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NAVIER-STOKES CALCULATIONS WITH A COUPLED STRONGLY IMPLICIT METHOD PART II: SPLINE SOLUTIONS*

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Abstract

The coupled strongly implicit method described in Part 1 of this study is combined with a deferred-corrector spline solver for the vorticity-stream function form of the Navier-Stokes equation. Solutions for cavity, channel and cylinder flows are obtained with the fourth-order spline 4 procedure. The strongly coupled spline corrector method converges as rapidly as the finite difference calculations of Part 1 and also allows for arbitrary large time increments for the Reynolds numbers considered, ($R_e \leq 1000$). In some cases fourth-order smoothing or filtering is required in order to suppress high frequency oscillations.

1. INTRODUCTION

In recent papers[(1-3)] the present authors have formulated various higher-order collocation techniques based on polynomial interpolation or Hermitian discretization procedures. Three-point formulations leading to fourth-order and sixth-order methods were considered. In order to obtain equal accuracy, these higher-order methods require fewer mesh points when compared with second-order accurate finite-difference techniques. This can mean a reduction in computer storage and time. In particular, the fourth-order accurate spline 4 method[(1)] generally requires one-quarter

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the number of points, in each coordinate direction, compared to finite-difference solutions of comparable accuracy.

In addition to improvements in accuracy, these methods have certain desirable properties. The discrete equations remain block-tridiagonal in character. In view of the Hermitian formulation, where the derivatives are treated as unknowns, the application of derivative boundary conditions is somewhat more straightforward, and with non-uniform grids the deterioration in accuracy is less severe than with conventional finite-difference methods. However, the resulting system of equations is more complex than those of the usual second-order accurate finite-difference procedures. For a single differential equation, finite-differences lead to a scalar tridiagonal system, whereas the higher order techniques are block tridiagonal; specifically, 2x2 blocks are required for the fourth order spline methods and 3x3 blocks for the sixth order Hermite systems. Consequently, for a given mesh the storage requirements become larger. The solution procedure therefore requires more operations and is more time consuming than the scalar algorithm usually applied for the tridiagonal finite-difference equations. However, for equal accuracy the spline methods require fewer points and therefore are still more efficient.

In a recent study[4] some of the complexities associated with computer storage and the general spline solution procedure have been eliminated. This was accomplished by "uncoupling" the higher-order correction from the coupled lower-order ("finite-difference") system of equations. This deferred-corrector method can therefore result in a significant reduction in computer storage. A 2x2 block tridiagonal system is reduced to two scalar tridiagonal systems; 3x3 blocks reduce to a tridiagonal and a 2x2 block tridiagonal system. In particular, the method can be programmed into a subroutine package to be used with existing finite-difference codes in order to render higher-order accuracy to the final solutions. This simplicity is achieved by treating the higher-order accurate terms as explicit correctors to a modified finite-difference equation. This correction is evaluated in such a manner that the resulting procedure is consistent and unconditionally stable.[4] Related non-iterative methods of deferred-correction have been discussed by several authors for ordinary differential equations.[5] If these procedures are extended so as to achieve full convergence, for ODE's or for PDE's in a temporal or spatial marching procedure, an instability can result. The analysis of reference (4) corrects this deficiency. Finally, it is noteworthy that the Khosla-Rubin difference method,[6] which provides an explicit central-difference corrector to an implicit upwind-difference scheme, is in fact a lower-order unconditionally stable adaptation of the deferred-corrector method.

It should be noted that by uncoupling the spline or Hermite corrector and thereby reducing the size of the matrix blocks, it is now possible to obtain higher-order solutions and still maintain a considerable degree of coupling between the dependent variables in the governing systems of equations. In effect, the equations are coupled solely through the implicit or finite-difference-like terms. This is particularly important for the Navier-Stokes calculations, as the rate of convergence is increased notably when the equations are coupled. The rate of convergence does not appear to be as sensitive to the uncoupling of the higher-order spline terms. In this way we have been able to develop a coupled solver for the vorticity (ω)-stream function (ψ) description of the Navier-Stokes equations. Only (2x2) blocks must be inverted in order to obtain fourth-order accuracy. This procedure has been used with ADI, predictor-corrector, direct solver and strongly implicit two-dimensional solution techniques.[4] In the present study the coupled strongly implicit method is used exclusively to treat a variety of problems.

In Part 1[7] of the present study the coupled strongly implicit method has been detailed. The importance of implicitly coupling the boundary conditions has been demonstrated. Second-order accurate central-finite-difference solutions have been obtained for (i) the flow in a driven cavity, (ii) the temperature and heat transfer in the cavity, (iii) a channel with a rearward facing step, and (iv) a circular cylinder with a splitter plate. In the second part of the analysis presented here, these geometries are reconsidered with the deferred-corrector spline adaption of the coupled strongly implicit method.

In section 2 the spline collocation equations and analysis are reviewed. In section 3 the deferred-corrector technique of reference 4 is described for both the K-R and spline formulations. In section 4 the coupled strongly implicit spline corrector method is reviewed for the ψ-ω Navier-Stokes equations. Finally, the spline solutions for the problems of Part 1 are presented.

2. Spline 4 Formulation

A variety of fourth and sixth-order spline and hermite procedures have been derived in references 1 and 2. The major advantages of these techniques over conventional three-point finite-difference methods are also discussed in these papers. In the present study we will be concerned only with the spline 4 procedure. The following two- and three-point spline relationships between the variable u_j and its first (ℓ_j) and second derivatives (L_j) in the y-coordinate direction (i fixed) are required for the execution of the iterative procedure.[1,2]

$$L_j = K_j + \frac{1+\sigma^3}{6\sigma(1+\sigma)^2}[K_{j+1}-(1+\sigma)K_j+\sigma K_{j-1}] \quad , \qquad \text{(1a)}$$

$$\begin{aligned} \ell_j &= \frac{h_j}{3}(K_j+.5K_{j-1}) + \frac{u_j-u_{j-1}}{h_j} \\ &= -\frac{h_{j+1}}{3}(K_j+.5K_{j+1}) + \frac{u_{j+1}-u_j}{h_{j+1}} \end{aligned} \qquad \text{(1b)}$$

$$\sigma\ell_{j-1}+2(1+\sigma)\ell_j+\ell_{j+1} = \frac{3}{h_j}\left(\frac{u_{j+1}}{\sigma} + \frac{\sigma^2-1}{\sigma}u_j-\sigma u_{j-1}\right) \qquad \text{(1c)}$$

$$\sigma K_{j+1}+2(1+\sigma)K_j+K_{j-1} = \frac{6}{\sigma h_j^2}[u_{j+1}-(1+\sigma)u_j+\sigma u_{j-1}] \quad , \qquad \text{(1d)}$$

$$\begin{aligned} K_j &= \frac{2}{h_j}(\ell_{j-1}+2\ell_j)-6\frac{u_j-u_{j-1}}{h_j^2} \\ &= \frac{-2}{h_{j+1}}(\ell_{j+1}+2\ell_j)+6\frac{u_{j+1}-u_j}{h_{j+1}^2} \quad , \end{aligned} \qquad \text{(1e)}$$

where $\sigma = h_{j+1}/h_j$ and h_j is the mesh width.

The truncation errors of the derivatives are given as:

$$\ell_j = (u_y)_j + \frac{\sigma(\sigma-1)}{2u} h_j^3 (u^{iv})_j + \frac{1-\sigma+\sigma^2}{18} h_j^4 (u^v)_j. \quad , \quad (2a)$$

$$L_j = (u_{yy})_j + \frac{7}{180} (1+\sigma^2)(\sigma-1) h_j^3 (u^v)_j.$$
$$+ [\frac{\sigma^2}{360} + \frac{(\sigma-1)^2}{1080} (7\sigma^2-2\sigma+7)] h_j^4 (u^{vl})_j \quad . \quad (2b)$$

Similar expressions are obtained for m_i, M_i representing (u_x), (u_{xx}), respectively.

