FINITE ELEMENT MESH CONFIGURATIONS USING ISOENERGETICS AND EQUALIZED ENERGY LEVELS

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SUMMARY

The concept of equalizing energy levels was shown to be a viable additional criterion in assisting the analyst in laying out finite element grids according to the isoenergetic discretization technique. In addition it is clear that similar problems specifically with respect to mesh refinement in piecewise approximation theory are being researched.

Further, common criteria in both areas are being developed in an effort to cope with the question of discretization for improved piecewise approximations.

INTRODUCTION

One of the most fundamental decisions which a finite element analyst must make is how to discretize the continuum. Research on optimum mesh configurations has been primarily based on the minimization of the potential energy functional with respect to both the nodal displacements and nodal co-ordinates [refs. 1,2,3,4,5]. This leads to the following set of non-linear equations

$$[k]\,\{\Delta\} - \{F\} = 0 \tag{1}$$

and

$$\tfrac{1}{2}\,\{\Delta\}^T \frac{\partial [k]}{\partial x_j}\,\{\Delta\} - \frac{\partial \{F\}^T}{\partial x_j}\,\{\Delta\} = 0 \tag{2}$$

where $\{\Delta\}$ is in general a column vector of unknown values of the displacement field and its derivatives at the nodal points

$[k]$ is the stiffness matrix

$\{F\}$ is a column vector of nodal forces

and x_j is the co-ordinate of node j

A solution of these equations will yield the best possible approximate solution for a given number of elements and initial topology. However, the computational effort in solving these equations is excessive and there is no guarantee that the global minimum has been achieved. As a result two alternative approaches have been employed.

One procedure is to approximately satisfy equation (2), that is until the left hand side is less than some prescribed tolerance ε [refs. 4,5]. This still involves the calculation of the derivatives of [k] and {F} with respect to the nodal co-ordinates.

The other approach is to develop a set of guidelines such that "near optimum" grids can be obtained [refs. 1,2,3,6,7] without explicitly dealing with equation (2). The fundamental concept in this method is based upon the observation that optimal grids align themselves along lines of constant strain energy density - so-called isoenergetics [refs. 2,8]. These contours provide the stress analyst with a picture of the location of stress concentrations, relative density of elements to be allocated to various regions and proper element orientation with respect to the critical stress gradients. This latter aspect is of major importance for the so-called simplex elements which are available in almost all finite element software packages and are used in complex non-linear analyses to reduce the computational costs.

The primary objective of this paper is to present an interactive isoenergetic discretization technique which incorporates the concept of equalizing energy levels as an additional criterion in assisting the analyst in laying out finite element grids. In addition, a mesh refinement approach employing a similar criterion in the piecewise approximate solutions of differential equations is presented. This suggests a possible refinement strategy in finite element analysis.

ISOENERGETIC DISCRETIZATION PROCEDURE

This procedure for generating efficient mesh configurations incorporates not only the necessary topological considerations but also the response characteristics of the problem. The steps involved are as follows:

Step (1) An initial course grid is sketched on a digitizer or produced by automatic mesh generation schemes whichever is appropriate.

Step (2) The grid is analyzed and the strain energy density values are calculated at the nodes and/or the integrating points if applicable. The number of contours to be plotted is selected by the user based upon the maximum energy density differences. Subsequently, the isoenergetics and initial grid are superimposed or displayed separately on the CRT screen.

Step (3) Given this display of information the stress analyst can

(a) Modify the initial grid such that the element gradation reflects the density of isoenergetics and element orientation, in the case of triangles, is directed along lines orthogonal to the contours, that is, in the direction of the maximum strain energy density gradient.

or (b) Select an arbitrary number of isoenergetic contours within which a new mesh can be defined consistent with the above characteristics of the isoenergetics.

Both these operations can be done at a display terminal or with a hard copy of the isoenergetics on a digitizing table.

Clearly, steps (2) and (3) can be repeated until two successive configurations, whose modification usually only involves a shifting of the isoenergetics, provide little or no improvement in the strain energy content.

EQUALIZING ENERGY LEVELS

One of the most interesting observations during the study of optimal grid configurations was that for a class of one dimensional problems the energy content in all the elements was equal [refs. 3,1]. This notion of equal energy levels for optimal meshes was arrived at independently by Prager [ref. 9]. Subsequently Masur (private communication) examined bars under varying axial forces and beams subjected to lateral loads, both with arbitrary variations of the cross section. He concluded that it appears to be futile to search for a universally valid optimality criterion in terms of the average energy in the entire element.

