\ell,0}$, then

$$\left\langle R^{z} \right\rangle_{/\bar{\lambda}} = -\frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{s=-\ell}^{\ell} \sum_{q=-\infty}^{\infty} J_\ell \left(\frac{a_e}{a}\right)^{\ell} \frac{(\ell-s)!}{\ell!} \tag{3-123}$$

$$\times\; P_{\ell,s}(0)\, S_{2\ell}^{0,s}\, Y_q^{-\ell-1,s} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{jq\bar{\lambda}}\, d\bar{\lambda}$$

Clearly,

$$\int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{jq\bar{\lambda}}\, d\bar{\lambda} = \delta_{q,0} = \begin{cases} 1 & \text{for } q = 0 \qquad \text{(3-124a)} \\ 0 & \text{for } q \neq 0 \qquad \text{(3-214b)} \end{cases}$$

[1] Short-period contributions which are propagated with large step sizes appear as spurious noise in the integrated results.

which agrees with Equations (3-102) and (3-106). Thus, the averaged zonal harmonic disturbing function simplifies to

$$\left\langle R^{z} \right\rangle_{\bar{\lambda}} = -\frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{s=-\ell}^{\ell} J_{\ell} \left(\frac{a_e}{a}\right)^{\ell} \frac{(\ell-s)!}{\ell!} P_{\ell,s}(0)\, S_{2\ell}^{0,s}\, Y_0^{-\ell-1,s} \tag{3-125}$$

In view of the discussion in Section 2.1.2.4, the summation over the index s for the range $-\ell \leq s \leq \ell$ is easily collapsed to a summation over the range $0 \leq s \leq \ell$ to give

$$\left\langle R^{z} \right\rangle_{\bar{\lambda}} = -\frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{s=0}^{\ell} \delta_s \epsilon^{s} J_{\ell} \left(\frac{a_e}{a}\right)^{\ell} \frac{(\ell-s)!}{\ell!} P_{\ell,s}(0)\, S_{2\ell}^{*\,0,s,\epsilon}\, Y_0^{-\ell-1,\epsilon s} \tag{3-126}$$

where

$$\delta_s = \begin{cases} 1/2 & \text{for } s=0 \\ 1 & \text{for } s \neq 0 \end{cases} \tag{3-127}$$

The inclination function is defined by Equation (3-64) or (3-67). Evaluation of Equations (3-64) for $m = 0$ yields

$$S_{2\ell}^{*\,0,s,\epsilon} = \epsilon^{\ell}\, 2^{-s} (\alpha - j\epsilon\beta)^{s}\, P_{\ell-s}^{s,s}(\epsilon\gamma) \qquad (\text{for } 0 \leq s \leq \ell) \tag{3-128}$$

Several simplifications can be made in Equations (3-126) and (3-128). First, from Equation (2-305),

$$Y_0^{-\ell-1,\epsilon s} = (k - j\eta h)^{|\epsilon s|}\, K_0^{-\ell-1,\epsilon s} \tag{3-129}$$

where

$$\eta = \operatorname{sgn}(-\epsilon s) = -\epsilon \tag{3-130}$$

Also, it follows from Equations (2-134) and (2-303) that

$$K_0^{-\ell-1,\epsilon s} = K_0^{-\ell-1,s} \qquad (\text{for } \epsilon = \pm 1) \tag{3-131}$$

Therefore,

$$Y_0^{-\ell-1,\epsilon s} = (k + j\epsilon h)^s \, K_0^{-\ell-1,s} \tag{3-132}$$

In addition, it follows from Equations (2-303) that

$$K_0^{-\ell-1,s} = e^{-s} \, X_0^{-\ell-1,s} \tag{3-133}$$

Thus

$$Y_0^{-\ell-1,\epsilon s} = (k + j\epsilon h)^s \, e^{-s} \, X_0^{-\ell-1,s} \tag{3-134}$$

Second, the Jacobi polynomial representation of the inclination function for the zonal harmonics case given in Equation (3-128) can be simplified since

$$P_{\ell-s}^{s,s}(\epsilon\gamma) = \epsilon^{\ell-s} \, P_{\ell-s}^{s,s}(\gamma) \tag{3-135}$$

and since it is easily shown (Reference 26) that

$$P_{\ell-s}^{s,s}(\gamma) = 2^s \frac{\ell!}{(\ell+s)!} (1-\gamma^2)^{-s/2} \, P_{\ell,s}(\gamma) \tag{3-136}$$

Therefore, the inclination function for the zonal harmonic terms takes the form

$$S_{2\ell}^{*0,s,\epsilon} = \epsilon^{-s} \frac{\ell!}{(\ell+s)!} \left(\frac{\alpha - j\epsilon\beta}{\sqrt{1-\gamma^2}} \right)^s P_{\ell,s}(\gamma) \tag{3-137}$$

Substituting Equations (3-134) and (3-137) into Equation (3-126) yields the expression

$$\begin{aligned} \left\langle R_z \right\rangle_{\bar{\lambda}} = -\frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell\pm s \text{ even}]}}^{\infty} \sum_{s=0}^{\ell} \sum_{\epsilon=\pm1} \delta_s J_\ell \left(\frac{a_e}{a}\right)^\ell \frac{(\ell-s)!}{(\ell+s)!} P_{\ell,s}(0)\, P_{\ell,s}(\gamma) \\ \times \left(\frac{\alpha - j\epsilon\beta}{\sqrt{1-\gamma^2}} \right)^s \left(\frac{k + j\epsilon h}{e} \right)^s X_0^{-\ell-1,s} \end{aligned} \tag{3-138}$$

Since

$$\operatorname{Re}\left\{ \frac{x + jy}{\sqrt{x^2+y^2}} \right\} = \operatorname{Re}\left\{ \frac{x - jy}{\sqrt{x^2+y^2}} \right\} \tag{3-139}$$

(where Re(z) designates the real part of z), then

$$\operatorname{Re}\left\{ \sum_{\epsilon=\pm1} \left(\frac{\alpha - j\epsilon\beta}{\sqrt{1-\gamma^2}} \right)^s \left(\frac{k + j\epsilon h}{e} \right)^s \right\} = \operatorname{Re}\left\{ 2 \left(\frac{\alpha - j\beta}{\sqrt{1-\gamma^2}} \right)^s \left(\frac{k + jh}{e} \right)^s \right\} \tag{3-140}$$

and Equation (3-138) is further simplified to the form

$$\left\langle R^2 \right\rangle_{/\bar{\lambda}} = -\frac{2\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{s=0}^{\ell} \delta_s J_\ell \left(\frac{a_e}{a}\right)^\ell \frac{(\ell-s)!}{(\ell+s)!} P_{\ell,s}(0) P_{\ell,s}(\gamma) \tag{3-141}$$

$$\times \left(\frac{\alpha - j\beta}{\sqrt{1-\gamma^2}}\right)^s \left(\frac{k+jh}{e}\right)^s X_0^{-\ell-1,s}$$

since only the real part of this expression is of importance. Furthermore, the possibility of vanishing divisors is eliminated through Equation (3-133) and the following definition

$$Q_{\ell,s}(\gamma) = (1-\gamma^2)^{-s/2} P_{\ell,s}(\gamma) \tag{3-142}$$

The function $Q_{\ell,s}(\gamma)$ is a polynomial in γ and is simply the sth derivative of the Legendre polynomial $P_\ell(\gamma)$ (Reference 26), i.e.,

$$Q_{\ell,s}(\gamma) = \frac{d^s}{d\gamma^s} P_\ell(\gamma) \tag{3-143}$$

Substituting Equations (3-133) and (3-142) into Equation (3-112) and defining

$$V_{\ell,s} = \frac{(\ell-s)!}{(\ell+s)!} P_{\ell,s}(0) \tag{3-144}$$

yields the following expression for the zonal harmonic disturbing function:

$$\left\langle R^z \right\rangle_{\bar{\lambda}} = -\frac{2\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{s=0}^{\ell} \delta_s J_\ell \left(\frac{a_e}{a}\right)^\ell V_{\ell,s} Q_{\ell,s}(\gamma) K_0^{-\ell-1,s} (\alpha - j\beta)^s (k+jh)^s \tag{3-145}$$

This form of the averaged zonal harmonic disturbing function will be used in the development of the averaged equations of motion presented in Section 3.4.1.

3.3.3 The Averaged Zonal and Nonresonant Tesseral Harmonic Disturbing Function[1]

The combined averaged zonal and nonresonant tesseral harmonic disturbing function consists of all nonresonant terms in the averaged nonspherical gravitational disturbing function given in Equation (3-91). For the purpose of this discussion, it is assumed that no resonance exists.[2] The disturbing function is

$$\left\langle R^{Z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{q=-\infty}^{\infty} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!}$$

$$\times\, P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, Y_q^{-\ell-1,s} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j(q\bar{\lambda}-m\theta)}\, d\bar{\lambda} \tag{3-146}$$

To avoid residual short-period terms in the averaged disturbing function, time-independent averaging is used. Therefore,

$$\int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j(q\bar{\lambda}-m\theta)}\, d\bar{\lambda} = \begin{cases} \delta_{q,0} & \text{(for } m = 0) \\ e^{-jm\theta_0} & \text{(for } q = 0) \\ 0 & \text{(for } q \neq 0,\ m \neq 0) \end{cases} \tag{3-147}$$

$$= \delta_{|q-m|}\, e^{-jm\theta_0}$$

[1] The averaged zonal and nonresonant tesseral harmonic disturbing functions are combined to facilitate computational efficiency when both models are requested.

[2] The combination of resonant and nonresonant spherical harmonic terms in the equations of motion is not generally warranted because of the relative insignificance of the nonresonant terms and the unacceptably small step sizes they impose on the numerical integration procedure.

Consequently, the time independently averaged zonal and nonresonant tesseral harmonic disturbing function takes the form

$$\left\langle R^{Z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j\, S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!} \times P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, Y_0^{-\ell-1,s}\, e^{-jm\theta_0} \tag{3-148}$$

since for $q \neq 0$

$$\delta_{|q-m|} = 0 \tag{3-149}$$

The averaging factor is unity for all nonvanishing terms and is an over estimate of the more realistic averaging factor obtained from the time-dependent averaging operation and which is given in Equation (3-109). This time-dependent averaging factor is used to obtain the correct amplitudes of the m-daily terms.

In view of Equations (3-86), (3-147), and (3-149), the collapsed expression for the averaged zonal and nonresonant tesseral harmonic disturbing function, including the correct averaging factor for the m-daily terms, takes the form

$$\left\langle R^{Z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell \pm s \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{\epsilon=\pm 1} \epsilon^{s}\, \delta_s \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j\, S_{\ell,m}\right) \frac{(\ell-s)!}{(\ell-m)!} \times P_{\ell,s}(0)\, S_{2\ell}^{m,s} (k + j\epsilon h)^{s}\, K_0^{-\ell-1,s} \left\{ \begin{matrix} 1 \\ \dfrac{\sin\left[\frac{\omega}{n} m\pi\right]}{\frac{\omega}{n} m\pi} \end{matrix} \right\} e^{-jm\theta_0} \tag{3-150}$$

where the sum over ν has been performed and where the inclination function is defined by Equation (3-64) to be

$$S^{m,s,\epsilon}_{\ell\ell} = I^m \epsilon^{\ell+m} \begin{cases} (-1)^{m-s} 2^{-m} \dfrac{(\ell+m)!}{(\ell+s)!} \dfrac{(\ell-m)!}{(\ell-s)!} (\alpha+jI\beta)^{m-\epsilon Is} (1+I\gamma)^{\epsilon Is} P^{m-s,m+s}_{\ell-m}(\epsilon\gamma) & (0 \le s \le m-1) \\ 2^{-s} (\alpha-j\epsilon\beta)^{s-\epsilon Im} (1+I\gamma)^{\epsilon Im} P^{s-m,s+m}_{\ell-s}(\epsilon\gamma) & (m \le s \le \ell) \end{cases} \tag{3-151}$$

Substituting this expression into Equation (3-150) yields the expression

$$\begin{aligned}
\left\langle R^{\ell+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\epsilon=\pm1} \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - jS_{\ell,m}) I^m \epsilon^m \left\{ \frac{\sin\left[\frac{\omega}{n} m\pi\right]}{\frac{\omega}{n} m\pi} \right\} e^{-jm\theta_0} \\
\times \Bigg[\sum_{\substack{s=0 \\ [\ell\pm s \text{ even}]}}^{m-1} \epsilon^{\ell+s} \delta_s (-1)^{m-s} 2^{-m} \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) \\
\times (\alpha+jI\beta)^{m-\epsilon Is} (k+j\epsilon h)^s K_0^{-\ell-1,s} (1+I\gamma)^{\epsilon Is} P^{m-s,m+s}_{\ell-m}(\epsilon\gamma) \\
+ \sum_{\substack{s=m \\ [\ell\pm s \text{ even}]}}^{\ell} \epsilon^{\ell+s} \delta_s 2^{-s} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) (\alpha-j\epsilon\beta)^{s-\epsilon Im} (k+j\epsilon h)^s \\
\times K_0^{-\ell-1,s} (1+I\gamma)^{\epsilon Im} P^{s-m,s+m}_{\ell-s}(\epsilon\gamma) \Bigg]
\end{aligned} \tag{3-152}$$

Since $\ell \pm s$ is restricted to take on only even values, the factor $\epsilon^{\ell+s}$ is always unity and can be deleted from the above expression. This form of the averaged disturbing function will be used in the development of the averaged equations of motion presented in Section 3.4.2.

3.3.4 The Averaged Resonant Tesseral Harmonic Disturbing Function

The averaged resonant tesseral harmonic disturbing function is isolated from the averaged nonspherical gravitational disturbing function given in Equation (3-91) using the near-resonance constraint

$$\frac{\omega}{\bar{n}} = \frac{N'}{N} - \delta \tag{3-153}$$

or

$$N'\bar{n} - N\omega = \delta \tag{3-154}$$

In order to avoid confusion with the notation $\epsilon = \pm 1$, the symbol δ is used in the above equations instead of the ϵ used in Section 3.3.1.2.

Since

$$\bar{\lambda} = \int_0^t \bar{n}\, dt + \bar{\lambda}_0 \tag{3-155}$$

and

$$\theta = \int_0^t \omega\, dt + \theta_0 \tag{3-156}$$

it then follows that

$$N'\lambda - N\theta = \mu(t) \tag{3-157}$$

where, most generally,

$$\mu(t) = \int_0^t \delta(t)\, dt + \mu_0 \tag{3-158}$$

and is referred to as the critical argument.

Consequently, the resonant harmonic terms in the averaged nonspherical gravitational disturbing function in Equation (3-91) are those for which

$$q = kN' \tag{3-159a}$$

$$m = kN \tag{3-159b}$$

are satisfied. Thus, the averaged resonant tesseral harmonic disturbing function is given by

$$\left\langle R^{RT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\substack{\ell=2 \\ [\ell+s \text{ even}]}}^{\infty} \sum_{s=-\ell}^{\ell} \sum_{k=1}^{k'} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,kN} - j S_{\ell,kN}\right) \frac{(\ell-s)!}{(\ell-kN)!}$$
$$\times P_{\ell,s}(0)\, S_{2\ell}^{kN,s}\, Y_{kN'}^{-\ell-1,s}\, e^{jk(N'\lambda_0 - N\theta_0)} \tag{3-160}$$

where k' is the integer part of ℓ/N.

The argument of the imaginary exponential function is constant for exact resonance. For near resonance, it assumes the role of the critical argument in Equation (3-157). The averaging factor for exact resonance, which has a value of unity, is assumed above. If the resonance is not a very deep resonance, the appropriate averaging factor

$$\frac{\sin \delta kN\pi}{\delta kN\pi} \tag{3-161}$$

(where δ is defined by Equation (3-153)) should be used. The inclination function is defined exactly as in Equation (3-63) or (3-66), with m replaced by kN.

In order to accommodate the user's desire to specify the particular terms of degree ℓ and order m to be included in the nonspherical gravitational disturbing function, it is convenient to express the averaged resonant disturbing function for a specific degree and order as follows

$$\left\langle R^{RT}_{\ell,m} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\substack{s=-\ell \\ [\ell+s \text{ even}]}}^{\ell} \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - j\, S_{\ell,m}) \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, S^{m,s}_{2\ell} \times Y^{-\ell-1,s}_{kN'} \frac{\sin(\delta kN\pi)}{\delta kN\pi}\, e^{j(kN'\lambda_0 - m\theta_0)} \tag{3-162}$$

where m satisfies Equation (3-159b). The averaging factor in Equations (3-161) (if used) assumes the form

$$\frac{\sin(\delta m\pi)}{\delta m\pi} \tag{3-163}$$

Substituting the expressions for the function $S_{2\ell}^{m,s}$ (Equation (3-63)), the modified Hansen coefficients, $Y_{kN'}^{-\ell-1,s}$, and the appropriate averaging factor given in Equation (3-161) into Equation (3-162) yields the result

$$
\begin{aligned}
\left\langle R_{\ell,m}^{RT} \right\rangle_{\bar{\lambda}} = {} & \frac{\mu}{a}\left(\frac{a_e}{a}\right)^{\ell}\left(C_{\ell,m} - j\,S_{\ell,m}\right)\frac{\sin(\delta m\pi)}{\delta m\pi}\, e^{j(kN'\lambda - m\theta)}\; I^{m} \\
& \times \Bigg[\sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{-m-1} (-1)^{m-s}\, 2^{s}\, \frac{(\ell-s)!}{(\ell-m)!}\, P_{\ell,s}(0)\,(\alpha + j\beta)^{Im-s} \\
& \times (k - j\eta h)^{|s-kN'|}\, K_{kN'}^{-\ell-1,s}\,(1+I\gamma)^{-Im}\, P_{\ell+s}^{m-s,-m-s}(\gamma) \\
& + \sum_{\substack{s=-m \\ [\ell \pm s \text{ even}]}}^{m} (-1)^{m-s}\, 2^{-m}\, \frac{(\ell+m)!}{(\ell+s)!}\, P_{\ell,s}(0)\,(\alpha + jI\beta)^{m-Is} \qquad (3\text{-}164) \\
& \times (k - j\eta h)^{|s-kN'|}\, K_{kN'}^{-\ell-1,s}\,(1+I\gamma)^{Is}\, P_{\ell-m}^{m-s,m+s}(\gamma) \\
& + \sum_{\substack{s=m+1 \\ [\ell \pm s \text{ even}]}}^{\ell} 2^{-s}\, \frac{(\ell-s)!}{(\ell-m)!}\, P_{\ell,s}(0)\,(\alpha - j\beta)^{s-Im}\,(k - j\eta h)^{|s-kN'|} \\
& \times K_{kN'}^{-\ell-1,s}\,(1+I\gamma)^{Im}\, P_{\ell-s}^{s-m,s+m}(\gamma) \Bigg]
\end{aligned}
$$

where $\eta = \mathrm{sgn}(kN' - s)$.

Expressing the first summation and the first half of the second summation in Equation (3-164) over positive values of the index s (using Equation (2-51)) yields the form

$$
\begin{aligned}
\left\langle R^{RT}_{\ell,m} \right\rangle_{\bar{\lambda}} = {} & \frac{\mu}{a}\left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j\,S_{\ell,m}\right) \frac{\sin(\delta m \pi)}{\delta m \pi}\, e^{j(kN'\lambda - m\theta)}\, I^m \\
& \times \Bigg[\sum_{\substack{s=m+1 \\ [\ell-s \text{ even}]}}^{\ell} (-1)^m\, 2^{-s} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(\alpha + jI\beta\right)^{m+Is} \\
& \times (k - j\eta h)^{|-s-kN'|}\, K^{-\ell-1,-s}_{kN'}\, (1+I\gamma)^{-Im}\, P^{m+s,\,s-m}_{\ell-s}(\gamma) \\
& + \sum_{\substack{s=1 \\ [\ell-s \text{ even}]}}^{m} (-2)^{-m} \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) \left(\alpha + jI\beta\right)^{m+Is} (k - j\eta h)^{|-s-kN'|} \\
& \times K^{-\ell-1,s}_{kN'}\, (1+I\gamma)^{-Is}\, P^{m+s,\,m-s}_{\ell-m}(\gamma) \qquad (3\text{-}165) \\
& + \sum_{\substack{s=0 \\ [\ell-s \text{ even}]}}^{m} (-2)^{-m} (-1)^s \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) \left(\alpha + jI\beta\right)^{m-Is} \\
& \times (k - j\eta h)^{|s-kN'|}\, K^{-\ell-1,s}_{kN'}\, (1+I\gamma)^{Is}\, P^{m-s,\,m+s}_{\ell-m}(\gamma) \\
& + \sum_{\substack{s=m+1 \\ [\ell-s \text{ even}]}}^{\ell} 2^{-s} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(\alpha - j\beta\right)^{s-Im} (k - j\eta h)^{|s-kN'|} \\
& \times K^{-\ell-1,s}_{kN'}\, (1+I\gamma)^{Im}\, P^{s-m,\,s+m}_{\ell-s}(\gamma) \Bigg]
\end{aligned}
$$

This equation can be simplified by defining $\epsilon = -1$ for $s \leq 0$ and $\epsilon = +1$ for $s \geq 0$, which yields the result

$$\left\langle R^{RT}_{\ell,m} \right\rangle_{\bar{\lambda}} = \sum_{\epsilon=\pm 1} \frac{\mu}{a} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j S_{\ell,m}\right) \frac{\sin(\delta m \pi)}{\delta m \pi} e^{j(kN'\lambda - m\theta)} \epsilon^{m} I^{m}$$

$$\times \Bigg[\sum_{\substack{s=0 \\ [\ell \pm s \text{ even}]}}^{m} \delta_s \cancelto{1}{\epsilon^{\ell+s}} (-1)^{m-s} 2^{-m} \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) \left(\alpha + j I \beta\right)^{m - \epsilon I s}$$

$$\times \left(k - j\eta h\right)^{|\epsilon s - kN'|} K_{kN'}^{-\ell-1,\epsilon s} \left(1 + I\gamma\right)^{\epsilon I s} P_{\ell-m}^{m-s,m+s}(\epsilon\gamma) \qquad (3\text{-}166)$$

$$+ \sum_{\substack{s=m+1 \\ [\ell \pm s \text{ even}]}}^{\ell} \cancelto{1}{\epsilon^{\ell+s}} 2^{-s} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(\alpha - j\epsilon\beta\right)^{s - \epsilon I m} \left(k - j\eta h\right)^{|\epsilon s - kN'|}$$

$$\times K_{kN'}^{-\ell-1,\epsilon s} \left(1 + I\gamma\right)^{\epsilon I m} P_{\ell-s}^{s-m,s+m}(\epsilon\gamma) \Bigg]$$

where $\eta = \text{sgn}(kN' - \epsilon s)$.

This result is identical in form to that of the nonresonant tesseral harmonic case given in Equation (3-152), with the exception of the factors

$$\left(k - j\eta h\right)^{|\epsilon s - kN'|} K_{kN'}^{-\ell-1,\epsilon s}$$

which are a generalization of the equivalent factors in the nonresonant case.

Equation (3-165) is used in the development of the averaged equations of motion for the case of resonant tesseral harmonics presented in Section 3.4.3.

3.4 THE FIRST-ORDER AVERAGED EQUATIONS OF MOTION

The first-order averaged equations of motion for the nonspherical gravitational perturbation are based on a modified version of Lagrange's Planetary Equations (discussed in Sections 2 and 3 of Reference 5). These equations expressed in terms of the mean equinoctial elements take the form

$$\frac{da}{dt} = \frac{2a}{A}\frac{\partial R}{\partial \lambda} \tag{3-167a}$$

$$\frac{dh}{dt} = \frac{B}{A}\left(\frac{\partial R}{\partial k} - \frac{h}{1+B}\frac{\partial R}{\partial \lambda}\right) + \frac{kC}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \tag{3-167b}$$

$$\frac{dk}{dt} = -\frac{B}{A}\left(\frac{\partial R}{\partial h} + \frac{k}{1+B}\frac{\partial R}{\partial \lambda}\right) - \frac{hC}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \tag{3-167c}$$

$$\frac{dp}{dt} = -\frac{pC}{2AB}\left(k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial k} + \frac{\partial R}{\partial \lambda}\right) + \frac{IC^2}{4AB}\frac{\partial R}{\partial q} \tag{3-167d}$$

$$\frac{dq}{dt} = -\frac{qC}{2AB}\left(k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial k} + \frac{\partial R}{\partial \lambda}\right) - \frac{IC^2}{4AB}\frac{\partial R}{\partial p} \tag{3-167e}$$

$$\frac{d\lambda}{dt} = n - \frac{2a}{A}\frac{\partial R}{\partial a} + \frac{B}{A(1+B)}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + \frac{C}{2AB}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right) \tag{3-167f}$$

where

$$A = na^2$$

$$B = \sqrt{1-h^2-k^2}$$

$$C = 1 + p^2 + q^2$$

and the elements (a, h, k, p, q, λ) are understood to be mean elements, n is the mean-mean motion, and R is the averaged disturbing function.

The disturbing functions developed in the previous sections of this report have been expressed in terms of the mean direction cosines $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ of the equatorial $\hat{z}$ axis with respect to the equinoctial reference frame $(\hat{f}, \hat{g}, \hat{w})$, rather than in terms of the equinoctial elements p and q. Consequently, expressions of the form

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial \alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial R}{\partial \beta}\frac{\partial \beta}{\partial p} + \frac{\partial R}{\partial \gamma}\frac{\partial \gamma}{\partial p}$$

$$\frac{\partial R}{\partial q} = \frac{\partial R}{\partial \alpha}\frac{\partial \alpha}{\partial q} + \frac{\partial R}{\partial \beta}\frac{\partial \beta}{\partial q} + \frac{\partial R}{\partial \gamma}\frac{\partial \gamma}{\partial q}$$

are required to modify Equations (3-167) in order to accommodate the particular form of the disturbing functions. The following results are demonstrated in Appendix C of this document:

$$\frac{\partial \alpha}{\partial p} = -\frac{2}{C}(q\beta I + \gamma) \tag{3-168a}$$

$$\frac{\partial \alpha}{\partial q} = \frac{2p\beta I}{C} \tag{3-168b}$$

$$\frac{\partial \beta}{\partial p} = \frac{2q\alpha I}{C} \tag{3-168c}$$

$$\frac{\partial \beta}{\partial q} = -\frac{2I}{C}(p\alpha - \gamma) \qquad \text{(3-168d)}$$

$$\frac{\partial \gamma}{\partial p} = \frac{2\alpha}{C} \qquad \text{(3-168e)}$$

$$\frac{\partial \gamma}{\partial q} = -\frac{2\beta I}{C} \qquad \text{(3-168f)}$$

$$\frac{\partial R}{\partial p} = \frac{2}{C}\left[\alpha \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \alpha} + qI\left(\alpha \frac{\partial R}{\partial \beta} - \beta \frac{\partial R}{\partial \alpha}\right)\right] \qquad \text{(3-169a)}$$

$$\frac{\partial R}{\partial q} = -\frac{2I}{C}\left[\beta \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \beta} + p\left(\alpha \frac{\partial R}{\partial \beta} - \beta \frac{\partial R}{\partial \alpha}\right)\right] \qquad \text{(3-169b)}$$

$$p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q} = \frac{2}{C}\left[p\left(\alpha \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \alpha}\right) - qI\left(\beta \frac{\partial R}{\partial \gamma} - \gamma \frac{\partial R}{\partial \beta}\right)\right] \qquad \text{(3-170)}$$

where C is defined as before. Substituting these expressions into Equations (3-167) yields the final form of the averaged Variation of Parameters (VOP) equations of motion used in the current investigation, i.e.,

$$\frac{da}{dt} = \frac{2a}{A}\frac{\partial R}{\partial \lambda} \qquad \text{(3-171a)}$$

$$\frac{dh}{dt} = \frac{B}{A}\frac{\partial R}{\partial k} + \frac{k}{AB}\left(pR_{\alpha,\gamma} - IqR_{\beta,\gamma}\right) - \frac{hB}{A(1+B)}\frac{\partial R}{\partial \lambda} \qquad (3\text{-}171b)$$

$$\frac{dk}{dt} = -\left[\frac{B}{A}\frac{\partial R}{\partial h} + \frac{h}{AB}\left(pR_{\alpha,\gamma} - IqR_{\beta,\gamma}\right) - \frac{kB}{A(1+B)}\frac{\partial R}{\partial \lambda}\right] \qquad (3\text{-}171c)$$

$$\frac{dp}{dt} = \frac{C}{2AB}\left[p\left(R_{h,k} - R_{\alpha,\beta} - \frac{\partial R}{\partial \lambda}\right) - R_{\beta,\gamma}\right] \qquad (3\text{-}171d)$$

$$\frac{dq}{dt} = \frac{C}{2AB}\left[q\left(R_{h,k} - R_{\alpha,\beta} - \frac{\partial R}{\partial \lambda}\right) - IR_{\alpha,\gamma}\right] \qquad (3\text{-}171e)$$

$$\frac{d\lambda}{dt} = n - \frac{2a}{A}\frac{\partial R}{\partial a} + \frac{B}{A(1+B)}R_{h,k} + \frac{1}{AB}\left(pR_{\alpha,\gamma} - IqR_{\beta,\gamma}\right) \qquad (3\text{-}171f)$$

where $n = \bar{n}$; A, B, and C are defined as in Equation (3-167); and

$$R_{x,y} = x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x} \qquad (3\text{-}172)$$

for any two variables x and y.

