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UNIFIED CONNECTED THEORY OF FEW-BODY REACTION MECHANISMS

IN N-BODY SCATTERING THEORY<sup>†</sup>

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# N-BODY SCATTERING THEORY

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## 1. Introduction

Most of our knowledge of elementary particle, nuclear, and atomic physics comes from collision experiments. When these collisions involve systems of many particles, exact solutions of the scattering equations are out of the question. Models that emphasize certain reaction mechanisms can successfully explain much of the available data, but as both experimental and numerical techniques get sharpened, there is an increasing need to understand how these models are embedded in an exact multiparticle theory. This would allow us to correct these models for excluded physical processes, and to evaluate the importance of these corrections.

For the case of bound states, there is a theory that provides a framework for understanding bound state calculations in the above sense. This is the Bethe-Breuckner-Goldstone theory [1]. This theory does not give a prescription for calculations, but rather provides a complete framework for systematically including, and evaluating corrections [2] to various bound state calculations.

Our goal in this paper is to construct an exact multiparticle reaction theory that does a similar thing for collision theory. We restrict our considerations to non-relativistic potential theory (though we make some comments on how both particle creation and relativistic kinematics might be included). The formulation of such a theory is difficult because the boundary conditions that come into a collision theory are much more complicated than those of a bound state theory. By working in momentum space these boundary condition problems can be reduced to problems of analytic structure.

In non-relativistic potential theory, there are many existing N-body theories based on mathematically well understood equations [3]. These theories guarantee the existence of formal solutions [4], and allow for practical calculations when  $N \leq 4$  [5]. They are also useful guides for constructing models of simple reaction mechanisms in more complicated problems. Unfortunately none of these theories have the kind of flexibility that we demand. A typical example that illustrates the type of flexibility that we desire would be to treat scattering of  ${}^6\text{Li}$  by  ${}^3\text{He}$ , treating the  ${}^6\text{Li}$  simultaneously as a bound  ${}^3\text{He}$ - ${}^3\text{H}$  system and a bound  ${}^2\text{H}$ - ${}^4\text{He}$  system, maintaining both unitarity across the  ${}^3\text{He}$ - ${}^3\text{H}$  and  ${}^2\text{H}$ - ${}^4\text{He}$  breakup cuts and compactness of the kernel of the integral equation. In addition we would like to be able to construct a first principles optical potential (from a compact kernel theory) describing the effect of the excluded channels [6]. Applications of this type are beyond the scope of existing N-body theories. In addition, since the structure of most N-body theories is dictated by compactness considerations having little to do with physics, it is not clear that truncations maintaining the structure of these equations will clearly relate to any physical process. There are other approaches that are oriented more towards practical calculations, such as distorted wave approximations and multiple scattering approaches [7], however these are not easily extendible to include all many body corrections.

The type of theory that we want should satisfy several constraints if it is going to be useful. First the theory should be based on a very flexible foundation that allows us to separate the important aspects

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of a reaction from the unimportant ones. We would like to do this in a way that relates directly to the physics, and is independent of the dynamical structure of any equation. In addition we would like to be able to calculate the contributions from both the important and unimportant parts of the dynamics and understand how they combine to give the full dynamics. A compact kernel approach is desirable, since solutions of compact kernel equations are well understood, admitting constructive solution techniques. We would like compactness to be maintained independent of how the dynamics is divided.

We would also like to be able to control the unitarity structure of our equations. By this we mean that we would like to know what parts of our dynamics are responsible for the presence of flux in various asymptotic channels, and how non-unitary corrections, due to loss of flux, may be included in a controlled manner.