It should be noted that with $\sigma = 1+0(h_j)$, even the variable grid representation leads to fourth-order accuracy. For given values of u_j, a single inversion of a scalar tridiagonal matrix {equation (1d)} is required for the evaluation of ℓ_j and L_j. If u_j is to be evaluated from the solution of a differential equation, the spline 4 procedure leads to a 3x3 block-tridiagonal system for $(u,\ell,K)_j$; this can always be reduced to a 2x2 system with the help of equations (1b). In references 1 and 2, this coupled procedure was used in the solution of a variety of flow problems. In reference 4, the spline 4 method was further simplified such that the 2x2 system for $(u,K)_j$ is reduced to the inversion of two scalar tridiagonal systems for u_j, K_j. This reduces the computer storage considerably and allows the higher-order improvement to be programmed as a simple corrector to a finite-difference code.

It is this deferred-corrector approach that is applied here for the solution of the $(\psi-\omega)$ Navier-Stokes equations. The procedure is completely general and can be extended to any higher-order method. In reference 3 solutions are obtained with the Hermite 6 procedure for turbulent boundary layers. The stability analysis is reported in reference 4.

3. Iterative Solution Procedure: ψ-ω Equations

A general deferred-corrector procedure for several higher-order numerical methods has been described in reference 4, where the necessary stability conditions have also been presented. In this study, we focus specifically on the spline 4 procedure,[1] applied as a deferred-corrector, for the (ψ-ω) form of the Navier-Stokes equations. This approach is unconditionally stable, differs from earlier methods of deferred-correction[5] and represents a natural extension of the second order K-R scheme[6] to higher-order differencing procedures. We are primarily concerned here with the steady-state solution procedure; however, the method is described for the general unsteady equations. For the applications described herein, large time steps ($\Delta \approx 10^6$) are used in all of the calculations.

For the (ψ-ω) Navier-Stokes equations, the general iterative procedure can be written, in the transformed variables (x,y), as follows:

Vorticity Transport Equation:

The difference approximation for the vorticity equation (3)

$$\frac{1}{J}\,\omega_t + (u\omega)_x + (v\omega)_y = \frac{1}{R_e}\,\nabla^2\omega \tag{3}$$

is

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{J\Delta t} + A[\mu_x \frac{(u\omega)_{i+1,j}-(u\omega)_{ij}}{\sigma_1 k_i} + (1-\mu_x)\frac{(u\omega)_{i,j}-(u\omega)_{i-1,j}}{k_i}$$

$$+ \mu_y \frac{(v\omega)_{i,j+1}-(v\omega)_{ij}}{\sigma_2 h_j} + (1-\mu_y)\frac{(v\omega)_{ij}-(v\omega)_{i,j-1}}{h_j}]^{n+1}$$

$$+ \frac{B}{R_e}[\frac{2}{\sigma_1(1+\sigma_1)k_i^2}\{\omega_{i+1,j}-(1+\sigma_1)\omega_{ij}+\sigma_1\omega_{i-1,j}\} + \tag{4a}$$

$$\frac{2}{\sigma_2(1+\sigma_2)h_j^2}\{\omega_{i,j+1}-(1+\sigma_2)\omega_{ij}+\sigma_2\omega_{i,j-1}\}]^{n+1}$$

$$= \varepsilon\,[CRX+XRY + \frac{1}{R_e}(VSX + VSY)]^{n} \quad ,$$

where

$$CRx = A\,[\mu_x \frac{(u\omega)_{i+1,j}-(u\omega)_{ij}}{\sigma_1 k_i} + (1-\mu_x)\frac{(u\omega)_{ij}-(u\omega)_{i-1,j}}{R_i} - \tilde{m}_{ij}], \tag{4b}$$

$$CRy = A\,[\mu_y \frac{(v\omega)_{i,j+1}-(v\omega)_{ij}}{\sigma_2 h_j} + (1-\mu_y)\frac{(v\omega)_{ij}-(v\omega)_{i,j-1}}{h_j} - \tilde{\ell}_{ij}], \tag{4c}$$

$$VSx = \frac{2B}{\sigma_1(1+\sigma_1)k_i^2}\{\omega_{i+1,j}-(1+\sigma_1)\omega_{ij}+\sigma_1\omega_{i-1,j}\}-M_{ij}, \tag{4d}$$

$$VSy = \frac{2B}{\sigma_2(1+\sigma_2)h_j^2}\{\omega_{i,j+1}-(1+\sigma_2)\omega_{ij}+\sigma_2\omega_{i,j-1}\}-L_{ij}, \tag{4e}$$

and

$$\mu_x = \mathrm{sgn}\left(\frac{u_{ij}}{|u_{ij}|}\right), \quad \mu_y = \mathrm{sgn}\left(\frac{v_{ij}}{|v_{ij}|}\right) \tag{4f,g}$$

$\ell, m, \tilde{\ell}, \tilde{m}, L, M$ are the spline approximations to $\omega_y, \omega_x, (v\omega)_y (u\omega)_x, \omega_{yy}$ and ω_{xx}, respectively, see section 2. J is the Jacobian of the mapping function to the (x,y) plane. The transformations will be reviewed for the various problems as they are presented, see Part 1.[7] The constant ε is prescribed as zero or unity; A and B are constants, which are chosen in order that the iterative procedure is unconditionally stable. When complete convergence is achieved, the terms involving A and B cancel, so that with $\varepsilon = 1$, a higher-order numerical solution of the Navier-Stokes equations is recovered.

The convective terms on the left-hand side of equation (4a) are of the form obtained with an upwind finite-difference discretization. For A=B=1 the left-hand side corresponds exactly with upwind differencing for convection and central differences for diffusion. The higher-order terms appear as explicit correctors on the righthand side. If the spline correctors were replaced with only second-order correctors the K-R method[6] is recovered.

This implicit-explicit splitting of the convective terms is not unique, and is prescribed here to provide the appropriate five-point formulas required for the coupled 2x2 algorithm presented in Part I of this study.[7] The novel feature of the present procedure is the introduction of the A, B splitting. This is not found in the usual deferred-corrector methods.[5] The final A,B values are specified from stability considerations. The stability analysis[4] provides the minimum relative weights of implicit convection (A) and diffusion (B) required to achieve unconditional stability. For second-order schemes, equal weighting with A=B=1 is unconditionally stable for $n \to \infty$; i.e., multiple iteration is convergent with the K-R method. On the contrary, in order to achieve higher-order accuracy, A and B must be non-equal and different from unity.[4] For the spline 4 method, A=2 and B=3 are the minimum integer values that lead to an unconditionally stable corrector.

In certain cases it is desirable to use the deferred-corrector in a step-like fashion. An initial first-order accurate solution is obtained with $A=B=1$, $\varepsilon=0$, i.e., upwind-differencing. This can be corrected to second-order accuracy with the K-R corrector. Finally, with $\varepsilon=1$, by selecting appropriate values of A and B, the order of the solution can be further upgraded. For spline 4, $A=2$, $B=3$ and the spline formulas of section 2 are applied. For some other higher-order methods the A,B values are given in reference 4. In this manner, a first-order accurate solution with considerable artificial viscosity is corrected to second-and then fourth-order. Alternatively, one could proceed directly from first to fourth-order or directly from second to higher-order, see reference 4. The higher-order solutions have little numerical diffusion.