These equations admit simplifications for certain cases. For example, it is shown that

$$\frac{da}{dt} \equiv 0 \qquad (3\text{-}173)$$

for the zonal and nonresonant tesseral (m-daily) harmonic perturbations. In addition, it is demonstrated that the zonal harmonics model admits the simplification

$$R_{h,k} - R_{\alpha,\beta} \equiv 0 \tag{3-174}$$

The averaged equations of motion for the zonal harmonic model are presented in Section 3.4.1; the averaged equations of motion for the combined zonal and nonresonant tesseral harmonics model are presented in Section 3.4.2; and the averaged equations of motion for the resonant tesseral harmonic model are presented in Section 3.4.3. The final form of the averaged equations of motion as given were implemented into the Research and Development (R&D) version of the Goddard Trajectory Determination System (GTDS).

3.4.1 The Averaged Equations of Motion for the Zonal Harmonic Model

Inspection of Equations (3-171) and (3-172) indicates that the averaged equations of motion require partial derivatives of the zonal harmonic disturbing function with respect to the elements $(a, h, k, \alpha, \beta, \gamma, \lambda)$ (the direction cosines α, β, γ are a redundant set of parameters). The averaged zonal harmonic disturbing function obtained in Section 3.3.2 (Equation (3-145)) is given by the real part of the expression

$$\left\langle R^{Z}\right\rangle_{\bar{\lambda}} = -\frac{2\mu}{a} \sum_{\substack{\ell=2 \\ [\ell+s \text{ even}]}}^{\infty} \sum_{s=0}^{\ell} \delta_s J_\ell \left(\frac{a_e}{a}\right)^{\ell} V_{\ell,s}\, Q_{\ell,s}(\gamma)\, K_0^{-\ell-1,s} (\alpha - j\beta)^s (k+jh)^s \tag{3-175}$$

The averaged disturbing function is rearranged into a more optimal form for evaluation purposes before the required partial derivatives are obtained.

Practical considerations require that the summation over ℓ be truncated at some finite value, i.e., $2 \le \ell \le L$. In addition, a closer inspection of the functions in Equation (3-175) indicates that, for $s = \ell$,

$$K_0^{-\ell-1,\ell} = 0$$

since (Equation (2-249))

$$X_0^{-\ell-1,\ell} = 0$$

Also, for $s = \ell - 1$,

$$P_{\ell,\ell-1}(0) = 0$$

Thus, the index s can be restricted to the range $0 \le s \le \ell - 2$, along with the condition that $\ell - s$ must be even.

It is often desirable to truncate on powers of $e \sin i$ for certain cases. Since

$$\left|(\alpha - j\beta)^s (k + jh)^s\right| = (e \sin i)^s \tag{3-176}$$

the range of the index s is truncated to $0 \le s \le S \le \ell - 2$, where S is the maximum power of $e \sin i$ retained.[1]

The exact order in which the two summations are performed depends on the recurrence relations used to evaluate the various quantities in the disturbing function. It is desirable that the recurrence relations be free of computational small divisors and that they use simple starting values. Clearly, the complex polynomials $(\alpha - j\beta)^s$ and $(k + jh)^s$ should be evaluated in order of ascending powers s in order to avoid a divide operation which could introduce a small divisor

---

[1] Higher powers of the eccentricity will usually be retained since the function $K_0^{-\ell-1,s}$ is a polynomial in $(1 - e^2)^{-1/2}$. Truncating on the complex polynomial $(k + jh)^s$ is equivalent to truncating on the D'Alembert characteristic $e^{|s|}$ in classical elements.

for small eccentricities or inclinations. Similarly, the parallax factor $(a_e/a)^{\ell}$ should be evaluated in ascending powers of ℓ. The real polynomial $Q_{\ell,s}(\gamma)$ possesses recurrence relations based on the recurrence relations for the associated Legendre polynomial, $P_{\ell,s}(x)$. These recurrence relations are obtained by substituting the relations

$$Q_{\ell,s}(\gamma) = (1-\gamma^2)^{-s/2} P_{\ell,s}(\gamma) = (-1)^s (1-\gamma^2)^{-s/2} P_{\ell}^s(\gamma) \qquad (3\text{-}177a)$$

$$P_{\ell}^s(\gamma) = \frac{1}{2}\left[\exp\left(j s \frac{\pi}{2}\right) P_{\ell}^s(\gamma + j0) + \exp\left(-j s \frac{\pi}{2}\right) P_{\ell}^s(\gamma - j0)\right] \qquad (3\text{-}177b)$$

$$(z^2-1)^{1/2} = \exp\left(\pm j \frac{\pi}{2}\right)(1-\gamma^2)^{1/2} \qquad (3\text{-}177c)$$

into Equations (2-232) through (2-240) (Reference 26). The resulting fixed-order(s) recurrence relations

$$Q_{\ell+1,s}(\gamma) = \frac{1}{\ell-s+1}\left[(2\ell+1)\gamma Q_{\ell,s}(\gamma) - (\ell+s) Q_{\ell-1,s}(\gamma)\right] \qquad (3\text{-}178)$$

can proceed in ascending values of ℓ without any small divi . . difficulty. The fixed-degree recurrence

$$Q_{\ell,s-1}(\gamma) = \frac{1}{(\ell-s+1)(\ell+s)}\left[2 s \gamma Q_{\ell,s}(\gamma) - (1-\gamma^2) Q_{\ell,s+1}(\gamma)\right] \qquad (3\text{-}179)$$

must proceed in descending values of the index s in order to avoid the possibly small divisor $1 - \gamma^2$. This direction is, however, not well suited for the complex polynomial recurrence formulas. This also applies to the recurrence relations for the function $K_0^{-\ell-1,s}$, since they are also governed by the associated

Legendre polynomial recurrence relations (in view of Equations (2-229) and (2-303)) and thus are better evaluated using the recurrence on ascending values of ℓ. In fact, it is shown below that these functions can be expressed in terms of the polynomials $Q_{\ell,s}(\gamma)$ and, thus, Equations (3-178) and (3-179) directly apply.

Consequently, the fixed-order (s) recurrence relations appear to be the best choice and require that the summation over ℓ must be performed before the summation of s. It is easily verified that

$$\sum_{\ell=2}^{L} \sum_{s=0}^{\ell-2} = \sum_{s=0}^{s \le L-2} \sum_{\ell=s+2}^{L}$$

The averaged disturbing function is then expressed as

$$\left\langle R^{z} \right\rangle_{\bar{\lambda}} = -\frac{2\mu}{a} \sum_{s=0}^{s \le L-2} \delta_s (\alpha - j\beta)^s (k + jh)^s \sum_{\substack{\ell = s+2 \\ [\ell \pm s \text{ even}]}}^{L} J_\ell \left(\frac{a_e}{a}\right)^\ell V_{\ell,s} Q_{\ell,s}(\gamma) K_0^{-\ell-1,s} \qquad (3\text{-}180)$$

For the purpose of software implementation, the following definitions are made:

$$G_s + j H_s = (\alpha - j\beta)^s (k + jh)^s \qquad (3\text{-}181)$$

and

$$H_\ell^s = \left(\frac{a_e}{a}\right)^\ell V_{\ell,s}(0)\, Q_{\ell,s}(\gamma)\, K_0^{-\ell-1,s} \qquad (3\text{-}182)$$

which is strictly a real function.

The real part of the averaged zonal harmonic disturbing function (Equation (3-180)) then takes the form

$$\left\langle R^{z} \right\rangle_{\bar{\lambda}} = -\frac{2\mu}{a} \sum_{s=0}^{S \le L-2} \delta_s G_s \sum_{\substack{\ell = s+2 \\ [\ell \pm s \text{ even}]}}^{L} J_\ell H_\ell^s \tag{3-183}$$

Before the necessary partial derivatives for the averaged equations of motion are presented, it is necessary to elaborate further on the function $K_0^{-\ell-1,s}$. As shown in the discussion in Section 2.2.1.3.4, the special Hansen coefficient $X_0^{n,s}$ can be expressed as (Equation (2-223))

$$X_0^{-\ell-1,s} = e^{|s|} K_0^{-\ell-1,s}$$

Furthermore, from Equation (2-229),

$$X_0^{-\ell-1,s} = \frac{(\ell-1)!}{(\ell+|s|-1)!} x^{\ell} P_{\ell-1}^{|s|}(x)$$

where

$$x = \frac{1}{\sqrt{1-e^2}}$$

Since only positive values of s are of interest, the absolute value notation is dropped. Clearly,

$$K_0^{-\ell-1,s} = \frac{(\ell-1)!}{(\ell+s-1)!} e^{-s} x^{\ell} P_\ell^s(x)$$

$$= \frac{(\ell-1)!}{(\ell+s-1)!} (-e)^{-s} x^{\ell} (1-x^2)^{s/2} Q_{\ell,s}(x) \tag{3-184}$$

in view of Equation (3-142). Since

$$(1-x^2)^{s/2} = \left(\frac{-e^2}{1-e^2}\right)^{s/2} = (-e)^s\, x^s$$

it follows that

$$K_0^{-\ell-1,s} = \frac{(\ell-1)!}{(\ell+s-1)!}\, x^{\ell+s}\, Q_{\ell,s}(x) \tag{3-185}$$

Thus, the partial derivative with respect to h and k will be obtained through the chain rule, i.e.,

$$\frac{\partial K_0^{-\ell-1,s}}{\partial h} = \frac{\partial K_0^{-\ell-1,s}}{\partial x}\,\frac{\partial x}{\partial h}$$

etc. Since

$$x = (1-e^2)^{-1/2} = (1-h^2-k^2)^{-1/2}$$

then

$$\frac{\partial x}{\partial h} = x^3 h \tag{3-186a}$$

and

$$\frac{\partial x}{\partial k} = x^3 k \tag{3-186b}$$

The following partial derivatives of the real part of the disturbing function given in Equation (3-183) are easily verified:

$$\frac{\partial \langle R^{*} \rangle_{\lambda}}{\partial a} = \frac{2\mu}{a^2} \sum_{s} \delta_s\, G_s \sum_{\ell} (\ell+1)\, J_\ell\, H_\ell^s \tag{3-187a}$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial h} = -\frac{2\mu}{a}\left[\sum_{s} \delta_s \frac{\partial G_s}{\partial h} \sum_{\ell} J_\ell H_\ell^s + h x^3 \sum_{s} \delta_s G_s \sum_{\ell} J_\ell \frac{\partial H_\ell^s}{\partial x}\right] \quad (3\text{-}187b)$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial k} = -\frac{2\mu}{a}\left[\sum_{s} \delta_s \frac{\partial G_s}{\partial k} \sum_{\ell} J_\ell H_\ell^s + k x^3 \sum_{s} \delta_s G_s \sum_{\ell} J_\ell \frac{\partial H_\ell^s}{\partial x}\right] \quad (3\text{-}187c)$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial \alpha} = -\frac{2\mu}{a} \sum_{s} \delta_s \frac{\partial G_s}{\partial \alpha} \sum_{\ell} J_\ell H_\ell^s \quad (3\text{-}187d)$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial \beta} = -\frac{2\mu}{a} \sum_{s} \delta_s \frac{\partial G_s}{\partial \beta} \sum_{\ell} J_\ell H_\ell^s \quad (3\text{-}187e)$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial \gamma} = -\frac{2\mu}{a} \sum_{s} \delta_s G_s \sum_{\ell} J_\ell \frac{\partial H_\ell^s}{\partial \gamma} \quad (3\text{-}187f)$$

$$\frac{\partial \langle R^{z} \rangle_{\bar{\lambda}}}{\partial \lambda} \equiv 0 \quad (3\text{-}187g)$$

where

$$\frac{\partial G_s}{\partial h} = \mathrm{Re}\left\{ (\alpha - j\beta)^s \, s j \, (k + jh)^{s-1} \right\} \quad (3\text{-}188a)$$

$$\frac{\partial G_s}{\partial k} = \mathrm{Re}\left\{ (\alpha - j\beta)^s \, s \, (k + jh)^{s-1} \right\} \quad (3\text{-}188b)$$

$$\frac{\partial G_s}{\partial \alpha} = \mathrm{Re}\left\{ s (\alpha - j\beta)^{s-1} (k + jh)^s \right\} \quad (3\text{-}188c)$$

$$\frac{\partial G_s}{\partial \beta} = \mathrm{Re}\left\{ -js(\alpha - j\beta)^{s-1}(k+jh)^s \right\} \tag{3-188d}$$

$$\frac{\partial H_\ell^s}{\partial x} = \left(\frac{a_e}{a}\right)^\ell V_{\ell,s}\, Q_{\ell,s}(\gamma)\, \frac{\partial K_0^{-\ell-1,s}}{\partial x}$$

(3-188e)

$$= \left(\frac{a_e}{a}\right)^\ell V_{\ell,s}\, Q_{\ell,s}(\gamma)\, \frac{(\ell-1)!}{(\ell+s-1)!}\left[(\ell+s)\, x^{\ell+s-1}\, Q_{\ell,s}(x) + x^{\ell+s}\,\frac{dQ_{\ell,s}(x)}{dx}\right]$$

$$\frac{\partial H_\ell^s}{\partial \gamma} = \left(\frac{a_e}{a}\right)^\ell V_{\ell,s}\, K_0^{-\ell-1,s}\, \frac{dQ_{\ell,s}(\gamma)}{dx} \tag{3-188f}$$

The summations are performed over the ranges $0 \le s \le S \le L-2$ and $s+2 \le \ell \le L$. The following recurrence relations are used to evaluate the above quantities:

<u>Recurrence Relations for the G and H Polynomials and Their Partial Derivatives</u>

$$G_s = (\alpha k + \beta h)G_{s-1} - (\alpha h - \beta k)H_{s-1} \tag{3-189a}$$

$$H_s = (\alpha h - \beta k)G_{s-1} + (\alpha k + \beta h)H_{s-1} \tag{3-189b}$$

$$\frac{\partial G_s}{\partial h} = s\beta\, G_{s-1} - s\alpha\, H_{s-1} \tag{3-189c}$$

$$\frac{\partial G_s}{\partial k} = s\alpha G_{s-1} + s\beta H_{s-1} \tag{3-189d}$$

$$\frac{\partial G}{\partial \alpha} = s k G_{s-1} - s h H_{s-1} \qquad (3\text{-}189e)$$

$$\frac{\partial G}{\partial \beta} = s h G_{s-1} + s k H_{s-1} \qquad (3\text{-}189f)$$

$$G_0 = 1 \qquad (3\text{-}189g)$$

$$H_0 = 0 \qquad (3\text{-}189h)$$

Recurrence Relations for the Quantity $V_{\ell,s}$

$$V_{\ell,s} = -\frac{(\ell-s-1)}{\ell+s} V_{\ell-2,s} \qquad (3\text{-}190a)$$

$$V_{s,s} = \frac{1}{2s} V_{s-1,s-1} \qquad (3\text{-}190b)$$

$$V_{0,0} = 1 \qquad (3\text{-}190c)$$

Recurrence Relations for the Polynomials $Q_{\ell,s}(\gamma)$ and $dQ_{\ell,s}(\gamma)/d\gamma$

$$Q_{\ell,s}(\gamma) = \frac{1}{\ell-s}\left[(2\ell-1)\gamma Q_{\ell-1,s}(\gamma) - (\ell+s-1) Q_{\ell-2,s}(\gamma)\right] \qquad (3\text{-}191a)$$

$$Q_{s+1,s}(\gamma) = (2s+1)\gamma Q_{s,s}(\gamma) \qquad (3\text{-}191b)$$

$$Q_{s,s}(\gamma) = (2s-1)!! \tag{3-191c}$$

$$\frac{dQ_{\ell,s}(\gamma)}{d\gamma} = Q_{\ell,s+1}(\gamma) \tag{3-191d}$$

$$\frac{dQ_{s,s}(\gamma)}{d\gamma} = 0 \tag{3-191e}$$

Recurrence Relations for the Function $K_0^{-\ell-1,s}$ and $dK_0^{-\ell-1,s}/dx$

For the software implementation, the notation

$$D_\ell^s(x) = K_0^{-\ell-1,s} \tag{3-192}$$

was used. In this notation, the recurrence relations take the form

$$D_{s+1}^s(x) = \frac{x^{2s+1}}{2^s} \tag{3-193a}$$

$$D_{s+2}^s(x) = (s+1)\,x^2\,D_{s+1}^s(x) \tag{3-193b}$$

$$D_\ell^s(x) = \frac{(\ell-1)\,x^2}{(\ell+s-1)(\ell-s-1)}\left[(2\ell-3)\,D_{\ell-1}^s(x) - (\ell-2)\,D_{\ell-2}^s(x)\right] \tag{3-193c}$$

$$\frac{dD_{s+1}^s(x)}{dx} = (2s+1)\,\frac{x^{2s}}{2^s} \tag{3-193d}$$

$$\frac{dD_{s+2}^s(x)}{dx} = (s+1)\,x^2\,(2s+3)\,\frac{x^{2s}}{2^s} \tag{3-193e}$$

$$\frac{dD_\ell^s(x)}{dx} = \frac{(\ell-1)\,x^2}{(\ell+s-1)(\ell-s-1)}\left[(2\ell-3)\frac{dD_{\ell-1}^s(x)}{dx} - (\ell-2)\frac{dD_{\ell-2}^s(x)}{dx}\right] + \frac{2}{x}D_\ell^s(x) \tag{3-193f}$$

Since the averaged disturbing function is independent of λ, it follows that (Equation (3-171a))

$$\frac{da}{dt} = \frac{2a}{A}\frac{\partial \bar{R}}{\partial \lambda} \equiv 0 \tag{3-194}$$

Also, it can now be demonstrated that

$$\bar{R}_{h,k} - \bar{R}_{\alpha,\beta} \equiv 0 \qquad \text{(Equation (3-174))}$$

From Equation (3-172),

$$\bar{R}_{h,k} = h\frac{\partial \bar{R}}{\partial k} - k\frac{\partial \bar{R}}{\partial h} \tag{3-195}$$

Multiplying Equations (3-187c) and (3-187b) by h and k, respectively, and taking the difference yields

$$h\frac{\partial \bar{R}}{\partial k} - k\frac{\partial \bar{R}}{\partial h} = -\frac{2\mu}{a}\sum_{s=0}^{S\le L-2}\delta_s\left(h\frac{\partial G_s}{\partial k} - k\frac{\partial G_s}{\partial h}\right)\sum_{\ell=s+2}^{L} J_\ell H_\ell^s \tag{3-196}$$

Similarly, multiplying Equations (3-187e) and (3-187d) by α and β, respectively, and taking the difference yields

$$\alpha\frac{\partial R}{\partial \beta} - \beta\frac{\partial R}{\partial \alpha} = -\frac{2\mu}{a}\sum_{s}\delta_s\left(\alpha\frac{\partial G_s}{\partial \beta} - \beta\frac{\partial G_s}{\partial \alpha}\right)\sum_{\ell=s+2}^{L} J_\ell H_\ell^s \tag{3-197}$$

Therefore,

$$R_{h,k} - R_{\alpha,\beta} = -\frac{2\mu}{a} \sum_{s} \delta_s \left[\left(h \frac{\partial G_s}{\partial k} - k \frac{\partial G_s}{\partial h} \right) - \left(\alpha \frac{\partial G_s}{\partial \beta} - \beta \frac{\partial G_s}{\partial \alpha} \right) \right] \times \sum_{\ell} J_\ell H_\ell^s \tag{3-198}$$

Multiplying Equations (3-188b) and (3-188a) by h and k, respectively, and taking the difference yields

$$h \frac{\partial G_s}{\partial k} - k \frac{\partial G_s}{\partial h} = (s\alpha h - s\beta k) G_{s-1} + (s\beta h + s\alpha k) H_{s-1} \tag{3-199}$$

Multiplying Equations (3-188d) and (3-188c) by α and β, respectively, and taking the difference yields

$$\alpha \frac{\partial G_s}{\partial \beta} - \beta \frac{\partial G_s}{\partial \alpha} = (s\alpha h - s\beta k) G_{s-1} + (s\alpha k + s\beta h) H_{s-1} \tag{3-200}$$

Comparison of Equations (3-199) and (3-200) shows that

$$h \frac{\partial G_s}{\partial k} - k \frac{\partial G_s}{\partial h} = \alpha \frac{\partial G_s}{\partial \beta} - \beta \frac{\partial G_s}{\partial \alpha} \tag{3-201}$$

and Equations (3-174) is verified.

In view of Equations (3-174) and (3-194), the final form for the averaged equations of motion for the zonal harmonic case is simplified to

$$\frac{da}{dt} = 0 \tag{3-202a}$$

$$\frac{dh}{dt} = \frac{B}{A}\frac{\partial R^z}{\partial k} + \frac{k}{AB}\left(pR^z_{\alpha,\beta} - IqR^z_{\beta,\gamma}\right) \qquad (3\text{-}202b)$$

$$\frac{dk}{dt} = -\frac{B}{A}\frac{\partial R^z}{\partial h} - \frac{h}{AB}\left(pR^z_{\alpha,\beta} - IqR^z_{\beta,\gamma}\right) \qquad (3\text{-}202c)$$

$$\frac{dp}{dt} = -\frac{C}{2AB}R^z_{\beta,\gamma} \qquad (3\text{-}202d)$$

$$\frac{dq}{dt} = -\frac{IC}{2AB}R^z_{\alpha,\gamma} \qquad (3\text{-}202e)$$

$$\frac{d\lambda}{dt} = n - \frac{2a}{A}\frac{\partial R^z}{\partial a} + \frac{B}{A(1+B)}R^z_{h,k} + \frac{1}{AB}\left(pR^z_{\alpha,\gamma} - IqR^z_{\beta,\gamma}\right) \qquad (3\text{-}202f)$$

where R^z is the averaged disturbing function, the elements are interpreted to be mean elements, and n designates the mean-mean motion.

All of the requisite partial derivatives have been obtained and are given in Equations (3-187) and (3-188).

3.4.2 The Averaged Equations of Motion for the Combined Zonal and Nonresonant Tesseral Harmonic Model[1]

The averaged disturbing function for the combined zonal and nonresonant tesseral harmonic model given in Equation (3-152) must be rearranged to maximize computational efficiency. First, the summation over ℓ is truncated at L, i.e., $2 \le \ell \le L$; the summation over m is allowed to vary through the range $0 \le m \le M \le L$; and the two summations over s are allowed to vary through the

[1]This assumes time-independent averaging, i.e., a stationary central body during the averaging operation, and, thus, this model contains only the m-daily contribution of the tesseral harmonic model.

ranges $0 \leq s \leq [m, s \leq L-2]$ (where [ ] denotes the minimum value) and $M+1 \leq s \leq S \leq L-2$ (where $S \leq L-2$ is the maximum power of $e \sin i$ retained in the expansion). The summation over s is bounded above by $L-2$ for the same reasons presented in the discussion for the averaged equations of motion of the zonal harmonic model. The averaged disturbing function for the combined zonal and nonresonant tesseral harmonic field is given by (Equation (3-152))

$$
\begin{aligned}
\left\langle R^{Z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\ell=2}^{L} \sum_{m=0}^{[\ell,M]} \sum_{\epsilon=\pm 1} \left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j S_{\ell,m}\right) I^m \epsilon^m & \left\{ \begin{matrix} 1 \\ \dfrac{\sin\left(\frac{\omega}{n} m\pi\right)}{\frac{\omega}{n} m\pi} \end{matrix} \right\} e^{-jm\theta_0} \\
\times \left\{ \sum_{\substack{s=0 \\ (\ell \pm s \text{ even})}}^{[m-1,S]} \delta_s (-1)^{m-s}\, 2^{-m} \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) \left(\alpha + jI\beta\right)^{m-\epsilon I s} \right. & \\
\times \left(k + j\epsilon h\right)^s K_0^{-\ell-1,s} \left(1 + I\gamma\right)^{\epsilon I s} P_{\ell-m}^{m-s,\,m+s}(\epsilon\gamma) & \qquad (3\text{-}203) \\
+ \sum_{\substack{s=m \\ (\ell \pm s \text{ even})}}^{S \leq L-2} \delta_s\, 2^{-s} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(\alpha - j\epsilon\beta\right)^{s-\epsilon I m} & \\
\left. \times \left(k + j\epsilon h\right)^s K_0^{-\ell-1,s} \left(1 + I\gamma\right)^{\epsilon I m} P_{\ell-s}^{s-m,\,s+m}(\epsilon\gamma) \right\} &
\end{aligned}
$$

where [ ] denotes the minimum value. The averaging factors 1 and $\dfrac{\sin\left(\frac{\omega}{n} m\pi\right)}{\frac{\omega}{n} m\pi}$ are appropriate for the cases $m = 0$ and $m > 0$, respectively.

Clearly, all real and complex polynomials should be computed recursively in order of ascending powers to avoid a divide operation and, hence, possible small divisors. The one exception is the evaluation of the real polynomials $(1 + I\gamma)^{\epsilon Is}$ and $(1 + I\gamma)^{\epsilon Im}$, since the order of increasing powers is dependent on the sign of the factor ϵI. However, since the quantity $I\gamma$ is always non-negative, i.e., $I = 1$ for $\gamma \geq 0$ and $I = -1$ for $\gamma < 0$, then $(1 + I\gamma) \geq 1$ is always satisfied and there is no possiblity of a small divisor. The required computational sequence for the functions $K_0^{-\ell-1,s}$ is the same as in the zonal harmonics case, i.e., the recurrence uses ascending values of ℓ and a fixed value of s. The final consideration is that of the computational sequence of the Jacobi polynomials.

Recurrence relations for the Jacobi polynomials can be found in References 26, 47, and 48. The particular recurrence relation

$$P_{n+1}^{a,b}(x) = \frac{1}{2(n+1)(n+\rho)(2n+\rho-1)} \times \left\{ (2n+\rho)\left[(2n+\rho-1)(2n+\rho+1)x + a^2 - b^2\right] P_n^{a,b}(x) - 2(n+a)(n+b)(2n+\rho+1) P_{n-1}^{a,b}(x) \right\} \tag{3-204}$$

where

$$\rho = a + b + 1$$

is well suited in that only a single parameter, i.e., the degree of the polynomial, n, varies. In addition, the starting values for the recurrence relation are quite simple, i.e.,

$$P_0^{a,b}(x) = 1 \tag{3-205}$$

$$P_1^{a,b}(x) = \frac{(a+b+2)x}{2} + \frac{a-b}{2} \tag{3-206}$$

and are easily verified from the definition of the Jacobi polynomial given in Equation (2-33). The requirement that the Jacobi polynomial recurrence be performed for increasing degree in Equation (3-205) requires that the index ℓ be varied in ascending order while holding m constant in the first sum over s, $0 \leq s \leq \min(m-1, S)$, and that ℓ be varied in ascending order for fixed s in the second sum over s where $m \leq s \leq [L-2, S]$. The net result of all of these considerations requires that the averaged disturbing function be expressed with the summation over ℓ as the innermost summation.

