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**LSA Project
Task Report**

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5101-73

**Determination of Bulk Diffusion
Lengths for Angle-lapped
Semiconductor Material via the
Scanning Electron Microscope—
A Theoretical Analysis**

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SEMICONDUCTOR MATERIAL VIA THE SCANNING
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Oldwig von Roos

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ABSTRACT

A standard procedure for the determination of the minority carrier diffusion length by means of a scanning electron microscope (SEM) consists in scanning across an angle-lapped surface of a P-N junction and measuring the resultant short circuit current I_{sc} as a function of beam position. A detailed analysis of the I_{sc} originating from this configuration is presented. It is found that, for a point source excitation, the I_{sc} depends very simply on x , the variable distance between the surface and the junction edge. The expression for the I_{sc} of a planar junction device is well known. If d , the constant distance between the plane of the surface of the semiconductor and the junction edge in the expression for the I_{sc} of a planar junction is merely replaced by x , the variable distance of the corresponding angle-lapped junction, an expression results which is correct to within a small fraction of a percent as long as the angle between the surfaces, $2\theta_1$, is smaller than 10° .

I. INTRODUCTION

In continuation of previous work (Reference 1), we are now addressing ourselves to the configuration shown in Figure 1, which displays a cross section through a solar cell together with typical dimensions. The bulk material, that part of the junction which shows maximum thickness $d \approx 200 \mu\text{m}$ in Figure 1, consists of uniformly-doped P material extending to the junction edge where the transition region between P and N material begins. The back surface of the cell has been partially cleared of the ohmic contact, polished, and lapped in such a manner that the plane of the back surface and the plane of the junction edge subtend an angle of $2 \theta_1$. An electron beam is directed toward the surface, excites electron-hole pairs, and produces a characteristic short-circuit current I_{sc} .

An analysis of the I_{sc} under the circumstances depicted in Figure 1 has been done by Hackett (Reference 2). His result can be stated in our notation as:

$$I_{sc} = e S_0 \left[e^{\xi \cos(2 \theta_1)/L} + \frac{1 - \eta/\cos(2 \theta_1)}{1 + \eta/\cos(2 \theta_1)} e^{-\xi \cos(2 \theta_1)/L} \right] e^{-x/L} \quad (1)$$

where e is the electronic charge, S_0 the strength of the point source of electron-hole pairs, L the diffusion length of the minority carriers, and η is given by (Reference 1):

$$\eta = L s/D \quad (2)$$

Equation (1) is valid only for $x/L \gg 1$. For moderately low surface recombination velocities, s , of the order of 10^4 cm/sec and taking $L = 50 \mu\text{m}$ and the diffusion constant for $10 \Omega \text{ cm}$ P material (Reference 3) $D = 27 \text{ cm}^2 \text{ sec}^{-1}$, η turns out to be 1.85. Since in this case, one and $\eta/\cos(2 \theta_1)$ are comparable in magnitude,