The theory that we develop is consistent with all of these criteria, and has several additional noteworthy features. First the approximate dynamics is calculated from an approximate Hamiltonian. This makes it easy to consider symmetry properties of our truncated system. Second, the theory allows for the inclusion of many body forces in a natural way. Since all particle creation effects can be implicitly described by energy dependent many body forces, the inclusion of relativistic kinematics would allow us to extend some of our techniques to higher energy collisions (the relativistic kinematics and energy dependent forces will alter our analytic considerations in an obvious way). Third, the basic input that comes into our theory are transition operators and bound state wave functions corresponding to fewer body problems. These are more

directly related to experimentally measurable quantities than are potentials. Fourth, we are able to maintain unitarity of the truncated equations without sacrificing analyticity (as compared with unitary K-matrix theories). This makes our theory compatible with dispersion theoretic techniques. Finally our theory is based on a description of the dynamics in terms of asymptotic channels. This permits us to make direct contact with experimentally observed quantities, and to keep careful track of how boundary conditions come into play.

In developing this theory we obtain two very useful results that should be of interest by themselves. The first is Theorem 1 which uses combinatoric techniques to construct a democratic expansion of the full Hamiltonian in terms of proper partition Hamiltonians. In this way we can easily understand N-body scattering in terms of propagation through the various non-orthogonal fewer body channels. To our knowledge this is a new result. The other result concerns the use of combinatoric techniques in treating connectivity. The key results are Theorem 4 and the decomposition (1). These results are not new [8], but their power, which exceeds the applications at hand, has not been fully appreciated. In particular they allow us to get connected kernel equations with completely arbitrary truncations of many body Hamiltonians.

This paper is divided into six sections. In the next section we describe background material that will be needed in subsequent sections. Our notation is established in this section. In the third section we give a precise definition of a reaction mechanism. We use combinatoric techniques to prove our Hamiltonian decomposition theorem. Dynamic operators associated with a given reaction mechanism are defined, and related to solutions of fewer body problems. In section IV we develop

compact kernel integral equations for operators corresponding to each reaction mechanism in terms of fewer body solutions. These operators are then used as input in another compact kernel integral equation for the full dynamic operators. In section V we prove various properties of these equations. All of our integral equations are shown to have connected kernels or connected iterated kernels. In addition we prove that the dynamic operators corresponding to the various reaction mechanisms satisfy a suitably truncated optical theorem.

## II. Background

In this section we establish our notation and introduce necessary background material. The material is divided into three parts. First we discuss the notion of a partition lattice. This will be important in handling the connectivity aspects of our theory. Then we introduce the notation and conventions that will be used in the remainder of the paper. Finally we collect some well-known spectral properties of subsystem operators which are needed as input in any N-body theory. We begin with an introduction to partition lattices.

### A. Partition Lattice

A partition,  $a$ , of  $N$  particles is a way of dividing the  $N$  particles into  $n_a$  disjoint groups of particles, called clusters. We use lower case Latin letters ( $a, b, c, \dots$ ) to denote partitions, and  $n_a, n_b, n_c, \dots$  to denote the number of clusters in the partition. For  $N$  particles there are two partitions that are uniquely specified once the number of clusters are given: the 1-cluster partition, and the  $N$ -cluster partition. We denote these by  $\underline{1}$  and  $\underline{0}$ , respectively. Given 2 partitions,  $a$  and  $b$ , we say  $a \subseteq b$  or  $b \supseteq a$  if  $a$  can be obtained from  $b$  by breaking up some (possibly 0) clusters of  $b$ . This concept is easily illustrated in the four-particle case where

$$a = (12)(34) \quad b = (123)(4) \quad c = (1)(2)(34)$$

$$c \subseteq a \quad c \not\subseteq b \quad a \not\subseteq b.$$

The relation  $\subseteq$  is a partial ordering on the set of partitions of  $N$  particles. We define least upper bounds (unions) and greatest lower bounds (intersections) relative to the partial ordering. Given two partitions  $a$  and  $b$ ,  $a \cup b$  is the finest partition (the one with most clusters) satisfying  $a \cup b \supseteq a$  and  $a \cup b \supseteq b$ . The intersection  $a \cap b$  is the coarsest partition (the

