NASA ~~Technical~~ Memorandum 78518

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(NASA-TM-78518) DIRECT NUMERICAL SOLUTION OF THE TRANSONIC PERTURBATION INTEGRAL EQUATION FOR LIFTING AND NONLIFTING AIRFOILS (NASA) 27 p HC A03/MF A01 CSCL 01A

N79-10045

Unclas

G3/05 33852

September 1978

National Aeronautics and
Space Administration

NASA Technical Memorandum 78518

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David Nixon, Ames Research Center, Moffett Field, California

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Ames Research Center
Moffett Field, California 94035

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SUMMARY

The linear transonic perturbation integral equation previously derived for nonlifting airfoils is formulated for lifting cases. In order to treat shock wave motions, a strained coordinate system is used in which the shock location is invariant. The tangency boundary conditions are either formulated using the thin airfoil approximation or by using the analytic continuation concept. A direct numerical solution to this equation is derived in contrast to the iterative scheme initially used, and results of both lifting and nonlifting examples indicate that the method is satisfactory.

I. INTRODUCTION

AN IMPORTANT problem in aerodynamics is the accurate prediction of the pressures on an airfoil that is oscillating harmonically with small amplitude in a transonic flow. In the limit of zero frequency this problem reduces to that of an airfoil that has undergone a perturbation equal to the magnitude of the oscillations. This simpler steady perturbation is therefore a good starting point to deriving a fundamental approach for unsteady flows or to gain experience in developing numerical procedures.

In transonic flows where there is a discontinuity, certain problems can arise in a perturbation solution in the region bounded by the extremities of the shock movement induced by the perturbation. Perturbation solutions assume that changes in the flow variables are of the same order of magnitude as the driving perturbation. In the region traversed by the shock during the perturbation the flow variables can jump from a pre-shock value to a post-shock value, and hence cannot be considered small perturbations.

Although little attention has been paid to the steady perturbation problem finite difference solutions are available for the similar harmonically decomposed unsteady transonic flow problem. Presently available methods of solving the harmonically decomposed problem include the finite difference calculation of Traci et al.[1]

and Weatherhill et al.[2] In these methods no account is taken of the shock motion, the shock being assumed to remain essentially at its steady state location. A serious drawback of these procedures is the occurrence of a severe numerical instability in the relaxation procedure used to solve the difference equations. This instability occurs at a critical Mach number dependent frequency beyond which the method diverges. A means of countering the difficulty is given by Hafez et al.;[3] however, the rate of convergence is slow. An alternative means of avoiding the instability is by direct solution of the difference equations, i.e., without iteration. However, the large number of mesh points makes this procedure not feasible especially for practical three-dimensional flows. All of these methods are shock capturing algorithms. Shock fitting procedures are given in Ref. 3 for the harmonic decomposition. A method of directly integrating the unsteady transonic small disturbance equation is the ADI method of Ballhaus and Goorjian[4].

An alternative to the finite difference solutions for transonic flow is the integral equation method[5,6]. The basic advantage of the integral equation method is that considerably fewer points are required for the discretization. In the present context this implies that direct inversion of a discretized equation is a more feasible proposition than for the finite difference methods. Also from evidence of the steady version of the integral equation method an integral equation solution of the perturbation or unsteady problem would be computationally faster than the direct integration method of Ref. 4. Furthermore the integral equation method gives automatic shock fitting. The main difficulty of the perturbation problem is in the treatment of the moving shock location.

A means of overcoming this difficulty is given by Nixon (7) in which a strained coordinate system is used. In these new coordinates the shock location remains constant and the magnitude of the straining, which is related to the shock movement, is found as part of the solution. In reference (7) the resulting linear perturbation equation for nonlifting flows is solved using the transonic integral equation (5) method with an iterative procedure. The magnitude of the shock movement, and hence the distortion of the coordinate system, is found by enforcing the regularity condition where the flow is continuous everywhere apart from the shock wave. Since the basic equation is linear it seems that a more direct solution of the perturbation equation is possible, and it is this approach that is investigated here with the principal objective being a future extension to oscillatory discontinuous transonic flows. Also, the perturbation integral equation for lifting flow is derived.

The integral equation is reduced by quadrature to a system of linear algebraic equations which incorporate the regularity condition of continuous flow except at the shock. These equations are solved by Gaussian elimination. Satisfactory results for lifting and nonlifting perturbations indicate that the procedure can be applied with confidence to the much more important oscillatory problems. Such an application is reported in reference (8).

The "base" solution used in the calculations requires a normal discontinuous representation of the shock. In the present work, the base solution is computed by a shock capturing finite difference program with a fine mesh since this is the most convenient means available. The normal, discontinuous shock requirement is obtained by suitable extrapolation of the velocities outside the shock capture region.