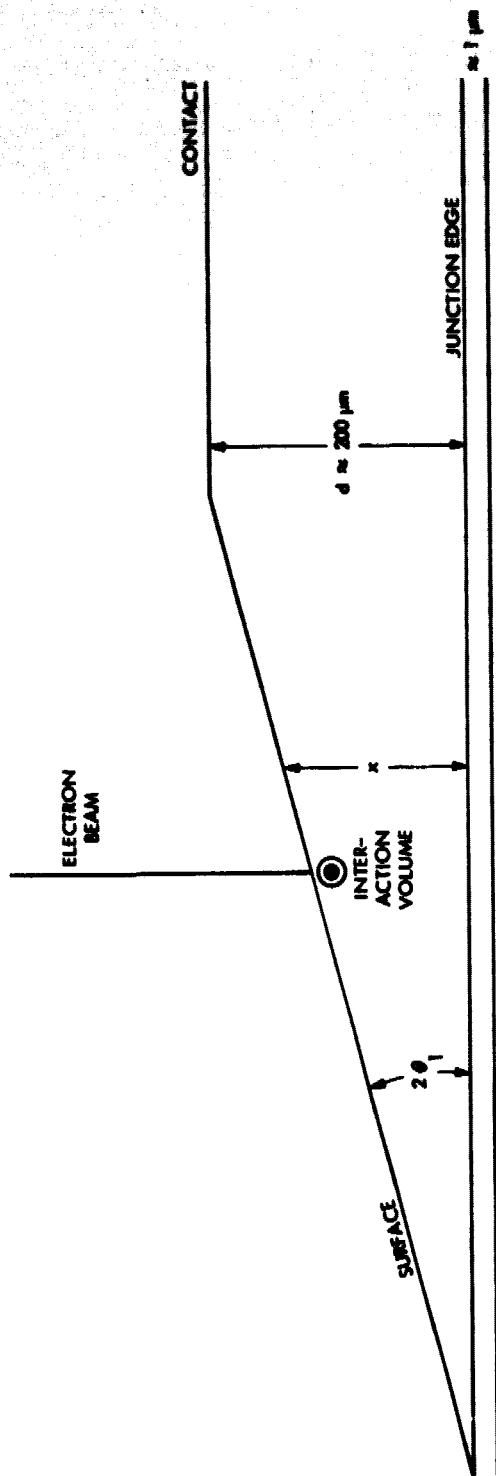


Figure 1. Cross Section of an Angle-lapped Solar Cell

the effect of increasing the angle between the two planes has an effect identical with an increase of the surface recombination velocity, s . Also, since ξ , the penetration depth of the electron beam (essentially the position at which maximum pair creation occurs) only depends on the beam energy, it is puzzling that, according to Equation (1), the effective penetration depth decreases with an increasing angle between the surface and the junction plane. These are strange features of expression (1) which must be explained. In the following pages we shall analyze the configuration represented by Figure 1 and show that the unphysical behavior of Equation (1) is due to the failure of not taking the boundary conditions, prevailing at the surface of the solar cell and the junction edge, properly into account. Although they will be shown in detail later on together with the limitations inherent to the model, we will give here the pertinent results and compare them with Equation (1).

Already implicit in Hackett's work (Reference 2) and also as derived by the author* is the expression for the I_{sc} generated by a point source excitation valid for a planar junction (a junction in which the surface and the junction or depletion layer edge form parallel planes a constant distance d apart, the material consisting of a uniformly-doped extrinsic semiconductor). It is given by:

$$I_{sc} = e S_0 \frac{\cosh (\xi/L) + \eta \sinh (\xi/L)}{\cosh (d/L) + \eta \sinh (d/L)} \quad (3)$$

The important result to be derived in the next section consists now of the following statement: If the surface plane of the semiconductor is tilted with respect to the plane of the junction edge as shown in Figure 1, we merely have

*See Reference 1 Part II, Equation (C2) for the general case.

to replace d the constant distance of the planar device, by the variable distance x of the angle-lapped device to obtain the corresponding I_{sc} . If, furthermore, $x/L \gg 1$, Equation (3) goes over into

$$I_{sc} = 2 e S_0 \left[\cosh(\xi/L) + n \sinh(\xi/L) \right] (1 + n)^{-1} e^{-x/L} \quad (4)$$

and it is this equation which has to be compared with Equation (1).

We notice that Equation (4) does not exhibit the unphysical features of Equation (1), and we also notice that Equation (4) and Equation (1) become identical if $\cos(2\theta_1) = 1$. In all practical cases the angle θ_1 is rather small, ranging from 5° to 0.5° , since solar cells consist of flat and thin wafers and the approximation $\cos(2\theta_1) = 1$ is a good one indeed. We shall however, show in the next section that the result (4) is still valid even when $\cos(2\theta_1) \neq 1$. However, we hasten to say that if θ_1 becomes larger than about 15° , the approximations inherent in the derivation of the simple result stated above become rapidly more and more unacceptable as the angle increases. Resorting to numerical analysis then becomes the only alternative, but fortunately, for the small angles encountered in practice, there is no need to deviate from the simple expression (3) (with d replaced by x of Figure 1).

II. ANALYSIS

As in previous work (Reference 1), the task at hand is to find a solution of the diffusion equation obeyed by the minority carriers, to wit:

$$\nabla^2 N - L^{-2} N = -D^{-1} S(x, y, z), \quad (5)$$

subject to the boundary conditions

$$-D \frac{\partial N}{\partial n} = s N, \quad (6a)$$

at the surface of the semiconductor and

$$N = 0, \quad (6b)$$

at the junction edge. The meaning of the various symbols in Equations (5) and (6) are identical with those used in previous papers of this series (Reference 1) but are explained again for the convenience of the reader: L is the diffusion length of the minority carriers, D is the diffusion constant, N is the number density of minority carriers, $S(x, y, z)$ is the source function or the number of excess carriers produced by the electron beam per cm^3 per second. Finally, s signifies the surface recombination velocity and n the outward normal to the surface. The negative sign in Equation (6a) is due to this choice. We like to emphasize again that Equations (5) and (6) are only valid if Shockley's junction theory applies (Reference 1). Low-level injection conditions are therefore assumed throughout.

