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DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

DEVELOPMENT OF A THREE-DIMENSIONAL
NAVIER-STOKES CODE ON CDC STAR-100 COMPUTER

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By

Veer N. Vatsa

Principal Investigator: G. L. Goglia

Final Report

For the period September 1, 1977 - October 31, 1978

Prepared for the

National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under

Research Grant NSG 1226
Dr. Julius E. Harris, Technical Monitor
High Speed Aerodynamics Division



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FOREWARD

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DEVELOPMENT OF A THREE-DIMENSIONAL NAVIER-STOKES CODE
ON CDC STAR-100 COMPUTER

By

Veer N. Vatsa¹

INTRODUCTION

The importance of a general purpose, three-dimensional Navier-Stokes solver cannot be overemphasized due to the inadequacy of boundary-layer equations and other approximate equation sets to solve complex flow phenomena such as those encountered in the vicinity of a wing-body junction. While no code apparently exists that can treat every flow situation, especially for turbulent flows, significant progress has recently been made by several investigators (refs. 1 to 4) using boundary-fitted coordinate systems (ref. 3) to treat complex geometry. Once the governing equations are cast in the body-fitted coordinate system, the approach is valid for different configurations and requires only the appropriate metric coefficients of transformation from a suitable code, such as that described in reference 3. Due to the availability of a special-purpose computer (CDC STAR-100) at NASA/LaRC, which is capable of achieving substantial speedup compared to conventional computers (refs. 5 to 6) for explicit type algorithms, an effort was initiated to develop a three-dimensional code in body-fitted coordinates using MacCormack's algorithm (ref. 7). The code is structured to be compatible with any general configuration, provided that the metric coefficients for the transformation are available. The governing equations are developed in primitive variables (ρ , u , v , w , p and T) in order to facilitate the incorporation of physical boundary conditions and turbulence-closure models. MacCormack's two-step, unsplit, time-marching algorithm (ref. 7) is used to solve the unsteady Navier-Stokes equations until steady-state solution is achieved.

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LIST OF SYMBOLS

a	speed of sound
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
C_x, C_y, C_z	coefficients used in damping terms
$C_1, \dots, C_4$	convenient nondimensional groupings, defined by equation (A-17)
e	internal energy
f	a twice differentiable function
F	column vector of flux given by equation (2a)
G	column vector of flux given by equation (2b)
H	column vector of flux given by equation (2c)
i	running index in ξ direction
$IMAX$	maximum number of grid points in ξ direction
$IVAR$	number of three-dimensional global variables
j	running index in η direction
J	Jacobian of transformation from (x, y, z) to (ξ, η, ζ) , given by equation (A-6)
J_{max}	maximum number of grid points in η direction
k	running index in ζ direction
K_{max}	maximum number of grid points in ζ direction
L_r	reference length
M_p	Mach number
p	pressure
q	heat flux vector
R	universal gas constant
Re	Reynolds number
S	contribution of fourth order damping terms, given by equations (11) and (12)

t	time
T	temperature
Δt	time step
u	velocity in x direction
U	column vector of dependent variables, given by equation (2a)
v	velocity in y direction
w	velocity in z direction
x, y, z	Cartesian coordinates
$\Delta x, \Delta y, \Delta z$	step sizes in Cartesian coordinates
$\beta_{11}, \dots, \beta_{33}$	convenient groupings of metric coefficients, given by equation (A-10)
γ	ratio of specific heats
Δ	dilatation term defined by equation (4)
λ	bulk viscosity coefficient
μ	viscosity coefficient
μ_1, μ_2	constants used in Sutherland's viscosity relation, given by equation (7)
ξ, η, ζ	transformed coordinates
$\Delta \xi, \Delta \eta, \Delta \zeta$	step sizes in transformed coordinates
ρ	density
σ	Prandtl number
τ	stress tensor whose components are defined by equation (3)

Subscripts

r	reference quantity
t	total or stagnation value
w	wall value
∞	free-stream value

Superscripts

n	quantity at time level t
n+1	quantity at time level t + Δt
*	dimensional variable
-	predictor value

GOVERNING EQUATIONS

The three dimensional, compressible Navier-Stokes equations in Cartesian coordinates are as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0 \quad (1)$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \end{bmatrix} \quad (2a)$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 - \tau_{xx} + p \\ \rho uv - \tau_{xy} \\ \rho uw - \tau_{zx} \\ \rho ue - u\tau_{xx} - u\tau_{yx} - w\tau_{zx} + up + q_x \end{bmatrix} \quad (2b)$$

$$G = \begin{bmatrix} \rho v \\ \rho uv - \tau_{xy} \\ \rho v^2 - \tau_{yy} + p \\ \rho vw - \tau_{yz} \\ \rho ve - u\tau_{yx} - v\tau_{yy} - w\tau_{zy} + vp + q_y \end{bmatrix} \quad (2c)$$

$$H = \begin{bmatrix} \rho w \\ \rho uw - \tau_{yz} \\ \rho vw - \tau_{xz} \\ \rho w^2 - \tau_{zz} + p \\ \rho we - u\tau_{xz} - v\tau_{yz} - w\tau_{zz} + wp + q_z \end{bmatrix} \quad (2d)$$

The stress and flux terms used in the above equations are given by

$$\begin{aligned} \tau_{xx} &= 2\mu \left(\frac{\partial u}{\partial x} \right) - \frac{2\mu}{3} \Delta \\ \tau_{yy} &= 2\mu \left(\frac{\partial u}{\partial y} - \frac{\Delta}{3} \right) \\ \tau_{zz} &= 2\mu \left(\frac{\partial w}{\partial z} - \frac{\Delta}{3} \right) \\ \tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\ \tau_{xz} &= \tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \tau_{yz} &= \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \\ q_x &= -k \frac{\partial T}{\partial x} ; q_y = -k \frac{\partial T}{\partial y} ; q_z = -k \frac{\partial T}{\partial z} \end{aligned} \quad (3)$$

where

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (4)$$

Note that Stoke's hypothesis ($\lambda = -\frac{2}{3}\mu$) is used in the stress terms. The internal energy e is given by

$$e = C_v T + \frac{1}{2}(u^2 + v^2 + w^2) \quad (5)$$

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The perfect gas law and Sutherland's viscosity equations are used to close the system

$$p = \rho RT \quad (6)$$

$$\mu = \frac{u_1 T^{3/2}}{(T + u_2)} \quad (7)$$

where u_1 and u_2 are constants.

The above equations are in dimensional form and are valid only in the Cartesian coordinate system (x, y, z) . These equations are then transformed to a new coordinate system (ξ, η, ζ) using implicit differentiation. Finally, the equations are nondimensionalized with appropriate reference quantities. Details of these transformation and final forms of the equations are presented in the Appendix.

NUMERICAL SCHEME

The CDC STAR-100 is a special type of computer which can do vector operations at a very fast speed. The word vector is used here to describe a set of up to 65000 data elements stored in consecutive storage locations in memory. This particular feature makes explicit methods very attractive for the STAR-100 computer. MacCormack (ref. 7) developed a two-step marching algorithm which has been used on a variety of problems and is especially suited to obtaining steady-state solution of the Navier-Stokes equations. To explain the algorithm, we recall (see Appendix) that governing equations can be written in the vector form as

$$\begin{aligned} \frac{\partial U}{\partial t} + \beta_{11} F_{\xi} + \beta_{21} G_{\xi} + \beta_{31} H_{\xi} + \beta_{12} F_{\eta} + \beta_{22} G_{\eta} + \beta_{32} H_{\eta} \\ + \beta_{13} F_{\zeta} + \beta_{23} G_{\zeta} + \beta_{33} H_{\zeta} = 0 \end{aligned} \quad (8)$$

where U , F , G , and H are vectors of length 5 and are given by equation (A-11) in the Appendix. Let superscript n denote the variables at a given time t . The variables are advanced to a new time $t + \Delta t$ through predictor and corrector steps that are given as:

Predictor step.

$$\begin{aligned}
 \overline{U}_{i,j,k}^{n+1} = & U_{i,j,k}^n - \frac{\Delta t}{\Delta \xi} \left[\beta_{11} (F_{i+1,j,k}^n - F_{i,j,k}^n) + \beta_{21} (G_{i+1,j,k}^n + G_{i,j,k}^n) \right. \\
 & \left. + \beta_{31} (H_{i+1,j,k}^n - H_{i,j,k}^n) \right] \\
 & - \left(\frac{\Delta t}{\Delta \eta} \right) \left[\beta_{12} (F_{i,j+1,k}^n - F_{i,j,k}^n) + \beta_{22} (G_{i,j+1,k}^n - G_{i,j,k}^n) \right. \\
 & \left. + \beta_{32} (H_{i,j+1,k}^n - H_{i,j,k}^n) \right] \\
 & - \left(\frac{\Delta t}{\Delta \zeta} \right) \left[\beta_{13} (F_{i,j,k+1}^n - F_{i,j,k}^n) + \beta_{23} (G_{i,j,k+1}^n - G_{i,j,k}^n) \right. \\
 & \left. + \beta_{33} (H_{i,j,k+1}^n - H_{i,j,k}^n) \right]
 \end{aligned} \tag{9}$$

The first derivative terms used in flux vectors F , G , H are evaluated using two-point backward differences.

Corrector step.

$$\begin{aligned}
 U_{i,j,k}^{n+1} = & \frac{1}{2} \left[U_{i,j,k}^n + \overline{U}_{i,j,k}^{n+1} \right] \\
 & - \left(\frac{\Delta t}{\Delta \xi} \right) \frac{1}{2} \left[\beta_{11} (\overline{F}_{i,j,k}^{n+1} - \overline{F}_{i-1,j,k}^{n+1}) + \beta_{21} (\overline{G}_{i,j,k}^{n+1} - \overline{G}_{i-1,j,k}^{n+1}) \right. \\
 & \left. + \beta_{31} (\overline{H}_{i,j,k}^{n+1} - \overline{H}_{i-1,j,k}^{n+1}) \right] \\
 & - \left(\frac{\Delta t}{\Delta \eta} \right) \frac{1}{2} \left[\beta_{12} (\overline{F}_{i,j,k}^{n+1} - \overline{F}_{i,j-1,k}^{n+1}) + \beta_{22} (\overline{G}_{i,j,k}^{n+1} - \overline{G}_{i,j-1,k}^{n+1}) \right. \\
 & \left. + \beta_{32} (\overline{H}_{i,j,k}^{n+1} - \overline{H}_{i,j-1,k}^{n+1}) \right] \\
 & - \left(\frac{\Delta t}{\Delta \zeta} \right) \frac{1}{2} \left[\beta_{13} (\overline{F}_{i,j,k}^{n+1} - \overline{F}_{i,j,k-1}^{n+1}) + \beta_{23} (\overline{G}_{i,j,k}^{n+1} - \overline{G}_{i,j,k-1}^{n+1}) \right. \\
 & \left. + \beta_{33} (\overline{H}_{i,j,k}^{n+1} - \overline{H}_{i,j,k-1}^{n+1}) \right]
 \end{aligned} \tag{10}$$

where the first derivative terms used in flux vectors F , G , H are evaluated using two-point forward differences.

Successive application of predictor and corrector step equations advances the solution by time Δt .

