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DIRECT METHOD FOR SOLVING TRANSFER EQUATION BY EXPANSION IN SPHERICAL HARMONICS: SCATTERING IN ATMOSPHERE WITH LAMBERTIAN LOWER BOUNDARY AND THERMAL RADIATION TRANSFER

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Translation of "Pryamoy Metod Resheniya Uravneniya Perenosa Razlozheniyem po Sfericheskin Garmonikam: Rasseyaniye v Atmosfere s Lambertovskoy Podstilayushchey Poverkhnost'yu i Perenos Teplovogo Izhucheniya," Academy of Sciences USSR, Institute of Space Research, Moscow, Report Pr-307, 1978, pp. 1-16.

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16. Abstract
The direct method for the solution of the spherical harmonics approximation to the equation of transfer of radiation is applied to the cases of 1) scattering of the solar radiation in the atmosphere with the Lamertian boundary and 2) thermal radiation transfer.

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DIRECT METHOD FOR SOLVING TRANSFER EQUATION BY EXPANSION IN SPHERICAL HARMONICS: SCATTERING IN ATMOSPHERE WITH LAMBERTIAN LOWER BOUNDARY AND THERMAL RADIATION TRANSFER

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Introduction /3*

In a recent series of studies [1-3] it was shown that the method of expansion in terms of spherical harmonics is a powerful tool which can be applied to solving the transfer equation in scattering media with scattering characteristics varying over the optical depth. Jeans [4] proposed the idea of this method; it was further developed during its application to the problem of the transfer of neutrons (corresponding bibliographical references are given in [1]). Canosa and Penafiel [1] applied the method to the transfer of radiation in anisotrotropically scattering homogeneous media and they pointed out an approach by which cumulative rounding errors can be eliminated during its application to media with great optical depth. Dave and Canosa [2] demonstrated the applicability of the method to inhomogeneous media. All these studies considered only the azumith--independent radiation intensity component (zero-th azimuth harmonic). Subsequently Dave [3] elaborated a generalization of the method which can be used to find higher azimuth harmonics.

This study has two objectives. First, all studies pertaining to the discussed method with which the author is familiar are based on using so-called "vacuum" boundary

*Numbers in the margin indicate pagination in the foreign text.

conditions on the upper and lower boundary of the scattering medium. When these conditions are applied to the atmosphere of a planet, they imply that the albedo of the lower boundary surface is assumed to be zero. Such a constraint is not justified. Therefore relations which allow us to apply the given method to more realistic lower boundaries are derived in this paper.

Second, all studies based on an application of the given method known to the author pertain to transfer of solar radiation, i.e. the assumption is made that the source function is determined by radiation from a source of parallel rays located outside the scattering medium. In this paper the case of thermal radiation transfer is also considered, i.e. it is assumed that the source function varies with depth and that it depends on the temperature of the atmosphere and the emissive power of the source. /4

I. Expansion in Spherical Harmonics

We present here without derivation the basic relations used in this method applied to the case of solar radiation transfer with "vacuum" boundary conditions. We follow studies [1, 2] and retain most of the notation used in them. Differences pertain only to a number of minor misprints and also to some changes made for more convenient electronic computer programming.

Let us consider a scattering and absorbing medium having optical depth τ_b whose scattering and absorption characteristics depend on the depth. These characteristics can be described by specifying the coefficients $\omega_0, \omega_1, \omega_L$ in the expansion of the scattering indicatrix in terms of Legendre polynomials

$$P(\cos\theta) = \sum_{l=0}^{L} \omega_l P_l(\cos\theta)$$

normalized in such a way that the quantity $_1$ represents the single scattering albedo. The selected number L is odd [1]. The original transfer equation together with the "vacuum" boundary conditions is expressed in the form:

$$u\frac{dI(\tau;u;\varphi)}{d\tau} + I(\tau;u;\varphi) =$$

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$$= \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P(\tau;u;\varphi;u',\varphi')\, I(\tau;u';\varphi')\, du'\, d\varphi' + \tag{1}$$

$$+ \frac{1}{4} F \exp\left(-\frac{\tau}{\mu_0}\right) P(\tau;u;\varphi;\mu_0,\varphi_0)$$

$$I(0;u;\varphi) = 0 \quad \text{when} \quad u > 0$$

$$I(\tau_0;u;\varphi) = 0 \quad \text{when} \quad u < 0$$

where I is the intensity of the scattered radiation;

τ is the optical thickness (depth) measured from the upper boundary of the atmosphere;

u, u' is the cosine of the zenith direction angle of scattered radiation measured from the positive direction of the τ axis (nadir);

P is the scattering indicatrix;

πF is the illumination intensity of an area beyond the atmosphere which is perpendicular to the direction of the Sun's rays;

μ_0 is the cosine of the Sun's zenith angle measured from the zenith;

ϕ_0 is the azimuth of the Sun.