Stream Function Equation

The stream function equation (5)

$$\nabla^2\psi = \omega/J \qquad (5)$$

is approximated by

$$\frac{2}{\sigma_1(1+\sigma_1)k_i^{\,2}}[\psi_{i+1,j}-(1+\sigma_1)\psi_{ij}+\sigma_1\psi_{i-1,j}]^{n+1}$$

$$+\frac{2}{\sigma_2(1+\sigma_2)h_j^{\,2}}[\psi_{i,j+1}-(1+\sigma_2)\psi_{ij}+\sigma_2\psi_{i,j-1}]^{n+1} \qquad (6)$$

$$=(1-\varepsilon)\frac{\omega}{J}+\frac{2\varepsilon}{\sigma_1(1+\sigma_1)k_i^{\,2}}[\psi_{i+1j}-(1+\sigma_1)\psi_{ij}+\sigma_1\psi_{i-1,j}]^n$$

$$+\frac{2\varepsilon}{\sigma_2(1+\sigma_2)h_j^{\,2}}[\psi_{i,j+1}-(1+\sigma_2)\psi_{ij}+\sigma_2\psi_{i,j-1}]^n+\varepsilon C\,[\tfrac{\omega}{J}-\tilde{L}_{ij}-\tilde{M}_{ij}]^n$$

$\tilde{L}_{ij}$ and $\tilde{M}_{ij}$ are the spline approximations to ψ_{yy} and ψ_{xx}, respectively. The velocities u and v are related to $\psi, \tilde{L}_{ij}$ and $\tilde{M}_{ij}$ by the the spline relationships given in section 2; the constant C plays a role similar to that of B in (4a) and ε is zero or one as before. For finite-differences C = 1 and for spline 4, $C \leq 1$. For sixth-order methods smaller C values are required.[4]

Computationally this iterative procedure has many advantages. The implicit inversion matrices are always strongly diagonally dominant. The initial spline correctors are obtained from reasonably smooth approximate solutions. In this way, it is possible to suppress spurious and bothersome oscillations in $\tilde{m}_{ij}$, $\tilde{M}_{j}$, etc. arising from the use of arbitrary initial conditions, or inaccurate coarse grid central-difference solutions. The final spline result on the same course grid may be quite reasonable.

One of the major advantages of this approach is improved computational efficiency. With the higher-order deferred-corrector, it is possible to minimize the number of arithmetic operations so that only a marginal increase over the lower-order finite-difference requirement results. The evaluation of the correctors are incorporated in an efficient and independent subroutine. Moreover, the corrector curve fits need not be evaluated every iteration. A systematic analysis to determine the optimum corrector update procedure has not been rigorously considered. It has been found that the number of over-all iterations for convergence is not significantly increased if the corrector and therefore the curve fits are evaluated every two to three iterations. It is through this optimization mechanism for achieving computational efficiency that the present application of higher-order deferred-correction techniques becomes even more attractive. It is interesting that the rate of convergence of this method is comparable to that found in the earlier coupled spline calculations.[1-3]

The higher-order approximations of the convective terms may sometimes lead to an increase in aliasing error and thus non-linear instability.[8] The present iterative procedure will not alleviate this form of instability. In such cases, additional considerations of smoothing and or filtering are required.[8] These will be introduced, as necessary, in the present study in order to insure that the aliasing error remains bounded. The spline 4 solution for the flow in a channel with a backward step is one example in which this mode of instability is found. The introduction of smoothing in the spline curve fits, maintains the over-all accuracy of the numerical scheme but eliminates the non-linear instability.[8] This procedure will be discussed for the channel problem.

4. Boundary conditions

For the spline calculations considered here, higher-order boundary conditions are required for ψ and ω; in addition, appropriate values for the corrector spline curve fits m_{ij}, M_{ij}, etc., must be specified. The various boundary conditions are as follows:

i) At a solid surface the stream function ψ is prescribed; the vorticity is evaluated from the no-slip condition, so that

$$\omega_{i,1} = \frac{3J}{h_2^{\,2}} (\psi_{i,2}-\psi_{i,1}) + J[-.5K_{i2} + \frac{1+\sigma^3}{6\sigma(1+\sigma)^2} \{K_{i,3} - (1+\sigma)K_{i2}+\sigma K_{i,1}\}] \tag{7a}$$

the portions of (7a) containing ψ are coupled directly into the 2x2 algorithm, see Part 1. The higher-order spline correction in (7a) is treated explicitly. This is consistent with the deferred-correction treatment of the equations.

ii) At a far-field boundary, the conditions $\omega_y \to 0$ and $\psi_y \to 1$ are specified. With equations (1) these conditions are of the form:

$$\psi_{iN} = h_N + \psi_{i,N-1} - \frac{h_N^2}{3} (K_{iN} + .5\ K_{i,N-1}) \tag{7b}$$

A similar expression obtains for ω_{iN}. As before, the K correctors are treated explicitly and the ψ terms implicitly.

iii) At the inflow boundary the values of ψ and ω are specified.

iv) At the outflow boundary the conditions $\psi_{xx} = \omega_{xx} = 0$ are imposed, either directly through extrapolation or indirectly by applying the boundary layer form of the governing equations. If extrapolation is used, the boundary values for ψ and ω are coupled implicitly into the solution algorithm. If the boundary-layer equations are assumed at the outflow, the spline 4 deferred-corrector procedure of section 3 is used to couple the solution of these equations to the interior flow. This is carried out by appropriate modification of the algorithm coefficients. There is no upstream influence from the outflow boundary with the boundary layer conditions.

Finally, the spline boundary conditions required for the curve fits of the M's, L's, etc., are obtained by satisfying the governing equations (3,5) at the surface; at the inflow/outflow the derivative conditions $K_{1j} = K_{2j}$ and $K_{Nj} = K_{N-1,j}$, etc., are specified. These conditions on the third derivative are less rigid than the direct specification of K. In many cases this leads to the suppression of high-frequency oscillations in the curve fits.

5. Examples

The various flows examined in Part 1 have been reinvestigated with the more accurate higher-order spline 4 deferred-corrector strongly implicit procedure. In all the cases, solutions are obtained directly with $\Delta t = 10^6$. For the flow in a channel with a backward facing step, the spline deferred correction solution exhibits oscillations that lead to a non-linear instability. This is an aliasing effect and can be eliminated by the introduction of filtering or smoothing of the spline curve fits in the axial directions. This effect will be detailed in a following section.

5.1 Flow in a Driven Cavity

This problem has been considered by many investigators using a variety of numerical techniques. It serves a particularly useful purpose here as the spline 4 solutions have already been obtained with an uncoupled time-dependent method [3] and therefore the effectiveness, stability and accuracy of the present coupled implicit corrector method can be assessed. Buneman's direct solver was applied in previous calculations[3] for the stream function and a predictor- corrector procedure for the vorticity equation was applied. Significantly, the present coupled strongly implicit method converges considerably faster than the uncoupled direct solver/predictor-corrector combination. An exact time comparison of the present procedure with the earlier spline 4 calculations is not possible as different computers were used. The uncoupled solutions typically required about 4 minutes on a CDC 6600 computer[3] for $R_e = 400$ and a 17x17 grid. The coupled strongly implicit deferred-corrector method converges in about 75 secs on an IBM 360/65; this is about 3 to 8 times slower than the 6600. Furthermore, the present analysis allows for a solution of the steady state form of the equations. The final results, for a uniform 17x17 grid, are identical with those presented in reference 3. The number of iterations for the higher-order corrector method is approximately the same as was required for the finite-difference solution of Part 1. Since the spline 4 calculations require only one-sixteenth the number of mesh points (in 2-dimensions) as the equally accurate finite-difference solutions of Part 1, the higher-order spline computations require much less computer time. Furthermore, the deferred-corrector method is relatively simple to implement and the spline corrector can be added as a subroutine to an existing finite-difference.

5.2 Heat Transfer in a Driven Cavity

Once converged velocity profiles are obtained, the solution of the energy equation

$$\frac{1}{J} T_t + uT_x + vT_y = \frac{1}{P_e} \nabla^2 T \qquad , \tag{7}$$

can be determined independently. The spline 4 discrete form of equation (7), utilizing the iterative procedure outlined in section 3, is the same as that for vorticity ω given by equation (4a); however, the non-conservation form of the energy equation is used here.