2. BASIC EQUATIONS

In a Cartesian coordinate system $(\bar{x},\bar{z})$ with the origin at the leading edge of the airfoil and the $\bar{x}$-axis aligned with the airfoil chord, the transonic small-disturbance equation for a free-stream Mach number M_∞ can be written as

$$\phi_{xx} + \phi_{zz} = \frac{1}{2} (\phi_x{}^2)_x \tag{1}$$

where, if $\beta = (1 - M_\infty{}^2)^{1/2}$, γ is the ratio of specific heats, and $k = k(\gamma,M_\infty)$ is a transonic similarity parameter, then ϕ is related to the perturbation velocity potential $\bar{\phi}$ by

$$\phi(x,z) = \frac{k}{\beta^2} \bar{\phi}(\bar{x},\bar{z}) ,$$

and the coordinate system (x,z) is related to $(\bar{x},\bar{z})$ by

$$\left.\begin{aligned} x &= \bar{x} \\ z &= \beta\bar{z} \end{aligned}\right\} . \tag{2}$$

In this transformed space, the perturbation velocities (u,w) in the x- and z-directions, respectively, are related to the physical perturbation velocities $(\bar{u},\bar{w})$ by

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{k}{\beta^2}\,\bar{u} = \frac{k}{\beta^2}\frac{\partial\bar{\phi}}{\partial\bar{x}} \\ w &= \frac{\partial\phi}{\partial z} = \frac{k}{\beta^3}\,\bar{w} = \frac{k}{\beta^3}\frac{\partial\bar{\phi}}{\partial\bar{z}} \end{aligned}\right\} . \tag{3}$$

The boundary conditions are that

a) the flow at the airfoil surface is tangential to the airfoil surface,

b) the velocity is finite an infinite distance from the airfoil, and

c) that for a subsonic trailing edge the Kutta condition of finite pressure at the trailing edge must be satisfied.

If the equations of the upper and lower surfaces of the airfoil are given by $\bar{z} = \bar{z}_u(\bar{x})$ and $\bar{z} = \bar{z}_L(\bar{x})$, respectively, then the tangency boundary condition in the transformed coordinate system (x,z) can be written as

$$\left.\begin{aligned} w(x,z_u) &= \bar{Z}_{u_x}(x)\left[1 + \frac{\beta^2}{k}\, u(x,z_u)\right] - \bar{\Lambda} \\ w(x,z_L) &= \bar{Z}_{L_x}(x)\left[1 + \frac{\beta^2}{k}\, u(x,z_L)\right] - \bar{\Lambda} \end{aligned}\right\} \quad , \qquad (4)$$

where if α is the angle of attack then

$$\left.\begin{aligned} &\bar{\Lambda} = \frac{k}{\beta^3}\,\alpha \\ &z_u(x) = \beta\bar{z}_u(\bar{x}) \ , \qquad \bar{Z}_u(x) = \frac{k}{\beta^3}\,\bar{z}_u(\bar{x}) \\ &z_L(x) = \beta\bar{z}_L(\bar{x}) \ , \qquad \bar{Z}_L(x) = \frac{k}{\beta^3}\,\bar{z}_L(\bar{x}) \end{aligned}\right\} \quad . \qquad (5)$$

For thick airfoils, the tangency boundary condition on the airfoil surface, equation (4), can be replaced by an equivalent boundary condition on the airfoil chord line, $z = \pm 0$, by using the idea of an analytic continuation [9] of the external flow inside the airfoil. This is obtained by using a Taylor series to expand

$w(x,z_u)$ and $w(x,z_L)$ to the chord line. Thus, on using equation (1), equation (4) gives, to second order in magnitude of the thickness/chord ratio of the airfoil,

$$\left.\begin{aligned} w(x,+0) &= -\bar{A} + \bar{Z}_{u_x}(x)\left[1 + \frac{\beta^2}{k}\, u(x,z_u)\right] + z_u(x)\Big[u(x,z_u) \\ &\qquad - \tfrac{1}{2}\, u^2(x,z_u)\Big]_x \\ w(x,-0) &= -\bar{A} + \bar{Z}_{L_x}(x)\left[1 + \frac{\beta^2}{k}\, u(x,z_L)\right] + z_L(x)\Big[u(x,z_L) \\ &\qquad - \tfrac{1}{2}\, u^2(x,z_L)\Big]_x . \end{aligned}\right\} \tag{6}$$

The usual thin airfoil boundary conditions can be recovered from equation (6) by neglecting all the second-order terms. This analytic continuation device is necessary if the integral equation method (5) is to be used in its commonly derived form in which the boundary conditions are required on the chord line rather than on the airfoil surface.

There is an apparent inconsistency in the use of a second order accurate boundary condition like Eq. (6) with a first order differential equation, Eq. (1). The small disturbance approximation breaks down in the region of the leading edge and leads to singular behavior. Generally this singular behavior is removed by including certain (fairly arbitrary) second-order terms. The boundary condition of Eq. (6) can be regarded as simply a more consistent way of introducing the necessary second-order terms.

The weak shock jump relations for equation (1) are

$$\left.\begin{aligned} \left[\phi_x - \frac{{\phi_x}^2}{2}\right]_-^+ + \tan\theta_s[\phi_z]_-^+ &= 0 \\ [\phi]_-^+ &= 0 \end{aligned}\right\} \tag{7}$$

where $[\ ]_{-}^{+}$ denotes a jump across a shock wave and θ_s is the angle the shock makes with the z-axis. If the shock wave can be assumed normal to the free stream, then

$$\left[\phi_x - \frac{\phi_x^2}{2}\right]_{-}^{+} = 0 \quad . \tag{8}$$

The problem under consideration is the calculation of the effect of a small change in the boundary terms $w(x,z_u)$, $w(x,z_L)$ of equation (4) on the pressure distribution. Thus, let equation (6) be written as

$$\left.\begin{aligned} w(x,+0) &= w_0(x,+0) + \varepsilon w_1(x,+0) \\ w(x,-0) &= w_0(x,-0) + \varepsilon w_1(x,-0) \end{aligned}\right\} \tag{9}$$

where ε is a small parameter.