The azimuth independent intensity component

$$I^0(\tau;u) = \frac{1}{2\pi}\int_{0}^{2\pi} I(\tau;u;\varphi)\, d\varphi \tag{2}$$

is sought in the form of an expansion in Legendre polynomials (henceforth the superscript "0" will be omitted)

$$I(\tau\,;\,u) = \sum_{l=0}^{L} \frac{2l+1}{2}\, f_l(\tau)\, P_l(u), \qquad (3)$$

where

$$f_l(\tau) = \int_{-1}^{1} I(\tau\,;\,u)\, P_l(u)\, du, \qquad (4)$$

and the problem is reduced to fin ing the vector

$$f(\tau) = \begin{pmatrix} f_0(\tau) \\ f_1(\tau) \\ \vdots \\ f_L(\tau) \end{pmatrix}$$

Equation (1) is reduced to a system of differential equations with boundary conditions, which in matrix notation, has the form: /6

$$\left.\begin{aligned} &A\,\frac{df(\tau)}{d\tau} + C(\tau)f(\tau) = s(\tau), \\ &f_o(0) = -Gf_e(0), \\ &f_o(\tau_0) = Gf_e(\tau_0), \end{aligned}\right\} \qquad (5)$$

where A and C(τ) are matrices having rank (L+1):

$$A = \begin{pmatrix} 0 & \beta_1 & & & & \\ \alpha_1 & 0 & \beta_2 & & \mathbf{0} & \\ & \alpha_2 & 0 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & \alpha_{L-1} & 0 & \beta_L \\ & & & & \alpha_L & 0 \end{pmatrix} \qquad (6)$$

$$C(\tau)=\begin{bmatrix} \sigma_0(\tau) & & & & & 0 \\ & \sigma_1(\tau) & & & & \\ & & \sigma_2(\tau) & & & \\ & & & \ddots & & \\ & & & & \sigma_{L-1}(\tau) & \\ 0 & & & & & \sigma_L(\tau) \end{bmatrix} \tag{7}$$

whose elements are

$$\alpha_l = \frac{l}{2l+1}; \qquad \beta_l = \frac{l}{2l-1}; \tag{8}$$

$$\sigma_l(\tau) = 1 - \frac{\omega_l(\tau)}{2l+1}, \tag{9}$$

respectively, and $s(\tau)$ is an L+1-dimensional vector with components

$$s_l(\tau) = \frac{F}{2(2l+1)} \exp\left(-\frac{\tau}{\mu_0}\right) \cdot \omega_l(\tau) P_l(\mu_0); \tag{10}$$

f_0 and f_e are (L+1)/2-dimensional vectors, whose components are respectively the odd and even components of the vector f:

$$f_0 = \begin{pmatrix} f_1 \\ f_3 \\ \vdots \\ f_L \end{pmatrix}; \qquad f_e = \begin{pmatrix} f_0 \\ f_2 \\ \vdots \\ f_{L-1} \end{pmatrix}$$

G is the matrix of boundary conditions found on the basis of the recurrence formulas /7

$$\left.\begin{aligned} G_{1,0} &= 0{,}5; \\ G_{1,2l} &= -\frac{(4l+1)(2l-3)}{2(4l-3)(l+1)} G_{1,2l-2}; \\ & \qquad l = 2, 3, \ldots\ldots (L+1)/2; \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} G_{2m+1,2l} &= -\frac{(2m+1)[2(l-m)+1]\cdot(l+m)}{2m[2(l-m)-1]\cdot(l+m+1)} G_{2m-1,2l}; \\ & \qquad m = 2, 3, \ldots\ldots (L+1)/2. \end{aligned}\right\}$$

Elements G_{ij} with even i and odd j are not defined, and the matrix is considered to have rank (L+1)/2.