The order of the summations over m and s depends on the range of s. The order of the summations is rearranged as follows. First, if the following summations are considered:

$$\sum_{\ell=2}^{L} \sum_{m=0}^{\ell} \sum_{s=0}^{[\ell-2,m]} \tag{3-207}$$

it is easily verified that

$$\sum_{m=0}^{\ell} \sum_{s=0}^{[\ell-2,m]} = \sum_{s=0}^{[\ell-2,M]} \sum_{m=s}^{[\ell,M]} \tag{3-208}$$

$$\sum_{\ell=2}^{L} \sum_{m=0}^{\ell} = \sum_{m=0}^{M\leq L} \sum_{\ell=\max(m,2)}^{L} \tag{3-209}$$

and

$$\sum_{\ell=2}^{L} \sum_{s=0}^{[\ell-2,M]} = \sum_{s=0}^{[L-2,M]} \sum_{\ell=\max(s+2,m)}^{L} \tag{3-210}$$

Thus,

$$\sum_{\ell=2}^{L} \sum_{m=0}^{\ell} \sum_{s=0}^{[\ell-2,m]} = \sum_{s=0}^{[L-2,M]} \sum_{m=s}^{M \leq L} \sum_{\ell=\max(s+2,m)}^{L} \tag{3-211}$$

Next, if the following summations are considered:

$$\sum_{\ell=2}^{L} \sum_{m=0}^{\ell} \sum_{s=m+1}^{\ell-2} \tag{3-212}$$

then it is straightforward to show

$$\sum_{\ell=2}^{L} \sum_{m=0}^{[\ell,M]} \sum_{s=m+1}^{\ell-2} = \sum_{m=0}^{[L,M]} \sum_{s=m+1}^{[S,L-2]} \sum_{\ell=s+2}^{L} \tag{3-213}$$

Thus, the net result of the above considerations on the ordering of the averaged disturbing function is

$$\left\langle R^{z+NT}\right\rangle_{\bar{\lambda}} = \frac{\mu}{a}\sum_{\epsilon=\pm 1}\left[\sum_{s=0}^{[S,M]}(-1)^s\,\delta_s\,(k+j\epsilon h)^s\,(1+I\gamma)^s\right.$$

$$\times\sum_{m=s}^{M\le L}\left(-\frac{\epsilon I}{2}\right)^m\left\{\frac{1}{\frac{\sin\left(\frac{\omega}{n}m\pi\right)}{\frac{\omega}{n}m\pi}}\right\}e^{-jm\theta_0}\,(\alpha+jI\beta)^{m-\epsilon Is}$$

$$\times\sum_{\substack{\ell=\max(m,s+2)\\(\ell\pm s\ \text{even})}}^{L}\left(\frac{a_e}{a}\right)^{\ell}(C_{\ell,m}-jS_{\ell,m})\,\frac{(\ell+m)!}{(\ell+s)!}\,P_{\ell,s}(0)\,K_0^{-\ell-1,s}\,P_{\ell-m}^{m-s,m+s}(\epsilon\gamma)$$

$$+\sum_{m=0}^{M\le L}(\epsilon I)^m\,(1+I\gamma)^{\epsilon Im}\left\{\frac{1}{\frac{\sin\left(\frac{\omega}{n}m\pi\right)}{\frac{\omega}{n}m\pi}}\right\}e^{-jm\theta_0} \tag{3-214}$$

$$\times\sum_{s=m+1}^{S}2^{-s}\,(\alpha-j\epsilon\beta)^{s-\epsilon Im}\,(k+j\epsilon h)^s$$

$$\left.\times\sum_{\substack{\ell=s+2\\(\ell\pm s\ \text{even})}}^{L}\left(\frac{a_e}{a}\right)^{\ell}(C_{\ell,m}-jS_{\ell,m})\,\frac{(\ell-s)!}{(\ell-m)!}\,P_{\ell,s}(0)\,K_0^{-\ell-1,s}\,P_{\ell-s}^{s-m,s+m}(\epsilon\gamma)\right]$$

where $S \le L-2$.

While the above ordering appears to satisfy the requirements of the various recurrence relations, it also imposes the recomputation of certain functions or, alternately, sufficient storage requirements to store these quantities. For the purpose of this investigation, it is assumed that auxiliary core storage resources are limited. In addition, the unnecessary recomputation of quantities will be avoided.

Inspection of the second series of summations in Equation (3-214) indicates that the functions $K_0^{-\ell-1,s}$ will have to be reevaluated each time the outer sum over m is increased by one. The alternative is to store all possible values for this function, which number $(L-1) \times S \leq (L-1)(L-2)$. The storage requirement can be minimized by rearranging the outer two summations, i.e.,

$$\sum_{m=0}^{M \leq L} \sum_{s=m+1}^{S \leq L-1} = \sum_{s=1}^{S \leq L-1} \sum_{m=0}^{[s-1, M]}$$

Equations (3-214) then take the form

$$
\begin{aligned}
\left\langle R^{z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{\epsilon=\pm 1} \Bigg[& \sum_{s=0}^{[S,M]} (-1)^s \delta_s (k+j\epsilon h)^s (1+I\gamma)^s \\
& \times \sum_{m=s}^{M\le L} \left(-\frac{\epsilon I}{2}\right)^m \begin{Bmatrix} 1 \\ \dfrac{\sin\left(\frac{\omega}{n} m\pi\right)}{\frac{\omega}{n} m\pi} \end{Bmatrix} e^{-jm\theta_0} (\alpha + jI\beta)^{m-\epsilon Is} \\
& \times \sum_{\substack{\ell=\max(m,S+2) \\ (\ell\pm s \text{ even})}}^{L} \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - jS_{\ell,m}) \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0)\, K_0^{-\ell-1,s}\, P_{\kappa-m}^{m-s,m+s}(\epsilon\gamma) \\
& + \sum_{s=1}^{S\le L-2} 2^{-s} (k+j\epsilon h)^s \\
& \times \sum_{m=0}^{[s-1,M]} (\epsilon I)^m (1+I\gamma)^{\epsilon Im} \begin{Bmatrix} 1 \\ \dfrac{\sin\left(\frac{\omega}{n} m\pi\right)}{\frac{\omega}{n} m\pi} \end{Bmatrix} e^{-jm\theta_0} (\alpha - j\epsilon\beta)^{s-\epsilon Im} \\
& \times \sum_{\ell=s+2}^{L} \left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - jS_{\ell,m}) \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, K_0^{-\ell-1,s}\, P_{\ell-s}^{s-m,s+m}(\epsilon\gamma) \Bigg]
\end{aligned}
\qquad (3\text{-}215)
$$

where $S \le L-2$.

As a result, the computation of the function $K_0^{-\ell-1,s}$ is not affected by the m summation and, hence, they are evaluated only once as they are needed. However, the complex polynomial $(\alpha - j\epsilon\beta)^{s-\epsilon Im}$ no longer proceeds in ascending powers for $\epsilon I = +1$. Consequently, the recurrence relation requires that they must be evaluated and stored prior to entering the second series of summations. The maximum number of quantities thus stored is $2S \le 2(I-2)$, which is generally significantly fewer in number than the total number of polynomials, $K_0^{-\ell-1,s}$.

Close inspection of the summations in Equation (2-215) indicates that

$$\sum_{s=0}^{[S,M]} \sum_{m=s}^{M\le L} + \sum_{s=1}^{S} \sum_{m=0}^{[s-1,M]} = \sum_{s=0}^{S} \sum_{m=0}^{M} \tag{3-216}$$

and the expression for the real part of the disturbing function in Equation (2-215) can be simplified to

$$\left\langle R^{z+NT} \right\rangle_{\bar{\lambda}} = \frac{\mu}{a} \sum_{s=0}^{S} \sum_{m=0}^{M} \left\{ \frac{1}{\frac{\sin[(\omega/n)\pi]}{(\omega/n)\pi}} \right\} \sum_{\epsilon=\pm 1} \Bigg[\delta_s \, G^{\epsilon}_{t,s,m}(\alpha,\beta,h,k,\theta)$$

$$\times \sum_{\ell=\max(m,s+2)}^{L} C_{\ell,m} \, E^{s}_{\ell,m}(a,x) \, R^{\epsilon}_{\ell,m,s}(\gamma) + \delta_s \, H^{\epsilon}_{t,s,m}(\alpha,\beta,h,k,\theta) \tag{3-217}$$

$$\times \sum_{\ell=\max(m,s+2)}^{L} S_{\ell,m} \, E^{s}_{\ell,m}(a,x) \, R^{\epsilon}_{\ell,m,s}(\gamma) \Bigg]$$

where

$$
G^{\epsilon}_{t,s,m} + H^{\epsilon}_{t,s,m} = \begin{cases} (\alpha - j\epsilon\beta)^{t} \left(\dfrac{k+j\epsilon h}{2}\right)^{s} (\epsilon I \cos\theta_0 - j\epsilon I \sin\theta_0)^{m} & (m < s) \quad (3\text{-}218a) \\ \\ (\alpha + jI\beta)^{t} (-k-j\epsilon h)^{s} \left(-\dfrac{\epsilon I}{2} \cos\theta_0 + j\,\dfrac{\epsilon I}{2} \sin\theta_0\right) & (m \geq s) \quad (3\text{-}218b) \end{cases}
$$

$$
t = \begin{cases} s - \epsilon I m & (m < s) \quad (3\text{-}219a) \\ m - \epsilon I s & (m \geq s) \quad (3\text{-}219b) \end{cases}
$$

$$
E^{s}_{\ell,m}(a,x) = V^{s}_{\ell,m}(a)\, K_0^{-\ell-1,s}(x) \qquad (3\text{-}220)
$$

$$
V^{s}_{\ell,m}(a) = \begin{cases} \left(\dfrac{a_e}{a}\right)^{\ell} \dfrac{(\ell-s)!}{(\ell-m)!}\, P_{\ell,s}(0) & (m < s) \quad (3\text{-}221a) \\ \\ \left(\dfrac{a_e}{a}\right)^{\ell} \dfrac{(\ell+m)!}{(\ell+s)!}\, P_{\ell,s}(0) & (m \geq s) \quad (3\text{-}221b) \end{cases}
$$

$$
R^{\epsilon}_{\ell,m,s}(\gamma) = \begin{cases} (1+I\gamma)^{\epsilon I m}\, P^{s-m,\,s+m}_{\ell-s}(\epsilon\gamma) & (m < s) \quad (3\text{-}222a) \\ \\ (1+I\gamma)^{\epsilon I s}\, P^{m-s,\,m+s}_{\ell-m}(\epsilon\gamma) & (m \geq s) \quad (3\text{-}222b) \end{cases}
$$

The following partial derivatives are required for the averaged equations of motion due to the combined zonal and nonresonant tesseral harmonic terms:

$$\frac{\partial\langle R\rangle}{\partial a}, \quad \frac{\partial\langle R\rangle}{\partial h}, \quad \frac{\partial\langle R\rangle}{\partial k}, \quad \frac{\partial\langle R\rangle}{\partial \alpha}, \quad \frac{\partial\langle R\rangle}{\partial \beta}, \quad \frac{\partial\langle R\rangle}{\partial \gamma}$$

Clearly,

$$\frac{\partial\langle R\rangle}{\partial \lambda} \equiv 0 \tag{3-223}$$

In order to obtain the partial derivatives with respect to h and k, it is necessary to take the partial derivative with respect to x. Thus, the complete partial derivatives with respect to h and k, respectively, are designated by

$$\frac{\partial R}{\partial h} = \left.\frac{\partial R}{\partial h}\right|_{k,x} + \left.\frac{\partial R}{\partial x}\right|_{h,k} \frac{\partial x}{\partial h} \tag{3-224a}$$

$$\frac{\partial R}{\partial k} = \left.\frac{\partial R}{\partial k}\right|_{h,x} + \left.\frac{\partial R}{\partial x}\right|_{h,k} \frac{\partial x}{\partial k} \tag{3-224b}$$

where the notation

$$\left.\frac{\partial R}{\partial h}\right|_{k,x}$$

denotes the partial derivative with respect to h while k and x are held constant. In view of Equations (3-186), then

$$\frac{\partial\langle R\rangle}{\partial h} = \left.\frac{\partial\langle R\rangle}{\partial h}\right|_{k,x} + h x^3 \left.\frac{\partial\langle R\rangle}{\partial x}\right|_{h,k} \tag{3-225a}$$

$$\frac{\partial\langle R\rangle}{\partial k} = \left.\frac{\partial\langle R\rangle}{\partial k}\right|_{h,x} + kx^3 \left.\frac{\partial\langle R\rangle}{\partial x}\right|_{h,k} \qquad (3\text{-}225b)$$

The partial derivatives of the disturbing function given in Equation (3-217) require the following partial derivatives:

$$\frac{\partial}{\partial a}\left(\frac{E^s_{\ell,m}(a,x)}{a}\right) = -\frac{\ell+1}{a^2} E^s_{\ell,m}(a,x) \qquad (3\text{-}226)$$

$$\frac{\partial G_{t,s,m}}{\partial k} = \begin{cases} \frac{s}{2} G_{t,s-1,m} & (m < s) \quad (3\text{-}227a) \\ -s\, G_{t,s-1,m} & (m \geq s) \quad (3\text{-}227b) \end{cases}$$

$$\frac{\partial H^{\epsilon}_{t,s,m}}{\partial k} = \begin{cases} \frac{s}{2} H^{\epsilon}_{t,s-1,m} & (m < s) \quad (3\text{-}228a) \\ -s\, H^{\epsilon}_{t,s-1,m} & (m \geq s) \quad (3\text{-}228b) \end{cases}$$

$$\frac{\partial G^{\epsilon}_{t,s,m}}{\partial h} = \begin{cases} -\frac{\epsilon s}{2} H_{t,s-1,m} & (m < s) \quad (3\text{-}229a) \\ \epsilon s\, H_{t,s-1,m} & (m \geq s) \quad (3\text{-}229b) \end{cases}$$

$$\frac{\partial H^{\epsilon}_{t,s,m}}{\partial h} = \begin{cases} \frac{\epsilon s}{2} G_{t,s-1,m} & (m < s) \quad (3\text{-}230a) \\ -\epsilon s G_{t,s-1,m} & (m \geq s) \quad (3\text{-}230b) \end{cases}$$

$$\frac{\partial G^{\epsilon}_{t,s,m}}{\partial \alpha} = t G^{\epsilon}_{t-1,s,m} \qquad (m < s) \qquad (3\text{-}231a)$$

$$\frac{\partial H^{\epsilon}_{t,s,m}}{\partial \alpha} = t H_{t-1,s,m} \qquad (m \geq s) \qquad (3\text{-}231b)$$

$$\frac{\partial G^{\epsilon}_{t,s,m}}{\partial \beta} = \begin{cases} \epsilon t H_{t-1,s,m} & (m < s) \quad (3\text{-}232a) \\ -It H_{t-1,s,m} & (m \geq s) \quad (3\text{-}232b) \end{cases}$$

$$\frac{\partial H^{\epsilon}_{t,s,m}}{\partial \beta} = \begin{cases} -\epsilon t G^{\epsilon}_{t-1,s,m} & (m < s) \quad (3\text{-}233a) \\ It G^{\epsilon}_{t-1,s,m} & (m \geq s) \quad (3\text{-}233b) \end{cases}$$

It is just as practical to compute the partial derivatives of the functions $K_0^{-\ell-1,s}$ and $R^{\epsilon}_{\ell,m,s}(\gamma)$ directly via recurrence relations than to express these partial derivatives in terms of the functions themselves.

In view of Equations (3-226) through (3-233), the partial derivatives take the form

$$\frac{\partial\langle R\rangle}{\partial a} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\frac{1}{\dfrac{\sin\left[\frac{\omega}{n} m\pi\right]}{\frac{\omega}{n} m\pi}}\right\}\sum_{\epsilon=\pm 1}\left[\delta_s G^{\epsilon}_{t,s,m}\right.$$

$$\times \sum_{\ell=\max(m,s+2)}^{L} (\ell+1)\, C_{\ell,m}\, E^{s}_{\ell,m}(a,x)\; R^{\epsilon}_{\ell,m,s}(\gamma) \tag{3-234a}$$

$$\left. + \delta_s H^{\epsilon}_{t,s,m} \sum_{\ell=\max(m,s+2)}^{L} (\ell+1)\, S_{\ell,m}\, E^{s}_{\ell,m}(a,x)\; R^{\epsilon}_{\ell,m,s}(\gamma)\right]$$

$$\left.\frac{\partial\langle R\rangle}{\partial h}\right|_{k,x} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\frac{1}{\dfrac{\sin\frac{\omega}{n} m\pi}{\frac{\omega}{n} m\pi}}\right\}\sum_{\epsilon=\pm 1}\left\{\begin{matrix} -\frac{\epsilon s}{2} \\ \epsilon s \end{matrix}\right\}\left[\delta_s H^{\epsilon}_{t,s-1,m}\right.$$

$$\times \sum_{\ell=\max(m,s+2)}^{L} C_{\ell,m}\, E^{s}_{\ell,m}(a,x)\, R^{\epsilon}_{\ell,m,s}(\gamma) \tag{3-234b}$$

$$\left. - \delta_s G^{\epsilon}_{t,s-1,m} \sum_{\ell=\max(m,s+2)}^{L} S_{\ell,m}\, E^{s}_{\ell,m}(a,x)\, R^{\epsilon}_{\ell,m,s}(\gamma)\right]$$

$$\left.\frac{\partial\langle R\rangle}{\partial k}\right|_{h,x} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\begin{matrix}1\\ \dfrac{\sin\frac{\omega}{n}m\pi}{\frac{\omega}{n}m\pi}\end{matrix}\right\}\sum_{\epsilon=\pm 1}\left\{\begin{matrix}\frac{s}{2}\\ -s\end{matrix}\right\}\Bigg[\delta_s G^{\epsilon}_{t,s-1,m}$$

$$\times\sum_{l=\max(m,s+2)}^{L} C_{l,m}\, E^{s}_{l,m}(a,x)\, R^{\epsilon}_{l,m,s}(\gamma) \tag{3-234c}$$

$$+\ \delta_s H^{\epsilon}_{t,s-1,m}\sum_{l=\max(m,s+2)}^{L} S_{l,m}\, E^{s}_{l,m}(a,x)\, R^{\epsilon}_{l,m,s}(\gamma)\Bigg]$$

$$\left.\frac{\partial R}{\partial x}\right|_{h,k} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\begin{matrix}1\\ \dfrac{\sin\frac{\omega}{n}m\pi}{\frac{\omega}{n}m\pi}\end{matrix}\right\}\sum_{\epsilon=\pm 1}\Bigg[\delta_s G^{\epsilon}_{t,s,m}$$

$$\times\sum_{l=\max(m,s+2)}^{L} C_{l,m}\, V^{s}_{l,m}(a)\,\frac{dK_0^{-l-1,s}}{dx}\, R^{\epsilon}_{l,m,s}(\gamma) \tag{3-234d}$$

$$+\ \delta_s H_{t,s,m}\sum_{l=\max(m,s+2)}^{L} S_{l,m}\, V^{s}_{l,m}(a)\,\frac{dK_0^{-l-1,s}}{dx}\, R^{\epsilon}_{l,m,s}(\gamma)\Bigg]$$

$$\frac{\partial\langle R\rangle}{\partial\alpha} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\begin{matrix}1\\ \dfrac{\sin\frac{\omega}{n}m\pi}{\frac{\omega}{n}m\pi}\end{matrix}\right\}\sum\left\{\begin{matrix}s-\epsilon Im\\ m-\epsilon Is\end{matrix}\right\}\Bigg[\delta_s\,G^{\epsilon}_{t,s,m}$$

$$\times\sum_{\ell=\max(m,s+2)}^{L} C_{\ell,m}\,E^{s}_{\ell,m}(a,x)\,R^{\epsilon}_{\ell,m,s}(\gamma) \qquad \text{(3-234e)}$$

$$+\ \delta_s\,H^{\epsilon}_{t,s,m}\sum_{\ell=\max(m,s+2)}^{L} S_{\ell,m}\,E^{s}_{\ell,m}(a,x)\,R^{\epsilon}_{\ell,m,s}(\gamma)\Bigg]$$

$$\frac{\partial\langle R\rangle}{\partial\beta} = \frac{\mu}{a}\sum_{s=0}^{S}\sum_{m=0}^{M}\left\{\begin{matrix}1\\ \dfrac{\sin\frac{\omega}{n}m\pi}{\frac{\omega}{n}m\pi}\end{matrix}\right\}\sum_{\epsilon=\pm1}(\epsilon s-Im)$$

$$\times\Bigg[\delta_s\,H^{\epsilon}_{t-1,s,m}\sum_{\ell=\max(m,s+2)}^{L} C_{\ell,m}\,E^{s}_{\ell,m}(a,x)\,R^{\epsilon}_{\ell,m,s} \qquad \text{(3-234f)}$$

$$-\ \delta_s\,G^{\epsilon}_{t-1,s,m}\sum_{\ell=\max(m,s+2)}^{L} S_{\ell,m}\,E^{s}_{\ell,m}(a,x)\,R^{\epsilon}_{\ell,m,s}(\gamma)\Bigg]$$

$$\frac{\partial \langle R \rangle}{\partial \gamma} = \frac{\mu}{a} \sum_{s=0}^{S} \sum_{m=0}^{M} \left\{ \begin{matrix} 1 \\ \dfrac{\sin \frac{\omega}{\bar{n}} m\pi}{\frac{\omega}{\bar{n}} m\pi} \end{matrix} \right\} \sum_{\epsilon=\pm 1} \left[\delta_s G^{\epsilon}_{t,s,m} \right.$$

$$\times \sum_{\ell=\max(m,s+2)}^{L} C_{\ell,m} E^{s}_{\ell,m}(a,x) \frac{d}{d\gamma} R^{\epsilon}_{\ell,m,s}(\gamma) \qquad (3\text{-}234g)$$

$$\left. + \delta_s H^{\epsilon}_{t,s,m} \sum_{\ell=\max(m,s+2)}^{L} S_{\ell,m} E^{s}_{\ell,m}(a,x) \frac{d}{d\gamma} R^{\epsilon}_{\ell,m,s}(\gamma) \right]$$

Two linear combinations of the indexes s and m appear in braces in Equations (3-234b), (3-234c), and (3-234e). The top quantity appearing inside the braces is valid for $m < s$ and the bottom quantity is valid for $m \geq s$. In addition, the averaging factor is unity for $m = 0$ or $\omega/\bar{n} \approx 0$ and is

$$\frac{\sin \frac{\omega}{\bar{n}} m\pi}{\frac{\omega}{\bar{n}} m\pi}$$

for $m \neq 0$ or $\omega/\bar{n} \not\approx 0$.

The following recurrence relations are used to evaluate the functions in the partial derivatives.

Recurrence Relations for the Polynomials $G_{t,s,m}$ and $H_{t,s,m}$

The general recurrence relations take the form

$$G^{\epsilon}_{t+a,\, s+b,\, m+c} = G^{\epsilon}_{t,s,m}\, G^{\epsilon}_{a,b,c} - H^{\epsilon}_{t,s,m}\, H^{\epsilon}_{a,b,c} \qquad (3\text{-}235a)$$

$$H^{\epsilon}_{t+a,\, s+b,\, m+c} = G^{\epsilon}_{t,s,m}\, H^{\epsilon}_{a,b,c} + H^{\epsilon}_{t,s,m}\, G^{\epsilon}_{a,b,c} \qquad (3\text{-}235b)$$

These recurrence relations will be represented by the notation

$$(t+a,\ s+b,\ m+c) \longleftarrow (t,s,m) \times (a,b,c) \qquad (3\text{-}236)$$

for $a \geq 0$, $b \geq 0$, and $c \geq 0$.

The recurrence relations used when $m < s$, for which $t = s - \epsilon Im$ are

$$(s-\epsilon Im,\ s,\ m) \longleftarrow (s-\epsilon Im-1,\ s-1,\ m) \times (1,1,0) \qquad (3\text{-}237a)$$

$$(s-\epsilon Im-1,\ s,\ m) \longleftarrow (s-\epsilon Im-1,\ s-1,\ m) \times (0,1,0) \qquad (3\text{-}237b)$$

$$(s-\epsilon Im,\ s-1,\ m) \longleftarrow (s-\epsilon Im-1,\ s-1,\ m) \times (1,0,0) \qquad (3\text{-}237c)$$

$$(s-\epsilon Is,\ s,\ s) \longleftarrow (s-1-\epsilon I(s-1),\ s-1,\ s-1) \times (1-\epsilon I,\ 1,\ 1) \qquad (3\text{-}237d)$$

where

$$G^{\epsilon}_{0,0,0} = 1 \qquad (3\text{-}238a)$$

$$H_{0,0,0} = 0 \qquad (3\text{-}238b)$$

$$G^{\epsilon}_{1,1,0} = \frac{\alpha k + \beta h}{2} \tag{3-239a}$$

$$H^{\epsilon}_{1,1,0} = \frac{\epsilon}{2}(\alpha h - \beta k) \tag{3-239b}$$

$$G^{\epsilon}_{0,1,0} = \frac{k}{2} \tag{3-240a}$$

$$H^{\epsilon}_{0,1,0} = \frac{\epsilon h}{2} \tag{3-240b}$$

$$G^{\epsilon}_{1,0,0} = \alpha \tag{3-241a}$$

$$H^{\epsilon}_{1,0,0} = -\epsilon\beta \tag{3-241b}$$

The term designated by

$$(1-\epsilon I, 1, 1) = \begin{cases} (0,1,1) & \text{for } \epsilon I = 1 & \text{(3-242a)} \\ (2,1,1) & \text{for } \epsilon I = -1 & \text{(3-242b)} \end{cases}$$

is easily obtained, since for $\epsilon I = 1$,

$$(0,1,1) \longleftarrow (0,1,0) \times (0,0,1) \tag{3-243}$$

where

$$G_{0,0,1} = \epsilon I \cos\theta_0 \tag{3-244a}$$

$$H_{0,0,1} = -\epsilon I \sin\theta_0 \tag{3-244b}$$

and for $\epsilon I = -1$,

$$(2,1,1) \leftarrow (0,1,1) \times (2,0,0) \tag{3-245}$$

and

$$(2,0,0) \leftarrow (1,0,0) \times (1,0,0) \tag{3-246}$$

The following recurrence relations are used for $m \geq s = 0$, for which $t = m - \epsilon Is = m$:

$$(m,0,m) \leftarrow (m-1,0,m-1) \times (1,0,1) \tag{3-247a}$$

$$(m-1,0,m) \leftarrow (m-1,0,m-1) \times (0,0,1) \tag{3-247b}$$

where

$$G_{0,0,0} = 1 \tag{3-248a}$$

$$H_{0,0,0} = 0 \tag{3-248b}$$

$$G_{1,0,1} = -\frac{\epsilon}{2}\left(I\alpha\cos\theta_0 + \beta\sin\theta_0\right) \tag{3-249a}$$

$$H_{0,0,1} = \frac{\epsilon}{2}\left(I\alpha\sin\theta_0 - \beta\cos\theta_0\right) \tag{3-249b}$$

$$G_{0,0,1} = -\frac{\epsilon I}{2}\cos\theta_0 \tag{3-250a}$$

$$H_{0,0,1} = \frac{\epsilon I}{2}\sin\theta_0 \tag{3-250b}$$

The following recurrence relations are used for the case $m \geq s > 0$:

$$(m-\epsilon I s,\ s,\ m) \longleftarrow (m-\epsilon I s-1,\ s-1,\ m-1) \times (1,1,1) \qquad (3\text{-}251a)$$

$$(m-\epsilon I s-1,\ s,\ m) \longleftarrow (m-\epsilon I s-1,\ s-1,\ m-1) \times (0,1,1) \qquad (3\text{-}251b)$$

$$(m-\epsilon I s,\ s-1,\ m) \longleftarrow (m-\epsilon I s-1,\ s-1,\ m-1) \times (1,1,0) \qquad (3\text{-}251c)$$

$$(s-\epsilon I s,\ s,\ s) \longleftarrow (s-1-\epsilon I(s-1),\ s-1,\ s-1) \times (1-\epsilon I,\ 1,\ 1) \qquad (3\text{-}252a)$$

$$(s-\epsilon I s-1,\ s,\ s) \longleftarrow (s-1-\epsilon I(s-1),\ s-1,\ s-1) \times (-\epsilon I,\ 1,\ 1) \qquad (3\text{-}252b)$$

$$(s-\epsilon I s,\ s-1,\ s) \longleftarrow (s-1-\epsilon I(s-1),\ s-1,\ s-1) \times (1-\epsilon I,\ 0,\ 1) \qquad (3\text{-}252c)$$