Damping Terms Used in Navier-Stokes Equations

Fourth order damping has been used in the Navier-Stokes equations to capture discontinuities (e.g. shock waves) in the solution. The particular form of damping chosen is from Tannehill et al. (ref. 8) and Holst (ref. 9), where a source term is added in both predictor and corrector steps of the solution. The main reason for selecting this type of damping is that it is easily put in vector form on the STAR-100 computer. The form of damping used is given below.

Predictor step.

$$\begin{aligned}
 S_{i,j,k}^n = & C_x \left(\frac{p_{i+1,j,k}^n - 2 p_{i,j,k}^n + p_{i-1,j,k}^n}{p_{i+1,j,k}^n + 2 p_{i,j,k}^n + p_{i-1,j,k}^n} \right) \\
 & \times \left[U_{i+1,j,k}^n - 2 U_{i,j,k}^n + U_{i-1,j,k}^n \right] \\
 & + C_y \left(\frac{p_{i,j+1,k}^n - 2 p_{i,j,k}^n + p_{i,j-1,k}^n}{p_{i,j+1,k}^n + 2 p_{i,j,k}^n + p_{i,j-1,k}^n} \right) \\
 & \times \left[U_{i,j+1,k}^n - 2 U_{i,j,k}^n + U_{i,j-1,k}^n \right] \\
 & + C_z \left(\frac{p_{i,j,k+1}^n - 2 p_{i,j,k}^n + p_{i,j,k-1}^n}{p_{i,j,k+1}^n + 2 p_{i,j,k}^n + p_{i,j,k-1}^n} \right) \\
 & \times \left[U_{i,j,k+1}^n - 2 U_{i,j,k}^n + U_{i,j,k-1}^n \right]
 \end{aligned} \tag{11}$$

Corrector step.

$$\begin{aligned}
 \overline{S}_{i,j,k}^{n+1} = & C_x \left(\frac{\overline{p}_{i+1,j,k}^{n+1} - 2 \overline{p}_{i,j,k}^{n+1} + \overline{p}_{i-1,j,k}^{n+1}}{\overline{p}_{i+1,j,k}^{n+1} + 2 \overline{p}_{i,j,k}^{n+1} + \overline{p}_{i-1,j,k}^{n+1}} \right) \\
 & \times \left[\overline{U}_{i+1,j,k}^{n+1} - 2 \overline{U}_{i,j,k}^{n+1} + \overline{U}_{i-1,j,k}^{n+1} \right]
 \end{aligned} \tag{12}$$

(cont'd)

$$\begin{aligned}
& + C_y \frac{\left| \overline{p_{i,j+1,k}^{n+1}} - 2 \overline{p_{i,j,k}^{n+1}} + \overline{p_{i,j-1,k}^{n+1}} \right|}{\left(\overline{p_{i,j+1,k}^{n+1}} + 2 \overline{p_{i,j,k}^{n+1}} + \overline{p_{i,j-1,k}^{n+1}} \right)} \\
& \quad \left[\overline{u_{i,j+1,k}^{n+1}} - 2 \overline{u_{i,j,k}^{n+1}} + \overline{u_{i,j-1,k}^{n+1}} \right] \\
& + C_z \frac{\left| \overline{p_{i,j,k+1}^{n+1}} - 2 \overline{p_{i,j,k}^{n+1}} + \overline{p_{i,j,k-1}^{n+1}} \right|}{\left(\overline{p_{i,j,k+1}^{n+1}} + 2 \overline{p_{i,j,k}^{n+1}} + \overline{p_{i,j,k-1}^{n+1}} \right)} \\
& \quad \left[\overline{u_{i,j,k+1}^{n+1}} - 2 \overline{u_{i,j,k}^{n+1}} + \overline{u_{i,j,k-1}^{n+1}} \right]
\end{aligned} \tag{12}$$

(concl'd)

Here C_x , C_y , C_z are the damping coefficients generally taken to be 0.1; however, the specific value is a function of the problem being solved. Damping can be completely removed by equating C_x , C_y , C_z to zero.

Arrangement of Data

Because of its special architecture, this section is devoted to the arrangement of data in the STAR-100 computer. The computer uses the "virtual memory" addressing system, which allows a code to have a data base larger than the central memory of the computer. However, when such a large code executes, the system can only accommodate 524,288 words of memory (distributed on 7 magnetic disks, referred to as pages); the rest of the required information is stored on peripheral disks (pages). When the code refers to some information that is not resident in the central memory, one of the pages (disks) from central memory becomes a peripheral unit to accommodate the required information in cores. This process is called "paging" and is a slow process (taking about 1/3 sec) due to hardware limitations of the system. Due to this feature of the system, the data in the code is arranged so that maximum use is made of this information residing in the memory at all times. Lambiotte (ref. 6) addressed this question in an earlier paper; following the procedure suggested in that work, we interleave all the global variables into one variable and arrange the data plane-by-plane from inflow plane to the downstream end. To give an example, let A be the interleaved global variable $A(\text{IMAX}, \text{IVAR}, \text{JMAX}, \text{KMAX})$

where IMAX, JMAX, and KMAX are the number of grid points in x, y, and z directions respectively and IVAR refers to different variables; for example, by assigning IVAR = 1 for density, A(IMAX, 1, JMAX, KMAX) would refer to the density. This notation is made semantic for readability (e.g. use IRHO for density, etc.) and applied to all global variables at two time levels. The advantage of such an approach is that at a given I location, all the required variables are grouped together permitting vector operations of size JMAX*KMAX at each plane. To minimize the paging even further, we perform the predictor step from I = 2 to IMAX - 1 and corrector step from I = IMAX - 1 to 2, so that near the downstream boundary (I = IMAX) we make best use of the available data. This approach has been very successful, and numerical experimentation with the computer code indicates that the resulting page faults were mainly due to the difference between data base and central memory and could be well predicted before the actual execution of code. For a 31x31x31 grid, the data base including coding was about 11 pages long, and it resulted in an average of 4 page faults ($11 - 7 = 4$) per iteration cycle.

RESULTS AND DISCUSSION

The results presented in this report are preliminary in nature, but nevertheless represent an important class of problems that could be used as test cases for future investigators involved with three-dimensional Navier-Stokes problems. The computational domain specialized to an axial corner is shown in figure 1. The inflow boundary is at $\xi = 0$ and outflow boundary at $\xi = \xi_{\max}$. The outer edge is formed by $\eta = \eta_{\max}$ and $\zeta = \zeta_{\max}$ boundaries, whereas the wall boundary is formed by $\eta = 0$ and $\zeta = 0$. The rest of the section is devoted to various test problems and their boundary conditions.

Case 1: Flat Plate in Supersonic Free Stream

To test conditions chosen are $M_{\infty} = 3.0$, $Re_{\infty} = 0.34 \times 10^5$ per foot; $T_{\infty,t} = 1092^{\circ}R$ and $T_w = 985.66^{\circ}R$. This test case is chosen to validate the computer code since accurate boundary-layer results are available for comparison. The flat plate is located at the $\eta = 0$ face of the computational box of figure 1. The following boundary conditions are applied.

Inflow condition.- At $\xi = 0$ all the flow variables are assumed at their free-stream value.

Wall condition.- At $\eta = 0$ ($K = 1$):

$$u = v = w = 0 \text{ (no slip, no injection)}$$

$$T = T_w$$

$$\frac{\partial p}{\partial \eta} = 0$$

At $\xi = 0$ ($J = 1$), continuation condition is used on all the flow variables, i.e. value of a variable at $J = 1$ is set equal to its value at $J = 2$.

Outer edge condition.- At $\eta = \eta_{\max}$ and $\xi = \xi_{\max}$, continuation condition is used on all variables.

Outflow condition.- At $\xi = \xi_{\max}$, all the variables are obtained using zeroth order extrapolation, which is equivalent to the continuation condition.

In figure 2, axial velocity results are shown for three different codes. SUPERMAC refers to MacCormack's Rapid Solver (ref. 10), a code released by the NASA Ames Research Center. Other data on figure 2 is from the unpublished results of Dave H. Rudy of NASA/LARC. Boundary-layer solution is also shown in this figure. The calculations here were done with $21 \times 6 \times 21$ mesh, thus only 21 nodes were used normal to the plate and 21 nodes along the longitudinal direction. The normal mesh is highly stretched to give good resolution in the viscous layer. From this figure, it is noted that all three codes basically give similar results. This test case simply was tried to give some credibility to the new code.

Case 2: Supersonic Flow Along an Axial Corner

The next test case chosen for this study is of supersonic flow along an axial corner. The test conditions are identical to those of case 1. The reason for selecting this case was that Dr. C.M. Hung of NASA Ames has already run this case and it was believed that a comparison with an independent code would add to the credibility of the present code.

The boundary conditions for the axial corner are as follows:

Inflow condition.- At $\xi = 0$ all the flow variables are assumed at their free-stream value.

Wall condition.- At $\eta = 0$ and $\zeta = 0$:

$$u = v = w = 0 \text{ (no slip, no injection)}$$

$$T = T_w$$

$$\left. \frac{\partial p}{\partial \eta} \right|_{\eta = 0} = 0 \text{ and } \left. \frac{\partial p}{\partial \zeta} \right|_{\zeta = 0} = 0$$

Outer edge condition.- Continuation condition is used on all variables at $\eta = \eta_{\max}$ and $\zeta = \zeta_{\max}$.

Outflow condition.- At $\zeta = \zeta_{\max}$, all the variables are obtained by extrapolation.

It should be noted that, in addition to the inflow plane, there are 2 more planes: $l = 2$ and $l = 3$ in the free stream. The corner starts at $l = 4$; therefore, at $l = 2$ and $l = 3$, appropriate symmetry and antisymmetry conditions are imposed at $\eta = 0$ and $\zeta = 0$. A $31 \times 31 \times 31$ mesh is used for all these computations.

The results are shown in figures 3 to 8. All the comparisons are done at $x = 0.95555'$. In figures 3 to 6, axial velocity, crossflow velocity, density and pressure profiles along the diagonal at $x = 0.09556'$ are shown. Except for the crossflow velocity, which is generally the most sensitive quantity, comparisons are in reasonable agreement. Note also that in Hung's scheme, difference equations are written in physical space (nonuniform mesh), whereas in the present code the difference equations are written in computational space (uniform mesh). Such different approaches have different truncation errors and could account for the slightly different results observed here.

In figures 7 and 8, velocity and density profiles are compared at $x = 0.09556'$ and at two fixed values of Y : $Y = 0.0162'$ is in the corner layer whereas $Y = Y_{\max}$ is far away from the corner where we expect to see a boundary-layer type behavior. Comparisons show a good agreement between the two schemes. On figure 7 boundary-layer solution is also shown, and it agrees very well with the Navier-Stokes solutions at $Y = Y_{\max}$ as expected. This set of results thus indicates that the present three-dimensional Navier-Stokes code is basically sound and would give correct results for degenerate cases (e.g. flat plate and axial corner).

Case 3: Subsonic Flow in an Axial Corner at $M_\infty = 0.95$.

The test case chosen here is of the subsonic flow in an axial corner at $M_\infty = 0.95$, for which asymptotic solutions in a corner layer have been presented by Weinburg and Rubin (ref. 11).