The problem is therefore to solve equation (1) subject to the boundary condition equation (9) assuming that the solution of equation (1) subject to the boundary conditions

$$\left.\begin{aligned} w(x,+0) &= w_0(x,+0) \\ w(x,-0) &= w_0(x,-0) \end{aligned}\right\} \tag{10}$$

is known.

It is shown in reference (7) that if shock waves are present in the flow then a distortion of the coordinate system is necessary in order to correctly formulate the perturbation problem. The shock location is invariant in this distorted coordinate system. It is a basic restriction of the theory that shock waves cannot be generated or destroyed during the perturbation. It is also assumed, for simplicity, that all shock waves can be considered normal to the free stream. This means that only the streamwise coordinate x need be strained. Following the general line of attack of reference (7) let there be N_u shock waves in the upper half plane, and N_L shock waves in the lower half plane. If x' denotes the strained streamwise coordinate, then let

$$x = x' + \epsilon x_1(x') + \cdots \tag{11}$$

where, following the suggestion of reference (7)

$$\left.\begin{aligned} x_1(x') &= \sum_{i=1}^{N_S} \delta x_{s_i} \hat{x}_i(x') \qquad 0 \le x' \le 1 \\ x_1(x') &= 0 \qquad x' < 0,\ x' > 1 \end{aligned}\right\} \tag{12}$$

where $N_S = N_u + N_L$.

In equation (12) δx_{S_i} is the movement of the ith shock wave and $\hat{x}_i(x')$ is some straining function. If the velocity potential $\phi(x,z)$ is expanded in the series

$$\phi(x,z) = \phi_0(x',z) + \epsilon\phi_1(x',z) + \cdots \tag{13}$$

then on substitution of equations (12) and (13) into equation (1) and equating coefficients of ϵ, the equations for ϕ_0 and ϕ_1 are obtained; thus

$$\phi_{0_{x'x'}} + \phi_{0_{zz}} = \phi_{0_{x'}}\phi_{0_{x'x'}} \tag{14}$$

with the boundary condition

$$\phi_{o_z}(x',\pm 0) = w_o(x',\pm 0) \tag{15}$$

and

$$\begin{aligned} \phi_{1_{x'x'}} + \phi_{1_{zz}} &= \left(\phi_{0_{x'}}\phi_{1_{x'}}\right) + \left[x_{1_{x'}}\left(\phi_{0_{x'}} - \phi^2_{0_{x'}}\right)\right]_{x'} \\ &\quad + \left[x_{1_{x'}}\left(\phi_{0_{x'}} - \frac{\phi^2_{0_{x'}}}{2}\right)\right]_{x'} \end{aligned} \tag{16}$$

with the boundary condition

$$\phi_{1_z}(x',\pm 0) = w_1(x',\pm 0) + x_1(x')w_{0_{x'}}(x',\pm 0) \quad . \tag{17}$$

It is assumed that the solution to the problem defined by equation (14), equation (15) is known.

The normal shock jump relations for equation (16) are

$$\left.\begin{aligned} \left[\phi_{1_{x'}} - \phi_{1_{x'}}\left(\phi_{0_{x'}} - x_{1_{x'}} \quad \phi_{0_{x'}} - \phi^2_{0_{x'}}\right)\right]^+_- &= 0 \\ [\phi_1]^+_- &= 0 \end{aligned}\right\} . \tag{18}$$

Eq. (16) and Eq. (18) can be written in integral form using Green's theorem. The domain of Green's theorem is the entire flow field with the exception of the slit $z = \pm 0$, any shock waves in the flow, and the field point (x,z). In this procedure the line integrals around the shock waves vanish because of the shock conditions, Eq. (18). If the general ideas of Ref. (6) are followed then the integral equation for the system defined by equations (16), (17), (18) is as follows.

$$\begin{aligned} u_1(x',z)\left[1 - u_0(x',z)\right] &= I_L(x',z) + I_S(x',z,x'_S) \\ &+ \sum_{i=1}^{N_S} \delta x_{S_i} I_{f_i}(x',z,x'_S) \end{aligned} \tag{19}$$

where

$$u_0(x',z) = \phi_{0_{x'}}(x',z) \ ; \quad u_1(x',z) = \phi_{1_{x'}}(x',z) \ , \tag{20}$$

and x'_S denotes the location of the N_S "base" shock waves. For $z \neq 0$

$$I_L(x',z) = \frac{1}{2\pi}\int_0^1 \frac{\Delta w_1(\xi)(x' - \xi)}{\left[(x - \xi)^2 + z^2\right]}\,d\xi + \frac{1}{2\pi}\int_0^1 \frac{\Delta u_1(\xi)z}{\left[(x' - \xi)^2 + z^2\right]}\,d\xi \tag{21}$$

$$\begin{aligned} I_S(x',z,x'_S) &= -\frac{1}{2\pi}\iint_S \psi_{\xi_{x'}}(x',\xi;z,\zeta)u_1(\xi,\zeta)u_0(\xi,\zeta)\,d\xi\,d\zeta \\ &= I_{S_T}(x',z,x'_S) \end{aligned} \tag{22}$$