Let us now look at Figure 1. The boundary conditions (6a) and (6b) have to be satisfied at the two inclined planes shown there. It is obvious then to

of the paper on which Figure 1 is shown to the reader and located at the intersection of the two planes depicted there, one being the surface, the other being the junction edge. The radial distance r from the z axis constitutes the second coordinate, and θ , the angle measured counterclockwise around the z axis, completes the specification of the coordinates. For convenience we define the zero angle $\theta = 0$ to be situated half way between the, by now notorious, planes defining the junction geometry (see Figure 1). Therefore, $\theta = \theta_1$ constitutes the equation for the plane of the semiconductor surface and $\theta = -\theta_1$ signifies the equation for the plane of the junction edge.

The diffusion length L for minority carriers is of the order of $100 \mu\text{m}$ for solar cells*. The distance x (defined in Figure 1) is of the same order of magnitude. A 20 keV electron beam possesses a range of $4 \mu\text{m}$ (Reference 1). The radius of the interaction volume produced by the beam is about a third of that (Reference 1). The penetration depth ξ is of the order of the range or, more likely, smaller. r_{max} is defined as the distance between the point of intersection of the two planes (see Figure 1) and the edge at which the angle-lapping was started and is given by $d/\sin(2\theta_1)$ which turns out to be for $d = 200 \mu\text{m}$ and $\theta_1 = 5^\circ$, $r_{\text{max}} = 1152 \mu\text{m}$. The magnitude of these numbers clearly indicates that a number of approximations may be introduced in turn, without undue harm to the analysis.

The first approximation to be introduced is the following:

$$S(x, y, z) = S_0 \delta(z) \delta(\theta - \theta_0) r_0^{-1} \delta(r - r_0) \quad (7)$$

*Solar cell grade semiconductor material can actually be defined that way.

The significance of this choice for the source function S is rather readily understood. A point source of strength, S_0 , (pairs created per second) is located at the position $z = 0$, $r = r_0$, and $\theta = \theta_0$ ($\theta_0 \leq \theta_1$) in our cylindrical coordinate system. The Dirac δ -function for the radial coordinate is defined by

$$\int_0^{\infty} r \, dr \, r_0^{-1} \delta(r - r_0) = 1, \quad (8)$$

as is customary.

Transcribing the boundary condition (6a) into our cylindrical coordinate system, it becomes:

$$\frac{1}{r} \frac{\partial N}{\partial \theta} = - \frac{s}{D} N, \quad \text{at } \theta = \theta_1 \quad (9)$$

This boundary condition together with the diffusion equation (5) leads to a system of equations which is not separable, and it is therefore impervious to a simple analytical solution. However, the choice (7) for the source function, dictated by the prevailing magnitudes of the parameters involved in this analysis, makes it rather obvious to introduce a second approximation, viz.:

$$\frac{\partial N}{\partial \theta} = - \frac{s \, r_0}{D} N, \quad \text{at } \theta = \theta_1, \quad (10)$$

with r_0 the radial position coordinate of the point source (7). In order to ascertain the significance of this second approximation, let us notice first that in the two extreme cases, $s = 0$ as well as $s = \infty$, the replacement of r by r_0 is immaterial, since then either $\partial N / \partial \theta = 0$ for $s = 0$, or $N = 0$ for $s = \infty$ independent of r . On the other hand, if s has an intermediate value, Equation (10) constitutes a true approximation. To see whether this approximation is not harmful to