Clearly,

$$(1-\epsilon I,\ 1,\ 1) = \begin{cases} (0,1,1) & \text{(for } \epsilon I = 1) \qquad (3\text{-}253a) \\ (2,1,1) & \text{(for } \epsilon I = -1) \qquad (3\text{-}253b) \end{cases}$$

$$(-\epsilon I,\ 1,\ 1) = \begin{cases} (-1,1,1) = 0 & \text{(for } \epsilon I = 1) \qquad (3\text{-}254a) \\ (1,1,1) & \text{(for } \epsilon I = -1) \qquad (3\text{-}254b) \end{cases}$$

$$(1-\epsilon I,\ 0,\ 1) = \begin{cases} (0,0,1) & \text{(for } \epsilon I = 1) \qquad (3\text{-}255a) \\ (2,0,1) & \text{(for } \epsilon I = -1) \qquad (3\text{-}255b) \end{cases}$$

These quantities are easily constructed from (0, 1, 0), (1, 0, 1), and (0, 0, 1). The quantities (1, 0, 1) and (0, 0, 1) are given by Equations (2-249) and (2-250). The quantity (0, 1, 0) uses the initialization values

$$G_{0,1,0} = -k \tag{3-256a}$$

$$H_{0,1,0} = -\epsilon h \tag{3-256b}$$

Recurrence Relations for the Functions $E^s_{\ell,m}(a)$, $V^s_{\ell,m}(a,x)$, $K_0^{-\ell-1,s}(x)$, and $dK_0^{-\ell-1,s}(x)/dx$

The structure imposed on the expressions for the disturbing function (Equation (3-217)) and its partial derivatives (Equations (3-226)) requires that recurrence relations for varying ℓ must be employed first, followed by the necessary relations for varying m. The recurrence relations for varying ℓ used to evaluate the functions $V^s_{\ell,m}(a)$, $K_0^{-\ell-1,s}(s)$, and $dK_0^{-\ell-1,s}(x)/dx$ require that each of these functions be evaluated separately and the necessary products then formed, i.e., $E^s_{\ell,m}(a,x)$ or, in the case of the partial derivative with respect to x, the product $V^s_{\ell,m}(a) \cdot dK_0^{-\ell-1,s}(x)/dx$. Each of these products is stored in a singly-dimensioned array to provide the necessary back values required for the recurrence relation used to evaluate the products for varying values of m. More specifically, the first time through the summations $s = 0$, $m = 0$, and $2 \le \ell \le L$. The recurrence relation

$$V^s_{\ell,0} = -\left(\frac{a_e}{a}\right)^2 \frac{(\ell+s-1)(\ell-s-1)}{\ell(\ell-1)} V^s_{\ell-2,s} \tag{3-257}$$

is used to evaluate the $V^s_{\ell,m}(a,x)$ function. The function $K_0^{-\ell-1,s}(x)$ is denoted by $D^s_\ell(x)$ in the software implementation and the necessary recurrence relations for $K_0^{-\ell-1,s}(x) \equiv D_\ell^s(x)$ and its derivatives are

$$D_\ell^s(x) = \begin{cases} \dfrac{x^{2s+1}}{2^s} & \text{(for } \ell = s+1) \quad (3\text{-}258a) \\[2ex] (s+1)\,\dfrac{x^{2s+3}}{2^s} & \text{(for } \ell = s+2) \quad (3\text{-}258b) \\[2ex] \dfrac{(\ell-1)\,x^2}{(\ell+s-1)(\ell-s-1)} \left[(2\ell-3)\, D_{\ell-1}^s - (\ell-2) D_{\ell-2}^s \right] & \text{(for } \ell > s+2) \quad (3\text{-}258c) \end{cases}$$

$$\frac{dD_\ell^s(x)}{dx} = \begin{cases} (2s+1)\,\dfrac{x^{2s}}{2^s} & \text{(for } \ell = s+1) \quad (3\text{-}259a) \\[2ex] (s+1)\,x^2\,(2s+3)\,\dfrac{x^{2s}}{2^s} & \text{(for } \ell = s+2) \quad (3\text{-}259b) \\[2ex] \dfrac{(\ell-1)\,x^2}{(\ell+s-1)(\ell-s-1)} \left[(2\ell-3)\,\dfrac{dD_{\ell-1}^s}{dx} - (n-2)\,\dfrac{dD_{\ell-2}^s}{dx} \right] + \dfrac{2}{x} D_\ell^s & \text{(for } \ell > s+2) \quad (3\text{-}259c) \end{cases}$$

For each value of ℓ, the products

$$E_{\ell,m}^s(a,x) = V_{\ell,m}^s(a)\, D_\ell^s(x)$$

$$\frac{\partial E_{\ell,m}^s(a,x)}{\partial x} = V_{\ell,m}^s(a)\, \frac{dD_\ell^s(x)}{dx}$$

are formed and stored as back values to be used in the recurrence relations

$$E^s_{\ell,m}(a,x) = \begin{cases} (\ell-m+1)\, E^s_{\ell,m-1}(a,x) & (m < s) \qquad (3\text{-}260a) \\ (\ell+m)\, E^s_{\ell,m-1}(a,x) & (m \geq s) \qquad (3\text{-}260b) \end{cases}$$

$$\frac{\partial E^s_{\ell,m}(a,x)}{\partial x} = \begin{cases} (\ell-m+1)\, \dfrac{\partial E^s_{\ell,m-1}(a,x)}{\partial x} & (m < s) \qquad (3\text{-}261a) \\ (\ell+m)\, \dfrac{\partial E^s_{\ell,m-1}(a,x)}{\partial x} & (m \geq s) \qquad (3\text{-}261b) \end{cases}$$

Each value of the functions $E^s_{\ell,m}(a,x)$ and $\partial E^s_{\ell,m}(a,x)\ \partial x$ are stored over the previous values $E^s_{\ell,m-1}(a,x)$ and $\partial E^s_{\ell,m-1}(a,x)\ \partial x$ as they are evaluated. After the summation over m $(0 \leq m \leq M)$ is performed, the value of s is increased by one, the value of m is reset to zero, and the recurrence relations over ℓ $[\max(m, s+2) \leq \ell \leq L]$ are repeated for the new value of s and the cycle repeats.

Recurrence Relations for the Polynomials $R^{\epsilon}_{\ell,m,s}(\Upsilon)$ and $dR^{\epsilon}_{\ell,m,s}(\Upsilon)\ d\Upsilon$

Recurrence Relations for the polynomials $R^{\epsilon}_{\ell,m,s}(\Upsilon)$ defined in Equations (3-222) are based on the recurrence relation for the Jacobi polynomial given in Equation (3-204). The recurrence is performed only over the index ℓ. Changes in the values of the indexes m and s are incorporated into the starting values for the recurrence over ℓ.

The recurrence relations for $m < s$ are

$$R^{\epsilon}_{s,m,s}(\gamma) = (1+I\gamma)^{\epsilon I m} \qquad (3\text{-}262)$$

$$R^{\epsilon}_{s+1,m,s}(\gamma) = \left[(s+1)\epsilon\gamma - m\right] R^{\epsilon}_{s,m,s}(\gamma) \qquad (3\text{-}262b)$$

$$R^{\epsilon}_{\ell,m,s}(\gamma) = \frac{1}{(\ell-1)(\ell+s)(\ell-s)} \Big\{ (2\ell-1)\left[\ell(\ell-1)\epsilon\gamma - ms\right] R^{\epsilon}_{\ell-1,m,s}(\gamma)$$

$$(\ell \geq s+2) \quad (3\text{-}262c)$$

$$- \ell(\ell+m-1)(\ell-m-1) R^{\epsilon}_{\ell-2,m,s}(\gamma) \Big\}$$

and

$$\frac{dR^{\epsilon}_{s,m,s}(\gamma)}{d\gamma} = \epsilon m (1+I\gamma)^{\epsilon I m - 1} = \frac{\epsilon I m}{(1+I\gamma)} R^{\epsilon}_{s,m,s}(\gamma) \qquad (3\text{-}263a)$$

$$\frac{dR^{\epsilon}_{s+1,m,s}(\gamma)}{\partial\gamma} = \left[(s+1)\epsilon\gamma - m\right] \frac{dR^{\epsilon}_{s,m,s}(\gamma)}{d\gamma} + \epsilon(s+1) R^{\epsilon}_{s,m,s}(\gamma) \qquad (3\text{-}263b)$$

$$\frac{dR^{\epsilon}_{\ell,m,s}(\gamma)}{d\gamma} = \frac{1}{(\ell-1)(\ell+s)(\ell-s)} \Big\{ (2\ell-1)\left[\ell(\ell-1)\epsilon\gamma - ms\right]$$

$$\times \frac{dR^{\epsilon}_{\ell-1,m,s}(\gamma)}{d\gamma} - \ell(\ell+m-1)(\ell-m-1)\frac{dR^{\epsilon}_{\ell-2,m,s}(\gamma)}{d\gamma} \quad (\ell \geq s+2) \quad (3\text{-}263c)$$

$$+ (2\ell-1)\ell(\ell-1)\epsilon R^{\epsilon}_{\ell-1,m,s}(\gamma) \Big\}$$

The recurrence relations for $m \geq s$ are

$$R^{\epsilon}_{m,m,s}(\gamma) = (1 + I\gamma)^{\epsilon I s} \quad \text{(3-264a)}$$

$$R^{\epsilon}_{m+1,m,s}(\gamma) = \left[(m+1)\epsilon\gamma - s\right] R^{\epsilon}_{m,m,s}(\gamma) \quad \text{(3-264b)}$$

$$R^{\epsilon}_{\ell,m,s}(\gamma) = \frac{1}{(\ell-1)(\ell+m)(\ell-m)} \Big\{ (2\ell-1)\left[\ell(\ell-1)\epsilon\gamma - ms\right] \quad (\ell \geq m+2) \quad \text{(3-264c)}$$

$$\times R^{\epsilon}_{\ell-1,m,s}(\gamma) - \ell(\ell+s-1)(\ell-s-1) R^{\epsilon}_{\ell-2,m,s}(\gamma)$$

and

$$\frac{dR^{\epsilon}_{m,m,s}(\gamma)}{d\gamma} = \epsilon s (1 + I\gamma)^{\epsilon I s - 1} = \frac{\epsilon I s}{(1 + I\gamma)} R^{\epsilon}_{m,m,s}(\gamma) \quad \text{(3-265a)}$$

$$\frac{dR^{\epsilon}_{m+1,m,s}(\gamma)}{d\gamma} = \left[(m+1)\epsilon\gamma - s\right] \frac{dR^{\epsilon}_{m,m,s}(\gamma)}{d\gamma} + \epsilon (m+1) R^{\epsilon}_{m,m,s}(\gamma) \quad \text{(3-265b)}$$

$$\frac{dR^{\epsilon}_{\ell,m,s}(\gamma)}{d\gamma} = \frac{1}{(\ell-1)(\ell+m)(\ell-m)} \Big\{ (2\ell-1)\left[\ell(\ell-1)\epsilon\gamma - ms\right]$$

$$\times \frac{dR^{\epsilon}_{\ell-1,m,s}(\gamma)}{d\gamma} - \ell(\ell+s-1)(\ell-s-1) \quad (\ell \geq m+2) \quad \text{(3-265c)}$$

$$\times \frac{dR^{\epsilon}_{\ell-2,m,s}(\gamma)}{d\gamma} + (2\ell-1)\ell(\ell-1)\epsilon R^{\epsilon}_{\ell-1,m,s}(\gamma) \Big\}$$

3.4.3 The Averaged Equations of Motion for the Resonant Tesseral Harmonics Model

The averaged equations of motion for the resonant tesseral harmonics model were implemented in the R&D version of GTDS in such a manner as to provide the flexibility for the user to specify any set of spherical harmonic terms of degree and order (ℓ, m) desired up to a maximum of 10 such terms. (This limit is reflected in the specific dimension statements and is easily modified.) No particular relationship is assumed among any of these terms. This can result in the recomputation of certain quantities common to two or more terms or in an increase in the cost of evaluating some of the necessary functions.

The form of the averaged disturbing function for the resonant tesseral harmonics model given in Equation (3-165) is used for the development of the averaged equations of motion. The expression for the averaged disturbing function for a single term of degree and order (ℓ, m) takes the form

$$\left\langle R^{RT}_{\ell,m} \right\rangle_{\bar{\lambda}} = R_0 \sum_{i=1}^{4} R_i \tag{3-266}$$

where

$$R_0 = \frac{\mu}{a}\left(\frac{a_e}{a}\right)^{\ell} \left(C_{\ell,m} - j\, S_{\ell,m}\right) \frac{\sin(\delta m \pi)}{\delta m \pi}\, e^{j(kN'\lambda - m\theta)}\, I^{m} \tag{3-267a}$$

$$R_1 = \sum_{\substack{s=m+1 \\ (\ell \pm s \text{ even})}}^{\ell} (-1)^{m}\, 2^{-s}\, C^{(1)}_{\ell,s}\, (\alpha + j\beta)^{Im+s}\, (k + jh)^{|-s-kN'|} \times K^{-\ell-1,-s}_{kN'}\, (1 + I\gamma)^{-Im}\, P^{m+s,\, m-s}_{\ell-s}(\gamma) \tag{3-267b}$$

since[1] $\eta = \mathrm{sgn}\,(s + kN') = +1$,

$$R_2 = \sum_{\substack{s=1 \\ (\ell \pm s \text{ even})}}^{m} (-2)^{-m} C^{(2)}_{\ell,s} (\alpha + jI\beta)^{m+Is} (k + jh)^{|-s-kN'|}$$

$$\times\, K_{kN'}^{-\ell-1,-s} (1+I\gamma)^{-Is} P_{\ell-m}^{m+s,m-s}(\gamma) \qquad (3\text{-}267c)$$

since[1] $\eta = \mathrm{sgn}\,(kN' + s) = +1$,

$$R_3 = \sum_{\substack{s=0 \\ (\ell \pm s \text{ even})}}^{m} (-2)^{-m} (-1)^{s} C^{(2)}_{\ell,s} (\alpha + jI\beta)^{m-Is} (k - j\eta h)^{|s-kN'|}$$

$$\times\, K_{kN'}^{-\ell-1,s} (1+I\gamma)^{Is} P_{\ell-m}^{m-s,m+s}(\gamma) \qquad (3\text{-}267d)$$

where $\eta = \mathrm{sgn}\,(kN' - s)$,

$$R_4 = \sum_{\substack{s=m+1 \\ (\ell \pm s \text{ even})}}^{\ell} 2^{-s} C^{(1)}_{\ell,s} (\alpha - j\beta)^{s-Im} (k - j\eta h)^{|s-kN'|} \qquad (3\text{-}267e)$$

$$\times\, K_{kN'}^{-\ell-1,s} (1+I\gamma)^{Im} P_{\ell-s}^{s-m,s+m}(\gamma)$$

where $\eta = \mathrm{sgn}\,(kN' - s)$

---

[1] The ratio $\dot{\lambda}/\dot{\theta} = N/N'$ is always positive if the retrograde equinoctial reference system is used for retrograde cases. Also, only nonnegative values of m appear in the disturbing function. Thus, it is easily shown that both k and kN' must be positive integers.

$$C^{(1)}_{\ell,s} = \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) = \begin{cases} (-1)^{(\ell-s)/2} \dfrac{(\ell-s-1)!!\,(\ell+s-1)!!}{(\ell-m)!} & (m+1 \le s < \ell) \quad (3\text{-}268a) \\[2em] \dfrac{(2\ell-1)!!}{(\ell+m)!} & (s = \ell) \quad (3\text{-}268b) \end{cases}$$

$$C^{(2)}_{\ell,s} = \frac{(\ell+m)!}{(\ell+s)!} P_{\ell,s}(0) = (-1)^{(\ell-s)/2} \frac{(\ell+m)!}{(\ell+s)!!\,(\ell-s)!!} \quad (s \le m) \qquad (3\text{-}269)$$

In some cases, it may be desirable to truncate on the eccentricity. Therefore, if all powers of the eccentricity through the nth power are retained, then the exponents of the (k, h) polynomials must satisfy the conditions

$$|-s-kN'| \le n \qquad (3\text{-}270)$$

or

$$-n-kN' \le s \le n-kN' \qquad (3\text{-}271)$$

in sums R_1 and R_2, and

$$|s-kN'| \le n \qquad (3\text{-}272)$$

or

$$kN'-n \le s \le kN'+n \qquad (3\text{-}273)$$

in sums R_3 and R_4. The summation limits for the four sums are then

$$R_1 \sim \sum_{s=m+1}^{[n-kN',\ell]} \tag{3-274a}$$

$$R_2 \sim \sum_{s=1}^{[n-kN',m]} \tag{3-274b}$$

$$R_3 \sim \sum_{s=\max(0,kN'-n)}^{[kN'+n,m]} \tag{3-274c}$$

$$R_4 \sim \sum_{s=\max(m+1,kN'-n)}^{[\ell,kN'+n]} \tag{3-274d}$$

where $[x, y]$ denotes the minimum value of x and y.

The four sums are arranged in order of increasing powers of the (α, β) polynomial. Since this exponent depends on the retrograde factor, I, the ordering of the sums is also dependent on the retrograde factor. For $I = 1$, the order chosen is

$$R = R_0(\underline{R_4} + R_3 + R_2 + R_1) \tag{3-275}$$

and for $I = -1$, the order is that given in Equation (3-266), i.e.,

$$R = R_0(\underline{R_1} + R_2 + R_3 + R_4) \tag{3-276}$$

In each case, the placement of the underlined sum in the ordering scheme is arbitrary because of the appearance of the complex conjugate of the (α, β) polynomials which appear in the other three sums. These conjugate polynomials are computed independently of the other polynomials in order to avoid the possibility of a small divisor.

Although the four sums are arranged in order of increasing powers of the (α, β) polynomial, inspection of Equations (3-267) indicates that the (α, β) polynomials are not ordered similarly inside all summations. Specifically, for $I = 1$, the exponent of the (α, β) polynomial progresses in decreasing order in the summation R_3. Similarly, for the retrograde case $I = -1$, the exponents of the (α, β) polynomial decrease in the summation R_2. Thus, these polynomials (the real and imaginary parts) are evaluated and stored prior to performing either of these particular summations.

The (k, h) polynomials were not arranged in order of ascending powers because of a conflict with the order of the (α, β) polynomials and because their exponents are not necessarily monotonic functions of the index s. These polynomials are evaluated and stored prior to the summation in which they appear.

The partial derivatives required for the equations of motion are developed next. Inspection of Equations (3-267) shows that

$$R_0 = R_0(\bar{a}, \bar{\lambda}) \tag{3-277}$$

and

$$R_i = R_i(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{h}, \bar{k}) \qquad (i = 1, 2, 3, 4) \tag{3-278}$$

The functions $K_{kN'}^{-\ell-1,s}$ are infinite power series in e^2 or, equivalently, in h^2-k^2. No finite form in $x = 1/(1-e^2)$ exists and, thus, it is not introduced in this model. The partial derivatives take the form

$$\frac{\partial R}{\partial a} = \frac{\partial R_0}{\partial a} \sum_{i=1}^{4} R_i = -\frac{(\ell+1)}{a} R_0 \sum_{i=1}^{4} R_i \tag{3-279a}$$

$$\frac{\partial R}{\partial \lambda} = \frac{\partial R_0}{\partial \lambda} \sum_{i=1}^{4} R_i$$

$$= -\frac{\mu}{a}\left(\frac{a_e}{a}\right)^{\ell} (C_{\ell,m} - jS_{\ell,m})\, jkN' \, \frac{\sin(\delta m\pi)}{\delta m\pi} \, e^{j(kN'\lambda - m\theta)} \, I^m \sum_{i=1}^{4} R_i \tag{3-279b}$$

$$= -jkN' R_0 \sum_{i=1}^{4} R_i$$

The partial derivatives of the disturbing function with respect to α, β, γ, h, k are of the general form

$$\frac{\partial R}{\partial y} = R_0 \sum_{i=1}^{4} \frac{\partial R_i}{\partial y_n} \tag{3-280}$$

The bar notation designating mean elements is dropped in the above and following equations. The elements are to be interpreted as mean elements and the disturbing function or its constituent sums are understood to be "averaged."

The partial derivatives of the first sum R_1 are

$$\frac{\partial R_1}{\partial \alpha} = \sum_{\substack{s=m+1 \\ (\ell \pm s \text{ even})}}^{[n-kN',\ell]} (-1)^m \, 2^{-s} \, C^{(1)}_{\ell,s} \, (Im+s)(\alpha+j\beta)^{Im+s-1} (k+jh)^{|-s-kN'|}$$

$$\times K^{-\ell-1,s}_{kN'} \, (1+I\gamma)^{-Im} \, P^{m+s,m-s}_{\ell-s}(\gamma) \tag{3-281a}$$

$$\frac{\partial R_1}{\partial \beta} = \sum_{\substack{s=m+1 \\ (\ell \pm s \text{ even})}}^{[n-kN',\ell]} (-1)^m \, 2^{-s} \, C^{(1)}_{\ell,s} \, (Im+s) j (\alpha+j\beta)^{Im+s-1} (k+jh)^{|-s-kN'|}$$

$$\times K^{-\ell-1,s}_{kN'} \, (1+I\gamma)^{-Im} \, P^{m+s,m-s}_{\ell-s}(\gamma) \tag{3-281b}$$

$$\frac{\partial R_1}{\partial \gamma} = \sum_{\substack{s=m+1 \\ (\ell \pm s \text{ even})}}^{[n-kN',\ell]} (-1)^m \, 2^{-s} \, C^{(1)}_{\ell,s} \, (\alpha+j\beta)^{Im+s} (k+jh)^{|-s-kN'|}$$

$$\times K^{-\ell-1,s}_{kN'} \left[-Im(1+I\gamma)^{-Im-1} \, P^{m+s,m-s}_{\ell-s}(\gamma) \right. \tag{3-281c}$$

$$\left. + (1+I\gamma)^{-Im} \frac{d}{d\gamma} P^{m+s,m-s}_{\ell-s}(\gamma) \right]$$

$$\frac{\partial R_1}{\partial h} = \sum_{\substack{s=m+1 \\ (l\pm s \text{ even})}}^{[n-kN',l]} (-1)^m\, 2^{-s}\, C^{(1)}_{l,s}\, (\alpha+j\beta)^{Im+s} (1+I\gamma)^{-Im}\, P^{m+s,m-s}_{l-s}(\gamma)$$

$$\times \left[j\,(s+kN')\,(k+jh)^{|-s-kN'|-1}\, K^{-l-1,s}_{kN'} \right. \qquad (s+kN' \geq 0) \qquad \text{(3-281d)}$$

$$\left. + (k+jh)^{|-s-kN'|} \frac{dK^{-l-1,s}_{kN'}}{dh} \right]$$

$$\frac{\partial R_1}{\partial k} = \sum_{\substack{s=m+1 \\ (l\pm s \text{ even})}}^{[n-kN',l]} (-1)^m\, 2^{-s}\, C^{(1)}_{l,s}\, (\alpha+j\beta)^{Im+s} (1+I\gamma)^{-Im}\, P^{m+s,m-s}_{l-s}(\gamma)$$

$$\times \left[(s+kN')\,(k+jh)^{|-s-kN'|-1}\, K^{-l-1,s}_{kN'} \right. \qquad (s+kN' \geq 0) \qquad \text{(3-281e)}$$

$$\left. + (k+jh)^{|-s-kN'|} \frac{dK^{-l-1,s}_{kN'}}{dk} \right]$$

If the real and imaginary parts of the complex polynomial $(\alpha + j\beta)^t$ and $(k + jh)^t$ are denoted by

$$A_t + jB_t = (\alpha+j\beta)^t \qquad \text{(3-282)}$$

$$G_r + jH_r = (k+jh)^r \qquad \text{(3-283)}$$

and, since $s - kN' \geq 0$ in R_1, Equations (2-281) are expressed as

$$\frac{\partial R_1}{\partial \alpha} = \sum_{s=m+1}^{[n-kN',\ell]} (-1)^m 2^{-s} C^{(1)}_{\ell,s} (Im+s)\left(A_{Im+s-1} + jB_{Im+s-1}\right)$$

(3-284a)

$$\times \left(G_{s+kN'} + jH_{s+kN'}\right) K^{-\ell-1,s}_{kN'} (1+I\gamma)^{-Im} P^{m+s,m-s}_{\ell-s}(\gamma)$$

$$\frac{\partial R_1}{\partial \beta} = \sum_{s=m+1}^{[n-kN',\ell]} (-1)^m 2^{-s} C^{(1)}_{\ell,s} (Im+s)\left(-B_{Im+s-1} + jA_{Im+s-1}\right)$$

(3-284b)

$$\times \left(G_{s+kN'} + jH_{s+kN'}\right) K^{-\ell-1,s}_{kN'} (1+I\gamma)^{-Im} P^{m+s,m-s}_{\ell-s}(\gamma)$$

$$\frac{\partial R_1}{\partial \gamma} = \sum_{s=m+1}^{[n-kN',\ell]} (-1)^m 2^{-s} C^{(1)}_{\ell,s} \left(A_{Im+s} + jB_{Im+s}\right)\left(G_{s+kN'} + jH_{s+kN'}\right)$$

(3-284c)

$$\times K^{-\ell-1,s}_{kN'} \left[-Im\,(1+I\gamma)^{-Im-1} P^{m+s,m-s}_{\ell-s}(\gamma) + (1+I\gamma)^{-Im} \frac{dP^{m+s,m-s}_{\ell-s}(\gamma)}{d\gamma} \right]$$

$$\frac{\partial R_1}{\partial h} = \sum_{s=m+1}^{[n-kN',\ell]} (-1)^m 2^{-s} C^{(1)}_{\ell,s} \left(A_{Im+s} + jB_{Im+s}\right)(1+I\gamma)^{-Im} P^{m+s,m-s}_{\ell-s}(\gamma)$$

$$\times (s+kN')\left(-H_{s+kN'-1} + jG_{s+kN'-1}\right) K^{-\ell-1,s}_{kN'} \tag{3-284d}$$

$$+ \left(G_{s+kN'} + jH_{s+kN'}\right) \frac{dK^{-\ell-1,s}_{kN'}}{dh}$$

$$\frac{\partial R_1}{\partial k} = \sum_{s=m+1}^{[n-kN',\ell]} (-1)^m 2^{-s} C^{(1)}_{\ell,s} \left(A_{Im+s} + jB_{Im+s}\right)(1+I\gamma)^{-Im} P^{m+s,m-s}_{\ell-s}(\gamma)$$

$$\times (s+kN')\left(G_{s+kN'-1} + jH_{s+kN'-1}\right) K^{-\ell-1,s}_{kN'} \tag{3-284e}$$

$$+ \left(G_{s+kN'} + jH_{s+kN'}\right) \frac{dK^{-\ell-1,s}_{kN'}}{dk}$$

The partial derivatives of the other three sums are similar in form and can be obtained in a straightforward manner.

Finally, the real parts of the partial derivatives are obtained as follows. If

$$U + jV = \sum_{i=1}^{4} R_i \tag{3-285a}$$

and

$$U_\xi + jV_\xi = \sum_{i=1}^{4} \frac{\partial R_i}{\partial \xi} \tag{3-285b}$$

where ξ is any of the parameters $(\alpha, \beta, \gamma, h, k)$, and, similarly, if

$$X + jY = R_0 \tag{3-286a}$$

and

$$X_\eta + Y_\eta = \frac{\partial R_0}{\partial \eta} \tag{3-286b}$$

where η is either a or λ, then the real part of the partial derivatives of the disturbing function take the general form

$$\frac{\partial \langle R^{RT} \rangle_\lambda}{\partial \eta} = X_\eta U - Y_\eta V \tag{3-287a}$$

$$\frac{\partial \langle R^{RT} \rangle_\lambda}{\partial \xi} = X U_\xi - Y V_\xi \tag{3-287b}$$

The recurrence relations used for evaluating the functions in the partial derivatives are determined next. Since this theory was implemented separately from the nonresonant theory in Section 3.4.2, some of the recurrence relations used may differ slightly.