The next step is the introduction of a grid of n integration points τ_i, with $\tau_1=0$ and $\tau_n=\tau_g$, and reduction of (5) to a system of matrix equations having the form

$$\left.\begin{array}{l} S_{i,i}f_i + S_{i,i+1}f_{i+1} = w_i , \\ \qquad i = 1, 2, \ldots n-1, \end{array}\right\} \tag{12}$$

$$(f_1)_o = - G(f_1)_e ; (f_n)_o = G(f_n)_e ,$$

where

$$S_{i,i} = -A + \frac{\tau_{i+1} - \tau_i}{2} \cdot C\left(\frac{\tau_i + \tau_{i+1}}{2}\right),$$

$$S_{i,i+1} = A + \frac{\tau_{i+1} - \tau_i}{2} \cdot C\left(\frac{\tau_i + \tau_{i+1}}{2}\right), \tag{13}$$

$$w_i = \frac{\tau_{i+1} - \tau_i}{2} \left[s(\tau_i) + s(\tau_{i+1}) \right].$$

Next, row and column operators are introduced, which interchange rows or columns of the matrix according to <u>eveness</u>, i.e. in such a way, that the first (L+1)/2 rows (columns) of the matrix are even (0, 2, ..., L-1) rows (columns) and the following (L+1)/2 rows (columns) are odd (1, 3,..., L) rows (columns). The system of matrix equations (12) is transformed to the form

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$$\left.\begin{array}{l} D_{i,i}g_i + D_{i,i+1}g_{i+1} = w_i , \\ \qquad i = 1, 2, \ldots n-1, \end{array}\right\} \tag{14}$$

$$g_1^b = - Gg_1^t ; \quad g_n^b = Gg_n^t , \tag{15}$$

where

$$D_{ij} = \text{Column } (S_{ij}),$$
$$g_i = \text{Row } (f_i). \tag{16}$$

The superscripts "t" and "b" denote respectively the top or bottom half of a vector or matrix. The initial system of equations for obtaining a solution is system (14, 15).

Canosa and Penafiel [1] presented methods for solving equations [14, 15] both for the case of moderate optical depth and for media whose optical depth is so great that it requires certain stabilizing transformations preventing an increase in rounding errors.

2.1. Medium with Moderate Optical Depth

A sequence is constructed from n-1 matrices having rank (L+1):

$$\left.\begin{aligned} \pi_i &= -(D_{i,i+1})^{-1} D_{i,i}\,; \\ i &= 1,\ 2,\ \ldots n-1. \end{aligned}\right\} \qquad (17)$$

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A sequence is constructed from n rectangular (L+1) x (L+1)/2 matrices

$$F_1 = \begin{pmatrix} I \\ -G \end{pmatrix}; \qquad \left.\begin{aligned} F_i &= \pi_{i-1} F_{i-1}\,, \\ i &= 2,\ 3,\ \ldots n \end{aligned}\right\} \qquad (18)$$

where I is the identity matrix.

A sequence is constructed from n (L+1)-dimensional vectors:

$$a_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \qquad \left.\begin{aligned} a_i &= (D_{i-1,i})^{-1} w_{i-1} + \pi_{i-1} a_{i-1}, \\ i &= 2,\ 3,\ \ldots n. \end{aligned}\right\} \qquad (19)$$

The (L+1)/2-dimensional vector $_1$ is determined from a solution of the system of (L+1)/2 equations having the following form in matrix notation:

$$(F_n^b - GF_n^t)l_1 = Ga_n^t - a_n^b. \tag{20}$$

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The sequence consisting of the n (L+1)-dimensional vectors

$$\left.\begin{array}{l} g_i = F_i l_1 + a_i, \\ i = 1, 2, \ldots n. \end{array}\right\} \tag{21}$$

is a solution of system (14, 15).

2.2. Medium with Great Optical Depth

The necessary number of so-called overdetermining points τ_{i_k} (k=1, 2. . , N_c) with i_1=1, i_{N_c}=n, is selected from the set of points τ_i. The points are selected on the basis of the requirement that the rounding errors cannot increase to an appreciable magnitude on an interval between neighboring over-determining points. Empirical selection rules for the position of the points τ_{i_k} are discussed in [2].