$$
\begin{aligned}
I_{f_i}(x',z,x_S') = \Bigg(& \frac{1}{\pi}\int_0^1 \frac{\hat{x}_i(\xi) w_{0_\xi}(\xi)(x'-\xi)}{[(x'-\xi)^2+z^2]}\,d\xi \\
& + \hat{x}_{i_{x'}}(x')\left[2u_0(x',z) - \frac{3}{2}u_0^2(x',z)\right] \\
& - \frac{1}{2\pi}\iint\limits_S \Big\{ \psi_{x'_\xi}(x',\xi;z,\zeta)\left[2u_0(\xi,\zeta) - \frac{3}{2}u_0^2(\xi,\zeta)\right]\hat{x}_{i_\xi}(\xi) \\
& + \psi_{x'}(x',\xi;z,\zeta)\left[u_0(\xi,\zeta) - \frac{1}{2}u_0^2(\xi,\zeta)\right]\hat{x}_{i_{\xi\xi}}(\xi)\Big\}\,d\xi\,d\zeta \Bigg) \\
= {} & I_{f_{T_i}}(x',z,x_S') + \hat{x}_{i_{x'}}(x')\left[2u_0(x',z) - \frac{3}{2}u_0^2(x',z)\right]
\end{aligned}
\tag{23}
$$

where the operator Δ is defined for a function $f(\xi,\zeta)$ by

$$
\Delta f(\xi) = f(\xi,+0) - f(\xi,-0) \ . \tag{24}
$$

For $z = \pm 0$, in equation (19)

$$
I_L(x',\pm 0) = \frac{1}{2\pi}\int_0^1 \frac{\Delta w_1(\xi)}{(x'-\xi)}\,d\xi \pm \frac{1}{\pi}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1 \frac{w_T(\xi)}{(x'-\xi)}\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \tag{25}
$$

$$
\begin{aligned}
I_S(x',\pm 0,x_S') = {} & \pm\frac{1}{\pi}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1 \frac{I_c(\xi,x_S')}{(x'-\xi)}\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \\
& + I_{S_T}(x',0,x_S')
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
I_{f_i}(x',\pm 0,x_S') = {} & \pm\frac{1}{\pi}\left(\frac{1-x'}{x'}\right)^{1/2}\int_0^1 \frac{I_{g_i}(\xi,x_S')}{(x'-\xi)}\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \\
& + \left[2\hat{x}_{i_{x'}}(x')u_0 - \frac{3}{2}u_0^2\hat{x}_{i_{x'}}(x')\right] + I_{f_{T_i}}(x',0,x_S')
\end{aligned}
\tag{27}
$$

where

$$I_c(x',x_S') = \frac{1}{2\pi} \iint_S \psi_{\xi_z}(x',\xi;0,\zeta)[u_1(\xi,\zeta)u_0(\xi,\zeta) - \hat{g}(\xi,\zeta)]d\xi\ d\zeta \tag{28a}$$

and

$$\hat{g}(\xi,\zeta) = \begin{cases} u_1(\xi,+0)u_0(\xi,+0)\ , & \zeta > 0 \\ u_1(\xi,-0)u_0(\xi,-0)\ , & \zeta < 0 \end{cases} \tag{28b}$$

$$\begin{aligned} Ig(x',x_S') = w_f(x') + \frac{1}{2\pi} \iint_S \Bigg(& \psi_{\xi_z}(x',\xi;0,\zeta)\Bigg\{\Big[2u_0(\xi,\zeta) - \frac{3}{2}u_0^{\,2}(\xi,\zeta)\Big] \\ & - \hat{v}(\xi,\zeta)\Bigg\}\hat{x}_{1_\xi}(\xi) + \psi_z(x',\xi;0,\zeta)\Big[u_0(\xi,\zeta) \\ & - \frac{u_0^{\,2}}{2}(\xi,\zeta)\Big]\hat{x}_{1_{\xi\xi}}(\xi)\Bigg)d\xi\ d\zeta \end{aligned} \tag{29}$$

where

$$\hat{v}(\xi,\zeta) = \begin{cases} 2u_0(\xi,+0) - \frac{3}{2}u_0^{\,2}(\xi,+0)\ , & \zeta > 0 \\ 2u_0(\xi,-0) - \frac{3}{2}u_0^{\,2}(\xi,-0)\ , & \zeta < 0\ . \end{cases}$$

The integral over S is defined by

$$\begin{aligned} \iint_S F\ d\xi\ d\zeta = \lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}} \Bigg(& \int_{-\infty}^{x'-\delta_1}\int_0^{\infty} F\ d\zeta\ d\xi + \int_{x'+\delta_1}^{x_{S_u}'-\delta_2}\int_0^{\infty} F\ d\zeta\ d\xi \\ & + \int_{x_{S_u}'+\delta_2}^{\infty}\int_0^{\infty} F\ d\zeta\ d\xi + \int_{-\infty}^{x_{S_L}'-\delta_2}\int_{-\infty}^{0} F\ d\zeta\ d\xi \\ & + \int_{x_{S_L}'+\delta_2}^{\infty}\int_{-\infty}^{0} F\ d\zeta\ d\xi \Bigg) \end{aligned} \tag{30}$$

where x'_{S_U} and x'_{S_L} represent all shocks in the upper and lower half planes respectively, and

$$w_T(\xi) = \frac{1}{2}\left[w_1(\xi,+0) + w_1(\xi,-0)\right]$$

$$w_f(x') = \frac{1}{2}\left[\phi_{1_z}(x',+0) + \phi_{1_z}(x',-0)\right] - w_T(x') .$$

It can be seen that equation (19) will give an infinite value of $u_1(x',z)$ when $u_0(x',z) = 1$ (the sonic line) unless

$$\sum_{i=1}^{N_S} \delta x_{S_i} I_{f_i}(x',z,x'_S)\Big|_{x'=x_{0_p}(z)} = -\left[I_L(x',z) + I_S(x',z,x'_S)\right]\Big|_{x'=x_{0_p}(z)} \tag{31}$$

where $x' = x_{0_p}(z')$ $(p = 1,N_S)$ denotes the location of the pth sonic line. If there are N_S shocks in the flow, then equation (31) gives the required N_S equations for the shock movements δx_{S_i} $(i = 1,N_S)$.