Recurrence Relations for the (α, β) Polynomials and the (k, h) Polynomials

Recurrence relations for the complex polynomials $(\alpha + j\beta)^t$ and $(k + j\eta(h))^r$ are straightforward and are special cases of the recurrence relations for the $G^{\epsilon}_{t,s,m}$ and $H^{\epsilon}_{t,s,m}$ polynomials discussed in the previous section. Specifically, if

$$A_t + jB_t = \begin{cases} (\alpha + j\beta)^t & \text{in } R_1 \quad (3\text{-}288a) \\ (\alpha + jI\beta)^t & \text{in } R_2 \text{ and } R_3 \quad (3\text{-}288b) \\ (\alpha - j\beta)^t & \text{in } R_4 \quad (3\text{-}288c) \end{cases}$$

then

$$A_t = A_{t-1}A_1 - B_{t-1}B_1 \tag{3-289a}$$

$$B_t = A_{t-1}B_1 + B_{t-1}A_1 \tag{3-289b}$$

where

$$A_0 = 1 \tag{3-290a}$$

$$B_0 = 0 \; . \tag{3-290b}$$

$$A_1 = \alpha \tag{3-291a}$$

$$B_1 = \begin{cases} \beta & \text{in } R_1 \quad (3\text{-}291b) \\ I\beta & \text{in } R_2 \text{ and } R_3 \quad (3\text{-}291c) \\ -\beta & \text{in } R_4 \quad (3\text{-}291d) \end{cases}$$

The recurrence relations for the (k, h) polynomial are identical in form; i.e., if

$$G_r + jH_r = \begin{cases} (k+jh)^r & \text{in } R_1 \text{ and } R_2 \quad (3\text{-}292a) \\ (k+j\eta h)^r & \text{in } R_3 \text{ and } R_4 \quad (3\text{-}292b) \end{cases}$$

then

$$G_r = G_{r-1}G_1 - H_{r-1}H_1 \tag{3-293a}$$

$$H_r = G_{r-1}H_1 + H_{r-1}G_1 \tag{3-293b}$$

where

$$G_0 = 1 \tag{3-294a}$$

$$H_0 = 0 \tag{3-294b}$$

$$G_1 = k \tag{3-295a}$$

$$H_1 = \begin{cases} h & \text{in } R_1 \text{ and } R_2 \quad (3\text{-}295b) \\ \eta h & \text{in } R_3 \text{ and } R_4 \quad (3\text{-}295c) \end{cases}$$

Recurrence Relation for the Jacobi Polynomial and Its Derivative

The recurrence relation for the Jacobi polynomial given in Equation (3-204) is used. The recurrence relation for the derivative of the Jacobi polynomial is obtained by differentiating Equation (3-204). The factors $(1 + I\Upsilon)^{-Im}$ and $(1 + I\Upsilon)^{-Is}$ and their derivatives are computed explicitly.

Evaluation of the Functions $K_{kN'}^{-\ell-1,\pm s}$, $dK_{kN'}^{-\ell-1,\pm s}/dh$, and $dK_{kN'}^{-\ell-1,\pm s}/dk$

These functions are evaluated using the Newcomb operator method discussed in Section 2.2.1.3.3. From Equation (2-303),

$$K_{kN'}^{-\ell-1,\pm s} = e^{|kN' \mp s|} X_{kN'}^{-\ell-1,\pm s} \tag{3-296}$$

In view of Equation (2-185),

$$X_{kN'}^{-\ell-1,\pm s} = \sum_{m=0}^{\infty} \Pi_{|kN' \mp s|}^{2m+|kN' \mp s|} e^{2m+|kN' \mp s|} \tag{3-297}$$

Therefore, through the nth power of the eccentricity,

$$K_{kN'}^{-\ell-1,\pm s} = \sum_{m=0}^{n/2} \Pi_{|kN' \mp s|}^{2m+|kN' \mp s|} e^{2m} \tag{3-298}$$

In view of Equation (2-191), this can be expressed as

$$K_{kN'}^{-\ell-1,\pm s} = \sum_{m=0}^{n/2} X_{\rho,\sigma}^{-\ell-1,\pm s} e^{2m} \tag{3-299}$$

where

$$\rho = m + \frac{|kN' \mp s| + (kN' \mp s)}{2} \tag{3-300a}$$

$$\sigma = m + \frac{|kN' \mp s| - (kN' \mp s)}{2} \tag{3-300b}$$

Also, since

$$e^2 = h^2 + k^2 \tag{3-301}$$

it follows that

$$\frac{\partial}{\partial h} K_{kN'}^{-\ell-1,\mp s} = \sum_{m=0}^{(n+1)/2} 2m\, X_{\rho,\sigma}^{-\ell-1,\mp s}\, h\, e^{2m-2} \tag{3-302}$$

and

$$\frac{\partial}{\partial k} K_{kN'}^{-\ell-1,\mp s} = \sum_{m=0}^{(n+1)/2} 2m\, X_{\rho,\sigma}^{-\ell-1,\mp s}\, k\, e^{2m-2} \tag{3-303}$$

(It should be noted that the integer k in the product kN' should not be confused with the equinoctial element k.)

The upper limit of the summation over m in Equations (3-302) and (3-303) is increased from $n/2$ to $(n+1)/2$ to guarantee that the partial derivatives with respect to h and k contain all terms through order e^n.

The Newcomb operators are computed with the recurrence relations given in Equations (2-196) and (2-198). They are stored in a singly-dimensioned array in the order in which they appear in the disturbing function.

SECTION 4 - EXPLICIT THEORY FOR THE DISTURBING THIRD-BODY PERTURBATION

This section presents the explicit development of the third-body disturbing function in terms of the equinoctial elements of the satellite. Also, the first-order averaged equations of motion for the special case of the nonresonant near-Earth (Moon) satellite are presented.

Section 4.1 discusses the development of the third-body disturbing function from the equations of motion for three mutually gravitating point masses. Section 4.2 describes the representation of the disturbing function in terms of the equinoctial elements of the satellite and the classical elements of the third body. In Sections 4.2.1 and 4.2.2, the rotation of the satellite-dependent and third-body-dependent spherical harmonic functions are presented. Generally, this formulation introduces two inclination functions, one for the satellite position and one for the third-body position. In Section 4.2.3, it is shown that appropriate selection of the coordinate reference system eliminates one of these inclination functions. The Fourier expansions in the mean longitude of the satellite and the mean anomaly of the third body are introduced in the disturbing function in Section 4.2.4.

Section 4.3 focuses on a special case which is applicable to nonresonant near-Earth satellites. In Section 4.3.1, the third-body disturbing function is developed in terms of the equinoctial elements of the satellite and the direction cosines of the third body. The relationship between this special case and the general case is explored. In Section 4.3.2, the Fourier series expansion in the mean longitude of the satellite is introduced in the disturbing function.

Section 4.4 discusses the averaged disturbing functions for the general and special cases (in Sections 4.4.1 and 4.4.2, respectively) and the averaged equations of motion for the special case are presented in Section 4.5. In addition, the algorithms which were implemented in the Research and Development (R&D) version of the Goddard Trajectory Determination System (GTDS) are described.

4.1 DEVELOPMENT OF THE THIRD-BODY DISTURBING FUNCTION

Generally, a Variation of Parameters (VOP) formulation of the third-body effects on the motion of a satellite assumes that the distance of the disturbing third body from the satellite is great relative to the satellite distance from the central body.[1] At such distances, the effects of the nonspherical gravitational field of the third body can be neglected and, thus, a point mass is assumed. The equations of motion describing the mutual gravitational attraction of three point masses in an arbitrary inertial reference system (see Figure 4-1) follows from the universal law of gravitation and Newton's Second Law of Motion. These equations of motion take the form

$$\ddot{\vec{r}}_c = -Gm_s \frac{(\vec{r}_c - \vec{r}_s)}{|\vec{r}_c - \vec{r}_s|^3} - Gm_T \frac{(\vec{r}_c - \vec{r}_T)}{|\vec{r}_c - \vec{r}_T|^3} \tag{4-1a}$$

$$\ddot{\vec{r}}_s = -Gm_c \frac{(\vec{r}_s \;\; \vec{r}_c)}{|\vec{r}_s - \vec{r}_c|^3} - Gm_T \frac{(\vec{r}_s - \vec{r}_T)}{|\vec{r}_s - \vec{r}_T|^3} \tag{4-1b}$$

$$\ddot{\vec{r}}_T = -Gm_c \frac{(\vec{r}_T - \vec{r}_c)}{|\vec{r}_T - \vec{r}_c|^3} - Gm_s \frac{(\vec{r}_T - \vec{r}_s)}{|\vec{r}_T - \vec{r}_s|^3} \tag{4-1c}$$

where r designates the position vector in an arbitrary inertial reference system and the subscripts c, s, and T designate the central body, satellite, and third body, respectively.

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[1] For the case of a close third body, a restricted three-body problem treatment (Reference 49) is more appropriate.

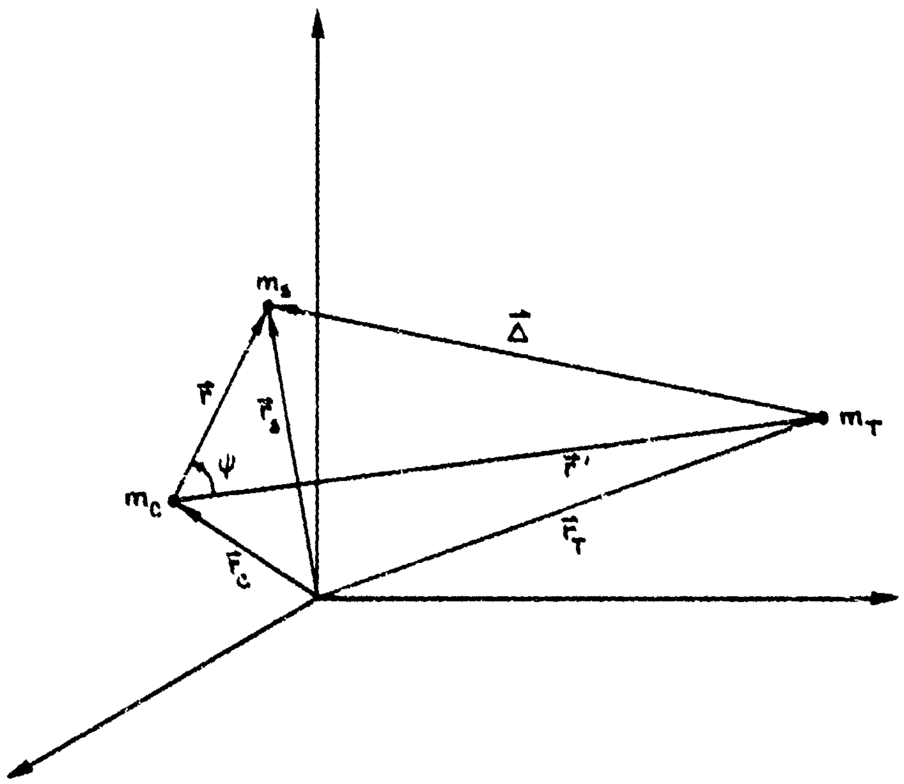

Figure 4-1. Three Point Masses Referred to an Inertial Reference Frame

It is desirable to transfer the origin of the reference system to the center of mass of the central body (see Figure 4-1). This is accomplished through the translation specified by the equations

$$\vec{r} = \vec{r}_s - \vec{r}_c \tag{4-2a}$$

$$\vec{r}\,' = \vec{r}_T - \vec{r}_c \tag{4-2b}$$

It follows that

$$\vec{r}_s - \vec{r}_T = \vec{r} - \vec{r}\,' = \vec{\Delta} \tag{4-3}$$

The equations of motion for the satellite and third body relative to the center of mass of the central body are obtained by subtracting Equation (4-1a) from Equations (4-1b) and (4-1c), respectively, which yields

$$\ddot{\vec{r}} = -G(m_c + \cancel{m_s}^{\,0})\frac{\vec{r}}{|\vec{r}|^3} - G m_T\left(\frac{\vec{\Delta}}{|\vec{\Delta}|^3} - \frac{\vec{r}\,'}{|\vec{r}\,'|^3}\right) \tag{4-4a}$$

$$\ddot{\vec{r}}\,' = -G(m_c + m_T)\frac{\vec{r}\,'}{|\vec{r}\,'|^3} - G \cancel{m_s}^{\,0}\left(\frac{\vec{\Delta}}{|\vec{\Delta}|^3} - \frac{\vec{r}}{|\vec{r}|^3}\right) \tag{4-4b}$$

Since the satellite mass, m_s, is negligible, the second term in Equation (4-4b) can be omitted. Thus, the motion of the disturbing third body is that of the classical two-body problem, which is known.[1] Consequently, this decoupling permits

[1] Equations (4-4) could have been formulated to account for the nonspherical gravitational attraction of the central body on both the satellite and the third body. However, this is not necessary since these effects appear only in the first term on the right-hand side of Equations (4-4) and since the purpose of this discussion is to obtain the disturbing function for the third-body disturbing acceleration represented by the second term in Equation (4-4a).

the solution of Equation (4-4a) without solving Equation (4-4b) simultaneously. Generally, an independently generated ephemeris for the third body is used in the evaluation of the disturbing acceleration in Equation (4-4a).

Since the disturbing acceleration is the result of a conservative force, it can be expressed as the gradient of a potential function. It is easily verified that

$$\frac{\vec{\Delta}}{|\vec{\Delta}|^3} - \frac{\vec{r}'}{|\vec{r}'|^3} = -\nabla\left(\frac{1}{|\vec{\Delta}|} - \frac{\vec{r}\cdot\vec{r}'}{|\vec{r}'|^3}\right) \tag{4-5}$$

where the gradient operator ∇ is defined in Equation (3-2). Thus, Equation (4-4a) can be expressed as

$$\ddot{\vec{r}} + Gm_c\frac{\vec{r}}{|\vec{r}|^3} = \nabla R_3 \tag{4-6}$$

where the disturbing function, R_3, is given by

$$R_3 = Gm_T\left(\frac{1}{|\vec{\Delta}|} - \frac{\vec{r}\cdot\vec{r}'}{|\vec{r}'|^3}\right) \tag{4-7}$$

The first term in the disturbing potential, $1/|\vec{\Delta}|$, is called the direct term and reflects the direct gravitational effects of the disturbing body on the satellite. The second term, referred to as the indirect term, reflects the indirect effects caused by the third body on the satellite through the perturbing effects of the third body on the central body. Specifically, the indirect term accounts for the effects on the satellite motion due to the motion of the central body about the barycenter of the central-body/third-body system caused by the gravitational attraction of the third body.

The expansion of Equation (4-7) is considered next. To obtain the expansion of the disturbing function in either the general case or the special case which is considered later in this section, it is necessary to expand the direct part of the disturbing function in powers of the ratio of the satellite and third-body distances, (r/r').[1] The vector equivalent of the Law of Cosines gives

$$|\vec{\Delta}| = |\vec{r} - \vec{r}'| = \sqrt{|\vec{r}|^2 + |\vec{r}'|^2 - 2\vec{r}\cdot\vec{r}'} \tag{4-8}$$

If the magnitude of a vector $\vec{x}$ is denoted by

$$x = |\vec{x}| \tag{4-9}$$

then, clearly,

$$\Delta = r'\sqrt{1 + \left(\frac{r}{r'}\right)^2 - 2\frac{r}{r'}\cos\psi} \tag{4-10}$$

since

$$\vec{r}\cdot\vec{r}' = r\,r'\cos\psi \tag{4-11}$$

where ψ is the elongation angle between the vectors r and r'. Thus,

$$\frac{Gm_T}{\Delta} = \frac{Gm_T}{r'}\,\frac{1}{\sqrt{1 + \left(\frac{r}{r'}\right)^2 - 2\frac{r}{r'}\cos\psi}} \tag{4-12}$$

[1] The discussion in this section has assumed that $r < r'$; however, this need not be the case. The following theory is equally valid for very distant satellites beyond the orbit of the disturbing third body, i.e., $r > r'$, provided that the expansion is performed in powers of (r'/r).

The second factor is the well-known generating function for the Legendre polynomials (Reference 42) defined in Equation (3-24), i.e.,

$$\left(\sqrt{1+\left(\frac{r}{r'}\right)^2 - 2\frac{r}{r'}\cos\psi}\right)^{-1} = \sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\psi) \tag{4-13}$$

and, therefore, the direct part of the disturbing function takes the form

$$\frac{Gm_T}{\Delta} = \frac{Gm_T}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\psi) \tag{4-14}$$

The first few Legendre polynomials are

$$P_0(x) = 1 \tag{4-15a}$$

$$P_1(x) = x \tag{4-15b}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \tag{4-15c}$$

and so forth. Consequently,

$$\frac{Gm_T}{\Delta} = \frac{Gm_T}{r'}\left[1 + \frac{r}{r'}\cos\psi + \left(\frac{r}{r'}\right)^2\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right) + \cdots\right] \tag{4-16}$$

Clearly,

$$\frac{Gm_T}{r'}\frac{r}{r'}\cos\psi = Gm_T\frac{\vec{r}\cdot\vec{r}'}{|\vec{r}'|^3} \tag{4-17}$$

Thus, the $P_1(\cos\Psi)$ term in the expansion of the direct part of the disturbing potential function cancels the indirect part of the disturbing potential function, and the expansion of the disturbing potential function in powers of (r/r') takes the form

$$R_3 = \frac{G m_T}{r'} \left[1 + \sum_{\ell=2}^{\infty} \left(\frac{r}{r'}\right)^{\ell} P_{\ell}(\cos\Psi) \right] \qquad (4\text{-}18)$$

The object in performing the expansion is to facilitate the development of the partial derivatives with respect to the satellite elements which are required for the Lagrange Planetary Equations. However, the point-mass term is completely independent of the satellite position. Thus, it does not contribute to the equations of motion for the satellite because the requisite partial derivatives are identically zero. Consequently, the above expression, with the point-mass term, $G m_T / r'$, deleted, will be used in the development of the disturbing functions for both the general and special cases.

4.2 A GENERAL EXPANSION OF THE DISTURBING FUNCTION IN EQUINOCTIAL ELEMENTS

The expansion of the disturbing function in terms of the equinoctial elements of the satellite and the classical elements of the third body requires that their distances from the central body and the cosine of the elongation angle, ψ, between them be expressed directly in the elements. The elongation angle is considered first.

It is assumed that the positions of the satellite and third body are given respectively by the spherical coordinates (r, θ, ϕ) and (r', θ', ϕ'), where $r \geq 0$, $0 \leq \theta \leq 2\pi$ and $(-\pi/2) \leq \phi \leq (\pi/2)$, etc., and are measured relative to some right-handed coordinate reference system with the origin located at the center of mass of the central body. For the purposes of discussion, the equatorial reference system is assumed. It follows from spherical trigonometry that

$$\cos\psi = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\theta - \theta') \tag{4-19}$$

Application of the Addition Theorem (Reference 30) for the Legendre polynomials yields

$$\begin{aligned} P_\ell(\cos\psi) &= P_\ell(\sin\phi)\, P_\ell(\sin\phi') \\ &\quad + 2\sum_{m=1}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(\sin\phi)\, P_{\ell,m}(\sin\phi') \cos m(\theta-\theta') \qquad (4\text{-}20) \\ &= 2\sum_{m=0}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(\sin\phi)\, P_{\ell,m}(\sin\phi') \cos m(\theta-\theta') \end{aligned}$$

where

$$\delta_m = \begin{cases} \frac{1}{2} & \text{(for } m = 0) \quad (4\text{-}21a) \\ 1 & \text{(for } m \quad 0) \quad (4\text{-}21b) \end{cases}$$

Since

$$Re \left\{ e^{jm(\theta-\theta')} \right\} = \cos m(\theta-\theta') \qquad (4\text{-}22)$$

Equation (4-20) can be expressed as

$$P_\ell(\cos\psi) = Re \left\{ 2 \sum_{m=0}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(\sin\phi) P_{\ell,m}(\sin\phi') e^{jm(\theta-\theta')} \right\} \qquad (4\text{-}23)$$

The complex variable notation will be used throughout the discussion. The real part of the expressions will be obtained as the final result. In addition, m' will denote the third-body mass in the remainder of this section.

Following Kaula (Reference 17), the following definition is made:

$$C_{\ell,m} + j S_{\ell,m} = \frac{Gm'}{r'^{\ell+1}} \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(\sin\phi') e^{jm\theta'} \qquad (4\text{-}24)$$

The functions $C_{\ell,m}$ and $S_{\ell,m}$ depend only on the disturbing third body since $Gm' = n'^2 a'^3$, where n' and a' are the mean motion and semimajor axis, respectively, of the third body. The disturbing function in Equation (4-18) minus the point-mass term can be expressed as

$$R_3 = 2 \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} r^\ell (C_{\ell,m} - j S_{\ell,m}) P_{\ell,m}(\sin\phi) e^{jm\theta} \qquad (4\text{-}25)$$

4.2.1 Rotation of the Satellite-Dependent Spherical Harmonic Functions

The surface harmonic functions $P_{\ell,m}(\sin\phi)\, e^{jm\theta}$ must be expressed in the equinoctial elements. In view of the discussion presented in Section 2.1.2 (Equation (2-24)), the appropriate transformation from the original reference system to the equinoctial reference system takes the form

$$P_{\ell,m}(\sin\phi)\, e^{jm\theta} = \sum_{\substack{s=-\ell \\ (\ell \pm s \text{ even})}}^{\ell} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0)\, S_{2\ell}^{m,s}\, e^{jsL} \qquad (4\text{-}26)$$

since the latitude of the satellite is identically zero in the equinoctial reference system. The longitude, L, is the true longitude of the satellite, measured from the origin of the longitudes in the equinoctial reference system.

The inclination function is given generally by Equations (2-45), where $(\Omega^*, i^*, \omega^*)$ are the Euler angles describing the orientation of the original coordinate system (equatorial) with respect to the equinoctial reference system. (The symbol * in the above expression is used to distinguish Euler angles from the classical elements Ω, i, ω.

If the original coordinate system is the equatorial system, the Euler angles are the same as those given in the discussion of the nonspherical gravitational perterbation which is given in Equations (3-53). Any of the definitions for the inclination function $S_{2\ell}^{m,s}$ given in Equations (3-55), (3-63), or (3-66) are appropriate. Substituting Equation (4-26) into Equation (4-25) yields the expression:

$$R_3 = 2 \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{\substack{s=-\ell \\ [\ell \pm s \text{ even}]}}^{\ell} \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(C_{\ell,m} - j S_{\ell,m}\right) r^{\ell}\, S_{2\ell}^{m,s}\, e^{jsL} \qquad (4\text{-}27)$$

4.2.2 Rotation of the Third-Body Dependent Spherical Harmonics

It may be desirable to express the functions $C_{\ell,m}$ and $S_{\ell,m}$ in terms of some set of orbital elements (e.g., classical, equinoctial, etc.). This is particularly true for cases involving resonance phenomena. There is no compelling reason to require a "nonsingular" set of elements for the third body since they are only parameters in the equations of motion for the satellite and do not introduce any singularities. Consequently, classical elements and the associated nodal reference frame will be used for the representation of the third body.

The nodal reference frame is defined by the right-handed orthogonal triad $(\hat{x}_N, \hat{y}_N, \hat{z}_N)$, where $\hat{x}_N$ points from the center of the central body to the ascending node of the third-body orbit, Ω', relative to the equatorial reference system. The unit vector $\hat{z}_N$ is the unit angular momentum vector of the third-body orbital motion, i.e.,

$$\hat{z}_N = \frac{\vec{r}' \times \dot{\vec{r}}'}{|\vec{r}' \times \dot{\vec{r}}'|}$$

The unit vector $\hat{y}_N$, which completes the orthogonal triad, lies in the third-body orbital plane, 90 degrees ahead (in the direction of the third-body motion) of the unit vector x_N. The relationship between the equatorial reference system $(\hat{x}, \hat{y}, \hat{z})$ and the third-body nodal reference system $(\hat{x}_N, \hat{y}_N, \hat{z}_N)$ is shown in Figure 4-2.

Applying the transformation in Equation (2-24) to the surface harmonics in Equation (4-24), i.e., rotating the surface harmonic functions to the nodal reference frame for the third body, yields

$$P_{\ell,m}(\sin\phi')\, e^{jm\theta'} = \sum_{\substack{p=-\ell \\ (\ell\pm p \text{ even})}}^{\ell} \frac{(\ell-p)!}{(\ell-m)!} P_{\ell,p}(0)\, S_{2\ell}^{m,p}\, e^{jp(\omega'+f')} \tag{4-28}$$

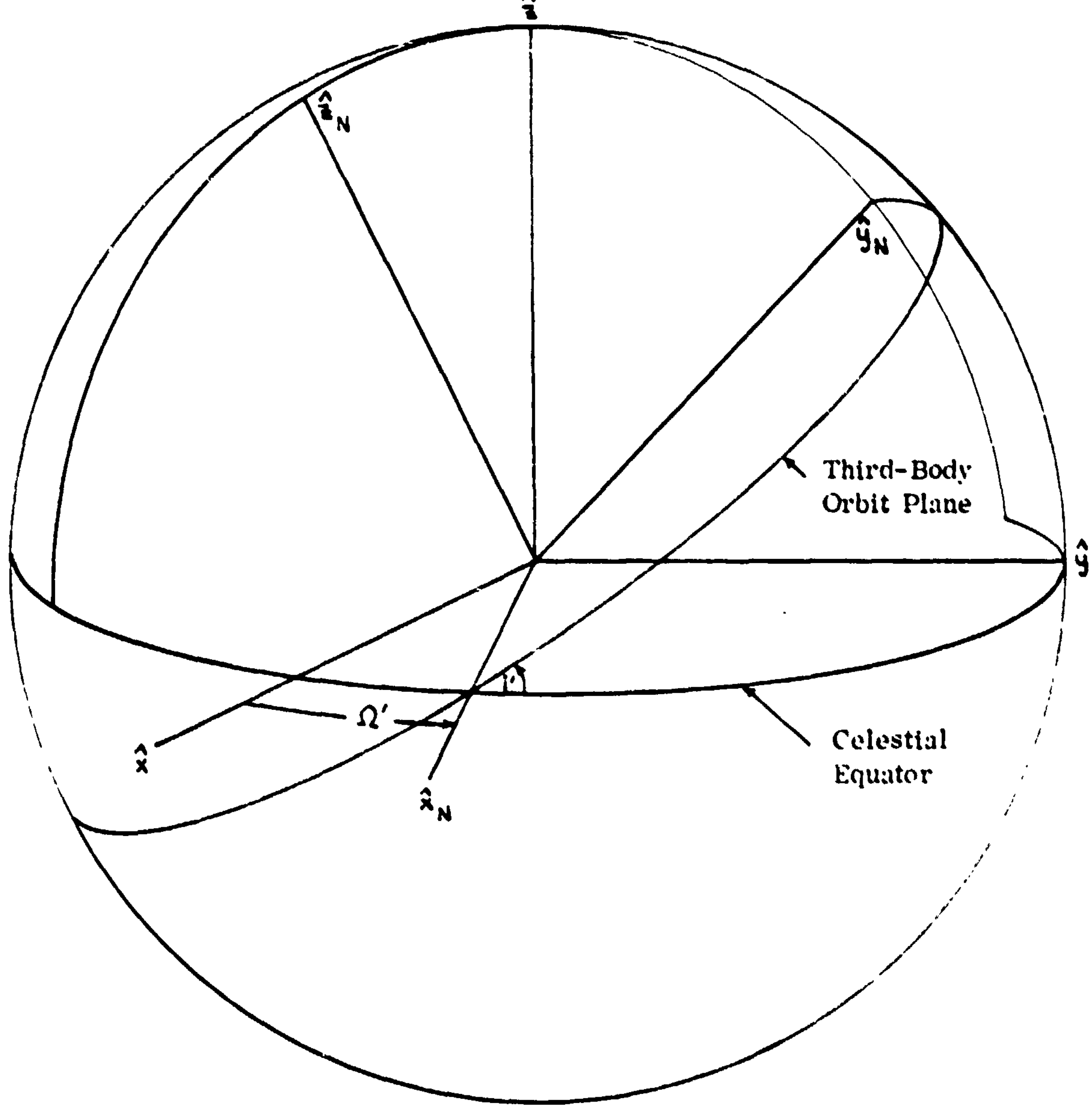

Figure 4-2. Relationship Between the Equatorial and Third-Body Nodal Reference Systems

where ω' and f' are the argument of pericenter and the true anomaly of the third body, respectively. Hence, it follows that

$$C_{\ell,m} + j S_{\ell,m} = \frac{G m'}{r'^{\ell+1}} \sum_{\substack{p=-\ell \\ (\ell \pm p \text{ even})}}^{\ell} \frac{(\ell-p)!}{(\ell+m)!} P_{\ell,p}(0)\, S_{2\ell}^{m,p}\, e^{jp(\omega'+f')} \tag{4-29}$$

Assuming the equatorial reference system as the original coordinate system, the Euler angles specifying the orientation of the equatorial system with respect to the nodal reference system are obtained from the transformation[1]

$$\hat{r}_N = R_3(0)\, R_1(i')\, R_3(\Omega')\, \hat{r}_E \tag{4-30}$$

where $\hat{r}_N$ and $\hat{r}_E$ are the positions of the satellite in the nodal reference frame and the equatorial reference frame, respectively, and i' and Ω' are the inclination and longitude of the ascending node of the third-body orbit referred to the equatorial reference system. The Euler angles corresponding to the inverse of the above transformation are required and are as follows:

$$\Omega^* = 0 \tag{4-31a}$$

$$i^* = i' \tag{4-31b}$$

$$\omega^* = -\Omega' \tag{4-31c}$$

---

[1]Clearly, the rotation $R_3(0)$ is an identity transformation.