Boundary conditions. - The only difference in this case compared to the supersonic case is in the outflow boundary condition at $\xi = \xi_{\max}$. In this case, pressure at the downstream end is specified at its free-stream value. Velocities and temperature are obtained by extrapolation, and density is then computed using equation of state (A-19) in the Appendix. The wall temperature is taken to be equal to free-stream static temperature.

The axial velocity and crossflow velocities along the diagonal at $x = 0.6'$ are shown in figures 9 and 10 in corner-layer coordinates (ref. 11). The Navier-Stokes code was run at Reynolds numbers, $R_e = 10538$ and $R_e = 42152$ to see the effect of Reynolds number on the solution. From figures 9 and 10 it is clear that as the Reynolds number increases, Navier-Stokes solution moves towards the asymptotic solution, which in this case is the solution to corner-layer equations for infinitely large R_e ; however it is still quite far from the asymptotic solution. On the basis of experience gained in similar studies with interacting boundary-layer solutions (refs. 12, 13), it is conjectured that a value of $R_e = O(10^7)$ may be required to reproduce the asymptotic solution. However, high Reynolds number flows are very time consuming as one can see from the stability criterion, the Courant-Friedrichs-Lewy (CFL)

Condition:

$$\Delta t \leq \frac{1}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + \frac{|w|}{\Delta z} + a \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}} \quad (13)$$

where a is the local speed of sound.

Note that as viscous layer becomes thin at higher values of R_e , smaller step sizes ($\Delta y, \Delta z$) are needed to resolve the viscous layer; hence a smaller value of allowable time step Δt results, thus taking longer time to converge. Because of insufficient time, higher Reynolds number cases were not tried; however, present results should provide a good reference point for future investigators who might be interested in resolving the details of corner layer.

In figure 11, axial velocity profiles at $z = z_{\max}$ are compared with the boundary-layer solution. It is clear from this figure that at $R_e = 42152$, the Navier-Stokes solution recovers the boundary-layer solution. Use of the same set of equations throughout the computational domain (corner-layer/boundary-layer/inviscid regions) and recovery of boundary-layer and inviscid solutions in their respective domains assure that the corner-layer solution so generated is the correct solution.

Case 4: Supersonic Flow in an Axial Corner at $M_\infty = 1.5$

The last test case presented is that of supersonic flow in an axial corner at $M_\infty = 1.5$, for which asymptotic corner-layer solutions are available in reference 11. The wall temperature is set equal to the free-stream static temperature. Boundary conditions are identical to those of test case 2.

The axial and crossflow velocities along the diagonal at $x = 0.4'$ are shown in figures 12 and 13. The corner-layer solutions are also plotted in these figures. It is noted that although the Navier-Stokes solution comes closer to the asymptotic solution as Reynolds number is increased, the shift is very slow compared to the subsonic case. Based on this limited data base, it seems very unlikely that Navier-Stokes solution will merge with asymptotic solution at finite values of Reynolds number. As in the previous case, it is seen from figure 14 that at $z = z_{\max}$ the Navier-Stokes solution has recovered the boundary-layer solution except near the edge of the boundary layer, where we do not expect good agreement due to leading-edge shock in the Navier-Stokes solutions.

It is felt here that the corner-layer solutions of Weinberg and Rubin (ref. 11) do not appropriately model the flow in the supersonic axial corner since no mechanism exists that can model the complex flow pattern that develops in the corner layer by interacting shocks. Figure 15 shows a constant density contour plot at $x = 0.4'$ obtained from the Navier-Stokes solution. The leading edge shocks and the resulting corner shock are identified on this figure. Such a shock system will have considerable influence on the viscous flow in the corner.

CONCLUSIONS AND RECOMMENDATIONS

A computer code for solving three-dimensional Navier-Stokes equations has been developed for the CDC STAR-100 vector-processing computer. The governing equations have been transformed through implicit differentiation; consequently the code can be applied to any general configuration provided the appropriate metric coefficients of transformation are available. The code has been tested and validated for several flows through comparison with known solutions.

Results for the subsonic axial corner, while encouraging, are not complete and require additional study. It is recommended that studies be made at higher Reynolds numbers in order to reach definite conclusions concerning the agreement between the corner-layer solution and Navier-Stokes solution.

For the supersonic axial corner, the comparison is not that good. It is conjectured that lack of modeling of complex, leading-edge shock interactions in the corner-layer equations is responsible for the observed differences. It is recommended to validate the solutions of Navier-Stokes equations for supersonic axial corner with experimental results when these become available.

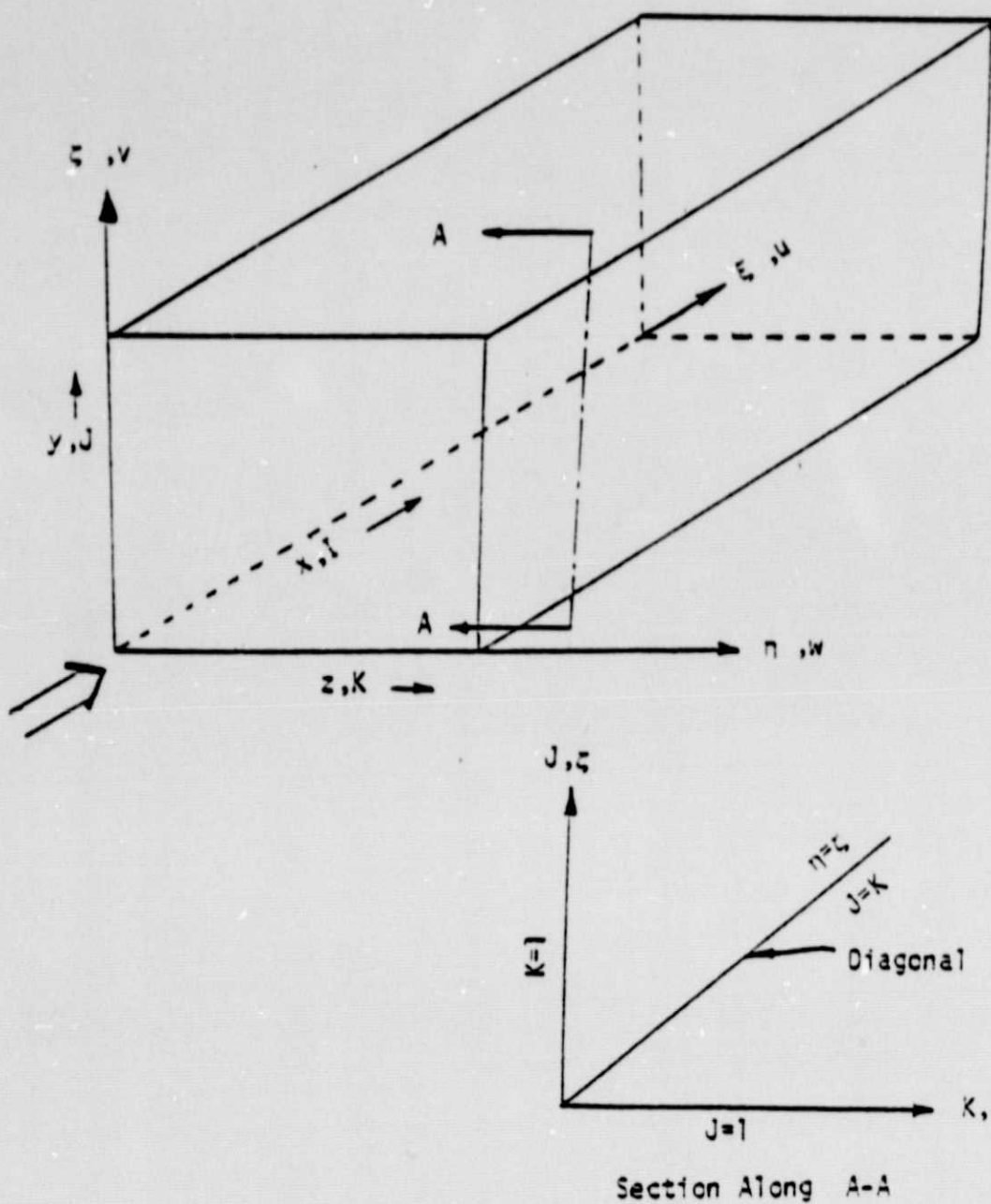


Figure 1. Schematic of axial corner flow.

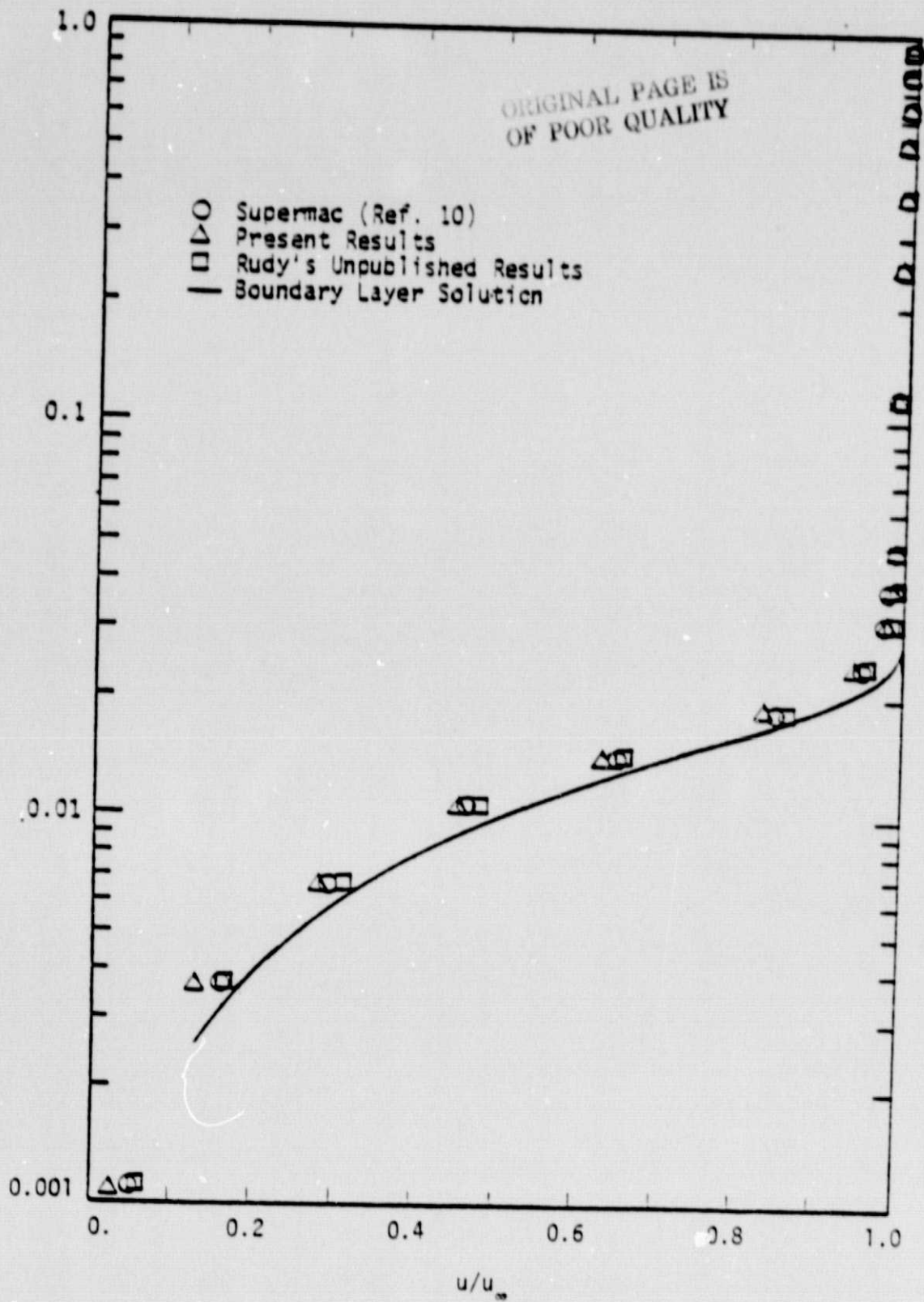


Figure 2. Axial velocity comparisons for flat plate at $M_\infty = 3.0$.