3. NUMERICAL SOLUTION PROCEDURE

The first step in the solution of the integral equation (19) is to discretize the integrals. The techniques used to discretize are outlined in Ref.[6] . Briefly, the flow field is divided into strips parallel to the x-axis, as shown in Fig. 1; and the variation of the flow variable part of the integrand, for example $u_1(\xi,\zeta)$, is approximated in terms of values on the strip edges. The field integrals can then be integrated with respect to ζ to give a line integral in terms of values on the strip edges, for example $u_0(\xi,z_j)u_1(\xi,z_j)$ $(j = 1,N_z)$ where z_j denotes the jth strip edge and N_z is the number of strips. The resulting line integral is then evaluated by quadrature in terms of specified values of the streamwise variable $x_k (k = 1,N_x)$ where N_x is the number of x stations. The values of $u_0(x_k,z_j)$, $u_1(x_k,z_j)$ are now denoted by u_{0_i}, u_{1_i} $(i = 1,N)$ respectively, where

$$i = k + (j - 1)N_x \quad \text{and} \quad N = N_xN_z$$

Using this notation the integral equation (19) can be written as

$$u_{1_j}(1 - u_{0_j}) = I_{LT_j} + \sum_{i=1}^{N_x} u_{1_i}\alpha_{ij} + \sum_{i=1}^{N} u_{1_i}u_{0_i}\beta_{i_j} + \sum_{q=1}^{N_s} \delta x_{s_q} I_{f_{qj}} \tag{32}$$

where

$$I_{f_{qj}} = I_{f_q}(x_k', z_j, x_s')$$

$$I_{LT_j} = I_{L_j} - \sum_{i=1}^{N_x} u_{1_i}\alpha_{i_j} .$$

The matrices α_{i_j}, β_{i_j} are evaluated using the ideas and influence functions given in Ref.[6] , and the δx_{s_i} $(i = 1,N_s)$ are given on each strip edge by equation (31). If a superscript (k) denotes values on the kth strip edge then from equation (31)

$$\delta x_{s_i} = - \left[I^{(k)}_{f_{ip}}\right]^{-1} \left[I^{(k)}_{L_p} + I^{(k)}_{s_p}\right] \tag{33}$$

where $I^{(k)}_{L_p}$, $I^{(k)}_{s_p}$ are the values of I_L, I_s, respectively, at the pth sonic line on the kth strip edge. The matrix $I^{(k)}_{f_{ip}}$ is the value of I_{f_i} at the pth sonic line on the kth strip edge and $\left[I_{f_{ip}}^{(k)}\right]^{-1}$ is the inverse.

Substitution of equation (33) into equation (32) gives

$$u_{1_j}\left(1 - u_{0_j}\right) = I_{LT_j} + \sum_{i=1}^{N_x} u_{1_i}\left\{\alpha_{i_j} - \sum_{q=1}^{N_s} \left[I_{f_{qp}}^{(k)}\right]^{-1} I_{f_{qj}} \alpha_{i_p}^{(k)}\right\}$$

$$+ \sum_{i=1}^{N} u_{1_i} u_{0_i}\left\{\beta_{i_j} - \sum_{q=1}^{N_s} \left[I_{f_{qp}}^{(k)}\right]^{-1} I_{f_{qj}} \beta_{i_p}^{(k)}\right\}$$

$$- \sum_{q=1}^{N_s} I_{f_{qj}} \left[I_{f_{ip}}^{(k)}\right]^{-1} I_{L_{T_p}}^{(k)} \tag{34}$$

where $\alpha_{ip}^{(k)}$, $\beta_{ip}^{(k)}$ are values at the pth sonic line on the kth strip edge. Generally these values are found by interpolation from the standard values α_{ij}, β_{ij}.

Equation (34) can be written in the simple form

$$A_{ji} u_{1_i} = B_j \tag{35}$$

where

$$A_{ji} = \left(\left(1 - u_{0_j}\right)\delta_{ij} - \mu_i\left\{\alpha_{ij} - \sum_{q=1}^{N_s} \left[I_{f_{qp}}^{(k)}\right]^{-1} I_{f_{qj}} \alpha_{ip}^{(k)}\right\}\right.$$

$$\left. - \left\{\beta_{ij} - \sum_{q=1}^{N_s} \left[I_{f_{qp}}^{(k)}\right]^{-1} I_{f_{qj}} \beta_{ip}^{(k)}\right\}\right) \tag{36}$$

$$B_j = I_{LT_j} - \sum_{q=1}^{N_s} I_{f_{qj}} \left[I_{f_{pq}}^{(k)}\right]^{-1} \left[I_{L_{T_p}}^{(k)}\right]. \tag{37}$$

δ_{ij} is the Kronecker delta, and

$$\begin{aligned} \mu_i &= 1 \quad i \le N_x \\ \mu_i &= 0 \quad i \ge N_x \; . \end{aligned} \tag{38}$$

Note that the matrix A_{ji} is singular if the x_i coincide with a sonic point. Hence the distribution of the computational mesh should be such that the sonic points on each strip edge do not lie on a mesh point. A_{ji} is a large dense matrix and in the present scheme the set of equations (35) is solved by Gaussian elimination.