Substituting these angles into the general expression for the function $S_{2\ell}^{m,p}$, given in Equations (2-45) (with s replaced by p), yields

$$S_{2\ell}^{m,p} = j^{p-m} e^{jm\Omega'}$$

$$\times \begin{cases} C_{i'/2}^{-m-p} \; S_{i'/2}^{m-p} \; P_{\ell+p}^{m-p,-m-p}(C_{i'}) & (-\ell \leq p \leq -m) \quad (4\text{-}32a) \\[2ex] \dfrac{(\ell+m)!\,(\ell-m)!}{(\ell+p)!\,(\ell-p)!} \; C_{i'/2}^{m+p} \; S_{i'/2}^{m-p} \; P_{\ell-m}^{m-p,m+p}(C_{i'}) & (-m \leq p \leq m) \quad (4\text{-}32b) \\[2ex] (-1)^{p-m} \; C_{i'/2}^{m+p} \; S_{i'/2}^{p-m} \; P_{\ell-p}^{p-m,p+m}(C_{i'}) & (m \leq p \leq \ell) \quad (4\text{-}32c) \end{cases}$$

since

$$C_{-x} = C_x$$

$$S_{-x} = -S_x$$

If the function $S_{2\ell}^{m,p}$ is expressed as

$$S_{2\ell}^{m,p} = j^{p-m} e^{jm\Omega'} U_{2\ell}^{m,p} \qquad (4\text{-}33)$$

where the definition of $U_{2\ell}^{m,p}$ follows from Equation (4-32), then Equation (4-29) takes the form

$$C_{\ell,m} + j S_{\ell,m} = \frac{G m'}{r'^{\,\ell+1}} \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(\sin \phi') \, e^{-jm\theta'}$$

$$= \frac{G m'}{r'^{\,\ell+1}} \sum_{\substack{p=-\ell \\ (\ell \pm p \text{ even})}}^{\ell} \frac{(\ell-p)!}{(\ell+m)!} P_{\ell,p}(0) \, j^{p-m} \, U_{2\ell}^{m,p} \, e^{j\left[p(\omega'+f')+m\Omega'\right]} \tag{4-34}$$

Comparison of this result with the result obtained by substituting Kaula's expression (given in Equation (3-10)) into Equation (4-23) yields the relation

$$F_{\ell,m,q}(i') = (-1)^q \, j^{\ell-m} \frac{(2q)!}{(\ell-m)!} P_{\ell,\ell-2q}(0) \, U_{2\ell}^{m,\ell-2q} = \frac{(2q-1)!!\,(2\ell-2q-1)!!}{(\ell-m)!} U_{2\ell}^{m,\ell-2q} \tag{4-35}$$

where

$$\ell - p = 2q \tag{4-36}$$

This completes the rotation of the third-body-dependent spherical harmonic functions to the nodal reference system of the third body.

Substituting the complex conjugate of Equation (4-34) and Equation (4-26) into Equations (4-25) yields the following express'on for the disturbing function:

$$R_3 = \frac{2Gm'}{r'} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{p=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell+m)!} \frac{(\ell-p)!}{(\ell-m)!} P_{\ell,s}(0) \, P_{\ell,p}(0) \left(\frac{r}{r'}\right)^{\ell}$$
$$[\ell \pm s \text{ even};\ \ell \pm p \text{ even}]$$

$$\times \, S_{2\ell}^{m,s} \, j^{p-m} \, U_{2\ell}^{m,p} \, e^{j\left\{sL - \left[p(\omega'+f')+m\Omega'\right]\right\}} \tag{4-37}$$

4.2.3 Elimination of One Spherical Harmonic Rotation

A judicious choice of the system of reference used for the decomposition of the elongation angle (Equation (4-19)) in the application of the Addition Theorem permits the elimination of the rotation of one set of spherical harmonic functions in Equation (4-20). The selection of the equinoctial reference system as the system of reference eliminates the rotation of the set of satellite-dependent spherical harmonic functions. Selection of the nodal reference frame, associated with the orbit of the third body, as the system of reference eliminates the rotation of the third-body-dependent spherical harmonic functions. In essence, the argument of the single remaining inclination function is the mutual inclination between the instantaneous planes of the satellite and third-body orbits, respectively.

Classically, the plane of the disturbing body has been adopted as the reference plane for those formulations involving expansions in the mutual inclination. This choice possesses the advantage of obviating the necessity of referring the orbital elements (position) of the third body (usually obtained from an independently generated ephemeris) to another reference system. Because of this consideration, the plane of the third body is adopted as the reference plane in the following discussion. The reference system, referred to as the nodal reference system, is shown in Figure 4-2. For the purposes of this discussion, the dynamical system of reference is assumed to be the equatorial reference system and is analogous to the original reference system in the preceding development involving two spherical harmonic rotations.

Clearly, the cosine of the elongation angle is independent of the particular reference system used to measure the latitude and longitude of the satellite, (ϕ, θ), and the latitude and longitude of the third body, (ϕ', θ'). In the nodal reference system, the angles ϕ', θ', describing the position of the third body on a unit sphere, are

$$\phi' = 0 \tag{4-38a}$$

$$\theta' = \omega' + f' \tag{4-38b}$$

where ω' and f' are the classical argument of pericenter and true anomaly of the third body, respectively. Thus, the general expression in Equation (4-19) reduces to

$$\cos\psi = \cos\phi \cos\left[\theta - (\omega' + f')\right] \tag{4-39}$$

where the satellite latitude and longitude are measured with respect to the nodal reference system. The Addition Theorem given by Equation (4-23) assumes the form

$$P_{\ell}(\cos\psi) = \mathrm{Re}\left\{ 2 \sum_{\substack{m=0 \\ [\ell \pm m \text{ even}]}}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(0)\, P_{\ell,m}(\sin\phi)\, e^{jm[\theta-(\omega'+f')]} \right\} \tag{4-40}$$

The spherical harmonic functions $P_{\ell,m}(\sin\phi)\, e^{jm\theta}$ must be rotated to the nodal reference frame. The Euler angles required for the specification of the function $S_{2\ell}^{m,s}$ are obtained from the rotation of the equinoctial reference frame $(\hat{f}, \hat{g}, \hat{w})$ (defined with respect to the equatorial reference frame) into the nodal reference frame, i.e.,

$$\hat{r}_N = R_3(-\tau')\, R_1(-\Delta i)\, R_3(I\Omega + \tau)\, \hat{r}_Q \tag{4-41}$$

where $\hat{r}_N$ and $\hat{r}_Q$ are the position vectors of the satellite referred to the nodal and equinoctial frames, respectively, Δi is the mutual inclination between the two orbital planes, τ' is the longitude of the ascending node of the satellite orbit referred to the nodal reference system, and τ is the angular distance in the satellite orbital plane between the longitude of the ascending nodes with respect to the equatorial and nodal reference systems of the satellite orbit. The above rotation follows immediately from inspection of Figure 4-3. Substituting the Euler angles

$$\omega^* = -\tau \tag{4-42a}$$

$$i^* = -\Delta i \tag{4-42b}$$

$$\Omega^* = I\Omega + \tau \tag{4-42c}$$

into Equation (2-45) yields

$$S_{2\ell}^{m,s} = j^{s-m} e^{-j[s(I\Omega+\tau) - m\tau']}$$

$$\times \begin{cases} (-1)^{m+s} C_{\Delta i/2}^{-m-s} S_{\Delta i/2}^{m-s} P_{\ell+s}^{m-s,-m-s}(C_{\Delta i}) & (-\ell \le s \le -m) \quad (4\text{-}43a) \\ (-1)^{s-m} \dfrac{(\ell+m)!}{(\ell+s)!} \dfrac{(\ell-m)!}{(\ell-s)!} C_{\Delta i/2}^{m+s} S_{\Delta i/2}^{m-s} P_{\ell-m}^{m-s,m+s}(C_{\Delta i}) & (-m \le s \le m) \quad (4\text{-}43b) \\ C_{\Delta i/2}^{m+s} S_{\Delta i/2}^{s-m} P_{\ell-s}^{s-m,s+m}(C_{\Delta i}) & (m \le s \le \ell) \quad (4\text{-}43c) \end{cases}$$

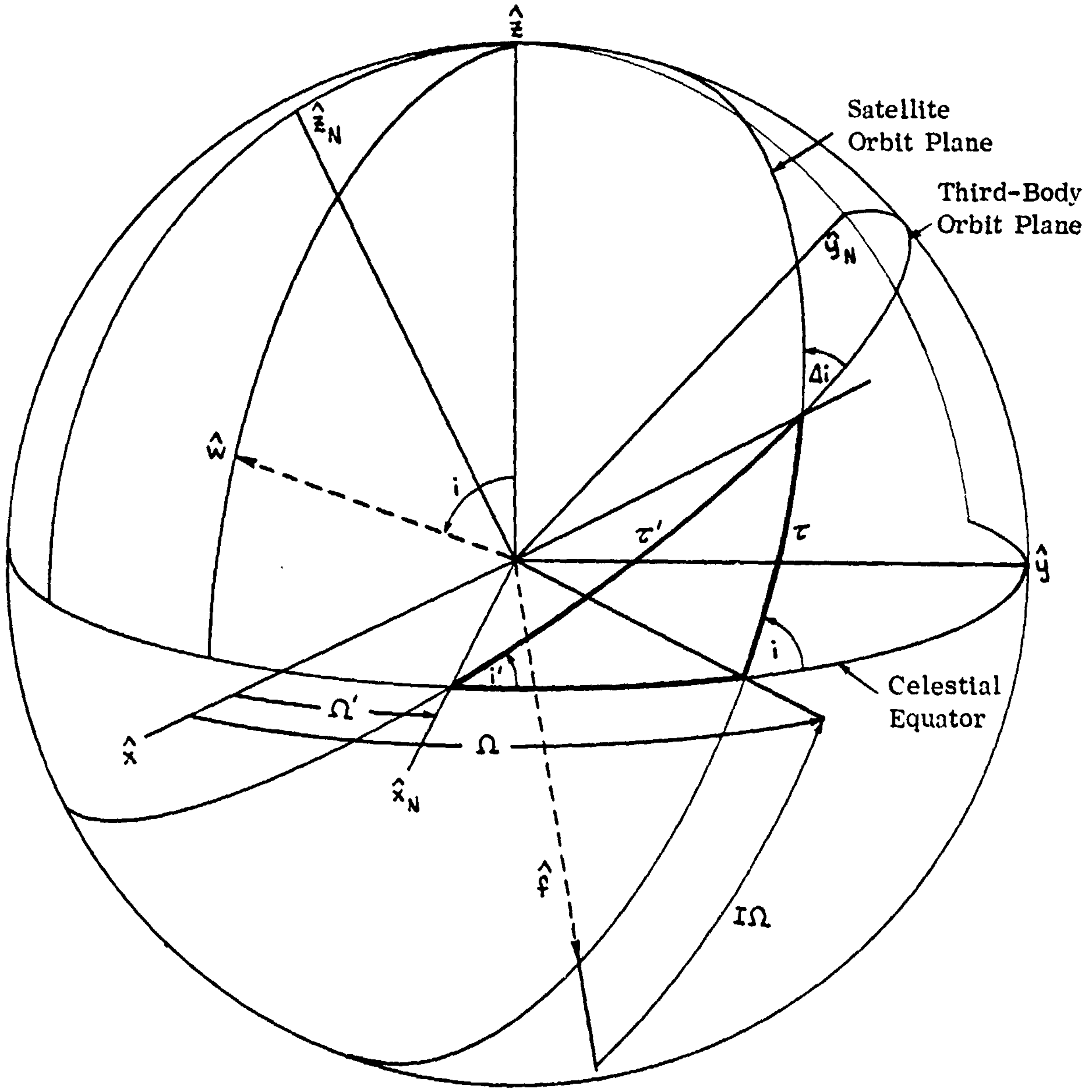

Figure 4-3. Relationship Between the Equinoctial and Third-Body Nodal Reference Systems

The expression for the disturbing function takes the form

$$R_3 = 2 \sum_{\substack{\ell=2 \\ (\ell \pm s \text{ even})}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \delta_m \frac{(\ell-s)!}{(\ell-m)!} P_{\ell,s}(0) \left(C_{\ell,m} - j S_{\ell,m}\right) r^{\ell} S_{2\ell}^{m,s} e^{jsL} \tag{4-44}$$

where

$$C_{\ell,m} + j S_{\ell,m} = \frac{Gm'}{r'^{\,\ell+1}} \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(0)\, e^{jm(\omega'+f')} \tag{4-45}$$

Substituting the explicit expressions for the functions $C_{\ell,m}$ and $S_{\ell,m}$ into Equations (4-44) yields

$$R_3 = \frac{2Gm'}{r'} \sum_{\substack{\ell=2 \\ (\ell \pm s \text{ even}) \\ (\ell \pm m \text{ even})}}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \frac{(\ell-s)!}{(\ell+m)!} P_{\ell,s}(0)\, P_{\ell,m}(0) \left(\frac{r}{r'}\right)^{\ell} S_{2\ell}^{m,s}\, e^{j[sL - m(\omega'+f')]} \tag{4-46}$$

Thus, a comparison with Equation (4-37) shows that one summation has been eliminated. However, this is accompanied by an increase in the complexity of evaluating the remaining $S_{2\ell}^{m,s}$ function.

Since the equations of motion are still referred to the equatorial system, the quantities τ', Δi, and τ, appearing in Equation (4-43), must be related to the equinoctial elements of the satellite and the classical elements of the third body (both of which are assumed to be referred to the equatorial reference system). The relationship between the classical elements of both the satellite and third body and the quantities τ', Δi, and τ are easily obtained. The relationship in terms of the equinoctial elements can then be constructed using the results given in Appendix A of Reference 5.

The relationship between $(\tau', \Delta i, \tau)$ and (i, Ω, i', Ω') can be obtained from the formulas of spherical trigonometry or as follows. In addition to the transformation given in Equation (4-41), it is clear from the inspection of Figure 4-3 that

$$\hat{x}_N = R_1(i')\, R_3(\Omega'-\Omega)\, R_1(-i)\, R_3(I\Omega)\, \hat{x}_Q \tag{4-47}$$

Therefore, a comparison of Equations (4-41) and (4-47) yields the relation

$$R_3(i')\, R_3(\Omega'-\Omega)\, R_1(-i)\, R_3(I\Omega) = R_3(-\tau')\, R_1(-\Delta i)\, R_3(I\Omega+\tau) \tag{4-48}$$

Clearly,

$$R_3(I\Omega+\tau) = R_3(\tau+I\Omega) = R_3(\tau)\cdot R_3(I\Omega) \tag{4-49}$$

Substituting this result into Equation (4-48) and postmultiplying each side of the resulting equation by $R_3^{-1}(I\Omega)$ yields

$$R_1(i')\, R_3(\Omega'-\Omega)\, R_1(-i) = R_3(-\tau')\, R_1(-\Delta i)\, R_3(\tau) \tag{4-50}$$

Performing the matrix multiplications in the above expression and comparing the respective elements of the two resulting matrices yields the relations

$$\cos(\Omega'-\Omega) = \cos\tau' \cos\tau + \cos\Delta i \sin\tau' \sin\tau \tag{4-51a}$$

$$\cos i \sin(\Omega'-\Omega) = \cos\tau' \sin\tau - \cos\Delta i \sin\tau' \cos\tau \tag{4-51b}$$

$$\sin i \ \sin(\Omega'-\Omega) = \sin\Delta i \ \sin\tau \tag{4-51c}$$

$$\cos i' \ \sin(\Omega'-\Omega) = \cos\Delta i \ \cos\tau' \sin\tau - \sin\tau' \cos\tau \tag{4-51d}$$

$$\cos i \ \cos i' \ \cos(\Omega'-\Omega) + \sin i' \sin i = \cos\Delta i \ \cos\tau' \cos\tau + \sin\tau' \sin\tau \tag{4-51e}$$

$$\sin\Delta i \ \cos\tau' = \sin i' \ \cos i - \cos i' \ \sin i \ \cos(\Omega'-\Omega) \tag{4-51f}$$

$$-\sin i \ \sin(\Omega'-\Omega) = -\sin\Delta i \ \sin\tau' \tag{4-51g}$$

$$\sin i' \ \sin(\Omega'-\Omega) = \sin\Delta i \ \sin\tau \tag{4-51h}$$

$$\cos\Delta i = \cos i \ \cos i' + \sin i \ \sin i' \ \cos(\Omega'-\Omega) \tag{4-51i}$$

If the equations of motion are referred to the nodal system also, a new set of equinoctial elements (a_1, h_1, k_1, p_1, q_1, λ_1) and a new equinoctial reference frame ($\hat{f}_1$, $\hat{g}_1$, $\hat{w}_1$) are introduced. In view of the definition of the equinoctial elements in terms of the classical elements (Reference 5, Appendix A), it follows that the equinoctial elements of the satellite, referred to the nodal reference system, are given by

$$a_1 = a \tag{4-52a}$$

$$h_1 = e \ \sin(\omega_1 + I\tau') \tag{4-52b}$$

$$k_1 = e \ \cos(\omega_1 + I\tau') \tag{4-52c}$$

$$p_1 = \tan^I\left(\frac{\Delta i}{2}\right) \sin\tau' \tag{4-52d}$$

$$q_1 = \tan^I\left(\frac{\Delta i}{2}\right)\cos\tau' \qquad (4\text{-}52e)$$

$$\lambda_1 = \ell + \omega_1 + I\tau' \qquad (4\text{-}52f)$$

where[1]

$$\omega_1 = \omega - \tau \qquad (4\text{-}53)$$

is the argument of pericenter referred to the nodal reference system and ω is the argument of pericenter referred to the equatorial reference system.

The equinoctial reference frame described by the orthogonal triad $(\hat{f}_1, \hat{g}_1, \hat{w}_1)$ is based on the orientation of the satellite orbital plane relative to the nodal system. (This is in contrast to the previous case where the equations of motion were referred to the equatorial system through the orthogonal triad $(\hat{f}, \hat{g}, \hat{w})$ describing the orientation of the satellite orbit relative to the equatorial reference system.)

The rotation from the nodal reference system to the new equinoctial reference system is (see Figure 4-4)

$$\hat{x}_{Q_1} = R_3(-I\tau')\, R_1(\Delta i)\, R_3(\tau')\, \hat{x}_N \qquad (4\text{-}54)$$

where I is the retrograde factor. This rotation is identical to that expressed in Equation (A-4) in Appendix A of Reference 5, if the symbol Ω is replaced by τ' and i is replaced by Δi.

---

[1] The only classical elements dependent on the system of reference are i, Ω, ω, which describe the orientation of the osculating ellipse with respect to the system of reference. The classical elements a, e, and ℓ are invariant with respect to a change in the reference system.

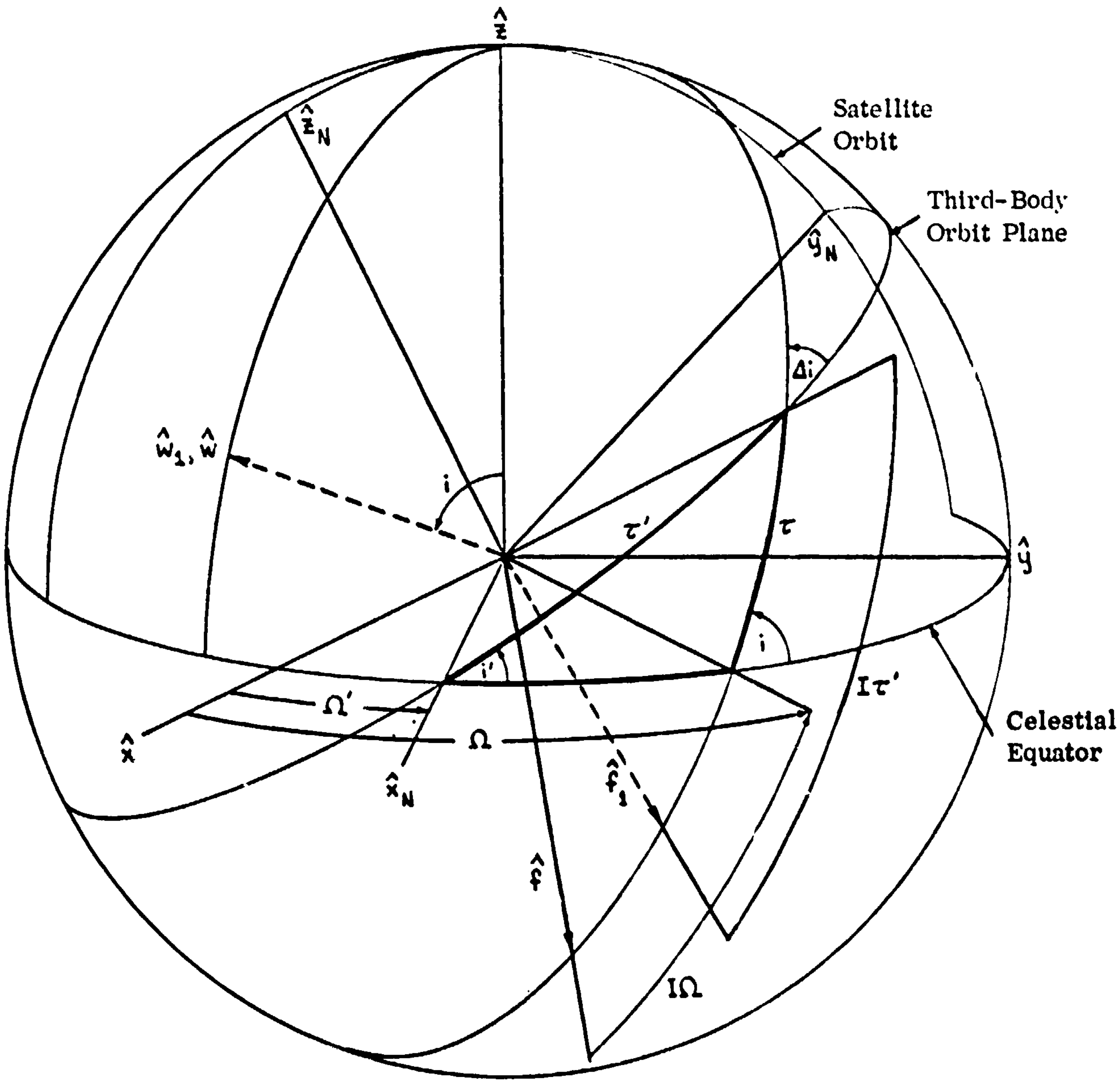

Figure 4-4. Relationship Between the Equinoctial Reference Frame Referred to the Equatorial Reference System ($\hat{f}$, $\hat{g}$, $\hat{w}$) and the Equinoctial Reference Frame Referred to the Third-Body Nodal Reference System ($\hat{f}_1$, $\hat{g}_1$, $\hat{w}_1$)

The Euler angles for the function $S_{22}^{m,s}$ correspond to the inverse of the above transformation and are

$$\omega^* = -\tau \tag{4-55a}$$

$$i^* = -\Delta i \tag{4-55b}$$

$$\Omega^* = I\tau' \tag{4-55c}$$

Substituting these into Equation (2-49) yields an expression for the function $S_{22}^{m,s}$ which is identical in form to that given in Equation (3-55), with, of course, the conditions

$$\Omega = \tau' \tag{4-56a}$$

$$i = \Delta i \tag{4-56b}$$

In addition, Equations (3-63) and (3-65) are also valid, provided that (α, β, γ) are understood to be replaced by $(\alpha_1, \beta_1, \gamma_1)$ in Equation (3-63) and that (p, q, γ) are replaced by (p_1, q_1, γ_1) in Equation (3-66), where

$$\alpha_1 = \hat{f}_1 \cdot \hat{z}_N = \frac{-2p_1 I}{1+p_1^2+q_1^2} = -I\, S_{\Delta i}\, S_{\tau'} \tag{4-57a}$$

$$\beta_1 = \hat{g}_1 \cdot \hat{z}_N = \frac{2q_1}{1+p_1^2+q_1^2} = S_{\Delta i}\, C_{\tau'} \tag{4-57b}$$

$$\gamma_1 = \hat{w}_1 \cdot \hat{z}_N = \frac{(1-p_1^2-q_1^2)I}{1+p_1^2+q_1^2} = C_{\Delta i} \tag{4-57c}$$

For this case in which the equations of motion are referred to the nodal reference frame, the disturbing function is of the same form as that given in Equation (4-46). The $S_{2\ell}^{m,s}$ function arguments $(\alpha_1, \beta_1, \gamma_1)$ or (p_1, q_1, γ_1) are consistent with the integrated elements, thus avoiding the transformations in Equations (4-51). Since the integrated equinoctial elements are referred to the nodal reference system, they must be transformed if equinoctial elements referred to the equatorial system are desired. (However, this transformation is applied only at the output points rather than at every integration step.) Equations (4-51) provide the basis for this transformation.

4.2.4 Introduction of the Fourier Series Expansions

The next step in the expansion of the disturbing function is the introduction of Fourier series expansions for the products

$$r^{\ell} e^{jsL} \; ; \qquad \frac{1}{r'^{\,\ell+1}} e^{jpf'}$$

for the satellite and third body, respectively. The third-body product need not necessarily be expanded. However, if a properly reduced force model for third-body resonance cases is desired, the expansion must be performed. The considerations on which the selection of the expansion variable is based are similar to those discussed in Section 3 for the nonspherical gravitational theory. Although the Fourier series expansion of the product $r^{\ell} e^{jsL}$ is finite in the eccentric longitude, it is of no significance for the averaged disturbing function and is not suitable for cases of resonance. Hence, the mean longitude is chosen. Also, the mean anomaly chosen is the expansion variable to facilitate the case of resonance. Substituting the expansions

$$\left(\frac{r}{a}\right)^{\ell} e^{jsL} = \sum_{q=-\infty}^{\infty} Y_q^{\ell,s} e^{jq\lambda} \tag{4-58}$$

where $Y_q^{\ell,s}$ is defined in Equation (2-304) and

$$\left(\frac{a'}{r'}\right)^{\ell+1} e^{jpf'} = \sum_{t=-\infty}^{\infty} X_t^{-\ell-1,p} e^{jt\ell'} \tag{4-59}$$

where ℓ' is the third-body mean anomaly, into Equation (4-37) yields the following expression for the disturbing function in Section 4.2.2:

$$R_3 = \frac{2Gm'}{a'} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{p=-\ell}^{\ell} \sum_{q=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \frac{(\ell-s)!}{(\ell+m)!} \frac{(\ell-p)!}{(\ell-m)!} P_{\ell,s}(0) P_{\ell,p}(0)$$
$$(\ell\pm s \text{ even } \& \ \ell\pm p \text{ even}) \tag{4-60}$$
$$\times \left(\frac{a}{a'}\right)^{\ell} S_{2\ell}^{m,s} \, j^{p-m} \, U_{2\ell}^{m,p} \, Y_q^{\ell,s} \, X_t^{-\ell-1,p} \, e^{j\left[q\lambda - t\ell' - p\omega' - m\Omega'\right]}$$

Substituting Equations (4-57) and (4-58) into Equation (4-46) yields the expression for the single-inclination-function form of the disturbing function obtained in Section 4.2.3, i.e.,

$$R_3 = \frac{2Gm'}{a'} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{q=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \frac{(\ell-s)!}{(\ell+m)!} P_{\ell,s}(0) P_{\ell,m}(0)$$
$$(\ell\pm s \text{ even } \& \ \ell\pm m \text{ even}) \tag{4-61}$$
$$\times \left(\frac{a}{a'}\right)^{\ell} S_{2\ell}^{m,s} \, Y_q^{\ell,s} \, X_t^{-\ell-1,m} \, e^{j\left[q\lambda - t\ell' - m\omega'\right]}$$

The expressions for the disturbing functions given in Equations (4-59) and (4-60) may be further improved; however, they suffice for the purpose of outlining the development of the disturbing function.