A sequence is constructed from N_c transformation matrices having rank (L+1)/2:

$$\left.\begin{array}{l} T_1 = I; \quad T_k = (F_{i_k} T_1 T_2 \cdots T_{k-1})^{-1}; \\ \qquad k = 2, 3, \ldots N_c. \end{array}\right\} \tag{22}$$

A sequence is constructed from n transformed (L+1)/2-dimensional vectors

$$\left.\begin{array}{ll} t_1=0; & t_k = a_{i_k}^t - F_{i_k}^t (T_1 T_2 t_2 + \ldots + T_1 T_2 \ldots T_{k-1} t_{k-1}); \\ t_2=a_{i_2}^t; & \qquad k = 3, 4, \ldots N_c. \end{array}\right\} \tag{23}$$

A sequence is constructed from n rectangular (L+1) x (L+1)/2 matrices:

$$\left.\begin{array}{l} \left.\begin{array}{l} U_i = F_i T_1 T_2 \cdots T_{k-1}, \\ i_{k-1} \leqslant i < i_k, \end{array}\right\} \\ k = 2, 3, \ldots N_c, \end{array}\right\} \tag{18-1}$$

$$U_n = F_n T_1 T_2 \cdots T_{N_c}.$$

A sequence is constructed from n (L+1)-dimensional vectors:

$$u_{i_k} = \left\{ \begin{pmatrix} 0 \\ a^b_{i_k} + F^b_{i_k}(T_1T_2t_2 + \dots + T_1T_2\dots T_kt_k), \end{pmatrix} \quad k = 2, 3, \dots N_c. \right\} \tag{19-1}$$

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The (L+1)/2-dimensional vector l_{N_c} is determined from a solution of the system of (L+1)/2 equations having the following form in matrix notation:

$$(U^b_n - GU^t_n)l_{N_c} = -u^b_n \tag{20-1}$$

The sequence consisting of N_c-1 (L+1)/2=dimensional vectors l_k:

$$\left. \begin{array}{l} l_k = T_{k+1}(l_{k+1} - t_{k+1}), \\ \quad k = N_c-1, N_c-2, \dots 1, \end{array} \right\} \tag{20-2}$$

is determined

The sequence of n (L+1)-dimensional vectors g_i representing a solution of system (14, 15) is determined:

$$\left. \begin{array}{l} \left. \begin{array}{l} g_i = U_il_k + u_i, \\ \quad i_k \leq i < i_{k+1}, \end{array} \right\} \\ \quad k = 1, 2, \dots N_c-1; \end{array} \right\} \tag{21-1}$$

$$g_n = U_nl_{N_c} + u_n.$$

3. Boundary Condition Matrix for Lambertian Lower Boundary

Formulas for elements of the matrix of boundary conditions (11) were presented above for the case when no scattered radiation impinges on the scattering medium from outside. When the medium is bounded (for example from below) by a boundary with nonzero albedo, such radiation is obviously present. Let us assume that the optical depth of the scattering atmosphere is sufficiently great and the illumination intensity of the lower boundary due to direct solar radiation is negligibly small. Then the boundary condition for the Lambertian boundary with albedo A can be written in the form: /11

$$I(\tau_g; u; \varphi) = \frac{A}{\pi} \int_0^1 \int_0^{2\pi} I(\tau_g; u'; \varphi') u' du' d\varphi'$$

This relation can be transformed to the condition for the azimuth-independent component expressed in the form:

$$I^0(\tau_g; u) = 2A \int_0^1 I^0(\tau_g; u') u' du' \tag{24}$$

taking into account (3), we have

$$\int_0^1 I^0(\tau_g; u') u' du' = \int_0^1 \sum_{l=0}^{L} \frac{2l+1}{2} f_l(\tau_g) P_l(u') u' du' =$$

$$= \sum_{l=0}^{L} \left[\frac{2l+1}{2} \int_0^1 P_1(u') P_l(u') du' \right] f_l(\tau_g) = \sum_{l=0}^{L} C_{1l} f_l(\tau_g)$$

Formulas for the C_{ml} are given, for instance, in [2]. For odd l, the following holds:

$$C_{ml} = \begin{cases} 0,5 \text{ when } m = l \\ 0 \text{ when } m \neq l \end{cases} \tag{25A}$$

For even l, the quantities C_{ml} and the elements of matrix G defined by formulas (11) are related by the simple relation

$$G_{ml} = 2C_{ml} \tag{25B}$$

Following the scheme proposed by Marschak (see for instance [2]), writing down expansion (3) for the left member of (24) and taking into account subsequent mathematical operations we obtain:

$$\sum_{l=0}^{L} \frac{2l+1}{2} f_l(\tau_g) P_l(u) = 2A \sum_{l=0}^{L} C_{1l} f_l(\tau_g) \tag{26}$$