Having obtained the u_{1_i} the total velocity $u_T(x_i,z_k)$ is given by

$$u_T(x_i,z_k) = u_0(x_i',z_k)\left[1 - \varepsilon \sum_{p=1}^{N_s} \delta x_{s_p} \hat{x}_{p_{x'}}(x_i')\right] + \varepsilon u_1(x_i',z_k) \tag{39}$$

where

$$x_i = x_i' + \varepsilon \sum_{p=1}^{N_s} \delta x_{s_p} \hat{x}_p(x_i') \tag{40}$$

and x_{s_p} is found from equation (33).

It can be seen from equation (39) that the total contribution of the perturbation variables is

$$\varepsilon \hat{u}_1(x_i',z_k) = \varepsilon u_1(x_i',z_k) - \left[\varepsilon \sum_{p=1}^{N_s} \delta x_{s_p} \hat{x}_{p_{x'}}(x_i')\right] u_0(x_i',z_k) \; . \tag{41}$$

In the preceding section the terms u_1 and δx_{s_p} are solved separately and it is possible that $\varepsilon\hat{u}_1(x_i,z_k)$ is small, whereas its components $\varepsilon u_1(x_i,z_k)$ and

$$\left[\varepsilon\sum_{p=1}^{N_s}\delta x_{s_p}\hat{x}_{p_{x'}}(x_i')\right]u_0(x_i',z_k)$$

may be relatively large. This situation can lead to large errors in $\varepsilon\hat{u}_1(x_i',z_k)$ even though the relative errors in its component parts are small. Consequently, it has been found in some cases that an alternative formulation in which $\hat{u}_1(x_i',z_k)$ rather than $u_1(x_i',z_k)$ is the dependent variable is more accurate. Thus equation (32) is replaced by

$$\hat{u}_{1_j}\left(1-u_{0_j}\right) = I_{L_{T_j}} + \sum_{i=1}^{N_x}\hat{u}_{1_i}\alpha_{i_j} + \sum_{i=1}^{N}\hat{u}_{1_i}u_{0_i}\beta_{i_j} + \sum_{p=1}^{N_s}\delta x_{s_p}\hat{I}_{f_{pj}} \tag{42}$$

where

$$\hat{I}_{f_{pj}} = I_{f_{pj}} + \sum_{i=1}^{N_x}u_{0_i}\alpha_{i_j}\hat{x}_{p_{x'}}(x_j') + \sum_{i=1}^{N}u_{0_i}^2\beta_{i_j}\hat{x}_{p_{x'}}(x_j') \ . \tag{43}$$

Apart from the definition of $\hat{I}_{f_{pj}}$ the numerical treatment is identical to the previous scheme. Having found $\hat{u}_{1_i}$ the total velocity is given by

$$u_T(x_i,z_k) = u_0(x_i',z_k) + \varepsilon\hat{u}_{1_i}(x_i',z_k) \tag{44}$$

and x_i is given by equation (40).

4. RESULTS

Two examples of perturbation flows calculated by the present method are shown in Figs. 2 and 3. The input for these examples, the base solution, is calculated by shock-capturing finite difference method using the transonic small disturbance equation, with a fine computational grid being used. The necessary normal shock solution is then estimated by the use of a suitable extrapolation. In both examples the transonic parameter k is given by

$$k = (\gamma + 1)M_\infty^{\ 2} \ .$$

For a single shock wave in each half plane the straining function

$$\hat{x}_1(x') = \frac{x'(1 - x')}{x'_S(1 - x'_S)}$$

is used.

The boundary conditions used in the first example are the usual thin airfoil boundary conditions. In the second example, two methods of eliminating the leading edge singularity are used; namely, the use of the analytic continuation boundary conditions of equation (6) and the use of the well known Riegels factor. In the latter, all the terms in equation (26) and equation (27) involving the factor $[(1 - x)/x]^{1/2}$ are divided by

$$\left\{1 + \frac{1}{\beta^2}\left[z_{L_{x'}}(x') - z_{L_{x'}}(x')\right]^2\right\}^{1/2}$$

which removes the $x^{-1/2}$ singularity for round-nosed airfoils.

In Fig. 2 the perturbed flow around a 10% biconvex airfoil at zero angle of attack for two different free-stream Mach numbers is shown. The base Mach number is 0.808 and in the present computations thin airfoil boundary conditions are used. Figures 2a and 2b show the pressure distribution for $M_\infty = 0.818$ and $M_\infty = 0.828$, respectively. Also shown are the results of direct calculations. It can be seen that the present method gives good agreement for the lower Mach number but that there is poorer agreement at the higher Mach number. This discrepancy is attributed to the linearization process leading to equation (16) being no longer a valid approximation.