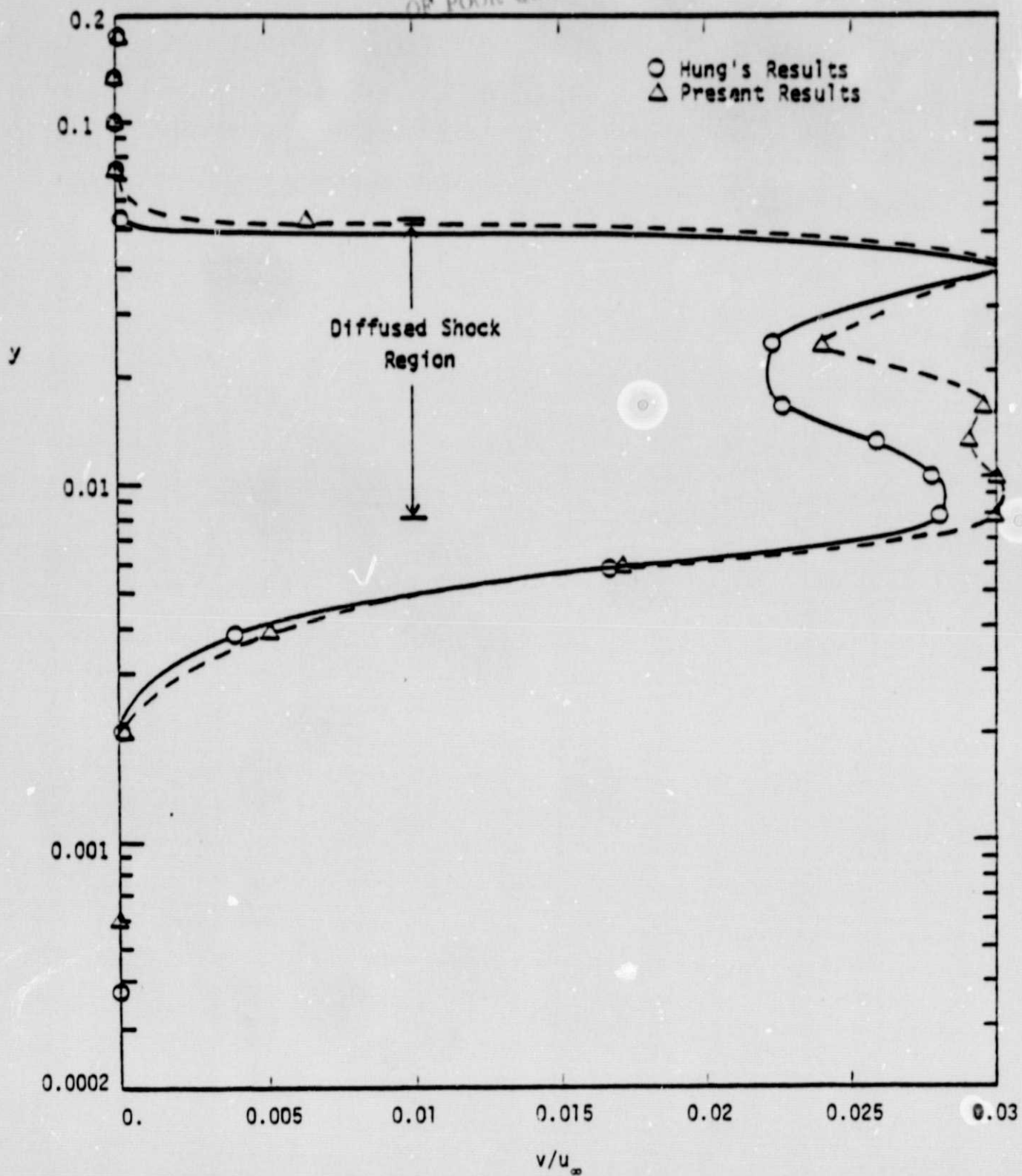


Figure 4. Crossflow velocity along diagonal ($M_\infty = 3.0$).

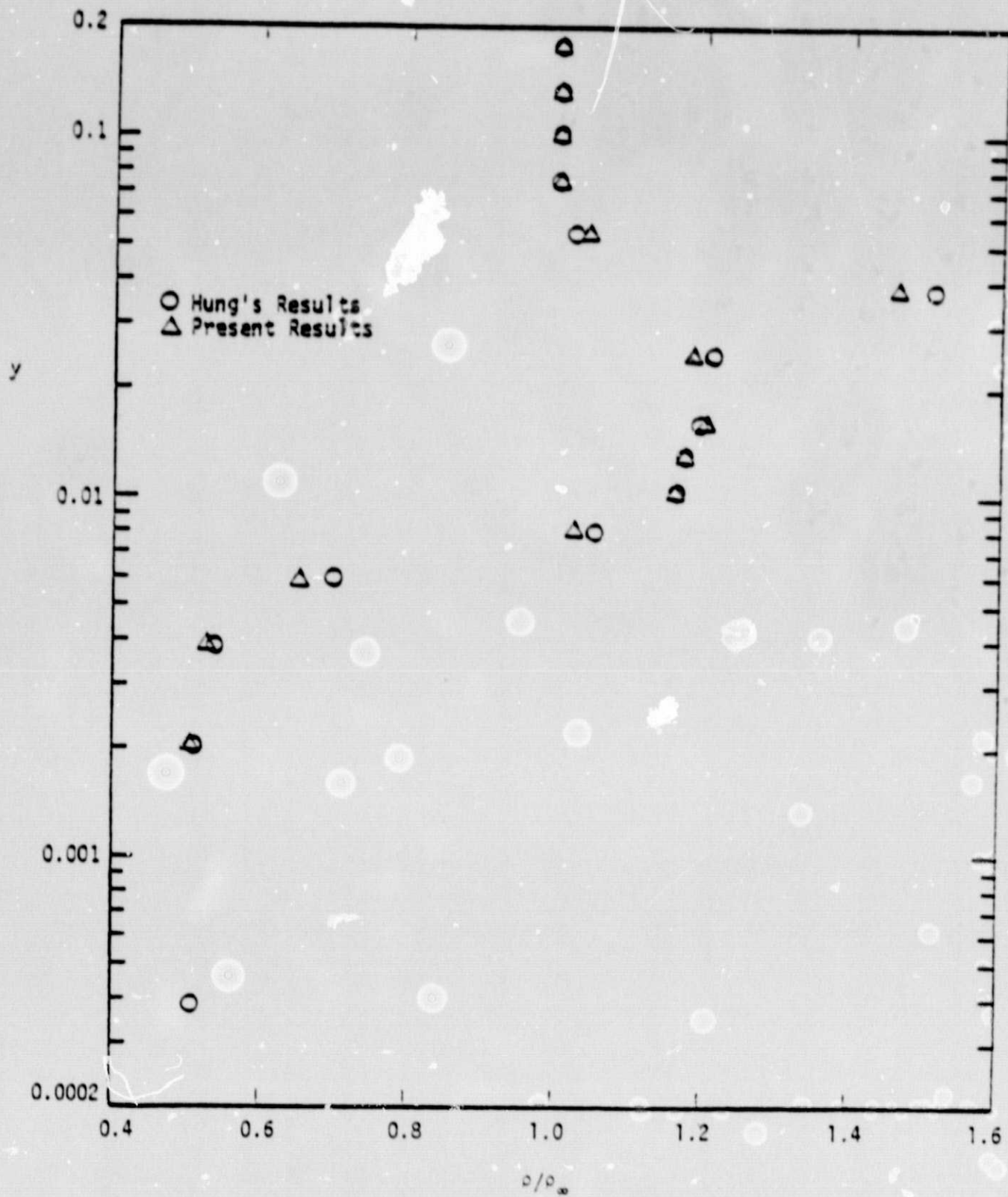


Figure 5. Density along diagonal ($M_\infty = 3.0$).

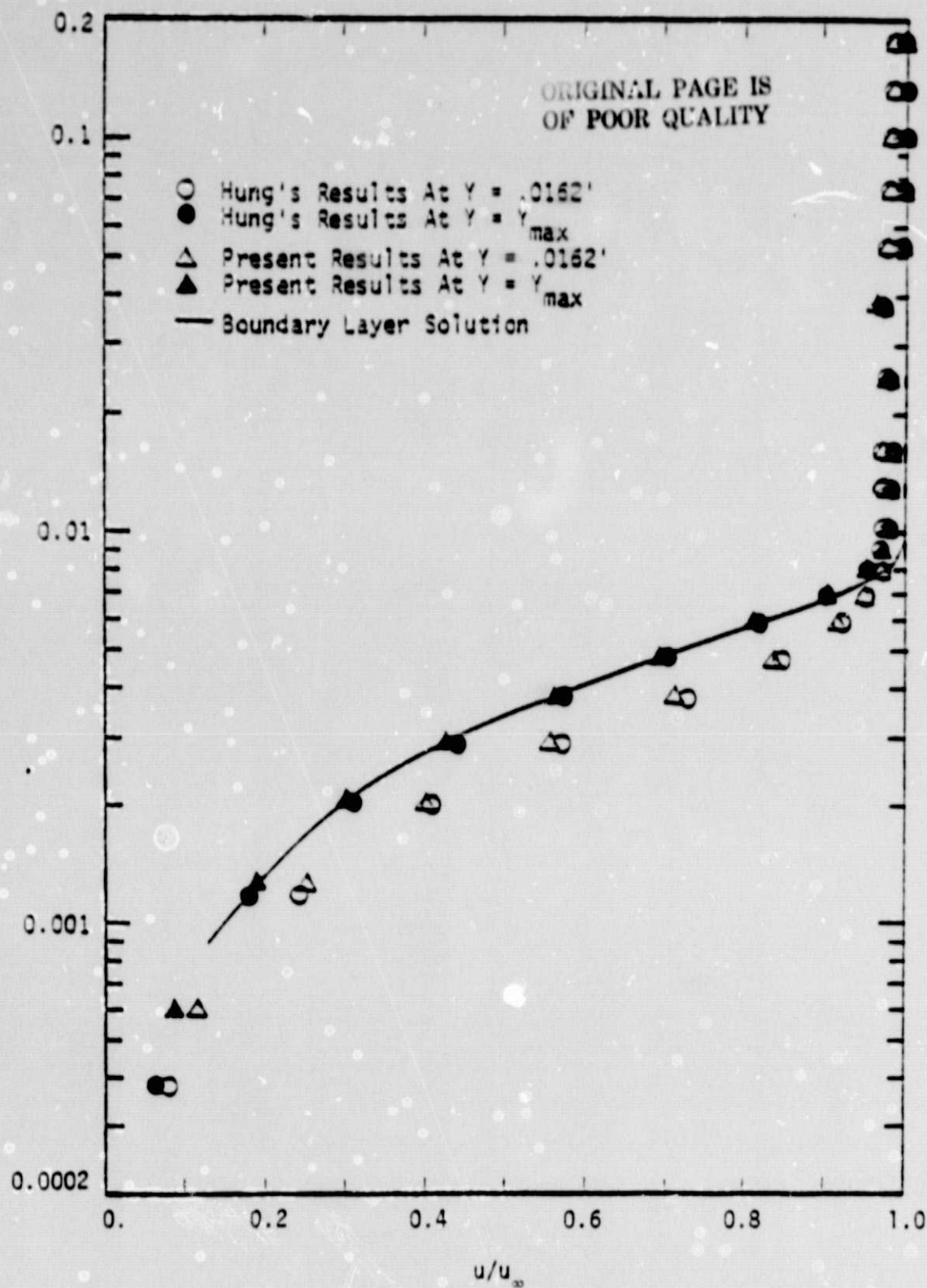


Figure 7. Axial velocity at fixed-Y-location ($M_{\infty} = 3.0$).