the subsequent analysis, let us consider the situation in more detail. The number density of excess carriers diffusing outward from the interaction volume has reached a value of roughly $e^{-2} = 0.15$ of its peak value at the interaction volume $2L$ or two diffusion lengths away. Those carriers which happen to reach the surface and are annihilated by traps residing there, two diffusion lengths away, will encounter a trap density which is slightly lower or higher than that prevailing at $r = r_0$ if the approximation (10) is made. But the number of carriers reaching the surface at a distance $2L$ away from the interaction volume is only a small fraction of those being collected by the junction. We must remember that $x < 200 \text{ } \mu\text{m}$ and $L = 100 \text{ } \mu\text{m}$ in our example, typical for solar cells. This state of affairs can be put in another way. The correct boundary condition (9) makes the product $s r$ variable as r is changed. The approximate boundary condition (10) insists on a constant product $s r_0$. As long as $r_0 \gg L$ the approximation (10) is excellent. We now realize that the approximation we are discussing is essentially a small angle approximation in the sense that

$$2L/r_0 = 2 \sin (2 \theta_1) L/x \ll 1, \quad (11)$$

must be satisfied in order that the approximation (10) is valid.*

Keeping in mind that the approximations (7) and (11) are usually quite well satisfied, the analysis proceeds along customary lines. First we find a complete orthonormal set of functions in the angular variable θ which satisfies the

*We note that condition (11) may well be satisfied for larger angles θ_1 provided that $L \ll x$.

boundary conditions (10) and (6b), the latter being $N = 0$ for $\theta = -\theta_1$, the junction edge. It is given by

$$F_n(\theta) = \left(\theta_1 - \frac{\sin 4 \ell_n}{4 \ell_n} \right)^{-1/2} \sin \left(\ell_n \frac{\theta_1 + \theta}{\theta_1} \right), \quad (12)$$

where the eigenvalues ℓ_n must satisfy the transcendental equation:

$$\tan(2 \ell_n) = - \frac{D}{s r_0 \theta_1} \ell_n, \quad (13)$$

for $n = 0, 1, 2, \dots$ etc. In terms of these functions the angular δ function occurring in Equation (7) can be written

$$\delta(\theta - \theta_0) = \sum_{n=0}^{\infty} F_n(\theta_0) F_n(\theta). \quad (14)$$

The choice for trigonometric functions for F_n is dictated by the structure of the Laplace operator ∇^2 in cylindrical coordinates. Continuing, we recall the fact that (Reference 5):

$$r_0^{-1} \delta(r - r_0) = \int_0^{\infty} k dk J_m(k r_0) J_m(k r), \quad (15)$$

where m is an arbitrary integer and J_m the Bessel function of order m . The source function of Equation (7) may now be written in terms of orthogonal functions as:

$$-D^{-1} S_0 \delta(z) \delta(\theta - \theta_0) r_0^{-1} \delta(r - r_0)$$

$$= -\frac{D^{-1} S_0}{2\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} d\ell \int_0^{\infty} k dk e^{i\ell z} J_m(k r_0) J_m(k r) F_n(\theta_0) F_n(\theta). \quad (16)$$

If we now adopt the "ansatz":

$$N(r, \theta, z) = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} d\ell \int_0^{\infty} k dk G_{nm}(k, \ell) J_m(k r) F_n(\theta) e^{i\ell z} \quad (17)$$

we note that the boundary conditions are automatically satisfied by virtue of the choice (12) for the angular functions F_n . All which is left to do is to satisfy the diffusion equation (Equation (5)) with the source term given by Equation (16). But this is a matter of simple algebra with the result

$$G_{nm}(k, \ell) = (\ell^2 + k^2 + L^{-2})^{-1} \frac{D^{-1} S_0}{2\pi} \left(\theta_1 - \frac{\sin 4 \ell_n}{4 \ell_n} \right)^{-1/2} J_m(k r_0) F_n(\theta_0) \delta_m, \ell_n / \theta_1. \quad (18)$$

The Kroneker δ symbol indicates that the integer m must be equal to the number l_n/θ_1 . In general of course, l_n/θ_1 is not an integer. On the other hand, Equation (15) is only valid if m is in fact an integer. But it is always possible to choose an angle θ_1 such that l_n/θ_1 becomes an integer for all n . To see this more clearly and at the same time to see that the selection of θ_1 in such a manner as to satisfy the requirement $l_n/\theta_1 = \text{integer}$ for all n does not unduely restrain the possible values θ_1 is allowed to possess, let us contemplate the case $s = 0$. Equation (13) reveals that the possible eigenvalues l_n are now given by:

$$l_n = \left(\frac{n}{2} + \frac{1}{4} \right) \pi, \quad n = 0, 1, 2, \dots \quad (19)$$