In Fig. 3 the flow around the upper surface of a NACA64A006 airfoil at $M_\infty = 0.854$ at two different angles of attack is shown. The base flow is nonlifting. In Fig. 3a the pressure distribution at 1/4° angle of attack is compared to the results of a direct finite difference calculation. It can be seen that the "analytic continuation" results give a better agreement with the direct calculation for the pressure than the "Riegels factor" calculation. However, the shock location predicted by the "Riegels factor" method is in better agreement than the "analytic continuation" method. The probable reason is that in the region of the shock motion the "Riegels factor" method gives essentially the same linear thin airfoil boundary conditions as for the direct-calculation, whereas the analytic continuation boundary conditions are nonlinear. It can also be seen that in both examples there is a discrepancy between the present results and the direct results especially in the leading-edge region. This is probably due to the different methods of treating the leading-edge region in the present method (analytic continuation or Riegels factor) and in the direct solution (thin airfoil boundary conditions with mesh adjustment). Similarly, the results of the higher angle of attack, $\alpha = 1/2°$, shown in Fig. 3b, are not in such good agreement as regards shock location. Again, this is attributed to the linearization process being no longer a valid approximation.

Generally therefore, the present method will adequately compute the perturbation flow for both lifting and nonlifting cases. For lifting flows some disagreement with direct results may be expected because of differences in the treatment of the boundary conditions. Also it would appear that for shock movements of more than 5%-6% of chord the linearization of the equations is no longer a valid approximation.

5. CONCLUSIONS

A direct numerical technique has been devised for the solution of perturbations of discontinuous transonic flows for both lifting and nonlifting configurations. Results of two examples indicate that the method is satisfactory provided the perturbations are small. The main purpose of the present paper is to test the numerical approach for future use in unsteady transonic flow problems.

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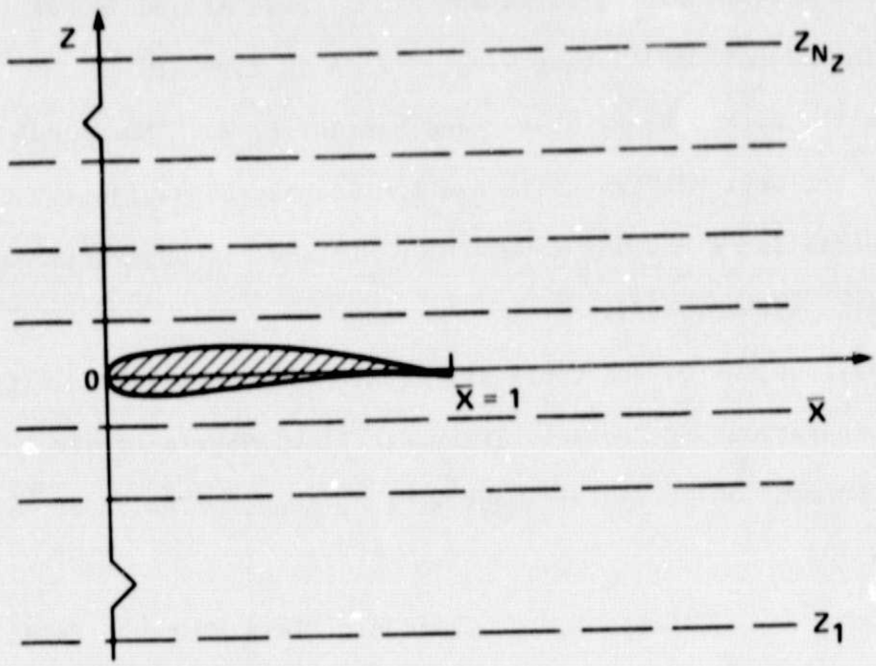
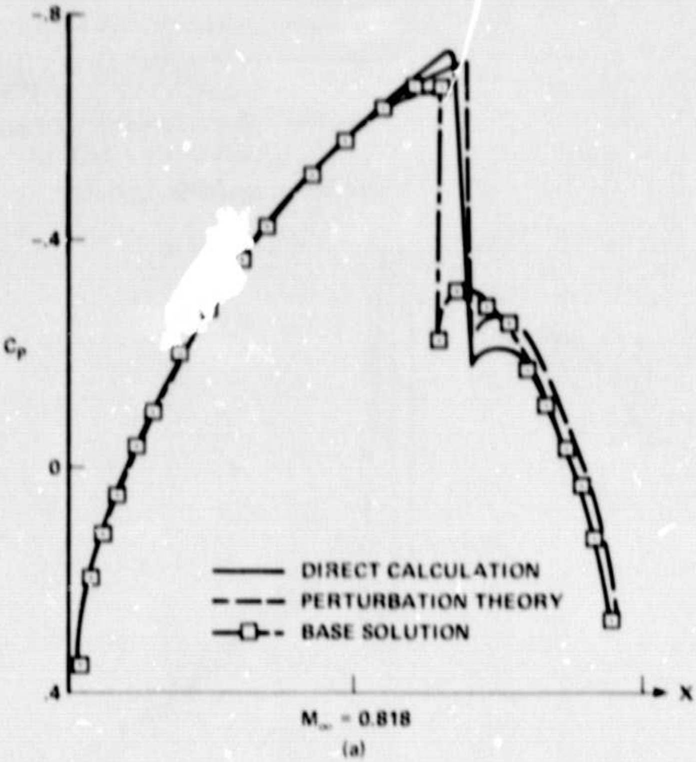


Figure 1.- Arrangement of strips.