The thermal boundary conditions and the origin of the prescribed velocity distribution are explained in Part 1. All of the thermal boundary conditions are implicitly coupled into Stone's solution algorithm. The spline boundary conditions are obtained by satisfying equation (7) at the walls of the cavity. The solution for a Peclet number P_e = 50 has been obtained on a 15x15 uniform grid. The heat transfer at the side walls is shown in figures 3,4 where the finite difference solution of Part 1 is also reproduced for comparison purposes. The temperature distribution at the upper and lower walls, as well as the mid-plane, is shown in figure 2. It can be seen that the spline 4 temperature distribution on the upper moving wall is considerably different from that obtained with the finite-difference procedure. Notably, there are no oscillations in the spline solutions.

5.3 Flow in a Channel with a Backward Facing Step

The conformal transformation that maps the step geometry into a straight channel, and the prescribed grid distribution are discussed in Fig. 5. The spline 4 iterative procedure is initiated after 10 to 20 iterations of the central finite-difference calculation. At the outflow boundary, the spline 4 solution to the ψ-ω form of the boundary-layer equations was coupled into the 2x2 strongly implicit algorithm for the Navier-Stokes equations. In order to test the higher-order accuracy of the spline 4 method the fully developed flow in a straight channel was evaluated. The change from second-to fourth-order accuracy was evident.

When the spline corrector was switched on, it was soon observed that the higher-order solution near the outflow boundary acquired a small oscillation. This subsequently developed into an instability. A number of reasons for this instability are possible. The most likely was the sudden transition from the fourth-order accurate representation of $(u\omega)_x$ in the interior to the first- order accurate boundary layer boundary condition at the outflow. In order to make this transition consistent a variety of boundary conditions for the spline curve fits were examined; however, the instability persisted. Subsequently, systematic-numerical experimentation indicated that the source of this instability was closely related to the ψ_{xx} curve fit for the evaluation of v and the curve fit of $(u\omega)_x$ required in the deferred corrector in equation (4a). Based on some previous studies[8] of non-linear instability it was apparent that the root cause was the aliasing error due to the higher-order representation of the convective terms along the channel and the extremely coarse grid. The instability was eliminated by introducing smoothing in the curve fits. For example, after $\tilde{K}_i$ is evaluated, a new $\tilde{K}_i$ is obtained from the three point smoothing formula

$$\tilde{K}_i = \frac{\tilde{K}_{i+1} + (1+\sigma_1)\tilde{K}_i + \sigma_1\tilde{K}_{i-1}}{2(1+\sigma_1)} \tag{8}$$

It can be shown that for a variable grid, formal third-order accuracy of the convective as well as diffusive terms results. For uniform grids the smoothing is fourth-order accurate for ψ,ω. Converged spline 4 solutions for $R_e = 1$, 100 and 1000 were now obtained. The vorticity along the lower and upper walls is shown in figs. 6,7 where the finite-difference solution on the same grid is also depicted.

With the present refined grid, the finite-difference solution is quite accurate. Outside of the recirculation region, the changes in the flow variables are not large and therefore the finite-difference approximation is also reasonable in the fully developed flow. In the recirculation region where the grid is very fine the spline 4 solution shows only a marginal difference. The peak wall vorticity is slightly larger. Also, the point of separation is slightly different from that of the finite-difference calculations. Finally, for the fully developed region the spline 4 results agree with the known exact solution to three decimal places. For $R_e = 1$, the flow separates below the step. The separation point moves towards the corner as R_e is increased.

SUMMARY

A spline 4 deferred corrector strongly implicit method for the ψ-ω form of the Navier-Stokes equation has been investigated. In this iterative approach, both convective and diffusive terms in the governing equations are reordered such that at each level of the iteration a modified finite-difference form of the governing equations is solved. The higher-order spline correction is therefore treated explicitly. This approach has many distinct advantages. For example, (i) the computer program does not involve any significant additional complexity other than would occur for a second-order finite-difference program. The implicit block size remains unchanged. ii) The additional computational effort is associated solely with the evaluation of the explicit spline curve fits. Therefore, the existing finite-difference codes can easily be upgraded to higher-order accuracy. iii) The over-all storage for the fourth-order method is considerably reduced. iv) Unlike the conditional stability of the method of deferred correction (ref. 5), the present technique is unconditionally stable.

Finally, with this coupled strongly implicit procedure, spline 4 "steady-state" solutions (with $\Delta t = 10^6$ have been obtained for a variety of flow problems. It has been found that the number of iterations for the spline 4 deferred corrector procedure is comparable to that required for the finite-difference solutions pre-

sented in part 1. The method of solution converges rapidly when compared with some of the earlier uncoupled methods applied for the solution of the Navier-Stokes equations.

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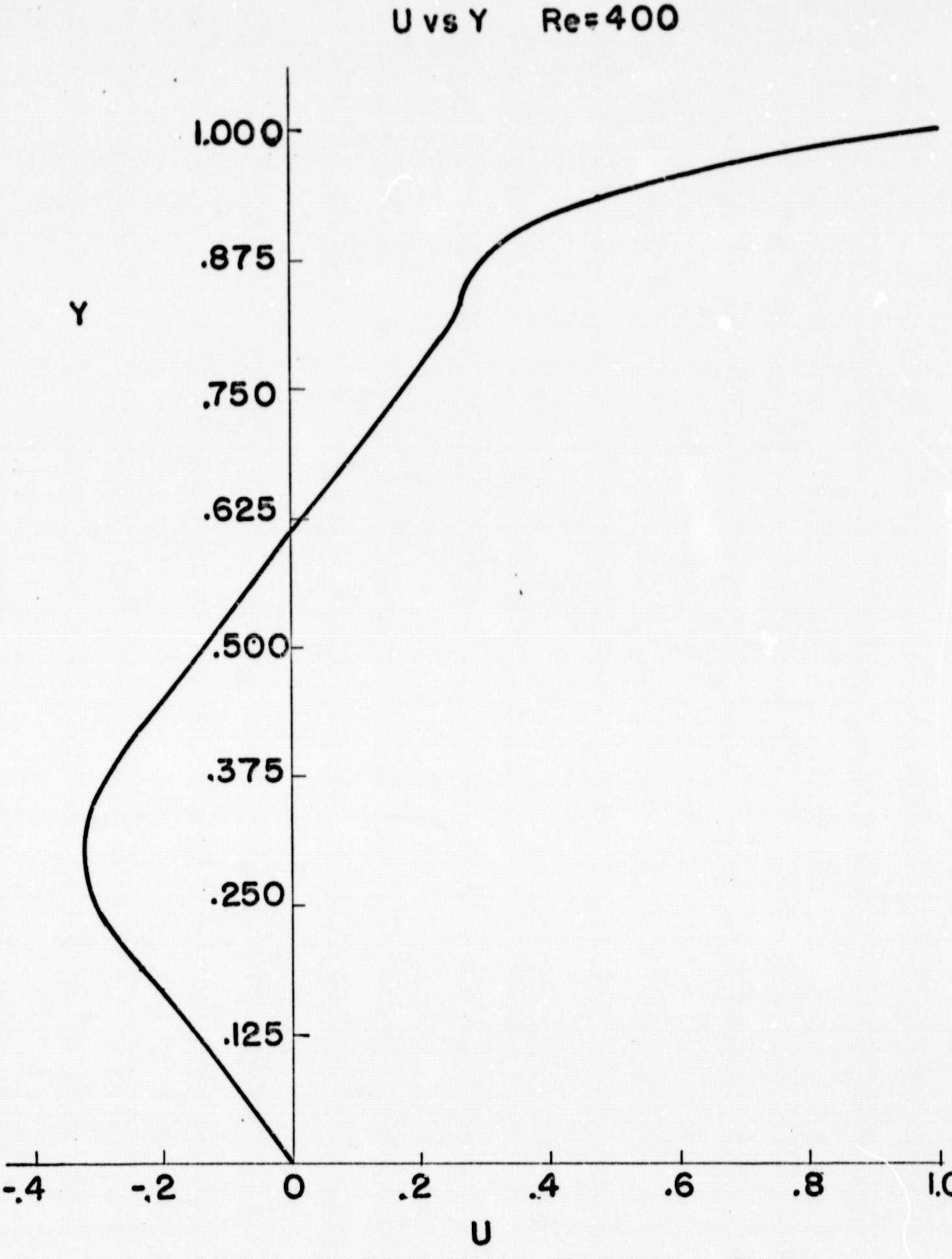