If we now choose $\theta_1 = \pi/4N$ with an arbitrary integer N , we are assured that $l_n/\theta_1 = (2n + 1) N$ is indeed integer for all n . Choosing the angle between the semiconductor planes to be 10° , for instance ($\theta_1 = 5^\circ$), we have $N = 9$; for 5° ($\theta_1 = 2.5^\circ$) we have $N = 18$ etc. It becomes obvious now that an analytic continuation performed on the index of the Bessel functions validates Equation (18) for arbitrary values of θ_1 . The excess minority carrier density N is now completely determined via Equations (18) and (17). But we are not particularly interested in this quantity since it is rather difficult to observe directly. Here, as in the previous papers (Reference 1), we are concerned with the short circuit current I_{sc} , a quantity which can be measured with ease. It is given by:

$$I_{sc} = e D \int_0^\infty dr \int_{-\infty}^\infty dz \, r^{-1} \frac{\partial N}{\partial \theta} \bigg|_{\theta = -\theta_1} \quad (20)$$

e the electronic charge is either positive for holes or negative for electrons depending on the material under investigation. In any case, using Equations (18), (16), and (20), performing the integrations over ℓ , r , and z gives this result:

$$I_{sc} = e S_0 \sum_{n=0}^{\infty} \int_0^{\infty} k dk (k^2 + L^{-2})^{-1} J_{\ell_n/\theta_1}(k r_0) \left(\theta_1 - \frac{\sin 4 \ell_n}{4 \ell_n} \right)^{-1} \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) \quad (21)$$

In the appendix it will be shown that expression (21) is equivalent to

$$I_{sc} = e S_0 \frac{\cosh [r_0 (\theta_1 - \theta_0)/L] + \eta \sinh [r_0 (\theta_1 - \theta_0)/L]}{\cosh [2 r_0 \theta_1/L] + \eta \sinh [2 r_0 \theta_1/L]} \quad (22)$$

if the smallest value for the index of the Bessel functions, ℓ_0/θ_1 , is not smaller than 5. The error introduced by identifying Equation (21) with Equation (22) will also be discussed, and it will be shown that the error is always small and becomes totally negligible as θ_1 approaches zero, as of course it should.

Realizing that for small angles

$$2 r_0 \theta_1 = x, \quad r_0 (\theta_1 - \theta_0) = \xi, \quad (23)$$

where x is the distance between the two inclined surfaces of Figure 1, and ξ is the penetration depth of the SEM beam, we see that Equation (22) goes over into Equation (3) with d replaced by x , thus proving our original claim.

III. SUMMARY

The result for the I_{sc} we have derived and which is given by Equation (22) is surprisingly simple, since it says that whether or not the pertinent surfaces of the semiconductor junction are plane parallel as in an ordinary solar cell or angle-lapped, and therefore inclined as shown in Figure 1, the same expression for the I_{sc} as a function of L etc. applies. This is of course subject to a number of approximations which we like to enumerate again. There are three approximations basic to our result other than the assumption of uniform doping and Shockley's low level injection theory. The first one is minor and is satisfied almost always. It is the assumption of a point source as the generator of excess electron-hole pairs. In fact, the radius of the interaction volume, being of the order of $1\text{ }\mu\text{m}$, is small compared to both L and x which are of the order of $100\text{ }\mu\text{m}$.^{*} The second approximation, the small angle approximation embodied in Equation (11), is always well satisfied for solar cells. The third approximation, the simplification of the integral (A6) of the appendix, is also a small angle approximation. The analysis shows that if the angle between the two planes of Figure 1 is less than 10° , the expression (22) for the I_{sc} is excellent.

^{*}This is to be compared with Reference 1, Part II, where this approximation was not possible.