4.3 EXPANSION OF THE THIRD-BODY DISTURBING FUNCTION - A SPECIAL CASE[1]

Expansion of the disturbing function in terms of the direction cosines of the third-body position vector, relative to the equinoctial reference frame, results in a striking simplification. However, the formulation does not lend itself to obtaining reduced force models for resonant cases. The following discussion assumes the equatorial reference frame for the equations of motion (i.e., the satellite equinoctial elements are referred to the equatorial reference system).

4.3.1 Representation of the Elongation Angle in Terms of Direction Cosines

The definition of $\cos\psi$ in terms of the dot product of the unit position vectors $\hat{r}$ and $\hat{r}'$, relative to the equinoctial reference frame, of the satellite and third-body, respectively, are considered. Clearly, for the satellite,

$$\phi = 0 \tag{4-62a}$$

$$\theta = L \tag{4-62b}$$

Therefore,

$$\hat{r} = \begin{bmatrix} \cos\phi \cos\theta \\ \cos\phi \sin\theta \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos L \\ \sin L \\ 0 \end{bmatrix} \tag{4-63}$$

and, for the third body,

$$\hat{r}' = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{4-64}$$

---

[1]This development is given by Cefola in Reference 10.

where α, β, and γ are the direction cosines of the third body and are defined by

$$\alpha = \cos\phi' \cos\theta' \tag{4-65a}$$

$$\beta = \cos\phi' \sin\theta' \tag{4-65b}$$

$$\gamma = \sin\phi' \tag{4-65c}$$

The quantities θ' and ϕ' are the longitude and latitude, respectively, of the third body relative to the equinoctial reference frame.

Since

$$\hat{r}\cdot\hat{r}' = \cos\psi \tag{4-66}$$

it then follows that

$$\cos\psi = \alpha\cos L + \beta\sin L \tag{4-67}$$

It also follows that $\cos\psi$ can be expressed as

$$\cos\psi = \mathrm{Re}\left\{(\alpha - j\beta)\, e^{jL}\right\} \tag{4-68}$$

where

$$\alpha + j\beta = \cos\phi'\, e^{j\theta'} \tag{4-69}$$

In view of Equations (4-62), (4-65), and (4-67), Equation (4-20) takes the form

$$P_\ell(\cos\Psi) = P_\ell(\alpha\cos L + \beta\sin L)$$

$$= \mathrm{Re}\left[2\sum_{\substack{\ell=2 \\ [\ell\pm m\ \text{even}]}}^{\infty}\sum_{m=0}^{\ell}\delta_m\frac{(\ell-m)!}{(\ell+m)!}P_{\ell,m}(0)\,P_{\ell,m}(\gamma)\,e^{jm(L-\theta')}\right] \tag{4-70}$$

It follows from Equation (4-69) that

$$e^{-jm\theta'} = \left(\frac{\alpha-j\beta}{\cos\phi'}\right)^m = \left(\frac{\alpha-j\beta}{\sqrt{1-\gamma^2}}\right)^m \tag{4-71}$$

Thus,

$$P_\ell(\cos\Psi) = \mathrm{Re}\left[2\sum_{\substack{\ell=2 \\ (\ell\pm m\ \text{even})}}^{\infty}\sum_{m=0}^{\ell}\delta_m\frac{(\ell-m)!}{(\ell+m)!}P_{\ell,m}(0)\,Q_{\ell,m}(\gamma)\,(\alpha-j\beta)^m e^{jmL}\right] \tag{4-72}$$

where

$$Q_{\ell,m}(\gamma) = (1-\gamma^2)^{-m/2}\,P_{\ell,m}(\gamma) = \frac{d^m P_\ell(\cos\Psi)}{d\gamma^m} \tag{4-73}$$

If

$$C_{\ell,m} - jS_{\ell,m} = (\alpha-j\beta)^m \tag{4-74}$$

then Equation (4-72) takes on the exact form given by Cefola and Broucke in Equation 29 of Reference 11.

In view of Equation (4-72), the disturbing function in complex variables takes the form

$$R_3 = \frac{2Gm'}{r'} \sum_{\substack{\ell=2 \\ [\ell \pm m \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{r}{r'}\right)^{\ell} P_{\ell,m}(0)\, Q_{\ell,m}(\gamma)\, (\alpha - j\beta)^m\, e^{jmL} \tag{4-75}$$

This formulation contains only two summations in contrast to Equations (4-37) and (4-46) which contain four and three summations, respectively. However, it is pointed out that the reduction in the number of summations from three to two requires some preprocessing of the ephemeris data to obtain the direction cosines of the third body (α, β, γ) relative to the equinoctial reference frame. This processing, however, is not very expensive, particularly if the ephemeris is in the form of Cartesian position components, since

$$\alpha = \hat{r}' \cdot \hat{f} \tag{4-76a}$$

$$\beta = \hat{r}' \cdot \hat{g} \tag{4-76b}$$

$$\gamma = \hat{r}' \cdot \hat{w} \tag{4-76c}$$

4.3.2 <u>Introduction of the Fourier Series Expansion</u>

The Fourier series expansion for the product $r^{\ell} e^{jmL}$ is required to complete the expansion of the disturbing function. No similar expansion is performed for the third body due to the specific formulation.

Since this formulation ultimately assumes no resonance phenomena, there is no compelling need to adopt the mean longitude as the expansion variable. The eccentric longitude provides a finite series representation which is useful for the representation of both the short-period and the long-period and secular contributions of the disturbing function. However, the eccentric longitude representation possesses no advantage over the mean longitude for the averaged disturbing function. Thus, for the sake of consistency with the previous developments, the mean longitude expansion is adopted. The resulting form of the disturbing function is

$$R_3 = \frac{2Gm'}{r'} \sum_{\substack{\ell=2 \\ (\ell \pm m \text{ even})}}^{\infty} \sum_{m=0}^{\ell} \sum_{q=-\infty}^{\infty} \delta_m \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{a}{r'}\right)^{\ell} P_{\ell,m}(0)\, Q_{\ell,m}(\gamma) \times (\alpha - j\beta)^m\, Y_q^{\ell,m}\, e^{jq\lambda} \tag{4-77}$$

4.4 THE AVERAGED DISTURBING FUNCTION

The averaged disturbing functions for both the general and special cases are developed in this section. The averaging operation is defined in Equation (3-89) and is repeated below for convenience:

$$\left\langle R_3 \right\rangle_{\bar{\lambda}} = \frac{1}{2\pi} \int_{\bar{\lambda}_0 - \pi}^{\bar{\lambda}_0 + \pi} R_3(\vec{\bar{a}}, \bar{\lambda})\, d\bar{\lambda} \tag{4-78}$$

4.4.1 The General Case

Application of the averaging operation to the disturbing function given in Equation (4-59) yields the expression

$$\begin{aligned} \left\langle R_3 \right\rangle_{\bar{\lambda}} = \frac{2Gm'}{r'} & \underset{(\ell \pm s \text{ even } \& \ \ell \pm p \text{ even})}{\sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \sum_{s=-\ell}^{\ell} \sum_{p=-\ell}^{\ell} \sum_{q=-\infty}^{\infty} \sum_{t=-\infty}^{\infty}} \frac{(\ell-s)!}{(\ell-m)!} \frac{(\ell-p)!}{(\ell+m)!} \\ & \times P_{\ell,s}(0)\, P_{\ell,p}(0) \left(\frac{a}{a'}\right)^{\ell} S_{2\ell}^{m,s}\, j^{p-m}\, U_{2\ell}^{m,p}\, Y_q^{\ell,s}\, X_t^{-\ell-1,p} \\ & \times e^{-j(p\omega' + m\Omega')} \frac{1}{2\pi} \int_{\lambda_0-\pi}^{\lambda_0+\pi} e^{j(q\lambda - t\ell')}\, d\lambda \end{aligned} \tag{4-79}$$

This expression considers the motion of the third body during the averaging interval. The following development is analogous to the corresponding development for the nonspherical gravitational disturbing function given in Section 3.3. Since

$$\lambda = nt + \lambda_0 \tag{4-80a}$$

and

$$\ell' = n't + \ell_0 \tag{4-80b}$$

eliminating the time in the above expressions yields the relation

$$\ell' = \frac{n'}{n}(\lambda - \lambda_0) + \ell'_0 \tag{4-81}$$

which is analogous to Equation (3-99) in the nonspherical case. Substituting this relation into the definite integral in Equation (4-79) and evaluating the result yields

$$\frac{1}{2\pi}\int_{\lambda_0-\pi}^{\lambda_0+\pi} e^{j(q\lambda - t\ell')}\,d\lambda = \begin{cases} 1 & \left(\text{for } q = t = 0 \text{ or } q = \frac{tn'}{n}\right) \quad (4\text{-}82a) \\[2ex] \dfrac{\sin\left[\left(q \cdot t\frac{n'}{n}\right)\right]}{\left(q - t\frac{n'}{n}\right)\pi}\, e^{j(q\lambda_0 - t\ell'_0)} & (\text{otherwise}) \quad (4\text{-}82b) \end{cases}$$

which is the third body equivalent of Equation (3-102).

The discussion of the averaging factor and the residual short-period terms given in Section 3.3 is also directly applicable for the general third-body case. Briefly, all the short-period contributions by the terms in the disturbing function for which $q \neq 0$, $t = 0$ in Equation (4-79) are completely eliminated in both time-dependently and time-independently averaged third-body cases. The t-monthly[1] terms, i.e., $q = 0$ and $t \neq 0$ in Equation (4-79), are partially suppressed in proportion to their frequency in the time-dependent case. For the time-independent case, the t-monthly terms are transparent to the averaging operation. This can result in an over estimate of the amplitude of each term and also in a phase error of π radians, as discussed in Section 3.3.1.2.2.1. For the case of exact resonance, the pure resonant terms survive the averaging operation, while the other quasi-isolated terms are completely suppressed. For cases of near resonance, the amplitudes of the "resonant terms" are reduced, depending on the shallowness of resonance. Also, the quasi-isolated short-period terms are only partially suppressed for near resonance. A more detailed discussion is given in Section 3.3. The analogy between the nonspherical gravitational disturbing function and the third-body disturbing function is straightforward.

In addition, the foregoing discussion also applies to the five-sum disturbing function given in Equation (4-61).

4.4.2 The Averaged Disturbing Function for the Special Case

Application of the averaging operation given in Equation (4-77) to the disturbing function given in Equation (4-76) yields the expression

$$\langle R_3 \rangle_{\bar{\lambda}} = \frac{2Gm'}{r'} \sum_{\substack{\ell=2 \\ [\ell \pm m \text{ even}]}}^{\infty} \sum_{m=0}^{\ell} \sum_{q=-\infty}^{\infty} \delta_m \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{a}{r'}\right)^{\ell} P_{\ell,m}(0)\, Y_q^{\ell,m} \times \frac{1}{2\pi} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} \frac{(\alpha - j\beta)^m}{r'^{\,\ell+1}} Q_{\ell,m}(\gamma)\, e^{jq\bar{\lambda}}\, d\bar{\lambda} \tag{4-83}$$

---

[1] The "t-monthly" terms in the lunar model are analogous to the m-daily terms contributed by the nonspherical tesseral harmonic model. The effects contributed by the Sun would be "t-yearly," etc.

This result recognizes the time dependence of the third-body direction cosines and the third-body distance. However, this form is apparently not well suited for evaluating the above integral analytically.

Kaufman (Reference 50) has developed series expansions for the direction cosines up through the fifth power in the ratio of the mean motions n'/n in order to model the time dependence. The resulting disturbing function is then analytically averaged. However, according to the discussion in Section 3.3 and its analogy to the third-body disturbing function, it is obvious that allowing third-body motion during the averaging operation introduces short-period terms, except in those cases of deep resonance. Thus, it seems that the utility of such expansions is limited to those cases of deep resonance. However, due to the treatment of the third-body position in Equation (4-83), the disturbing function for the resonance case is, in essence, an embedded resonant term. This also appears to be true for Kaufman's series representation of the direction cosines. Hence, the averaging operation defined in Equation (3-88), centered at $\bar{\lambda}_0$, must be used.

Of course, the direction cosine formulation could be used with the numerical averaging method for cases involving deep resonance, provided the averaging operation (Equation (3-88)) for embedded resonant terms is used. However, this is clearly an expensive procedure. In addition, for such applications, the disturbing function in Equation (4-83) is more expensive to evaluate than the disturbing acceleration given by the left-hand side of Equation (4-5).[1]

[1]The disturbing acceleration must be used in a Gaussian VOP formulation of the equations of motion.

The time-independent averaging assumption applied to Equation (4-83) requires that the direction cosines and third-body distance are constant over the averaging interval. Thus, Equation (4-83) can be expressed as

$$
\left\langle R_3 \right\rangle_{\bar{\lambda}} = \frac{2Gm'}{r'} \sum_{\substack{\ell=2 \\ (\ell \pm m \text{ even})}}^{\infty} \sum_{m=0}^{\ell} \sum_{q=-\infty}^{\infty} \delta_m \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{a}{r'}\right)^{\ell} P_{\ell,m}(0)\, Y_q^{\ell,m} \tag{4-84}
$$

$$
\times\, (\alpha - j\beta)^m\, Q_{\ell,m}(\gamma)\, \frac{1}{2\pi} \int_{\bar{\lambda}_0-\pi}^{\bar{\lambda}_0+\pi} e^{j\bar{\lambda}}\, d\bar{\lambda}
$$

which simplifies to

$$
\left\langle R_3 \right\rangle_{\lambda} = \frac{2Gm'}{r'} \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} \delta_m \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{a}{r'}\right)^{\ell} P_{\ell,m}(0)\, Y_0^{\ell,m}\, (\alpha - j\beta)^m\, Q_{\ell,m}(\gamma) \tag{4-85}
$$

Implicitly, the absolute value of the averaging factor has been set to unity and the disturbing function contains only the long-period and secular contributions and the contributions from the "t-monthly" terms of the disturbing function given by Equation (4-79). This is quite satisfactory for satellites with orbital periods that are much smaller than the third-body orbital period. However, for satellites with longer periods, the absolute value of the true averaging factor will depart significantly from unity. For the "t-monthly" terms, the averaging factor is given by

$$
\left| \frac{\sin\left(\frac{n'}{n} t\pi\right)}{\frac{n'}{n} t\pi} \right| \leq 1 \tag{4-86}
$$

where t is the index in the Fourier expansion in the third-body mean anomaly used for the general approach. Thus, the effects of this theory, when used on satellites of longer periods, will be to exaggerate these "t-monthly" effects.

4.5 THE FIRST-ORDER AVERAGED EQUATIONS OF MOTION FOR THE SPECIAL CASE OF NEAR-EARTH NONRESONANT SATELLITES

The first-order averaged equations of motion for the third-body effects are given by Equation (3-167), where the elements are interpreted as mean elements. The disturbing function is the averaged disturbing function given in Equation (4-85) and, thus, the equations of motion are valid for nonresonant cases only. The equinoctial elements p and q do not appear in the averaged disturbing function. The equivalent information is contained in the direction cosines of the third-body position vector, in view of Equations (4-7) and the definition of the vectors $(\hat{f}, \hat{g}, \hat{w})$ given in Appendix A of Reference 5, i.e.,

$$f = f(p, q) \tag{4-87}$$

etc. Thus, the equations of motion require the partial derivatives

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial \alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial R}{\partial \beta}\frac{\partial \beta}{\partial p} + \frac{\partial R}{\partial \gamma}\frac{\partial \gamma}{\partial p} \tag{4-88}$$

and similarly for q. This exactly parallels the development in the nonspherical gravitation theory, which uses the direction cosines of the inertial $\hat{z}$ axis with respect to the equinoctial reference system. In fact, Equations (3-168) apply also to the third-body averaged equations, with the exception, of course, that the definition of the direction cosines differ. This result is demonstrated in Appendix C. Consequently, Equations (3-171) also apply to the third-body case. In addition, a simplification of the equations of motion for the nonspherical gravitational zonal harmonic terms in Equations (3-73) and (3-74) also appears in this formulation for the third-body perturbation. Specifically,

$$R_{h,k} - R_{\alpha,\beta} \equiv 0 \tag{4-89}$$

where $R_{x,y}$ is defined in Equation (3-172). This simplification is demonstrated below.

Before the requisite partial derivatives for the equations of motion are obtained, the disturbing function is reordered as follows to accommodate the recurrence relations for the functions $K_0^{\ell,m}$ and $Q_{\ell,m}(\gamma)$:

$$\langle R_3 \rangle_{\bar{\lambda}} = \frac{2Gm'}{r'} \sum_{m=0}^{M \le L} \delta_m (\alpha - j\beta)^m (k + jh)^m \times \sum_{\substack{\ell = \max(2,m) \\ (\ell \pm m \text{ even})}}^{L} \frac{(\ell-m)!}{(\ell+m)!} \left(\frac{a}{r'}\right)^{\ell} P_{\ell,m}(0)\, K_0^{\ell,m}\, Q_{\ell,m}(\gamma) \tag{4-90}$$

If the following definition is made,

$$G_m + jH_m = (\alpha - j\beta)^m (k + jh)^m \tag{4-91}$$

then the real part of the disturbing function is given by

$$\langle R_3 \rangle_{\bar{\lambda}} = \frac{2Gm'}{r'} \sum_{m=0}^{M \le L} \delta_m G_m \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m \tag{4-92}$$

where

$$F_\ell^m = \left(\frac{a}{r'}\right)^{\ell} V_{\ell,m}\, Q_{\ell,m}(\gamma)\, K_0^{\ell,m} \tag{4-93}$$

and

$$V_{\ell,m} = \frac{(\ell-m)!}{(\ell+m)!} P_{\ell,m}(0) \tag{4-94}$$

The partial derivatives of the disturbing function are

$$\frac{\partial \langle R_3 \rangle}{\partial a} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m G_m \sum_{\substack{\ell = \max(2,m) \\ (\ell \pm m \text{ even})}}^{L} \frac{\partial F_\ell^m}{\partial a} \tag{4-95a}$$

$$\frac{\partial \langle R_3 \rangle}{\partial h} = \frac{2Gm'}{r'} \left[\sum_{m=0}^{M} \delta_m \frac{\partial G_m}{\partial h} \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m + 2h \sum_{m=0}^{M} \delta_m G_m \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} \frac{\partial F_\ell^m}{\partial e^2} \right] \tag{4-95b}$$

$$\frac{\partial \langle R_3 \rangle}{\partial k} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \frac{\partial G_m}{\partial k} \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m + 2k \sum_{m=0}^{M} \delta_m G_m \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} \frac{\partial F_\ell^m}{\partial e^2} \tag{4-95c}$$

$$\frac{\partial \langle R_3 \rangle}{\partial \alpha} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \frac{\partial G_m}{\partial \alpha} \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m \tag{4-95d}$$

$$\frac{\partial \langle R_3 \rangle}{\partial \beta} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \frac{\partial G_m}{\partial \beta} \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m \tag{4-95e}$$

$$\frac{\partial \langle R_3 \rangle}{\partial \gamma} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m G_m \sum_{\substack{\ell = \max(m,2) \\ [\ell \pm m \text{ even}]}}^{L} \frac{\partial F_\ell^m}{\partial \gamma} \tag{4-95f}$$

where

$$\frac{\partial F_\ell^m}{\partial a} = \frac{\ell}{a} F_\ell^m \tag{4-96}$$

$$\frac{\partial G_m}{\partial h} = m\beta G_{m-1} - m\alpha H_{m-1} \tag{4-97a}$$

$$\frac{\partial G_m}{\partial k} = m\alpha G_{m-1} + m\beta H_{m-1} \tag{4-97b}$$

$$\frac{\partial G_m}{\partial \alpha} = mk G_{m-1} - mh H_{m-1} \tag{4-97c}$$

$$\frac{\partial G_m}{\partial \beta} = mh G_{m-1} + mk H_{m-1} \tag{4-97d}$$

$$\frac{\partial F_\ell^m}{\partial e^2} = \left(\frac{a}{r'}\right)^\ell V_{\ell,m}\, Q_{\ell,m}(\gamma)\, \frac{\partial K_0^{\ell,m}}{\partial e^2} \tag{4-98}$$

$$\frac{\partial F_\ell^m}{\partial \gamma} = \left(\frac{a}{r'}\right)^\ell V_{\ell,m}\, K_0^{\ell,m}\, \frac{dQ_{\ell,m}(\gamma)}{d\gamma} \tag{4-99}$$

It follows from Equations (2-230) and (2-303) that

$$K_0^{\ell,m} = (-1)^m \frac{(\ell-m+1)!}{(\ell+1)!} \frac{1}{x^{\ell+1} e^m} P_{\ell+1}^m(x) \qquad (m \geq 0) \tag{4-100}$$

$$= \frac{(\ell-m+1)!}{(\ell+1)!} \frac{1}{x^{\ell+1} e^m} P_{\ell+1,m}(x)$$

Since

$$(-e)^{m} = \frac{(1-x^{2})^{m/2}}{x^{m}} \tag{4-101}$$

it follows that

$$\begin{aligned} K_{0}^{\ell,m} &= (-1)^{m} \frac{(\ell-m+1)!}{(\ell+1)!} x^{m-n-1} (1-x^{2})^{-m/2} P_{\ell+1,m}(x) \\ &= (-1)^{m} \frac{(\ell-m+1)!}{(\ell+1)!} x^{m-n-1} Q_{\ell,m}(\gamma) \end{aligned} \tag{4-102}$$

For the software implementation, the definition

$$A_{\ell+2}^{m} = \frac{(\ell-m+1)!}{(\ell+1)!} x^{m-n-1} Q_{\ell+1,m}(x) \tag{4-103}$$

is made.[1] Thus,

$$K_{0}^{\ell,m} = (-1)^{m} A_{\ell+2}^{m} \tag{4-104}$$

and

$$\frac{dK_{0}^{\ell,m}}{de^{2}} = (-1)^{m} \frac{dA_{\ell+2}^{m}}{de^{2}} \tag{4-105}$$

This treatment of the function $K_0^{\ell,m}$ is somewhat different from the treatment given in Section 3. It is presented in this manner because it reflects the

[1] This notation is used by Cefola in Reference 11.

software implementation for the third-body model. For the software implementation, the factor $(-1)^m$ is included in the definition of δ_m, i.e.,

$$\delta_m = \begin{cases} \frac{1}{2} & \text{(for } m = 0) \quad (4\text{-}106a) \\ (-1)^m & \text{(for } m \geq 0) \quad (4\text{-}106b) \end{cases}$$

Because of the commonality of the special functions between this third-body model and the nonspherical gravitational zonal harmonics model, the recurrence relations given in Equations (3-189) through (3-191) apply. The only exception is the recurrence relations for Cefola's $A^m_{\ell+2}$ functions and their derivatives which are

$$A^m_{\ell+2} = \begin{cases} \frac{(2m-1)!!}{m!} & (\ell = m-1) \quad (4\text{-}107a) \\ \frac{2m+1}{m+1} A^m_{m+1} & (\ell = m) \quad (4\text{-}107b) \\ \frac{2\ell+1}{\ell+1} A^m_{\ell+1} - \frac{(\ell+m)(\ell-m)}{\ell(\ell+1)} A^m_{\ell} & (\ell > m) \quad (4\text{-}107c) \end{cases}$$

and

$$\frac{d}{de^2} A^m_{\ell+2} =$$

$$\begin{cases} \dfrac{2m+1}{(m+1)(m+2)} A^m_{m+1} & (\ell = m-1) \quad (4\text{-}108a) \\[2ex] \dfrac{2\ell+1}{\ell+1} \dfrac{d}{de^2} A^m_{\ell+1} + \dfrac{(\ell+m)(\ell-m)}{\ell(\ell+1)} A^m_\ell & (\ell = m) \quad (4\text{-}108b) \\[2ex] \dfrac{2\ell+1}{\ell+1} \dfrac{d}{de^2} A^m_{\ell+1} - \dfrac{(\ell+m)(\ell-m)}{\ell(\ell+1)} \left[(1-e^2) \dfrac{d}{de^2} A^m_\ell - A^m_\ell \right] & (\ell > m) \quad (4\text{-}108c) \end{cases}$$

The identity

$$R_{h,k} - R_{\alpha,\beta} \equiv 0 \tag{4-109}$$

which results in the simplification of the equations of motion, is demonstrated below.

It follows from Equations (4-95) that

$$R_{h,k} = h \frac{\partial R_3}{\partial k} - k \frac{\partial R_3}{\partial h} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \left(h \frac{\partial G_m}{\partial k} - k \frac{\partial G_m}{\partial h} \right) \times \sum_{\substack{\ell = \max(2,m) \\ [\ell+m \text{ even}]}}^{L} F^m_\ell \tag{4-110}$$

and

$$R_{\alpha,\beta} = \alpha \frac{\partial R_3}{\partial \beta} - \beta \frac{\partial R_3}{\partial \alpha} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \left(\alpha \frac{\partial G_m}{\partial \beta} - \beta \frac{\partial G_m}{\partial \alpha} \right)$$

$$\times \sum_{\substack{\ell = \max(2,m) \\ [\ell \pm m \text{ even}]}}^{L} F_\ell^m \tag{4-111}$$

Therefore,

$$R_{h,k} - R_{\alpha,\beta} = \frac{2Gm'}{r'} \sum_{m=0}^{M} \delta_m \left[(G_m)_{h,k} - (G_m)_{\alpha,\beta} \right] \sum_{\ell = \max(m,2)}^{L} F_\ell^m \tag{4-112}$$

However, from Equations (4-98), it follows that

$$\begin{aligned} (G_m)_{h,k} &= h \frac{\partial G_m}{\partial k} - k \frac{\partial G_m}{\partial h} \\ &= m\alpha h G_{m-1} + m\beta h H_{m-1} - m\beta k G_{m-1} + m\alpha k H_{m-1} \end{aligned} \tag{4-113}$$

and

$$\begin{aligned} (G_m)_{\alpha,\beta} &= \alpha \frac{\partial G_m}{\partial \beta} - \beta \frac{\partial G_m}{\partial \alpha} \\ &= m\alpha h G_{m-1} + m\alpha k H_{m-1} - m\beta k G_{m-1} + m\beta h H_{m-1} \end{aligned} \tag{4-114}$$

and, thus,

$$(G_m)_{h,k} - (G_m)_{\alpha;\beta} \equiv 0 \qquad (4\text{-}115)$$

Equation (4-109) follows immediately from this identity.

APPENDIX A - DERIVATION OF VON ZEIPEL'S PARTIAL DIFFERENTIAL EQUATION[1]

The derivation of Von Zeipel's partial differential equation (Equation (2-192)) given by

$$(1-e^2)\, e \frac{\partial X^{n,k}}{\partial e} + (1+e^2)^{3/2}\, z \frac{\partial X^{n,k}}{\partial z}$$

$$= \left\{ k\left[1-(1-e^2)^{3/2}\right] + (k-n)\frac{e^2}{2} + (2k-n)\, ex + (k-n)\frac{e^2}{2} x^2 \right\} X^{n,k} \tag{A-1}$$

where

$$X^{n,k} = \left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^k \tag{A-2}$$

obviously requires that expressions for the partial derivatives $e\, \partial X^{n,k}/\partial e$ and $z\, \partial X^{n,k}/\partial z$ be obtained.