FIG. 1 VELOCITY DISTRIBUTION THROUGH THE VORTEX CENTER

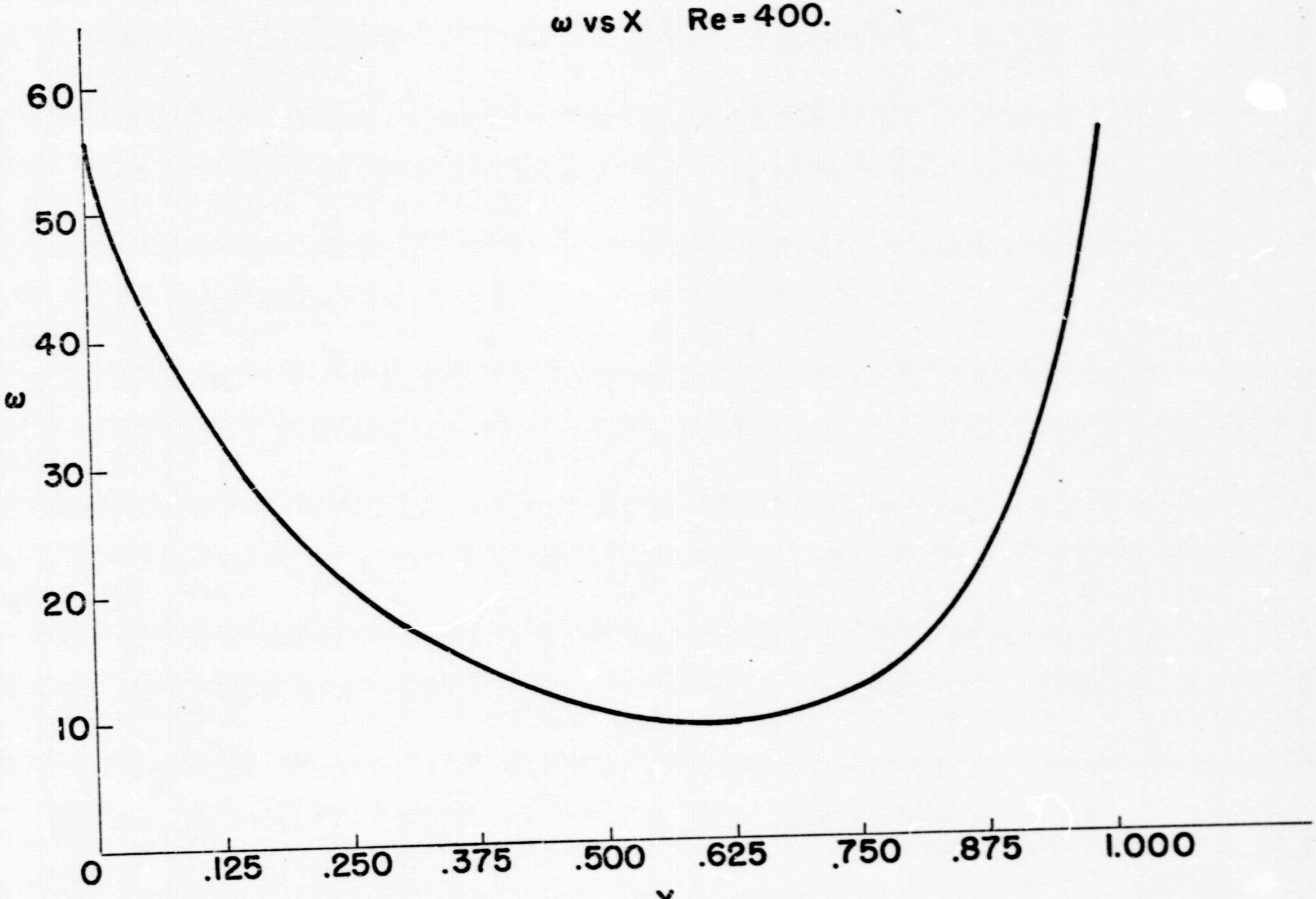


FIG. 2 VORTICITY DISTRIBUTION ON THE MOVING WALL

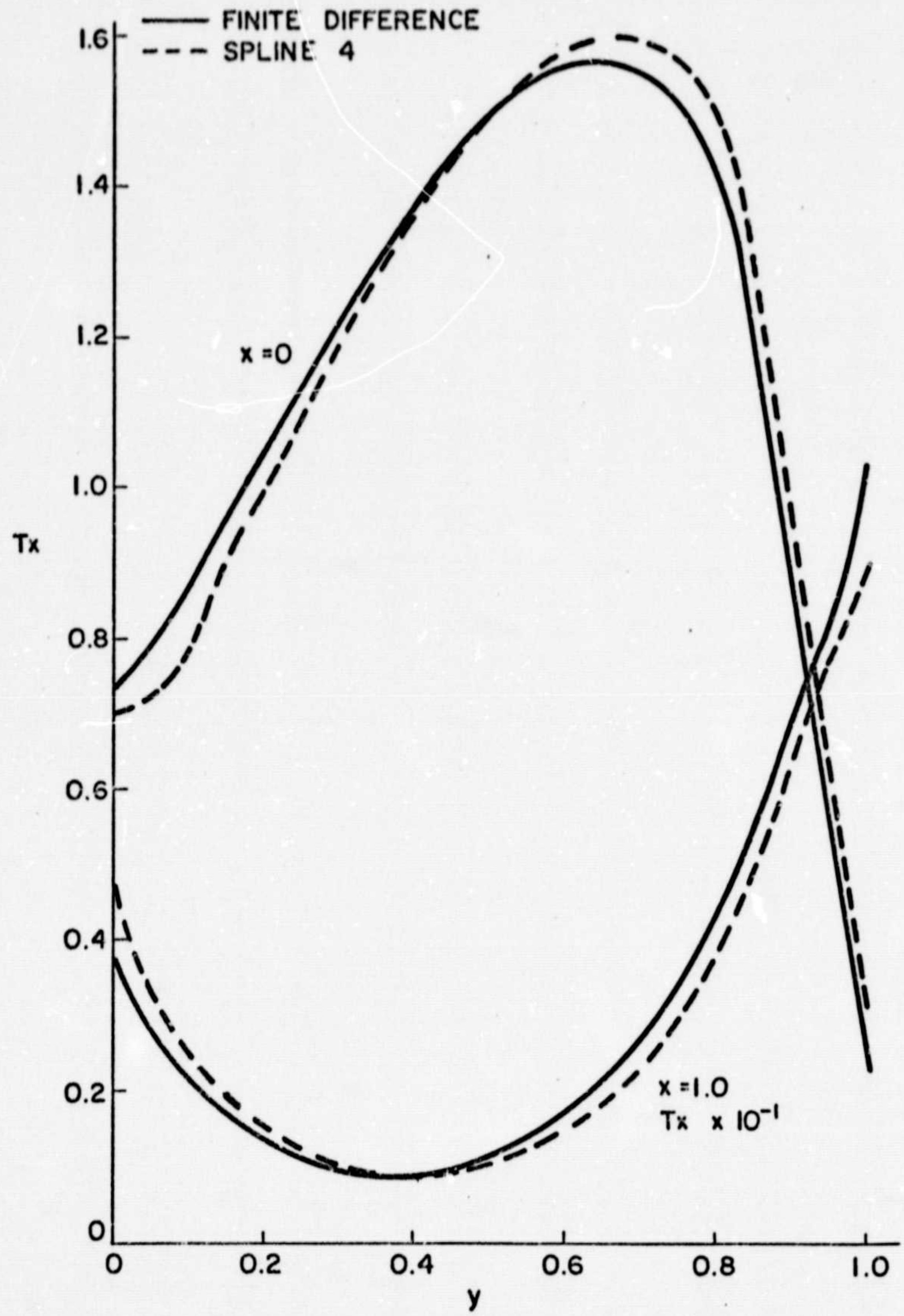


FIG. 3 HEAT TRANSFER ON SIDE WALLS OF CAVITY