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APPENDIX

PROOF OF THE EQUIVALENCE OF EQUATIONS (21) AND (22) OF THE TEXT

For the convenience of the reader we repeat Equation (21) here.

$$I_{sc} = e S_0 \sum_{n=0}^{\infty} \int_0^{\infty} k dk (k^2 + L^{-2})^{-1} J_{\ell_n/\theta_1}(k r_0) \left(\theta_1 - \frac{\sin 4 \ell_n}{4 \ell_n} \right)^{-1} \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) \quad (A1)$$

Consider the integral

$$\begin{aligned} I_m(r_0/L) &= \int_0^{\infty} k dk (k^2 + L^{-2})^{-1} J_m(k r_0) \\ &= \int_0^{\infty} x dx (x^2 + r_0^2/L^2)^{-1} J_m(x) \end{aligned} \quad (A2)$$

This may be rewritten as:

$$I_m = \int_0^{\infty} dx \int_0^{\infty} dt e^{-xt} J_m(x) \cos[(r_0/L)t] \quad (A3)$$

interchanging the order of integration and performing the integration over x yields:

$$I_m = \int_0^{\infty} dt (1 + t^2)^{-1/2} \left[(1 + t^2)^{1/2} - t \right]^m \cos[(r_0/L)t] \quad (A4)$$

With the substitution

$$t = \sinh y \quad (A5)$$

the integral (A4) goes over into:

$$I_m = \int_0^\infty dy e^{-my} \cos [(\sinh y) r_0/L] \quad (A6)$$

For small enough angles, certainly for those angles which satisfy the approximation (11) of the main text, r_0/L tends to be large. Therefore, only small values of y may be considered. Otherwise the cos term oscillates so rapidly that little contributions toward the integral arise.* Furthermore, for large values of y ($y > 1$) the exponential cuts down the amplitude of the integrand tremendously particularly for large m ($m > 5$).** Therefore, the argument of the cosine in Equation (A6) may comfortably be replaced by $y r_0/L$ and the value of the integral becomes

$$I_m(r_0/L) = \frac{m}{m^2 + (r_0/L)^2} \quad (A7)$$

Identifying m with ℓ_n/θ_1 as suggested by Equation (A1), we obtain now for the I_{sc} the following expression:

$$I_{sc} = e S_0 \sum_{n=0}^{\infty} \frac{\ell_n \theta_1}{\ell_n^2 + (r_0 \theta_1/L)^2} \left(1 - \frac{\sin 4 \ell_n}{4 \ell_n} \right)^{-1} \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) \quad (A8)$$

*The integral (A6) converges even for negative m less than one!

** m is of course given by ℓ_n/θ_1 from Equation (A1), and, therefore, a large m again signifies a small angle θ_1 .

If we let $\theta_1 \rightarrow 0$ and at the same time let $r_0 \rightarrow \infty$ in such a manner that the product $r_0 \theta_1$ stays finite, the sum (A8) must go over into the expression for the I_{sc} corresponding to the planar case (plane parallel surfaces of the untreated junction) given by Equation (3) of the text. Furthermore, we notice from Equation (12) of the main text that Equation (A8) is nothing else but an expansion of I_{sc} into a complete set of eigenfunctions $F_n(\theta_0)$. We strongly suspect therefore that Equation (A8) and Equation (22) of the main text are identical. In fact, the following identities can be proven trivially:

$$\begin{aligned} \int_{-\theta_1}^{\theta_1} d\theta_0 \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) \cosh [(\theta_1 - \theta_0) r_0/L] \\ = \frac{\ell_n \theta_1}{\ell_n^2 + (r_0 \theta_1/L)^2} [\cosh (2 \theta_1 r_0/L) - \cos (2 \ell_n)] , \end{aligned} \quad (A9)$$

and

$$\begin{aligned} \int_{-\theta_1}^{\theta_1} d\theta_0 \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) \sinh [(\theta_1 - \theta_0) r_0/L] \\ = \frac{1}{\ell_n^2 + (r_0 \theta_1/L)^2} \left[\theta_1 \ell_n \sinh (2 \theta_1 r_0/L) - \frac{r_0 \theta_1^2}{L} \sin (2 \ell_n) \right] \end{aligned} \quad (A10)$$

If we now add Equation (A9) to n times Equation (A10) and observe Equation (13) of the text, we obtain the result:

$$\int_{-\theta_1}^{\theta_1} d\theta_0 I_{sc}(\theta_0) \sin \left(\ell_n \frac{\theta_1 + \theta_0}{\theta_1} \right) = \frac{\ell_n \theta_1}{\ell_n^2 + (r_0 \theta_1 / L)^2} \quad (A11)$$

and this fact completes the proof that Equation (A8) and Equation (22) are indeed identical. That Equations (21) and (22) are equivalent rests on the approximation $\sinh y = y$.