Clearly,

$$e \frac{\partial X^{n,k}}{\partial e} = e \frac{\partial}{\partial e}\left[\left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^k\right] = e z^{-k} \frac{\partial}{\partial e}\left[\left(\frac{r}{a}\right)^n x^k\right] \tag{A-3}$$

since $z = e^{j\ell}$ is independent of the eccentricity, e. Then,

$$e z^{-k} \frac{\partial}{\partial e}\left[\left(\frac{r}{a}\right)^n x^k\right] = e z^{-k}\left[n\left(\frac{r}{a}\right)^{n-1} x^k \frac{\partial}{\partial e}\left(\frac{r}{a}\right) + k\left(\frac{r}{a}\right)^n x^{k-1} \frac{\partial X}{\partial e}\right] \tag{A-4}$$

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[1]This derivation is based in part on notes contributed by P. Cefola.

Substituting the relations

$$\frac{\partial}{\partial e}\left(\frac{r}{a}\right) = -\frac{x + x^{-1}}{2} \tag{A-5}$$

$$\frac{\partial x}{\partial e} = x\left[\frac{a}{r} + \frac{1}{1-e^2}\right]\frac{x - x^{-1}}{2} \tag{A-6}$$

into Equation (A-4) yields (in view of Equation (A-3))

$$\begin{aligned} e\,\frac{\partial X^{n,k}}{\partial e} = & -\frac{ne}{2}\left(\frac{r}{a}\right)^{n-1}\left(\frac{x}{z}\right)^{k}(x+x^{-1}) \\ & + \frac{ke}{2}\left(\frac{r}{a}\right)^{n-1}\left(\frac{x}{z}\right)^{k}(x-x^{-1}) \\ & + \frac{ke}{2(1-e^2)}\left(\frac{r}{a}\right)^{n}\left(\frac{x}{z}\right)^{k}(x-x^{-1}) \end{aligned} \tag{A-7}$$

With the aid of the identity

$$\frac{a}{r} = \frac{1 + \frac{e}{2}(x+x^{-1})}{1-e^2} \tag{A-8}$$

Equation (A-7) can be expressed (after some algebraic manipulation) as

$$e\frac{\partial X^{n,k}}{\partial e} = \frac{X^{n,k}}{1-e^2}\left[-n\frac{e^2}{2} - (2k+n)\frac{e}{2}x^{-1} + (2k-n)\frac{e}{2}x - (k+n)\frac{e^2}{4}x^{-2} + (k-n)\frac{e^2}{4}x^2\right] \tag{A-9}$$

The expression for the partial derivative $z\frac{\partial X^{n,k}}{\partial z}$ is now obtained. Clearly, from Equation (A-2),

$$\frac{\partial X^{n,k}}{\partial z} = z^{-k}\frac{\partial}{\partial z}\left[\left(\frac{r}{a}\right)^n x^k\right] - k\left(\frac{r}{a}\right)^n x^k z^{-(k+1)} \tag{A-10}$$

Also, from Equation (2-265)

$$z\frac{\partial}{\partial z}\left[\left(\frac{r}{a}\right)^n x^k\right] = \frac{(n+k)e}{2\sqrt{1-e^2}}\left(\frac{r}{a}\right)^{n-1}x^{k-1} - \frac{(n-k)e}{2\sqrt{1-e^2}}\left(\frac{r}{a}\right)^{n-1}x^{k+1} + \frac{k}{\sqrt{1-e^2}}\left(\frac{r}{a}\right)^n x^k \tag{A-11}$$

Therefore, it follows that

$$z \frac{\partial X^{n,k}}{\partial z} = z^{1-k} \frac{\partial}{\partial z}\left[\left(\frac{r}{a}\right)^n x^k\right] - k\left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^k$$

$$= \frac{(n+k)e}{2\sqrt{1-e^2}} \left(\frac{r}{a}\right)^{n-1}\left(\frac{x}{z}\right)^k x^{-1} - \frac{(n-k)e}{2\sqrt{1-e^2}} \left(\frac{r}{a}\right)^n\left(\frac{x}{z}\right)^k x \qquad \text{(A-12)}$$

$$- \frac{k}{\sqrt{1-e^2}} \left(\frac{r}{a}\right)^{n-1}\left(\frac{x}{z}\right)^k - k\left(\frac{r}{a}\right)^n \left(\frac{x}{z}\right)^k$$

In view of the identity given in Equation (A-8), Equation (A-12) can be expressed (after some simplification) as

$$z \frac{\partial X^{n,k}}{\partial z} = \frac{X^{n,k}}{(1-e^2)^{3/2}} \Bigg\{ (2k+n) \frac{e}{2} x^{-1} + (2k-n) \frac{e}{2} x$$

$$+ k \frac{e^2}{2} + k\left[1-(1-e^2)^{3/2}\right] \qquad \text{(A-13)}$$

$$+ (n+k) \frac{e^2}{4} x^{-2} - (n-k) \frac{e^2}{4} x^2 \Bigg\}$$

Inspection of Equations (A-9) and (A-13) shows that the expressions for

$$(1-e^2)\, e \frac{\partial X^{n,k}}{\partial e} \quad \text{and} \quad (1-e^2)^{3/2} z \frac{\partial X^{n,k}}{\partial z}$$

contain similar terms. Summing these expressions yields Von Zeipel's partial differential equation given in Equation (A-1).

APPENDIX B - A JACOBI POLYNOMIAL REPRESENTATION FOR THE FOURIER SERIES COEFFICIENTS $W_t^{\pm n,s}$

The coefficients (Equation (2-130b)) given below are considered first:

$$W_t^{n,s} = (1+\beta^2)^{-n}(-\beta)^{|t-s|}\binom{n-s}{\frac{|t-s|+t-s}{2}}\binom{n+s}{\frac{|t-s|-(t-s)}{2}} \tag{B-1}$$

$$\times F\left[\frac{|t-s|+(t-s)}{2}-n+s,\ \frac{|t-s|-(t-s)}{2}-n-s,\ |t-s|+1;\ \beta^2\right]$$

Application of the linear transformations

$$F(a,b,c;x) = \begin{cases} (1-x)^{-a}\, F\left(a, c-b, c; \frac{x}{x-1}\right) & \text{(B-2a)} \\ (1-x)^{-b}\, F\left(b, c-a, c; \frac{x}{x-1}\right) & \text{(B-2b)} \end{cases}$$

to the hypergeometric series in Equation (B-1) yields

$$F\left[\frac{|t-s|+t-s}{2}-n+s,\ \frac{|t-s|-(t-s)}{2}-n-s,\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n-s-(|t-s|+t-s)/2}\, F\left[\frac{|t-s|+t-s}{2}-n+s,\ \frac{|t-s|+t-s}{2}+n+s+1,\ |t-s|+1;\ \frac{\beta^2}{\beta^2-1}\right] & \text{(B-3a)} \\ (1-\beta^2)^{n+s-[|t-s|-(t-s)]/2}\, F\left[\frac{|t-s|-(t-s)}{2}-n-s,\ \frac{|t-s|-(t-s)}{2}+n-s+1,\ |t-s|+1;\ \frac{\beta^2}{\beta^2-1}\right] & \text{(B-3b)} \end{cases}$$

Using the relationship between the hypergeometric series and the Jacobi polynomial (Reference 26), i.e.,

$$F(-m,\ \alpha+\gamma+m+1,\ \alpha+1;\ x) = \frac{m!}{(\alpha+1)_m} P_m^{\alpha,\gamma}(1-2x) \tag{B-4}$$

it follows that Equations (B-3) can be expressed as

$$F\left[\frac{|t-s|+(t-s)}{2} - n+s,\ \frac{|t-s|-(t-s)}{2} - n-s,\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n-s-(|t-s|+t-s)/2} \dfrac{\{n-s-[|t-s|+(t-s)]/2\}!}{(|t-s|+1)_{n-s-[|t-s|+t-s]/2}} P^{|t-s|,\,t+s}_{n-s-(|t-s|+t-s)/2}(x) & \text{(B-5a)} \\[2ex] (1-\beta^2)^{n+s-[|t-s|-(t-s)]/2} \dfrac{\{n+s-[|t-s|-(t-s)]/2\}!}{(|t-s|+1)_{n+s-[|t-s|-(t-s)]/2}} P^{|t-s|,\,-(t+s)}_{n+s-[|t-s|-(t-s)]/2}(x) & \text{(B-5b)} \end{cases}$$

where $t+s \geq 0$ in Equation (B-5a) and $t+s \leq 0$ in Equation (B-5b), and

$$x = \frac{1+\beta^2}{1-\beta^2} = \frac{1}{\sqrt{1-e^2}} \tag{B-6}$$

Equations (B-5) can be simplified by considering the sign of the quantity $t - s$. For Equation (B-5a), where $t - s \geq 0$,

$$(1-\beta^2)^{n-s-(|t-s|+t-s)/2} \frac{\{n-s-[|t-s|+(t-s)]/2\}!}{(|t-s|+1)_{n-s-(|t-s|+t-s)/2}} P^{|t-s|,\, t+s}_{n-s-(|t-s|+t-s)/2}(x)$$

$$= \begin{cases} (1-\beta^2)^{n-t} \dfrac{(n-t)!}{(t-s+1)_{n-t}} P^{t-s,\, t+s}_{n-t}(x) & \begin{matrix}(t-s \geq 0)\\ (t+s \leq 0)\end{matrix} \Big\} \; (t \geq |s|) \quad \text{(B-7a)} \\[2ex] (1-\beta^2)^{n-s} \dfrac{(n-s)!}{(s-t+1)_{n-s}} P^{s-t,\, t+s}_{n-s}(x) & \begin{matrix}(t-s \leq 0)\\ (t+s \geq 0)\end{matrix} \Big\} \; (|t| \leq s) \quad \text{(B-7b)} \end{cases}$$

Similarly, Equation (B-5b) (where $t - s \leq 0$) simplifies to

$$(1-\beta^2)^{n+s-[|t-s|-(t-s)]/2} \frac{\{n+s-[|t-s|-(t-s)]/2\}!}{(|t-s|+1)_{n+s-[|t-s|-(t-s)]/2}} P^{|t-s|,\, -(t+s)}_{n+s-[|t-s|-(t-s)]/2}(x)$$

$$= \begin{cases} (1-\beta^2)^{n+s} \dfrac{(n+s)!}{(t-s+1)_{n+s}} P^{t-s,\, -(t+s)}_{n+s}(x) & \begin{matrix}(t-s \geq 0)\\ (t+s \leq 0)\end{matrix} \Big\} \; (s \leq -|t|) \quad \text{(B-8a)} \\[2ex] (1-\beta^2)^{n+t} \dfrac{(n+t)!}{(s-t+1)_{n+t}} P^{s-t,\, -(t+s)}_{n+t}(x) & \begin{matrix}(t-s \leq 0)\\ (t+s \geq 0)\end{matrix} \Big\} \; (t \leq -|s|) \quad \text{(B-8b)} \end{cases}$$

Substituting Equations (B-8) and (B-7) into Equations (B-5) yields the Jacobi polynomial representation of Equation (B-3), i.e.,

$$F\left[\frac{|t-s|+(t-s)}{2}-n+s,\ \frac{|t-s|-(t-s)}{2}-n-s,\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n-t}\,\dfrac{(n-t)!\,(t-s)!}{(n-s)!}\,P_{n-t}^{t-s,\,t+s}(x) & \left.\begin{matrix}(t-s\geq 0)\\(t+s\geq 0)\end{matrix}\right\} \Rightarrow (t\geq |s|\geq 0) \quad \text{(B-9a)} \\[2ex] (1-\beta^2)^{n-s}\,\dfrac{(n-s)!\,(s-t)!}{(n-t)!}\,P_{n-s}^{s-t,\,t+s}(x) & \left.\begin{matrix}(s-t\geq 0)\\(s+t\geq 0)\end{matrix}\right\} \Rightarrow (s\geq |t|\geq 0) \quad \text{(B-9b)} \\[2ex] (1-\beta^2)^{n+s}\,\dfrac{(n+s)!\,(t-s)!}{(n+t)!}\,P_{n+s}^{t-s,\,-(t+s)}(x) & \left.\begin{matrix}(t-s\geq 0)\\(t+s\leq 0)\end{matrix}\right\} \Rightarrow (s\leq -|t|\leq 0) \quad \text{(B-9c)} \\[2ex] (1-\beta^2)^{n+t}\,\dfrac{(n+t)!\,(s-t)!}{(n+s)!}\,P_{n+t}^{s-t,\,-(t+s)}(x) & \left.\begin{matrix}(s-t\geq 0)\\(t+s\leq 0)\end{matrix}\right\} \Rightarrow (t\leq -|s|) \quad \text{(B-9d)} \end{cases}$$

Substituting Equations (B-9) into Equation (B-1) and simplifying the result yields

$$W_t^{n,s} = \left(\frac{1-\beta^2}{1+\beta^2}\right)^n (-\beta)^{|t-s|} \begin{cases} (1-\beta^2)^{-t} \, P_{n-t}^{t-s,\,t+s}(x) & (t \ge |s| \ge 0) \quad \text{(B-10a)} \\ \dfrac{(n+s)!\,(n-s)!}{(n+t)!\,(n-t)!} (1-\beta^2)^{-s} \, P_{n-s}^{s-t,\,t+s}(x) & (s \ge -|t| \ge 0) \quad \text{(B-10)} \\ \dfrac{(n+s)!\,(n-s)!}{(n+t)!\,(n-t)!} (1-\beta^2)^{s} \, P_{n+s}^{t-s,\,-(t+s)}(x) & (s \le -|t| \le 0) \quad \text{(B-10c)} \\ (1-\beta^2)^{t} \, P_{n+t}^{s-t,\,-(t+s)}(x) & (t \le -|s| \le 0) \quad \text{(B-10d)} \end{cases}$$

This expression is further simplified to

$$W_t^{n,s} = \left(\frac{1-\beta^2}{1+\beta^2}\right)^n (-\beta)^{|t-s|} \begin{cases} \dfrac{(n+s)!\,(n-s)!}{(n+t)!\,(n-t)!} (1-\beta^2)^{-|s|} \, P_{n-|s|}^{|t-s|,\,|t+s|}(x) & (|s| \ge |t|) \quad \text{(B-11a)} \\ (1-\beta^2)^{-|t|} \, P_{n-|t|}^{|t-s|,\,|t+s|}(x) & (|s| \le |t|) \quad \text{(B-11b)} \end{cases}$$

This completes the derivation of the Jacobi polynomial representation for the Fourier series coefficients, $W_t^{n,s}$.

The coefficients $W_t^{-n,s}$ also admit a Jacobi polynomial representation. This is demonstrated using the definition given in the footnote on page 2-36, i.e.,

$$W_t^{-n,s} = (1-\beta^2)^{-(n-1)} \cos^{-n}\phi \, \beta^{|t-s|} \begin{pmatrix} n+s+\frac{|t-s|+(t-s)}{2}-1 \\ \frac{|t-s|+(t-s)}{2} \end{pmatrix} \begin{pmatrix} n-s+\frac{|t-s|-(t-s)}{2}-1 \\ \frac{|t-s|-(t-s)}{2} \end{pmatrix}$$

$$\times F\left[1-n-s+\frac{|t-s|-t-s}{2},\ 1-n+s+\frac{|t-s|+t-s}{2},\ |t-s|+1;\ \beta^2\right] \qquad \text{(B-12)}$$

Application of the linear transformation given in Equations (B-2) to the hypergeometric series in Equation (B-12) yields

$$F\left[1-n-s+\frac{|t-s|-(t-s)}{2},\ 1-n+s+\frac{|t-s|+(t-s)}{2},\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n+s-1-[|t-s|-(t-s)]/2} F\left[1-n-s+\frac{|t-s|-(t-s)}{2},\ n-s+\frac{|t-s|-(t-s)}{2},\ |t-s|+1;\ \frac{\beta^2}{\beta^2-1}\right] & \text{(B-13a)} \\ (1-\beta^2)^{n-s-1-[|t-s|+(t-s)]/2} F\left[1-n+s+\frac{|t-s|+t-s}{2},\ n+s+\frac{|t-s|+t-s}{2},\ |t-s|+1;\ \frac{\beta^2}{\beta^2-1}\right] & \text{(B-13b)} \end{cases}$$

which, in turn, admits the Jacobi polynomial representation

$$F\left[1-n-s+\frac{|t-s|-(t-s)}{2},\ 1-n+s+\frac{|t-s|+(t-s)}{2},\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n+s-1-[|t-s|-(t-s)]/2} \dfrac{\{n+s-1-[|t-s|-(t-s)]/2\}!}{(|t-s|+1)_{n+s-1-[|t-s|-(t-s)]/2}} P^{|t-s|,-(t+s)}_{n+s-1-[|t-s|-(t-s)]/2}(x) & \text{(B-14a)} \\ (1-\beta^2)^{n-s-1-[|t-s|+t-s]/2} \dfrac{\{n-s-1-[|t-s|+t-s]/2\}!}{(|t-s|+1)_{n-s-1-(|t-s|+t-s)/2}} P^{|t-s|,t+s}_{n-s-1-(|t-s|+t-s)/2}(x) & \text{(B-14b)} \end{cases}$$

where $t+s \leq 0$ in Equation (B-14a) and $t+s \geq 0$ in Equation (B-14b). This result can be simplified to yield

$$F\left[1-n-s+\frac{|t-s|-(t-s)}{2},\ 1-n+s+\frac{|t-s|+(t-s)}{2},\ |t-s|+1;\ \beta^2\right]$$

$$= \begin{cases} (1-\beta^2)^{n+s-1} \dfrac{(n+s-1)!\,(t-s)!}{(n+t-1)!} P^{t-s,-(t+s)}_{n+s-1}(x) & \begin{matrix}(t-s \geq 0)\\(t+s \leq 0)\end{matrix} \Rightarrow (s \leq -|t|) & \text{(B-15a)} \\ (1-\beta^2)^{n+t-1} \dfrac{(n+t-1)!\,(s-t)!}{(n+s-1)!} P^{s-t,-(t+s)}_{n+t-1}(x) & \begin{matrix}(t-s \leq 0)\\(t+s \leq 0)\end{matrix} \Rightarrow (t \leq -|s|) & \text{(B-15b)} \\ (1-\beta^2)^{n-t-1} \dfrac{(n-t-1)!\,(t-s)!}{(n-s-1)!} P^{t-s,t+s}_{n-t-1}(x) & \begin{matrix}(t-s \geq 0)\\(t+s \geq 0)\end{matrix} \Rightarrow (t \geq |s|) & \text{(B-15c)} \\ (1-\beta^2)^{n-s-1} \dfrac{(n-s-1)!\,(s-t)!}{(n-t-1)!} P^{s-t,t+s}_{n-s-1}(x) & \begin{matrix}(t-s \leq 0)\\(t+s \geq 0)\end{matrix} \Rightarrow (|t| \leq s) & \text{(B-15d)} \end{cases}$$

Substituting this last result into Equation (B-12) and simplifying yields

$$W_t^{-n,s} = \left(\frac{1-\beta^2}{1+\beta^2}\right)^{-n} \beta^{|t-s|} \begin{cases} (1-\beta^2)^{s} \, P_{n+s-1}^{t-s,-(t+s)}(x) & (s \le -|t|) \quad \text{(B-16a)} \\[2ex] \dfrac{(n-t-1)!\,(n+t-1)!}{(n-s-1)!\,(n+s-1)!} (1-\beta^2)^{t} \, P_{n+t-1}^{s-t,-(t+s)}(x) & (t \le -|s|) \quad \text{(B-16b)} \\[2ex] \dfrac{(n+t-1)!\,(n-t-1)!}{(n+s-1)!\,(n-s-1)!} (1-\beta^2)^{-t} \, P_{n-t-1}^{t-s,t+s}(x) & (t \ge |s|) \quad \text{(B-16c)} \\[2ex] (1-\beta^2)^{-s} \, P_{n-s-1}^{s-t,t+s}(x) & (s \ge |t|) \quad \text{(B-16d)} \end{cases}$$

which further simplifies to

$$W_t^{-n,s} = \left(\frac{1-\beta^2}{1+\beta^2}\right)^{-n} \beta^{|t-s|} \begin{cases} \dfrac{(n+t-1)!\,(n-t-1)!}{(n+s-1)!\,(n-s-1)!} (1-\beta^2)^{-|t|} \, P_{n-|t|-1}^{|t-s|,|t+s|}(x) & (|t| \ge |s|) \quad \text{(B-17a)} \\[2ex] (1-\beta^2)^{-|s|} \, P_{n-|s|-1}^{|t-s|,|t+s|}(x) & (|s| \ge |t|) \quad \text{(B-17b)} \end{cases}$$

A comparison of Equations (B-17) with Equations (B-11) shows that

$$W_t^{-(n+1),s} = (-1)^{|t-s|} \left(\frac{1-\beta^2}{1+\beta^2}\right)^{-2n-1} \frac{(n+t)!\,(n-t)!}{(n+s)!\,(n-s)!}\, W_t^{n,s} \tag{B-18}$$

In addition, it is also apparent that Equations (B-11) and (B-16) satisfy the conditions (see page 2-30)

$$W_t^{\pm n,s} = W_{-t}^{\pm n,-s} \tag{B-19}$$

The Jacobi polynomial representations derived above provide a new source of recurrence relations for the evaluation of the coefficients $W_t^{\pm n,s}$.

APPENDIX C - DERIVATION OF THE PARTIAL DERIVATIVES $\partial(\alpha, \beta, \gamma)/\partial(p, q)$

The direction cosines (α, β, γ) with respect to the equinoctial reference frame $(\hat{f}, \hat{g}, \hat{w})$ of an arbitrary unit vector $\hat{\nu}$ is considered, i.e.,

$$\alpha = \hat{f} \cdot \hat{\nu} \tag{C-1a}$$

$$\beta = \hat{g} \cdot \hat{\nu} \tag{C-1b}$$

$$\gamma = \hat{w} \cdot \hat{\nu} \tag{C-1c}$$

The expression for the unit vectors $(\hat{f}, \hat{g}, \hat{w})$ were derived in Appendix A of Volume I of this report (Reference 5) and are given by

$$\hat{f} = \frac{1}{1+p^2+q^2} \begin{bmatrix} 1-p^2+q^2 \\ 2pq \\ -2pI \end{bmatrix} \tag{C-2a}$$

$$\hat{g} = \frac{1}{1+p^2+q^2} \begin{bmatrix} 2pqI \\ (1+p^2-q^2)I \\ 2q \end{bmatrix} \tag{C-2b}$$

$$\hat{w} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 2p \\ -2q \\ (1-p^2-q^2)I \end{bmatrix} \tag{C-2c}$$

The unit vector $\hat{\upsilon}$ is independent of the equinoctial elements p and q. (For the applications in this report, the unit vector $\hat{\upsilon}$ is either the inertial $\hat{z}$ axis for the nonspherical gravitational theory or the unit position vector of the third body, $\hat{r}'$, in the third-body theory.) It follows from Equations (C-1) that

$$\frac{\partial \alpha}{\partial(p,q)} = \frac{\partial \hat{f}}{\partial(p,q)} \cdot \hat{\upsilon} \tag{C-3a}$$

$$\frac{\partial \beta}{\partial(p,q)} = \frac{\partial \hat{g}}{\partial(p,q)} \cdot \hat{\upsilon} \tag{C-3b}$$

$$\frac{\partial \gamma}{\partial(p,q)} = \frac{\partial \hat{w}}{\partial(p,q)} \cdot \hat{\upsilon} \tag{C-3c}$$

Taking the partial derivative with respect to p of the unit vector $\hat{f}$ yields

$$\frac{\partial \hat{f}}{\partial p} = \frac{-2p}{\left[1+p^2+q^2\right]^2}\begin{bmatrix} 1-p^2+q^2 \\ 2pq \\ -2pI \end{bmatrix} + \frac{1}{1+p^2+q^2}\begin{bmatrix} -2p \\ 2q \\ -2I \end{bmatrix} \tag{C-4}$$

or

$$\frac{\partial \hat{f}}{\partial p} = \frac{-2}{1+p^2+q^2} \begin{bmatrix} p\,\dfrac{(1-p^2+q^2)}{1+p^2+q^2} + p \\[2ex] \dfrac{2p^2 q}{1+p^2+q^2} - q \\[2ex] \dfrac{-2p^2 I}{1+p^2+q^2} + I \end{bmatrix} \tag{C-5}$$

which simplifies to

$$\frac{\partial \hat{f}}{\partial p} = \frac{-2}{1+p^2+q^2} \begin{bmatrix} \dfrac{2p+2pq^2}{1+p^2+q^2} \\[2ex] \dfrac{-q+p^2 q-q^3}{1+n^2+q^2} \\[2ex] \dfrac{I-p^2 I+q^2 I}{1+p^2+q^2} \end{bmatrix} \tag{C-6}$$

Using Equations (C-2), it is easily verified that

$$\frac{\partial \hat{f}}{\partial p} = \frac{-2}{1+p^2+q^2}\left[q I \hat{g} + \hat{w}\right] \tag{C-7a}$$

Similarly, it can be shown that

$$\frac{\partial \hat{f}}{\partial q} = \frac{2Ip}{1+p^2+q^2}\,\hat{g} \tag{C-7b}$$

and

$$\frac{\partial \hat{g}}{\partial p} = \frac{2Iq}{1+p^2+q^2}\,\hat{f} \tag{C-8a}$$

$$\frac{\partial \hat{g}}{\partial q} = \frac{-2I}{1+p^2+q^2}\,(p\hat{f} - \hat{w}) \tag{C-8b}$$

$$\frac{\partial \hat{w}}{\partial p} = \frac{2}{1+p^2+q^2}\,\hat{f} \tag{C-9a}$$

$$\frac{\partial \hat{w}}{\partial q} = \frac{-2I}{1+p^2+q^2}\,\hat{g} \tag{C-9b}$$

Substituting Equations (C-7), (C-8), and (C-9) into Equations (C-3a), (C-3b), and (C-3c), respectively, and simplifying the result yields

$$\frac{\partial \alpha}{\partial p} = \frac{-2}{1+p^2+q^2}\left(qI\hat{g}+\hat{w}\right)\cdot\hat{v} = \frac{-2}{1+p^2+q^2}\left[qI\beta+\gamma\right] \tag{C-10a}$$

$$\frac{\partial \alpha}{\partial q} = \frac{2I}{1+p^2+q^2}\, p\hat{g}\cdot\hat{v} = \frac{2Ip}{1+p^2+q^2}\,\beta \tag{C-10b}$$

$$\frac{\partial \beta}{\partial p} = \frac{2Iq}{1+p^2+q^2}\,\hat{f}\cdot\hat{v} = \frac{2Iq}{1+p^2+q^2}\,\alpha \tag{C-11a}$$

$$\frac{\partial \beta}{\partial q} = \frac{-2I}{1+p^2+q^2}\left(p\hat{f}-\hat{w}\right)\cdot\hat{v} = \frac{-2I}{1+p^2+q^2}\left(p\alpha-\gamma\right) \tag{C-11b}$$

$$\frac{\partial \gamma}{\partial p} = \frac{2}{1+p^2+q^2}\,\hat{f}\cdot\hat{v} = \frac{2}{1+p^2+q^2}\,\alpha \tag{C-12a}$$

$$\frac{\partial \gamma}{\partial q} = \frac{-2I}{1+p^2+q^2}\,\hat{g}\cdot\hat{v} = \frac{-2I}{1+p^2+q^2}\,\beta \tag{C-12b}$$

Using Equations (C-10) through (C-12), it is straightforward to show that if

$$F = F(p,q) \tag{C-13}$$

and

$$p = p(\alpha,\beta,\gamma) \tag{C-14a}$$

$$q = q(\alpha,\beta,\gamma) \tag{C-14b}$$

(where α, β, γ are a redundant set), then

$$\frac{\partial F}{\partial p} = \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial p} + \frac{\partial F}{\partial \gamma}\frac{\partial \gamma}{\partial p}$$

$$= \frac{2}{1+p^2+q^2}\left[\alpha\frac{\partial F}{\partial \gamma} - \gamma\frac{\partial F}{\partial \alpha} - qI\left(\alpha\frac{\partial F}{\partial \beta} - \beta\frac{\partial F}{\partial \alpha}\right)\right] \tag{C-15a}$$

and, similarly,

$$\frac{\partial F}{\partial q} = \frac{-2I}{1+p^2+q^2}\left[\beta\frac{\partial F}{\partial \gamma} - \gamma\frac{\partial F}{\partial \beta} + p\left(\alpha\frac{\partial F}{\partial \beta} - \beta\frac{\partial F}{\partial \alpha}\right)\right] \tag{C-15b}$$

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