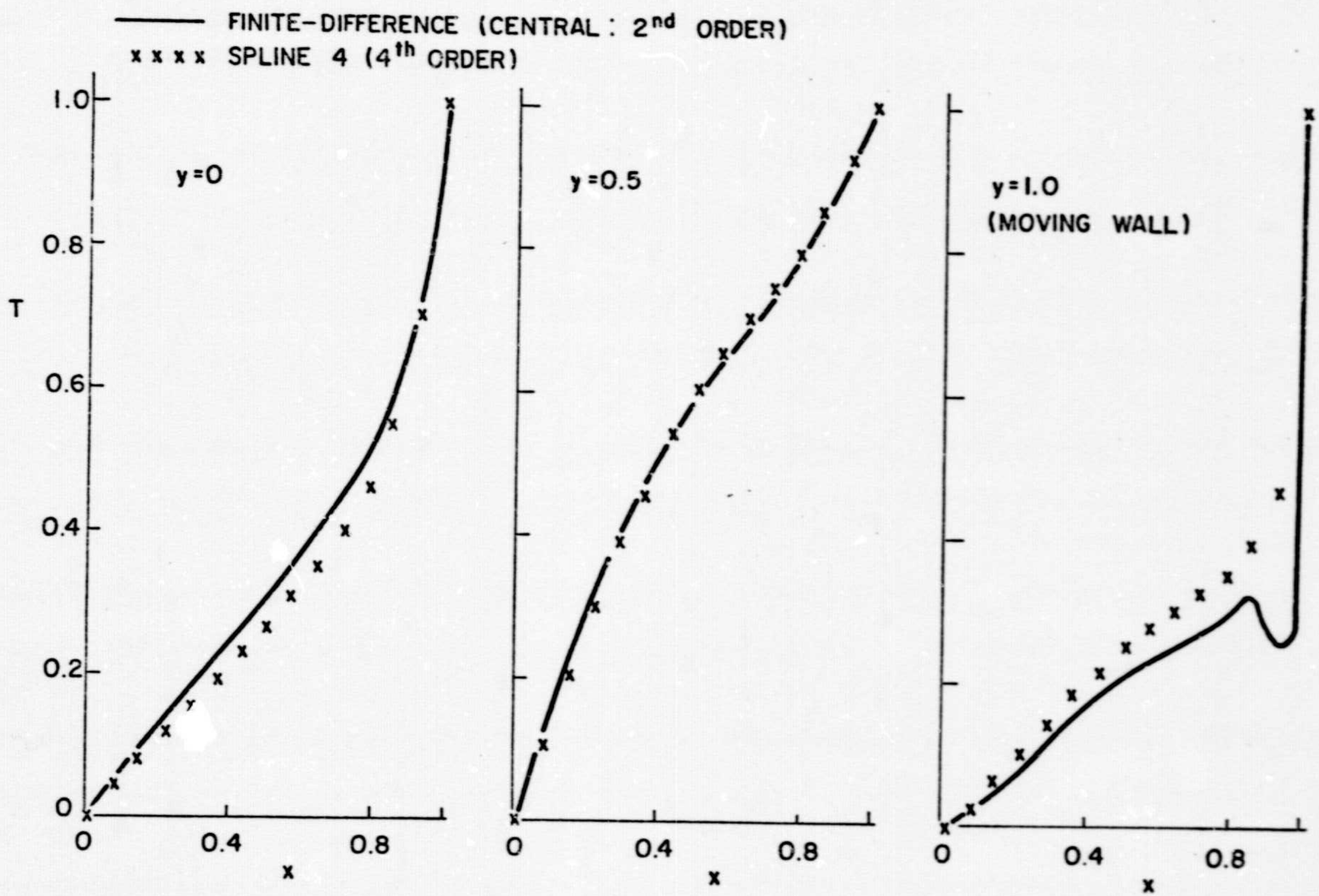
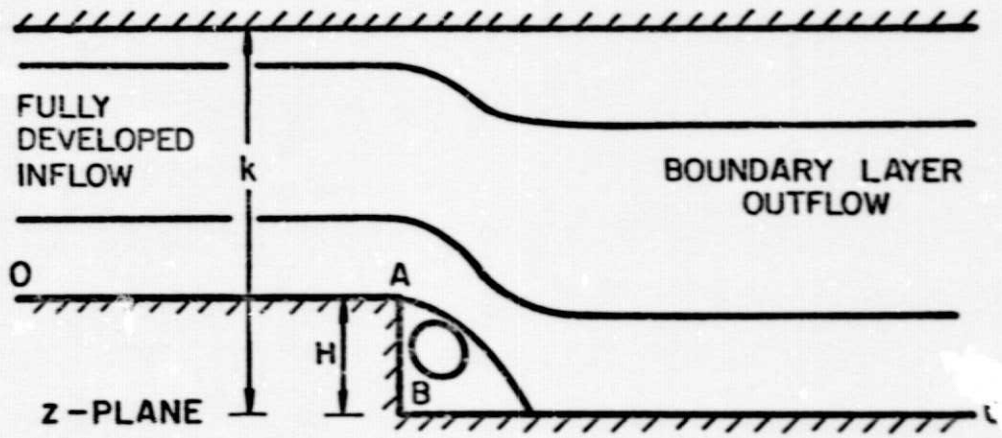


FIG. 4 TEMPERATURE PROFILES FOR DRIVEN CAVITY : PE = 50.



$$z = \frac{k}{\pi}\cosh^{-1}\left(\frac{2w-c-1}{c-1}\right) - \frac{k}{\pi\sqrt{c}}\cosh^{-1}\left(\frac{(c+1)w-2c}{(c-1)w}\right)$$