However, for all the various one and two dimensional problems studied, there was a very definite trend for the total energy content between the isoenergetic contour levels in the high strain gradient regions of the optimal grid to be equalized. Figure 1 shows clearly that for the special case of a linearly varying tapered bar subjected to a constant load P the optimized grid yields an equal amount of energy in each element. This is not exactly true for the parabolic taper; however there is clearly a tendency for the optimal element energy contents to be equalized relative to the unoptimized grids. Figures 2 and 3 demonstrate the same general trend, but in these two dimensional plane stress examples rather than dealing with a specific element the energy content between contours is calculated and then compared to the other bands of elements or contour levels of elements. The calculation of the energy content between contours is simply found by summing the product of the element strain energy density and the corresponding volume for each element within the particular contour. The results show that there is a general trend to equalizing the energy content in each contour level of the optimal grid relative to the unoptimized mesh.

It should be noted that this equalization effect is most noticeable in the immediate vicinity of the high strain gradients both for the one and two dimensional problems.

This result has proven extremely useful in providing an additional criterion for evaluating the effectiveness of the already established isoenergetic discretization procedure.

Several examples were re-examined after using the above mentioned isoenergetic discretization procedure to see if the near optimum grids had relatively equalized energy levels. A typical result is shown in figure 4 which clearly indicates the above trend. Consequently, this observation was introduced into the discretization procedure as an additional indicator along with the shifting of the isoenergetic contours and the variation in the total strain energy content to assist the analyst in establishing when further mesh modifications are required.

SIMILARITIES IN APPROXIMATION THEORY

In reviewing the mathematical literature dealing with splines having variable knots the following interesting results were found.

In a paper by De Boor [ref. 10] on the topic of good approximations by splines with variable knots it is suggested that in approximating a function f by elements of S_N^k, the N knots t_1, t_2 ... t_N should be chosen such that

$$\int_{t_i}^{t_{i+1}} | f^k (r) |^{1/k} dr$$

is approximately constant as a function of i, where

f is the function being approximated or a solution of a boundary value problem

S_N^k is a spline of order k (or, degree < k) with N knots

This concept has been tried by Dodson [ref. 11] in a scheme for the adaptive solution of ordinary differential equations. The procedure is as follows: using a current piecewise polynomial approximation of order less than k to the solution f, a piecewise constant approximation g to $f^{(k)}$ is assumed. Then a new knot is selected so as to equalize

$$\int | g(r) |^{1/k} dr$$

over the subintervals.

Subsequently Sewell [ref. 12] extended Dodson's results to piecewise polynomial approximations in two dimensions. In particular Dodson gave an algorithm for the automatic partitioning of an interval which provided the basic ideas for an automatic triangulation algorithm proposed by Sewell. From an error bound theorem developed in this work it is recommended that for a good approximation the integral

$$\iint \max_{i+j=k} \left| D_x^{\,i} D_y^{\,j} f \right|^{\sigma} dA$$

should be distributed as evenly as possible among the grid of triangles, where

$$\sigma = \frac{2}{k+2}$$

Throughout the presentation a polynomial of degree less than k is assumed in each triangle; consequently an estimate of the above integrand must be made. This is accomplished by approximating each k^{th} derivative within a triangle by a piecewise constant function G_j.

As a result the following elemental product

$$G_j^{\,\sigma} \; A_j$$

should be distributed as evenly as possible among the elements having areas A_j. In order to achieve this a continual mesh refinement process was carried out by subdividing those elements with the highest values of $G_j^{\,\sigma} A_j$. This clearly ensures the necessary grid refinement near the high gradients or singularities of the solution or function being approximated.

Referring back to the notion of equalizing energy levels, a similar criterion has been suggested in this paper for mesh modification; specifically the energy within each contour should be distributed as evenly as possible between successive contour levels in the vicinity of the high gradients of the solution field; that is,

$$U_1 \simeq U_2 \simeq U_3 \ldots$$

where $$U_m = \sum_j U^0_{m\,j} \; V_{m\,j}$$

in which $U^0_{m\,j}$ is the piecewise constant strain energy density function of element j in level m

$V_{m\,j}$ is the volume of element j in level m

and U_m is the total energy in contour level m

An important point to note here is that the above criterion is concerned with mesh modification rather than mesh refinement which Dodson and Sewell presented.