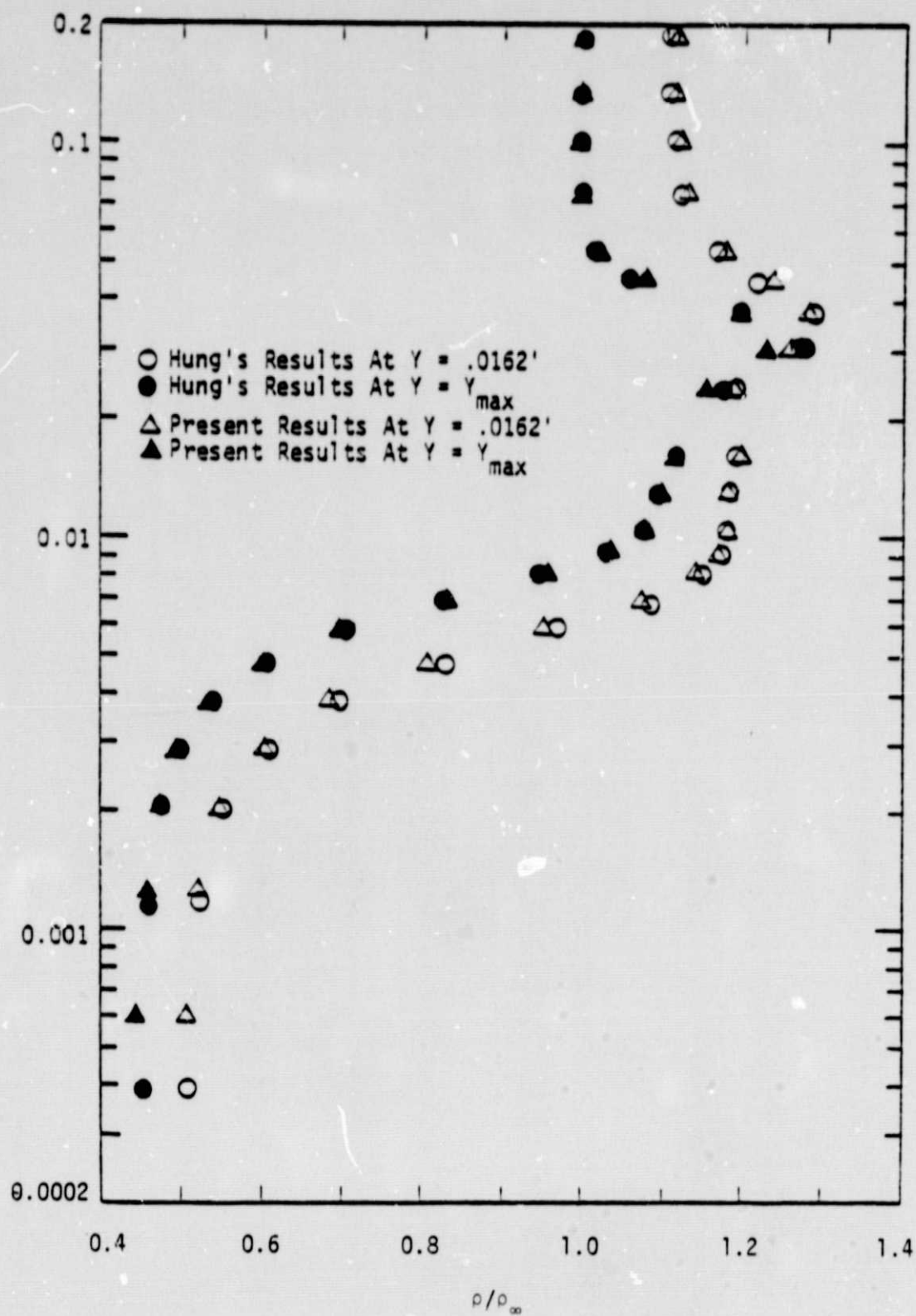


Figure 8. Density profiles at fixed Y-location ($M_\infty = 3.0$).

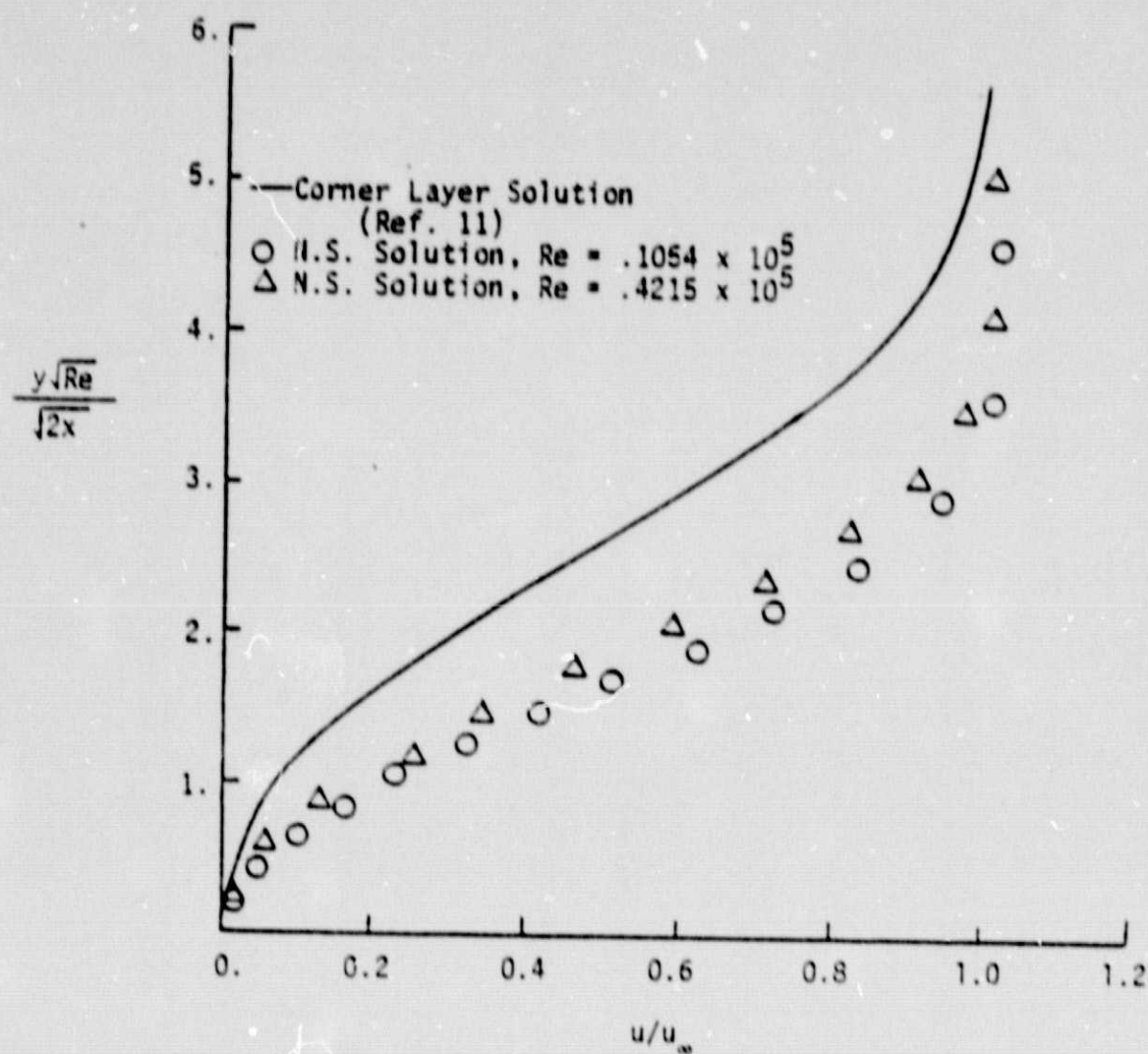


Figure 9. Axial velocity along diagonal ($M_\infty = 0.95$).