$$k = c^{1/2}H$$

$$w = e^{\zeta}$$

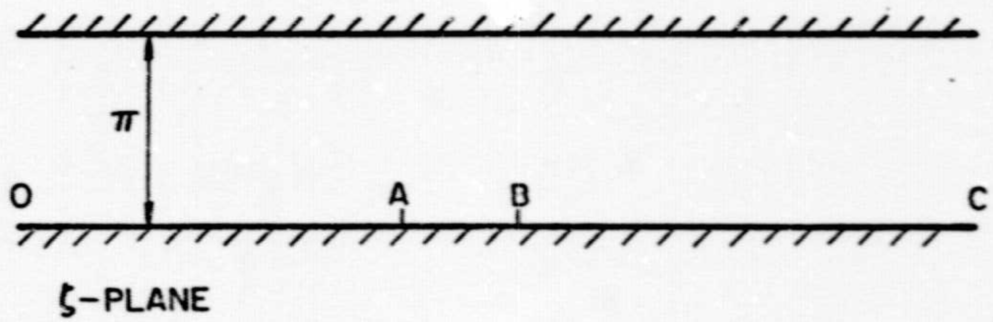


FIG. 5 STEP GEOMETRY IN PHYSICAL AND COMPUTATIONAL PLANES

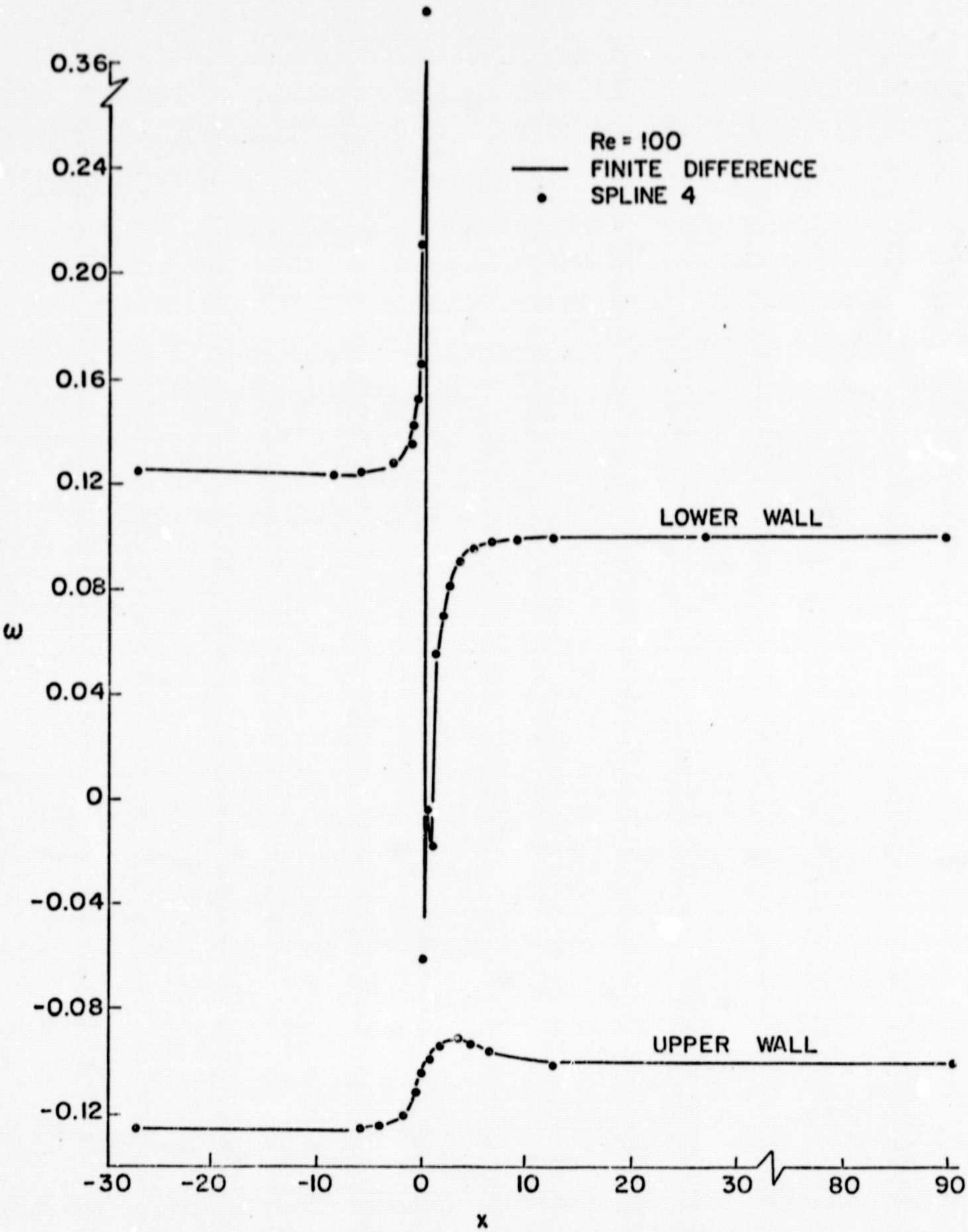


FIG. 6a WALL VORTICITY IN A CHANNEL