Further, from the research studied to date this criterion is best applied to contour levels rather than an examination of the individual elements. This suggests that future mesh modification and refinement schemes should consider bands of elements to be repositioned rather than individual finite elements.

CONCLUSIONS

The concept of equalizing energy levels was shown to be a viable additional criterion in assisting the analyst in laying out finite element grids according to the isoenergetic discretization technique. In addition it is clear that similar problems specifically with respect to mesh refinement in piecewise approximation theory are being researched.

Further, common criteria in both areas are being developed in an effort to cope with the question of discretization for improved piecewise approximations.

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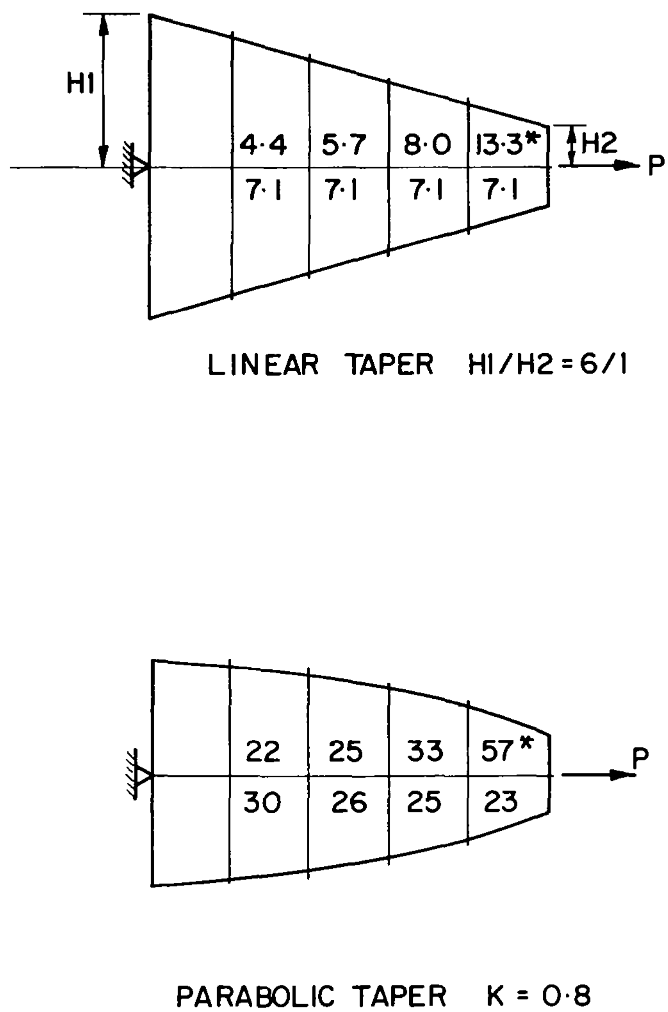

* UNOPTIMIZED STRAIN ENERGY LEVEL VALUES

Figure 1.- Strain energy levels for 2 one-dimensional problems.