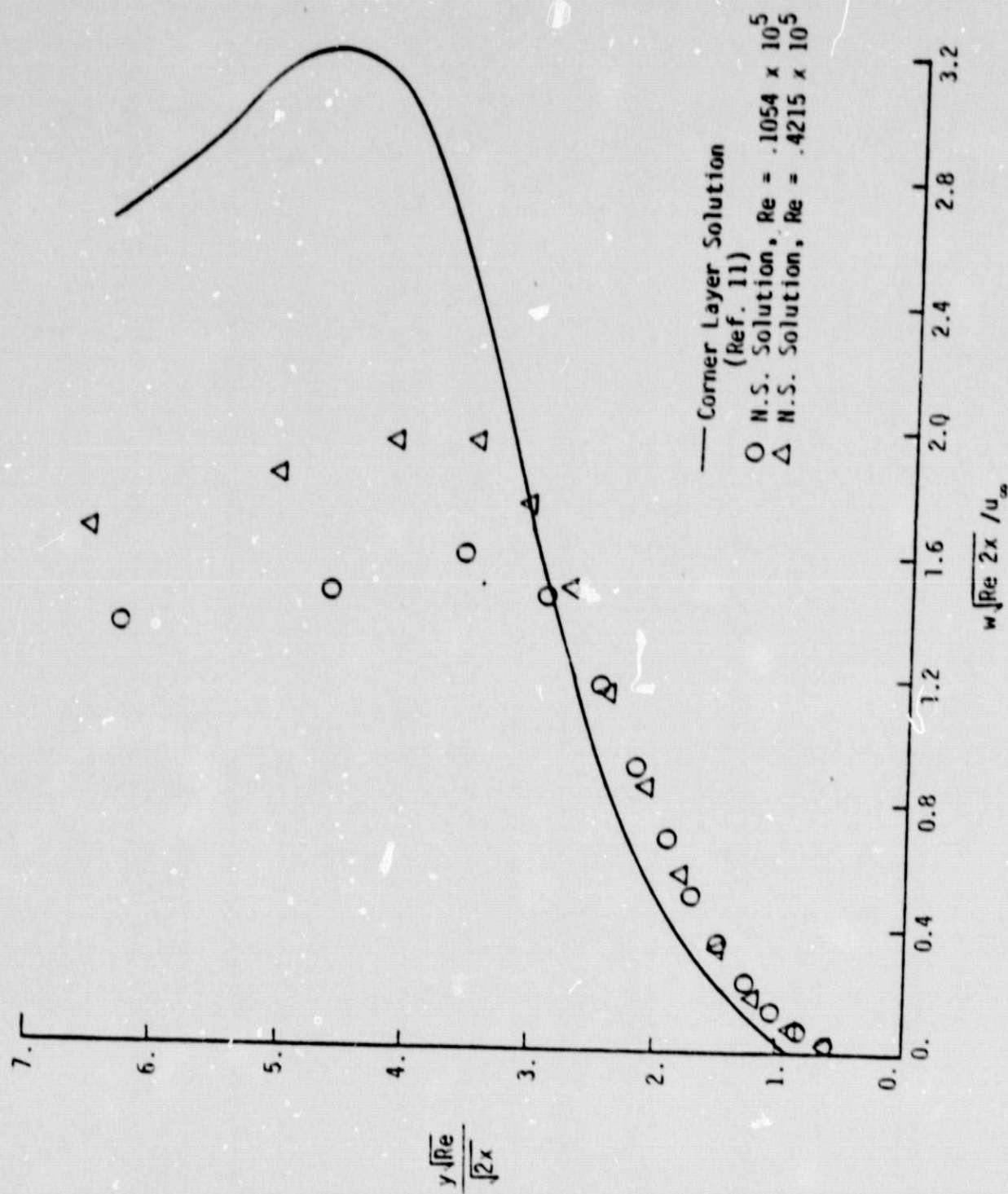


Figure 10. Crossflow velocity along diagonal ($M_\infty = 0.95$).

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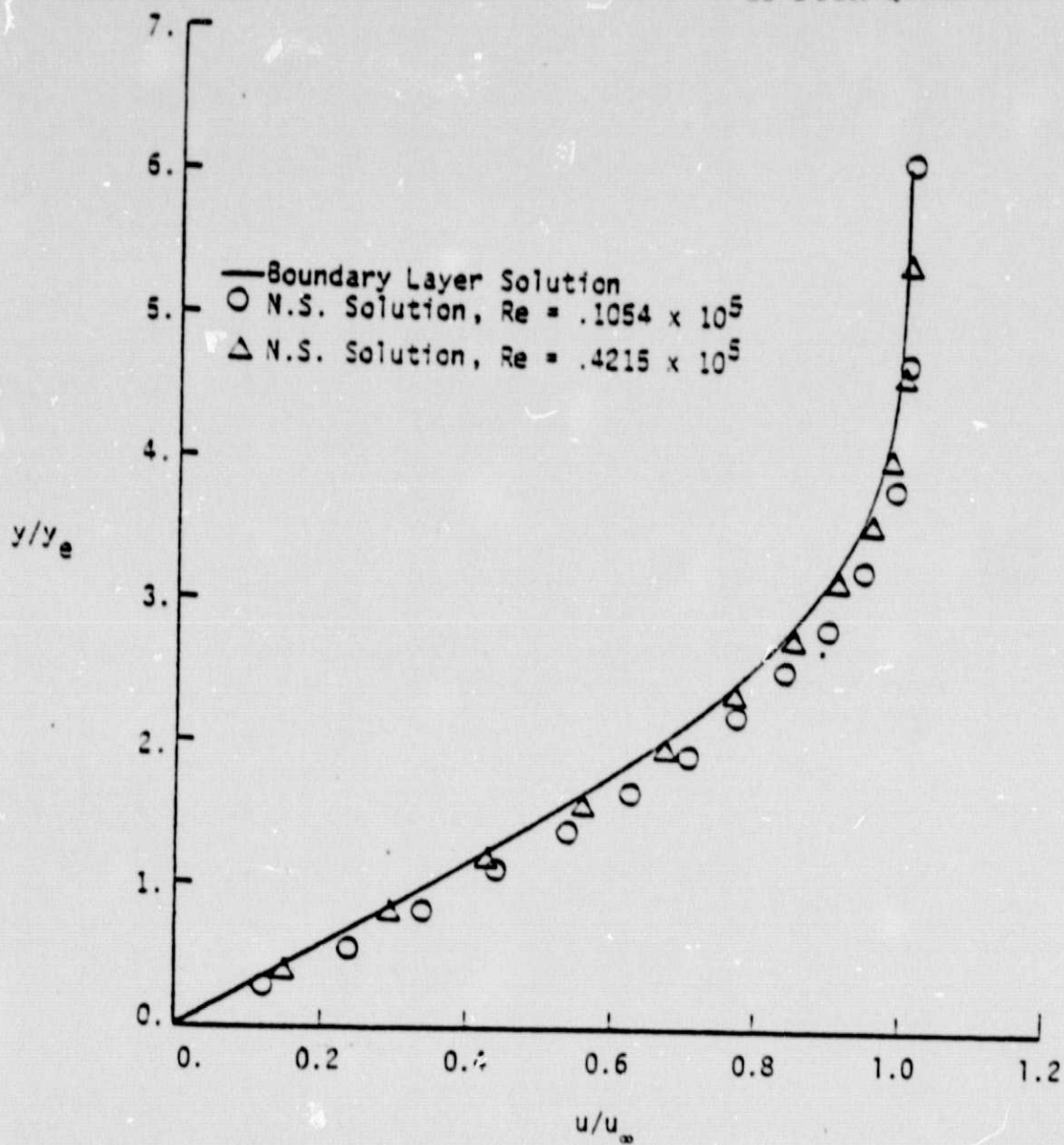


Figure 11. Axial velocity at $z = z_{\max}$ ($M_\infty = 0.95$).

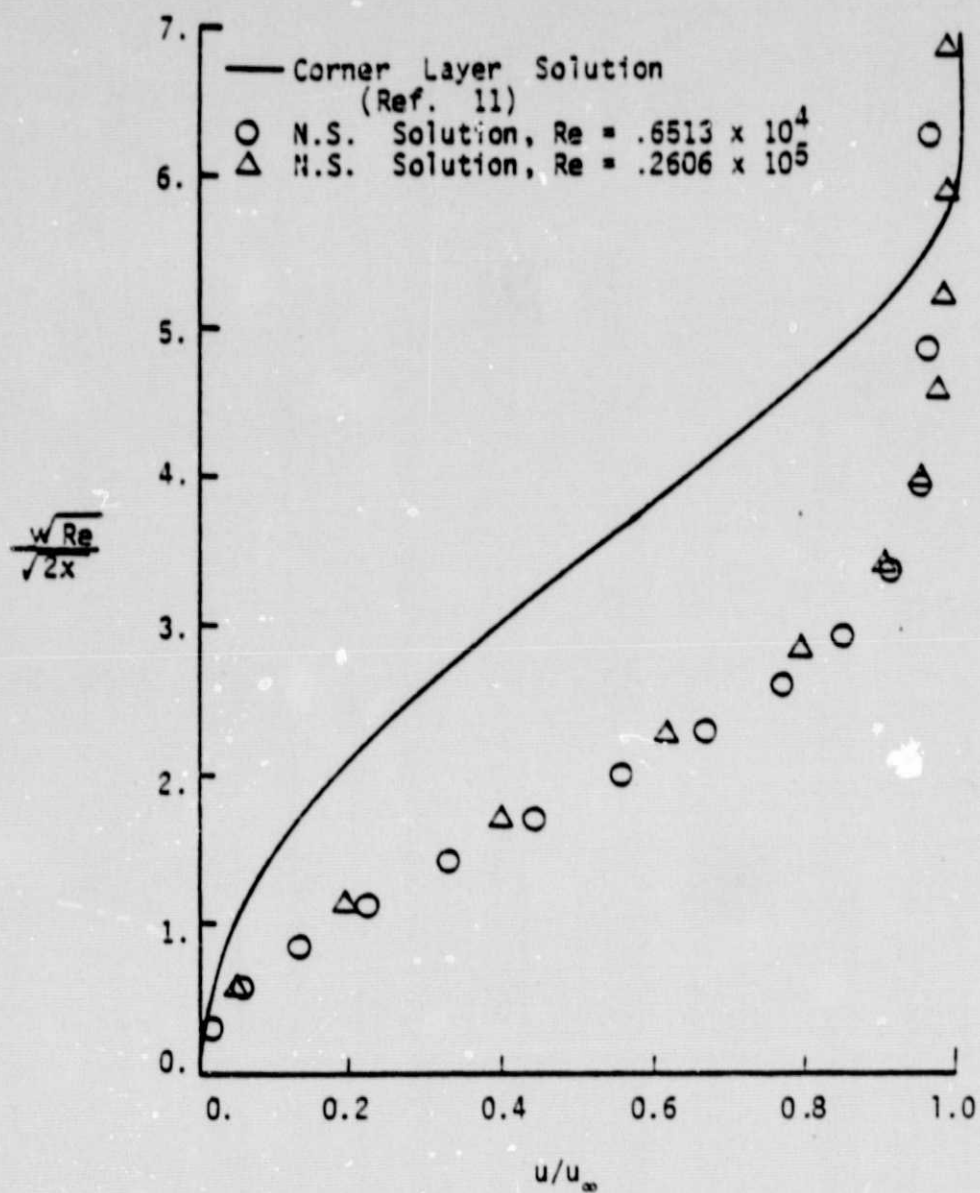


Figure 12. Axial velocity along diagonal ($M_\infty = 1.5$).