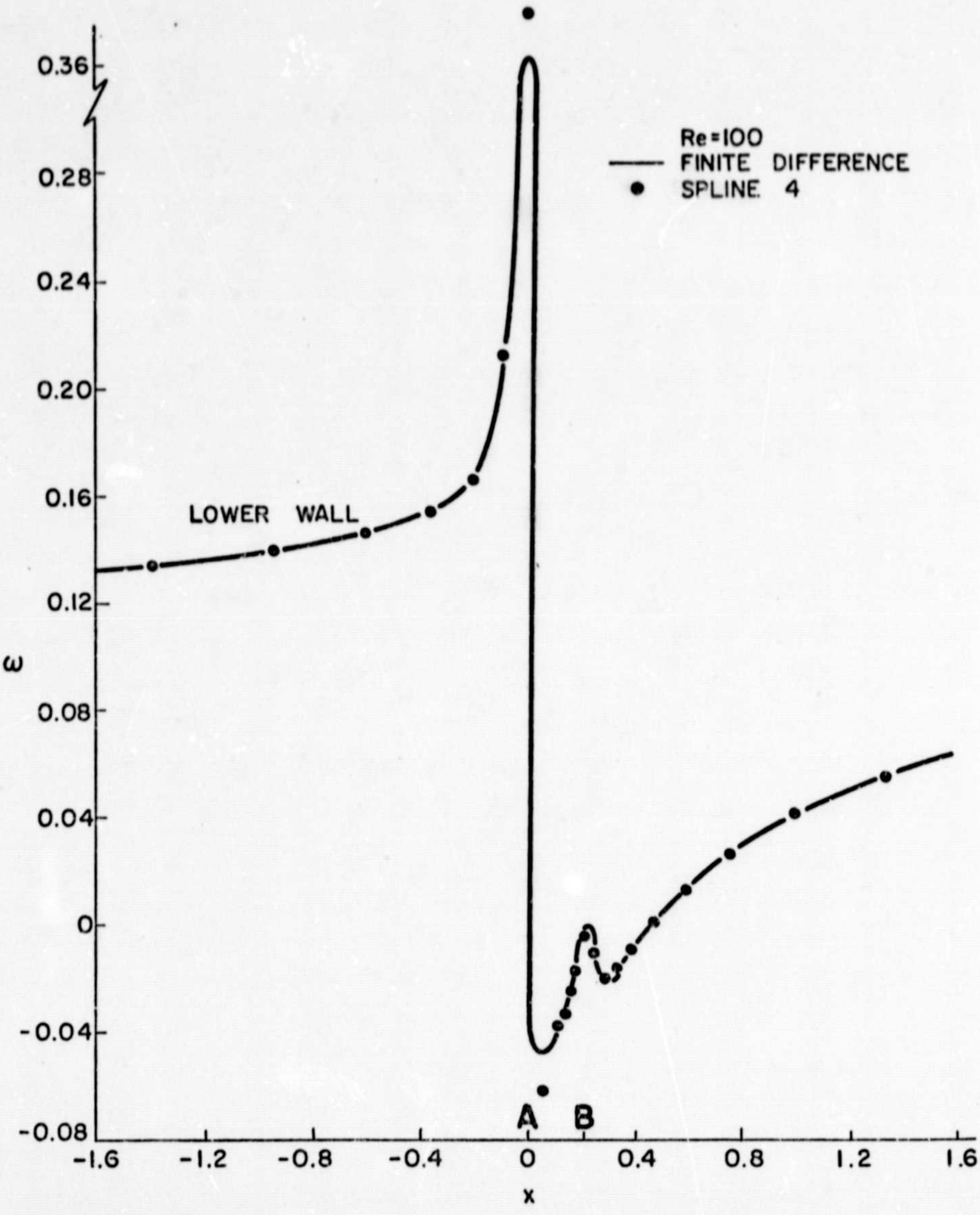


FIG. 6b WALL VORTICITY IN A CHANNEL

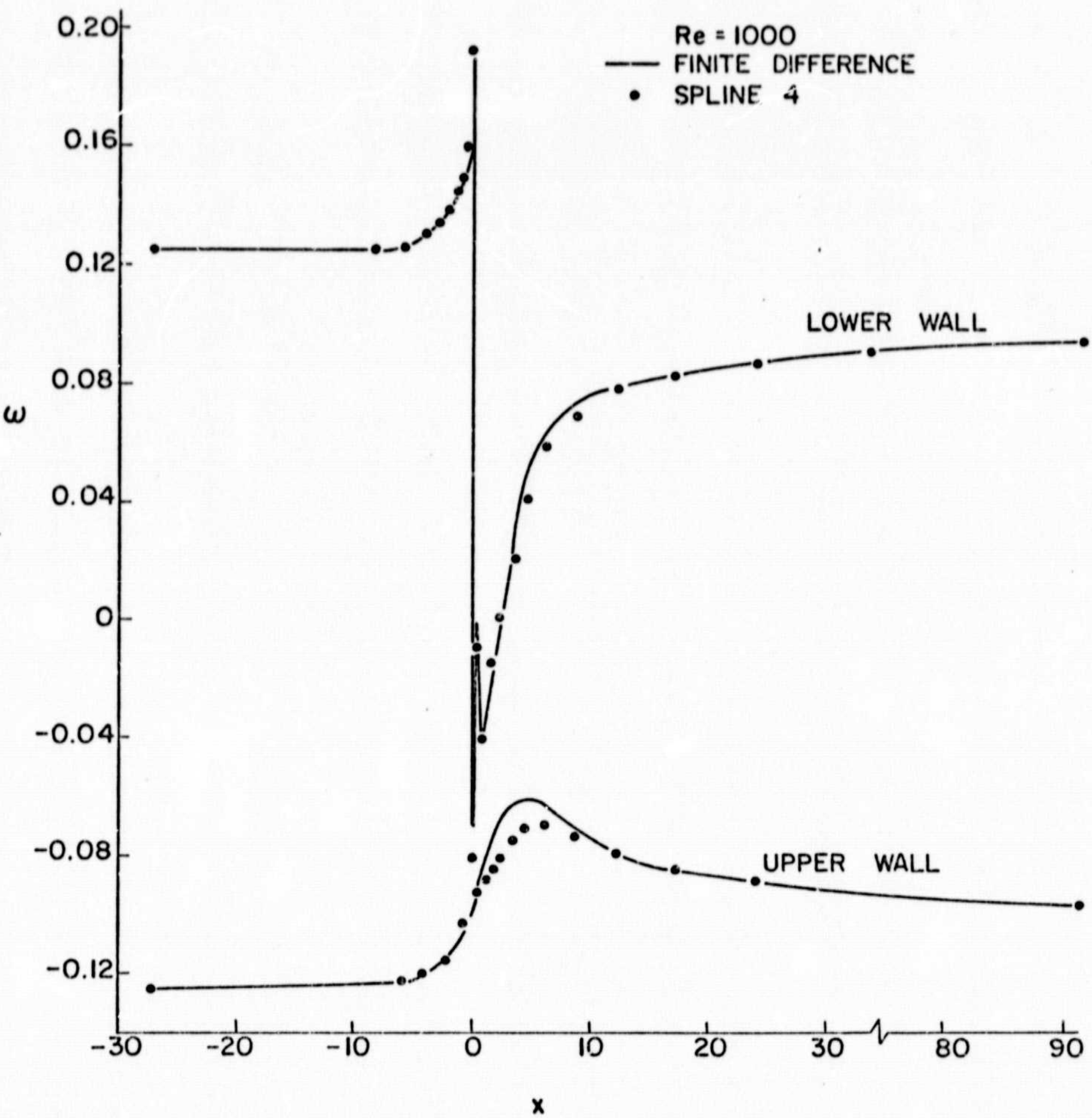


FIG. 7a WALL VORTICITY IN A CHANNEL

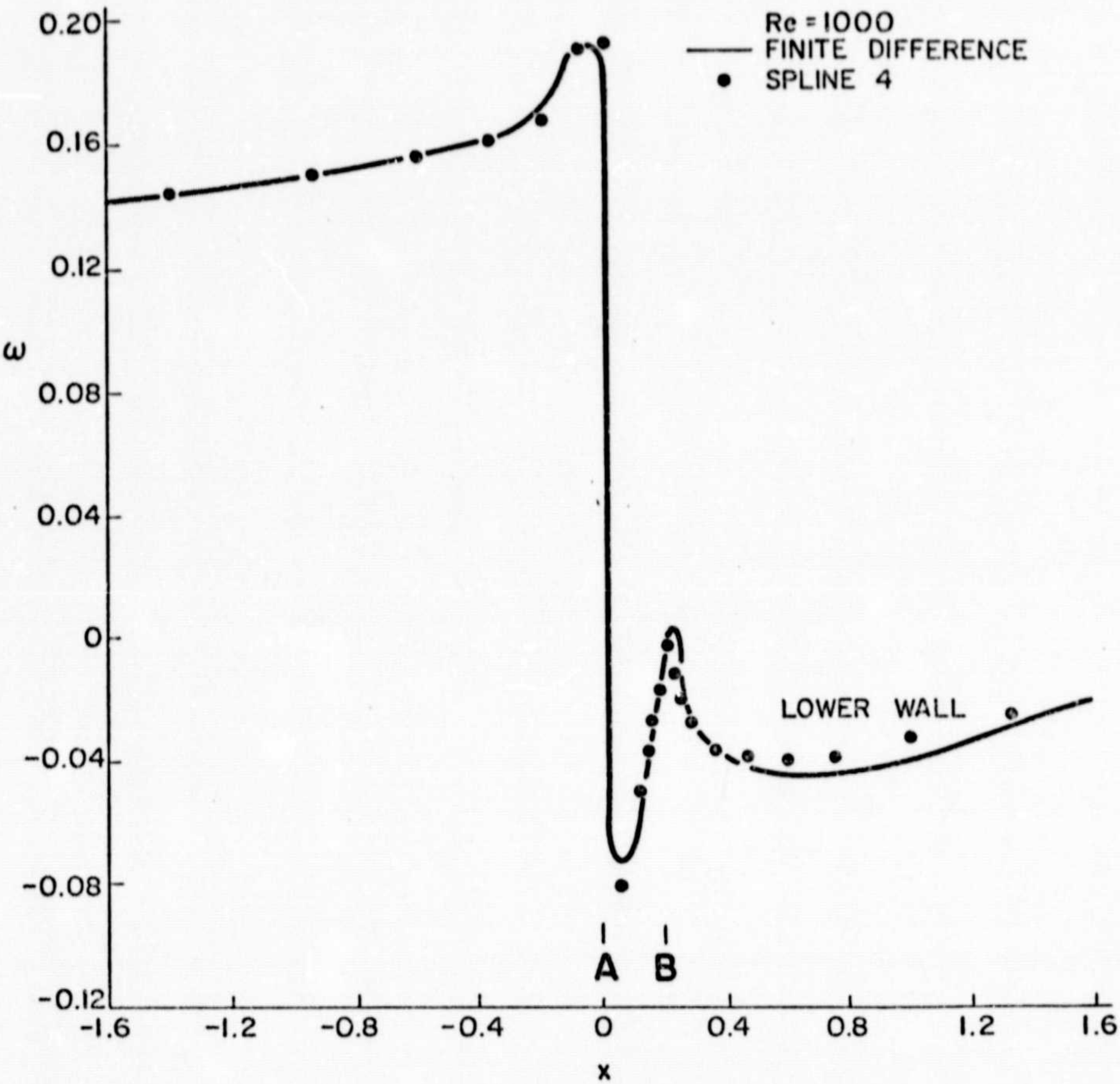


FIG. 7b WALL VORTICITY IN A CHANNEL