Since (26) can only be satisfied for a finite number of points in [-1, 0], following Marschak, we multiply both sides of (26) by $P_m(u)$ (m=1, 3,..., L), integrate over the interval [-1, 0], and require that these equalities be satisfied for all m. We have

$$\int_{-1}^{0} \frac{2l+1}{2} f_l(\tau_g)\, P_l(u)\, P_m(u)\, du = 2A \int_{-1}^{0} \sum_{l=0}^{L} C_{1l} f_l(\tau_g)\, P_m(u)\, du.$$

After obvious transformations, writing

$$D_{ml} = \frac{2l+1}{2} \int_{-1}^{0} P_m(u)\, P_l(u)\, du$$

we obtain: /12

$$\left.\begin{array}{c} \sum_{l=0}^{L} D_{ml} f_l(\tau_g) = 4AD_{m0} \sum_{l=0}^{L} C_{1l} f_l(\tau_g) \\ m = 1,\ 3,\ \ldots\ L \end{array}\right\} \quad (27)$$

Performing obvious operations it can be shown that

$$D_{ml} = \begin{cases} -C_{ml} & \text{for even } l \\ C_{ml} & \text{for odd } l \end{cases}$$

Taking into account (25A), we rewrite (24) in the form

$$\left.\begin{array}{c} \frac{1}{2} f_m - \sum_{l=0,2,\ldots}^{L-1} C_{ml} f_l = -4AC_{m0} \left(\frac{1}{2} f_1 + \sum_{l=0,2,\ldots}^{L-1} C_{1l} f_l\right) \\ m = 1,\ 3,\ \ldots\ L \end{array}\right\} \quad (28)$$

Let us first consider the case m=1. Substituting C_{10}=0.25, after obvious transformations and taking into account (25B), we obtain:

$$f_1 = \frac{1-A}{1+A} \sum_{l=0,2,\ldots}^{L-1} G_{1l} f_l \quad (29)$$

Next, let us consider the case of arbitrary odd m. Substituting (23) in (28) and taking into account (26) we obtain:

$$f_m = \sum_{\ell=0,2,\ldots}^{L-1} (G_{ml} - \frac{4A}{1+A} G_{m0}G_{11}) f_1 \quad (30)$$

It can be easily shown that (29) is a special case of (30) for m=1. Relation (30) holds for m=1, 3, ..., L, and it can be rewritten in matrix form:

$$f_o(\tau_g) = G_A f_e(\tau_g) \quad (31)$$

where the elements of matrix G_A have the form:

$$(G_A)_{ml} = G_{ml} - \frac{4A}{1+A} G_{m0}G_{11} \quad (32)$$

We obtained the sought form of the boundary condition for the Lambertian boundary surface. The matrix G_A must be substituted in equations (20) and (21) instead of the matrix G. It is easily seen that

$$G_A \to G \text{ as } A \to 0$$

Another obvious property follows: when A=1, having performed elementary operations, we have:

$$(G_A)_{11} = 0 \quad (33)$$

This corresponds to the fact that at a boundary with A=1, we must have zero radiation flow. Indeed, it can be easily seen that the magnitude of flow at the lower boundary is $2\pi f_1(\tau_g)$. Combining (31) and (33), we obtain /13

$$f_1^-(\tau_g) = 0$$

i.e. the flow is indeed zero.

4. Thermal Radiation Transfer

The thermal radiation transfer equation can be written in the following form (see for instance [5]):

$$u\frac{dI(\tau\,;\,u)}{d\tau} + I(\tau\,;\,u) =$$
$$= \frac{1}{2}\int_{-1}^{1} P(\tau\,;\,u;\,u')\,I(\tau\,;\,u')\,du' + \left[1-\omega_0(\tau)\right]B(T) \qquad (34)$$

where B(T) is the radiation intensity of an ideal black body. As before, the upper boundary condition is a vacuum condition, and the lower boundary condition can be written in the form

$$I(\tau_g;\,u) = 2A\int_{0}^{1} I(\tau_g;\,u')\,u'du' + (1-A)B(T_s) \qquad (35)$$

where 1-A, T_s is the emissive power and temperature of the lower boundary respectively.