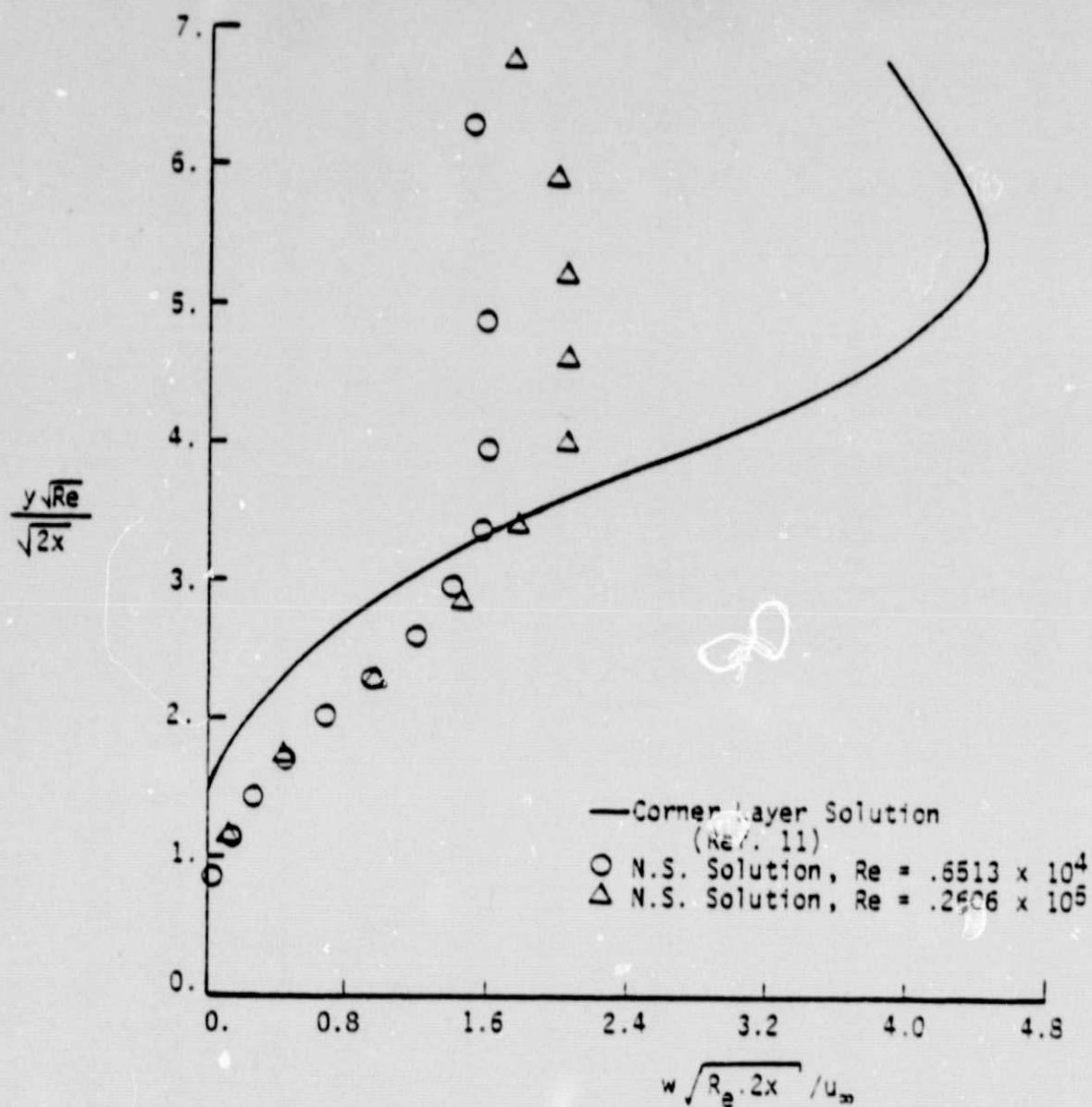


Figure 13. Crossflow velocity along diagonal ($M_\infty = 1.5$).

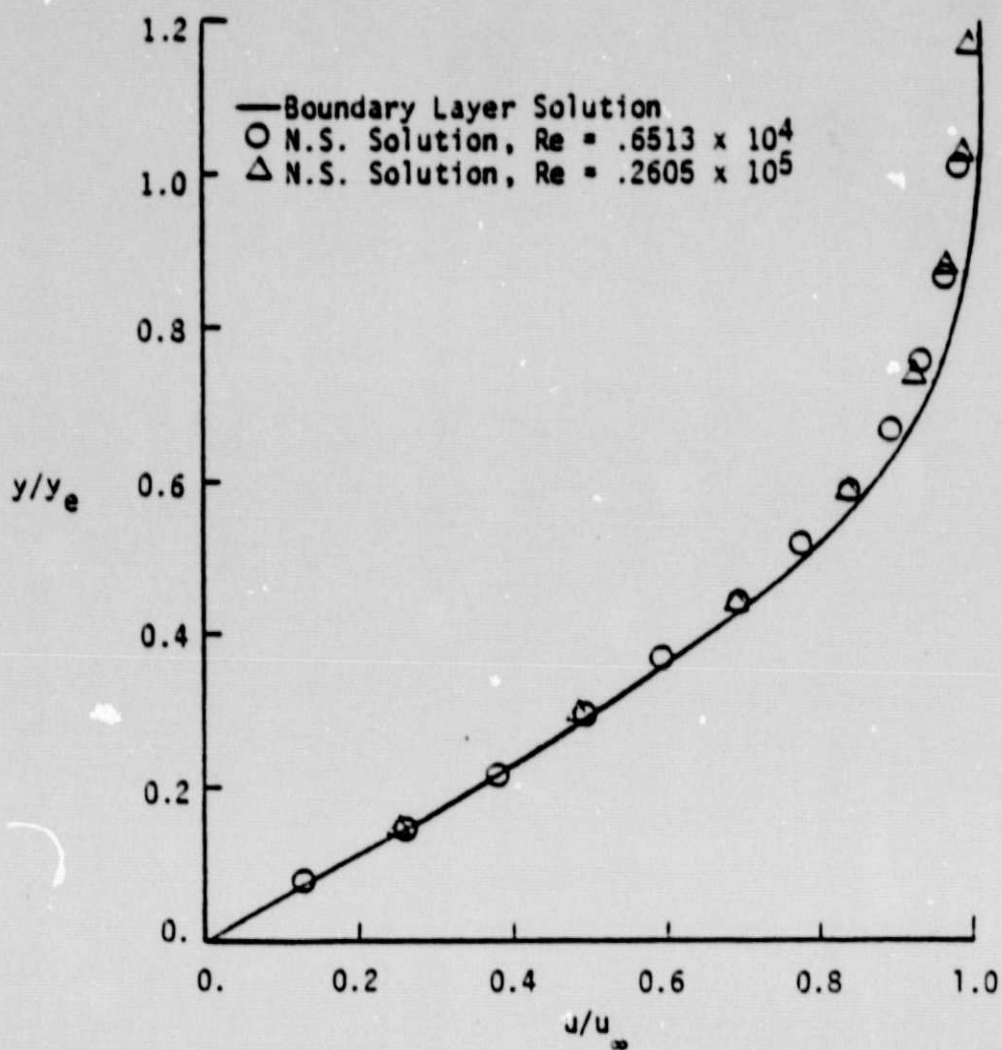


Figure 14. Axial velocity at $z = z_{\max}$ ($M_{\infty} = 1.5$).

CONTOUR FROM .98000 TO 1.19000 CONTOUR INTERVAL OF .000005-01

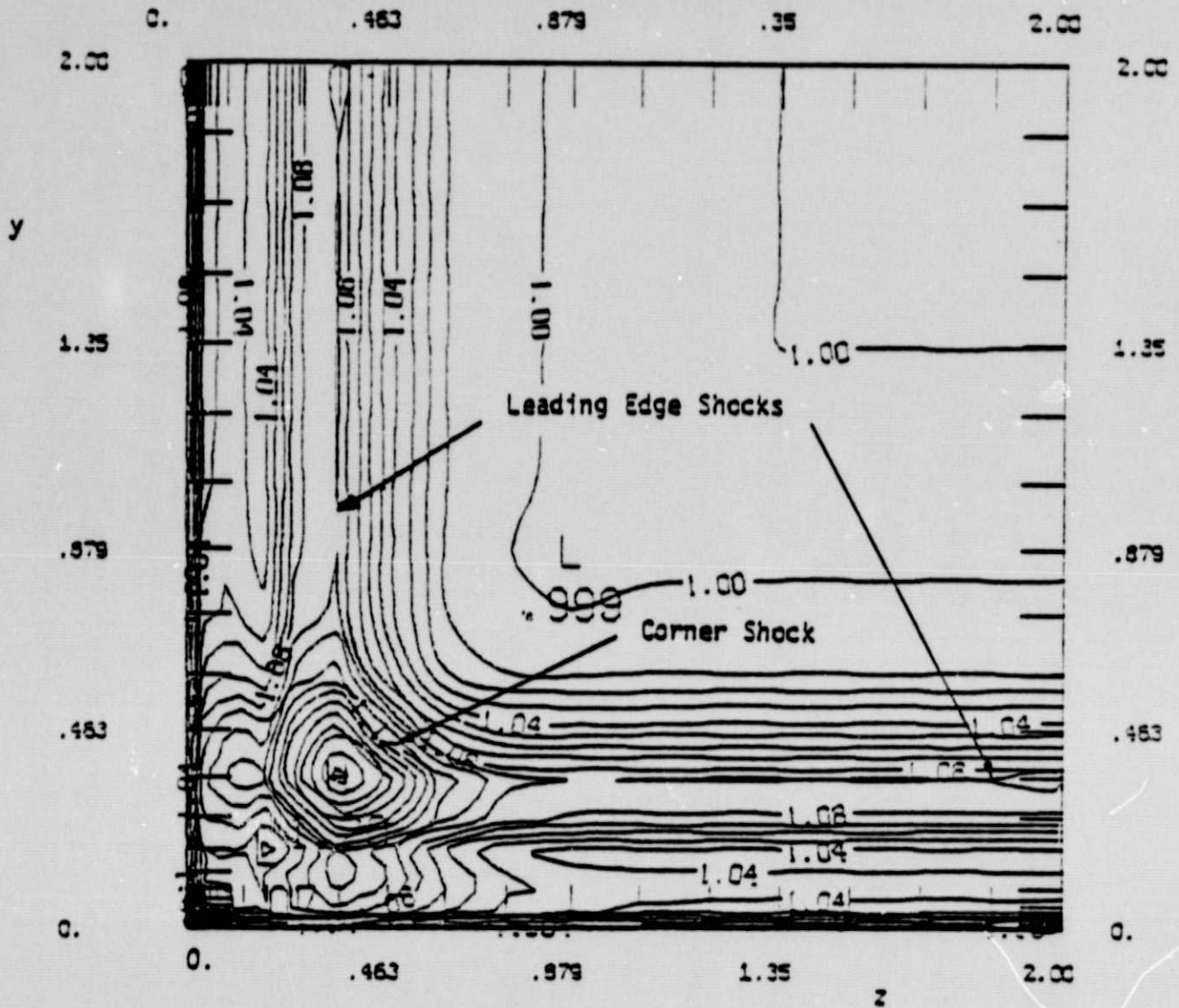


Figure 15. Density contour plot for axial corner ($M_\infty = 1.5$).

APPENDIX

TRANSFORMATION OF GOVERNING EQUATIONS

The set of governing equations (eqs. 1 to 7) are written in the Cartesian coordinate system. These are now transformed to a new set of independent variables (ξ, η, ζ) that form the computational space by using implicit differentiation. The new set of independent variables is completely general and forms a parallelepiped-shaped computational domain.

Let f be a function to be differentiated.