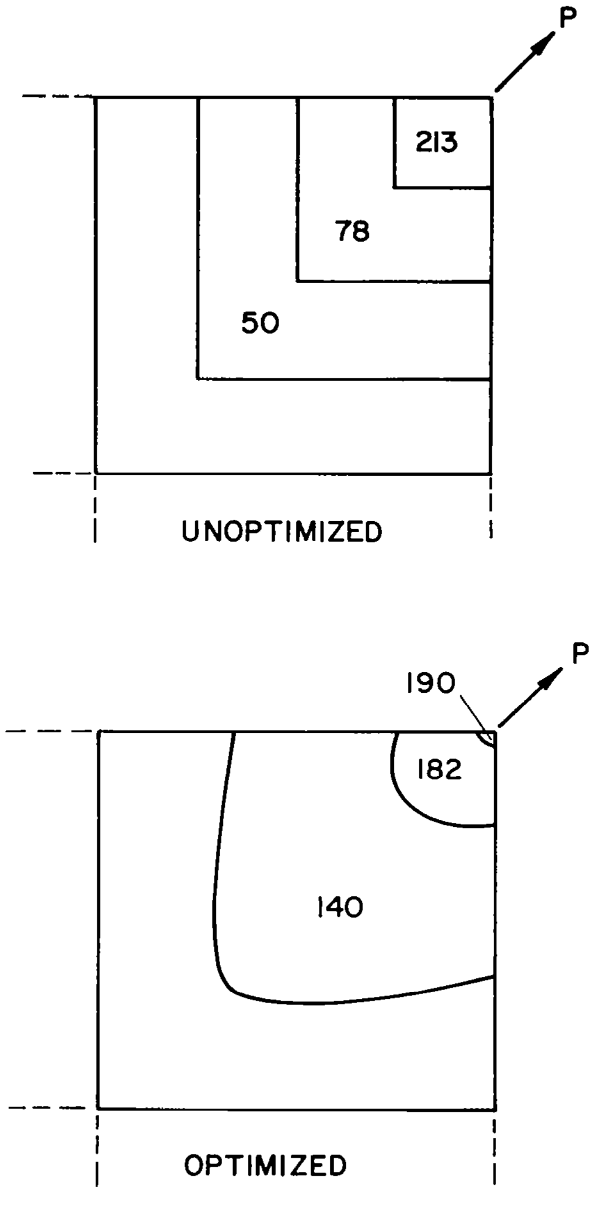

Figure 2.- Strain energy levels for a two-dimensional corner load proglem.

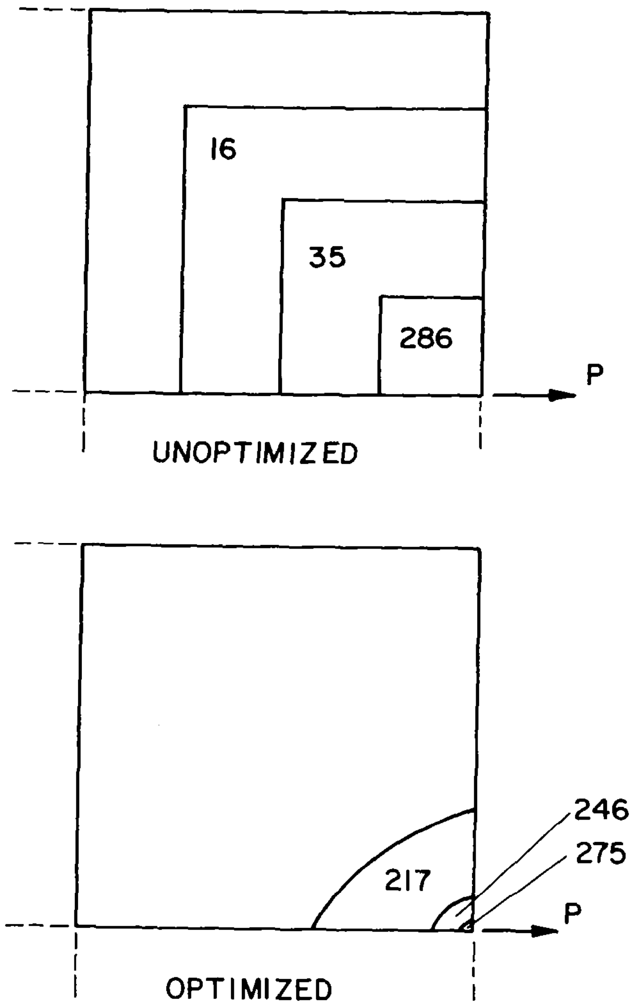

Figure 3.- Strain energy levels for a two-dimensional mid-edge load problem.

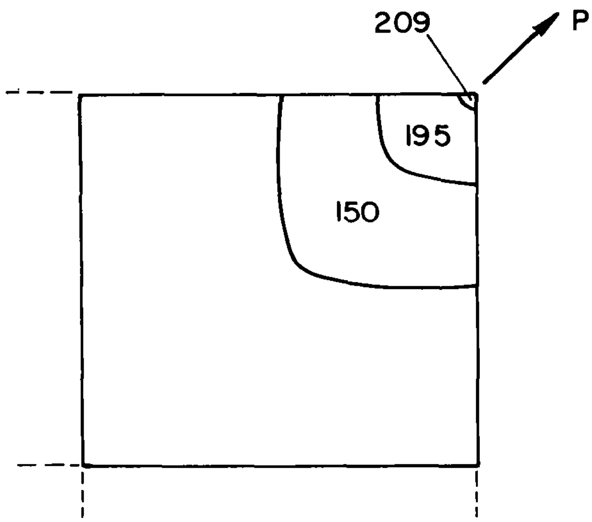

Figure 4.- Strain energy levels in a near optimized grid using isoenergetics.