Next, we follow the general idea of the spherical harmonics method. Applying the summation theorem for Legendre polynomials (see for instance [6]) and changing to the variable τ in the function B(T), we rewrite (34) in the form:

$$u\frac{dI(\tau\,;\,u)}{d\tau} + I(\tau\,;\,u) =$$
$$= \frac{1}{2}\sum_{l=0}^{L}\omega_l(\tau)\,P_l(u)\int_{-1}^{1} P_l(u')\,I(\tau\,;\,u')du' + \left[1-\omega_0(\tau)\right]B(\tau)$$

Multiplying the above equation by $P_k(u)$ (k=0, 1, ..., L) and integrating from -1 to 1, we obtain after obvious transformations

$$\left.\int_{-1}^{1}\frac{dI}{d\tau}P_k(u)\,u\,du + \left[1-\frac{\omega_k(\tau)}{2k+1}\right]\int_{-1}^{1}I(\tau\,;\,u)\,P_k(u)du = \frac{2}{2k+1}b_k(\tau)\right\} \quad k = 0,\,1,\,\ldots\,L$$ /14

where

$$b_k(\tau) = \begin{cases}\left[1-\omega_0(\tau)\right]\cdot B(\tau) & \text{for } k = 0\\ 0 & \text{for } k = 1,\,2,\ldots L\end{cases}$$

Applying the recurrence relation

$$(2k+1)\, u\, P_k(u) = (k+1)\, P_{k+1}(u) + kP_{k-1}(u)$$

(see for instance [7]) and replacing the subscript k by l, we obtain the system of equations

$$\left.\begin{aligned} \alpha_l \frac{df_{l-1}}{d\tau} + \beta_{l+1} \frac{df_{l+1}}{d\tau} + \sigma_l(\tau) f_l = s_l(\tau) \\ l = 0, 1, \ldots L \end{aligned}\right\} \qquad (36)$$

The validity of the system written in this form can be verified by keeping in mind that $P_l=0$ for all $l<0$ and taking $f_{L+1}=0$. Here α_l, β_l and $\sigma_l(\tau)$ are determined on the basis of formulas (8, 9), and for $s_l(\tau)$ werhave:

$$s_l(\tau) = \frac{2}{2l+1}\, b_l(\tau) = \begin{cases} 2\left[1 - \omega_0(\tau)\right]\cdot B(\tau)\, \Lambda & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2, \ldots L \end{cases}$$

System of (36) can be written in matrix notation in a form which is identical with the form of the differential equation in system (5). Thus, as was to be expected, the differenc from the case of solar radiation transfer is the source function vector $s(\tau)$ which is now determined from formula (37) instead of (10).

Let us now consider the lower boundary condition (35). The right member involves a constant which is independent of $I(\tau_g; u)$ within the context of the given formulation of the problem. Denoting for conciseness this constant by b, and proceeding analologously as in section 3, we write:

$$\sum_{l=0}^{L} \frac{2l+1}{2}\, f_l(\tau_g)\, P_l(u) = 2A \sum_{l=0}^{L} c_{1l} f_l(\tau_g) + b$$

For m=1, 3, ..., L we have:

$$\sum_{l=0}^{L} D_{ml} f_l = 4AD_{m0} \sum_{l=0}^{L} C_{1l} f_l + 2D_{m0} b$$

$$\frac{1}{2} f_m - \sum_{l=0,2,\ldots}^{L-1} C_{ml} f_l = -4AC_{m0}\left(\frac{1}{2} f_1 + \sum_{l=0,2,\ldots}^{L-1} C_{1l} f_l\right) - 2C_{m0} b$$

$$f_m = \sum_{l=0,2\ldots}^{L-1} (G_A)_{ml} f_l - \frac{2}{1+A} G_{m0} b$$

From the above we obtain the sought expression for the boundary condition in matrix form:

$$f_o(\tau_g) = G_A f_e(\tau_g) - \frac{2}{1+A} G_0 (1-A) B(T_g)$$

Here G_0 is the i-th column of the matrix G.

Clearly, as $A \to 0$ and $B(T_s) \to 0$, (38) becomes the "vacuum" boundary condition for τ_g in system (5).

5. Conclusions

It was shown above that the spherical harmonics method can be generalized to the case of a lower boundary with albedo different from zero and that it can also be applied to the case of thermal radiation transfer. In both cases the effect of the lower boundary can be taken into account by modifying the lower boundary condition matrix. In addition, in the case of thermal radiation transfer, it is necessary to make an appropriate replacement of the source function vector. The above-mentioned changes are minor and they can be carried out as additional modifications of existing computer programs using the spherical harmonics method.

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