Since

$$\begin{aligned}\xi &= \xi(x, y, z) \\ \eta &= \eta(x, y, z) \\ \zeta &= \zeta(x, y, z)\end{aligned}\tag{A-1}$$

one can write

$$\begin{aligned}f_{\xi} &= x_{\xi} f_x + y_{\xi} f_y + z_{\xi} f_z \\ f_{\eta} &= x_{\eta} f_x + y_{\eta} f_y + z_{\eta} f_z \\ f_{\zeta} &= x_{\zeta} f_x + y_{\zeta} f_y + z_{\zeta} f_z\end{aligned}\tag{A-2}$$

solving for f_x , f_y , f_z by Cramer's rule, one obtains

$$J \frac{\partial f}{\partial x} = f_{\xi} (y_{\eta} z_{\zeta} - y_{\zeta} z_{\eta}) + f_{\eta} (y_{\zeta} z_{\xi} - y_{\xi} z_{\zeta}) + f_{\zeta} (y_{\xi} z_{\eta} - y_{\eta} z_{\xi})\tag{A-3}$$

$$J \frac{\partial f}{\partial y} = f_{\xi} (x_{\zeta} z_{\eta} - x_{\eta} z_{\zeta}) + f_{\eta} (x_{\xi} z_{\zeta} - x_{\zeta} z_{\xi}) + f_{\zeta} (x_{\eta} z_{\xi} - x_{\xi} z_{\eta})\tag{A-4}$$

$$J \frac{\partial f}{\partial z} = f_{\xi} (x_{\eta} y_{\zeta} - x_{\zeta} y_{\eta}) + f_{\eta} (x_{\zeta} y_{\xi} - x_{\xi} y_{\zeta}) + f_{\zeta} (x_{\xi} y_{\eta} - x_{\eta} y_{\xi})\tag{A-5}$$

where

$$J = x_{\xi}(y_{\eta}z_{\zeta} - y_{\zeta}z_{\eta}) + x_{\eta}(y_{\zeta}z_{\xi} - y_{\xi}z_{\zeta}) + x_{\zeta}(y_{\xi}z_{\eta} - y_{\eta}z_{\xi}) \quad (A-6)$$

is the Jacobian of transformation. Rearranging the terms, the first derivatives of f can be rewritten as

$$\frac{\partial f}{\partial x} = \beta_{11}f_{\xi} + \beta_{12}f_{\eta} + \beta_{13}f_{\zeta} \quad (A-7)$$

$$\frac{\partial f}{\partial y} = \beta_{21}f_{\xi} + \beta_{22}f_{\eta} + \beta_{23}f_{\zeta} \quad (A-8)$$

$$\frac{\partial f}{\partial z} = \beta_{31}f_{\xi} + \beta_{32}f_{\eta} + \beta_{33}f_{\zeta} \quad (A-9)$$

where

$$\begin{aligned} \beta_{11} &= (y_{\eta}z_{\zeta} - y_{\zeta}z_{\eta})/J \\ \beta_{12} &= (y_{\zeta}z_{\xi} - y_{\xi}z_{\zeta})/J \\ \beta_{13} &= (y_{\xi}z_{\eta} - y_{\eta}z_{\xi})/J \\ \beta_{21} &= (x_{\zeta}z_{\eta} - x_{\eta}z_{\zeta})/J \\ \beta_{22} &= (x_{\xi}z_{\zeta} - x_{\zeta}z_{\xi})/J \\ \beta_{23} &= (x_{\eta}z_{\xi} - x_{\xi}z_{\eta})/J \\ \beta_{31} &= (x_{\eta}y_{\zeta} - x_{\zeta}y_{\eta})/J \\ \beta_{32} &= (x_{\zeta}y_{\xi} - x_{\xi}y_{\zeta})/J \\ \beta_{33} &= (x_{\xi}y_{\eta} - x_{\eta}y_{\xi})/J \end{aligned} \quad (A-10)$$

Using equations (A-7) to (A-9), equation (1) can be transformed to

$$\begin{aligned} \frac{\partial u}{\partial t} + \beta_{11}F_{\xi} + \beta_{21}G_{\xi} + \beta_{31}H_{\xi} + \beta_{12}F_{\eta} + \beta_{22}G_{\eta} + \beta_{32}H_{\eta} \\ + \beta_{13}F_{\zeta} + \beta_{23}G_{\zeta} + \beta_{33}H_{\zeta} = 0 \end{aligned} \quad (A-11)$$

The shear terms in F, G, H (eqs. 3 to 4) must be transformed to the ξ , η , ζ system also. These are given below:

$$\begin{aligned}
 \tau_{xx} &= 2\mu \left[\beta_{11}u_{\xi} + \beta_{12}u_{\eta} + \beta_{13}u_{\zeta} - \frac{\Delta}{3} \right] \\
 \tau_{yy} &= 2\mu \left[\beta_{21}v_{\xi} + \beta_{22}v_{\eta} + \beta_{23}v_{\zeta} - \frac{\Delta}{3} \right] \\
 \tau_{zz} &= 2\mu \left[\beta_{31}w_{\xi} + \beta_{32}w_{\eta} + \beta_{33}w_{\zeta} - \frac{\Delta}{3} \right] \\
 \tau_{xy} &= \tau_{yx} = \mu \left[\begin{aligned} &\beta_{11}v_{\xi} + \beta_{12}v_{\eta} + \beta_{13}v_{\zeta} \\ &+ \beta_{21}u_{\xi} + \beta_{22}u_{\eta} + \beta_{23}u_{\zeta} \end{aligned} \right] \\
 \tau_{xz} &= \tau_{zx} = \mu \left[\begin{aligned} &\beta_{31}u_{\xi} + \beta_{32}u_{\eta} + \beta_{33}u_{\zeta} \\ &+ \beta_{11}w_{\xi} + \beta_{12}w_{\eta} + \beta_{13}w_{\zeta} \end{aligned} \right] \\
 \tau_{yz} &= \tau_{zy} = \mu \left[\begin{aligned} &\beta_{21}w_{\xi} + \beta_{22}w_{\eta} + \beta_{23}w_{\zeta} \\ &+ \beta_{31}v_{\xi} + \beta_{32}v_{\eta} + \beta_{33}v_{\zeta} \end{aligned} \right] \\
 q_x &= -k \left[\beta_{11}T_{\xi} + \beta_{12}T_{\eta} + \beta_{13}T_{\zeta} \right] \\
 q_y &= -k \left[\beta_{21}T_{\xi} + \beta_{22}T_{\eta} + \beta_{23}T_{\zeta} \right] \\
 q_z &= -k \left[\beta_{31}T_{\xi} + \beta_{32}T_{\eta} + \beta_{33}T_{\zeta} \right]
 \end{aligned} \tag{A-12}$$

$$\begin{aligned}
 \Delta &= \beta_{11}u_{\xi} + \beta_{12}u_{\eta} + \beta_{13}u_{\zeta} \\
 &\quad + \beta_{21}v_{\xi} + \beta_{22}v_{\eta} + \beta_{23}v_{\zeta} \\
 &\quad + \beta_{31}w_{\xi} + \beta_{32}w_{\eta} + \beta_{33}w_{\zeta}
 \end{aligned} \tag{A-13}$$

Nondimensionalization Scheme

The equations presented so far are dimensional. However, it is generally more convenient to work with a nondimensional form of the equations, especially since important parameters such as the Reynolds number explicitly show up in the nondimensional form. The dependent and independent variables are nondimensionalized according to the following scheme:

$$\begin{aligned}
u &= \frac{u^*}{u_r} , & v &= \frac{v^*}{u_r} , & w &= \frac{w^*}{u_r} \\
p &= \frac{p^*}{p_r} , & T &= \frac{T^*}{T_r} , & \rho &= \frac{\rho^*}{\rho_r} \\
t &= \frac{t^*}{(L_r/u_r)} , & \mu &= \frac{\mu^*}{\mu_r} , & e &= \frac{e^*}{u_r^2} \\
x &= \frac{x^*}{L_r} , & y &= \frac{y^*}{L_r} , & z &= \frac{z^*}{L_r}
\end{aligned} \tag{A-14}$$

where * superscript refers to a dimensional quantity. Note that this superscript has not been used in the equations presented so far in the discussion to avoid the cumbersome notation. However, it has to be made clear that the equations up to equation (A-13) in the Appendix and up to equation (7) in the main text are in the dimensional form. The remainder of the equations are in nondimensional form. Letting

$$Re = \frac{\rho_r u_r L_r}{\mu_r} \tag{A-15}$$

be the Reynolds number, the nondimensional forms of equations (2a) to (2d) are as follows:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \end{bmatrix} \tag{A-16a}$$

$$F = \begin{bmatrix} \rho u \\ \rho u^2 - \frac{1}{Re} \tau_{xx} + C_1 p \\ \rho uv - \frac{1}{Re} \tau_{xy} \\ \rho uw - \frac{1}{Re} \tau_{zx} \\ \rho ue - \frac{1}{Re} u \tau_{xx} - \frac{1}{Re} v \tau_{yx} - \frac{1}{Re} w \tau_{zx} + C_1 u p \\ - \frac{C_2}{Re} \frac{u}{\sigma} \frac{\partial T}{\partial x} \end{bmatrix} \tag{A-16b}$$

$$G = \begin{bmatrix} \rho v \\ \rho uv - \frac{1}{R_e} \tau_{xy} \\ \rho v^2 - \frac{1}{R_e} \tau_{yy} + C_1 p \\ \rho vw - \frac{1}{R_e} \tau_{yz} \\ \rho ve - \frac{1}{R_e} u \tau_{yx} - \frac{1}{R_e} v \tau_{yy} - \frac{1}{R_e} w \tau_{zy} + C_1 vp - \frac{C_2}{R_e} \frac{\mu}{\sigma} \frac{\partial T}{\partial y} \end{bmatrix} \quad (A-16c)$$

$$H = \begin{bmatrix} \rho w \\ \rho uw - \frac{1}{R_e} \tau_{xz} \\ \rho vw - \frac{1}{R_e} \tau_{yz} \\ \rho w^2 - \frac{1}{R_e} \tau_{zz} + C_1 p \\ \rho we - \frac{1}{R_e} u \tau_{xz} - \frac{v}{R_e} \tau_{yz} - \frac{1}{R_e} w \tau_{zz} + C_1 wp - \frac{C_2}{R_e} \frac{\mu}{\sigma} \frac{\partial T}{\partial z} \end{bmatrix} \quad (A-16d)$$

where

$$\begin{aligned} C_1 &= \frac{P_r}{\sigma_r u_r^2}, & C_2 &= \frac{C_p T_r}{u_r^2} \\ C_3 &= \frac{C_v T_r}{u_r^2}, & C_4 &= \frac{\rho_r T_r C_p}{P_r} \end{aligned} \quad (A-17)$$

Also, we have

$$e = C_3 T + \frac{u^2 + v^2 + w^2}{2} \quad (A-18)$$

$$p = \left(\frac{\gamma-1}{\gamma} \right) C_4 \rho T \quad (A-19)$$

To summarize, the governing equations in nondimensional form are the same as equation (A-1) where the vectors U , F , G , and H are given by equations

(A-16a) to (A-16d). The shear stress terms in nondimensional form are identical to equations (A-12) to (A-13), provided a nondimensional form of variables is used in the definition of these quantities.

Two types of nondimensional schemes are optional in the computer code:

- (1) The first scheme is based on flat plate variables and uses the following reference quantities

$$\begin{aligned} u_r &= u_\infty, & p_r &= p_\infty, & T_r &= T_\infty \\ \rho_r &= \rho_\infty, & L_r &= 1, & u_r &= u(T_\infty) \end{aligned}$$

This is most suitable for flows that have straight flat regions at leading edge.

- (2) The second scheme is based on Van Dyke's variables. Here, the following reference quantities are used

$$\begin{aligned} u_r &= u_\infty, & p_r &= \rho_\infty u_\infty^2, & T_r &= \frac{u_\infty^2}{C_v} \\ \rho_r &= \rho_\infty, & L_r &= 1, & u_r &= u(u_\infty^2/C_v) \end{aligned}$$

This set of variables is most suitable for blunt body flows with bow shock ahead of the body. Any other nondimensionalization scheme can be easily incorporated into the code by suitably modifying the constants C_1 , C_2 , C_3 , C_4 defined by equation (A-17).

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