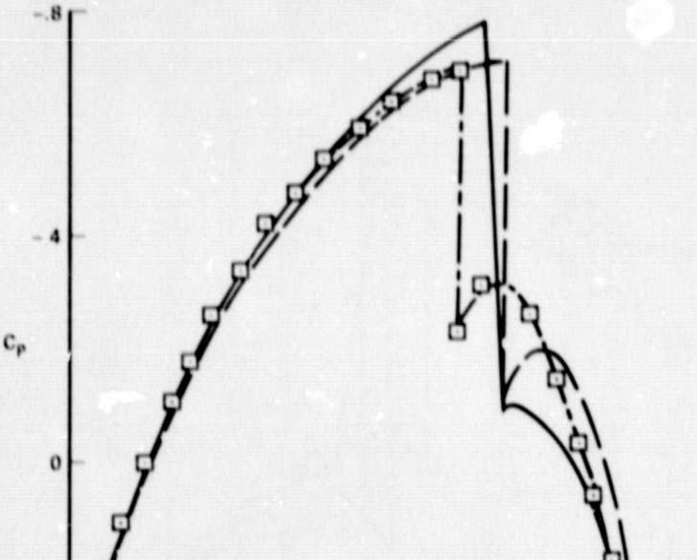


Figure 2.- Pressure distribution around a 10% biconvex airfoil; base solution M_∞ = 0.808.

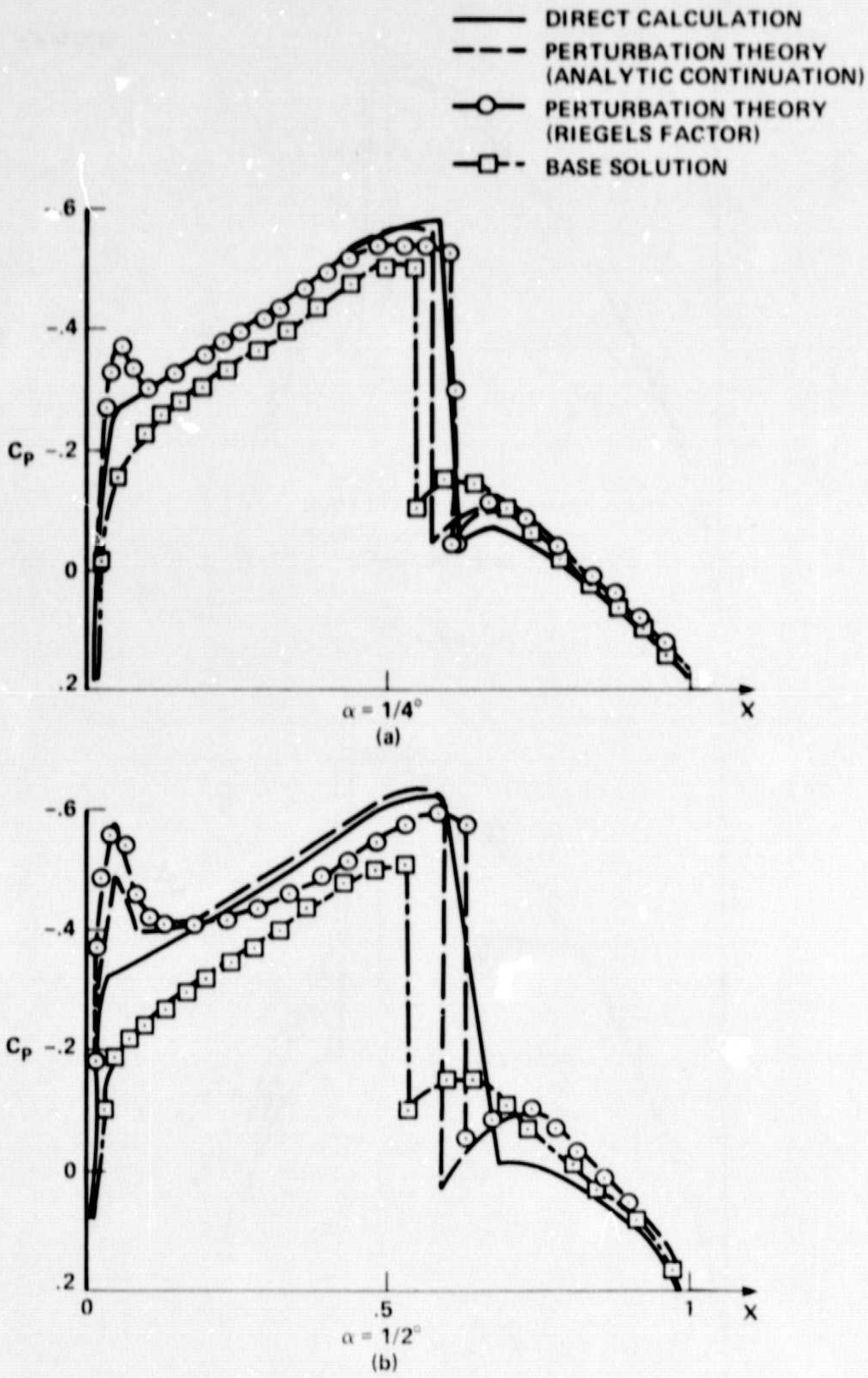


Figure 3.- Pressure distribution around a NACA 64A006 airfoil at $M_\infty = 0.854$; base solution $\alpha = 0°$.