one with the least number of clusters) satisfying  $a \cap b \subseteq a$  and  $a \cap b \subseteq b$ . The unions and intersections defined above both exist and are unique. The operations  $\cup$  and  $\cap$  endow the set of partitions with a lattice structure [9]. In order to understand  $\cup$  and  $\cap$  we give some examples in the four-particle partition lattice:

$$(12)(3)(4) \cup (12)(34) = (12)(34)$$

$$(123)(4) \cup (1)(234) = (1234)$$

$$(12)(3)(4) \cap (1)(234) = (1)(2)(3)(4)$$

$$(123)(4) \cap (13)(24) = (13)(2)(4).$$

The partition lattice is important in N-body theories because of its relation to the notion of the connectivity of N-body operators. Given a partition,  $a$ , and a linear operator,  $B$ , on our N-particle Hilbert space, we say B has connectivity a if  $B$  commutes with the  $3n_a$  parameter unitary group of translations that describe the motion of the clusters of  $a$ ; and these are the only translations with which  $B$  commutes. This means that on taking momentum space matrix elements of  $B$ , the only conserved momenta are the total CM momentum and the relative momenta between the various clusters of  $a$ .

The aim of all N-body theories is to recast the Schrödinger equation in the form of an integral equation with compact kernel. Compact kernel integral equations are well understood and admit constructive existence theorems [4]. Disconnected operators are generally never compact. One expects that if operators of definite connectivity satisfy sufficiently restrictive boundedness conditions, any product of these operators that is connected will be compact<sup>1</sup>. We therefore define a fiber compactness assumption (F.C.A.). This assumption puts sufficient conditions on an  $a$ -connected operator  $B$ , so that whenever a string,  $S$ , of such operators has overall connectivity 1,  $S$  is compact after

the CM motion is factored out. In this work we assume that we have a suitable F.C.A. operating<sup>2</sup>. In particular, whenever we mention connectivity, we mean connectivity plus some suitable F.C.A. We remark that an F.C.A. must be compatible with the structure of the equations to which it is applied. Since our main interest is in developing and motivating a new N-body formalism, we will not consider the nature of the F.C.A. We refer the interested reader to Ref. [11].

The important connection between connectivity and the partition lattice is the following: Let  $A$  and  $B$  be operators with connectivities  $a$  and  $b$ , respectively. If we consider  $AB$ , the product only commutes with those translation operators that simultaneously commute with both  $A$  and  $B$ . Since  $c \supseteq a$ ,  $c \supseteq b$  imply that the  $3n_c$  parameter unitary groups of translations of the clusters of  $c$  are also among the unitary group of translations of the clusters of both  $a$  and  $b$ , and since any such  $c$  must satisfy  $c \supseteq a \cup b$ , it follows that the unitary group of translations commuting with both  $A$  and  $B$  are precisely those describing the motion of the clusters of  $a \cup b$ . Thus we get the important result: If  $A$  has connectivity  $a$ , and  $B$  has connectivity  $b$ , then  $AB$  has connectivity  $a \cup b$ . When  $a \cup b = 1$ , the product  $AB$  will then be compact.

## B. Notation

Consider  $N$  nonrelativistic mutually interacting particles with a Hamiltonian of the form

$$H = K + V,$$

where  $K$  is the  $N$  particle kinetic energy operator, and  $V$  describes the interaction between particles. Typically  $V$  is of the form  $V = \sum_{i < j}^N V_{ij}$  where  $V_{ij}$  is a short-ranged force between particles  $i$  and  $j$ . Our formalism also allows the inclusion of 3, ...,  $N$  body forces. The case of  $N$ -body forces in the  $N$ -body



problem requires some modification; we discuss their inclusion in Appendix A. In everything that follows we assume that  $V$  consists only of 2, ..., N-1 body forces. External forces are easily included by assuming they are tied to an additional fictitious particle of infinite mass.

In general the operators we deal with are sums of operators of definite connectivity. For an operator  $B$  we define  $[B]_a$  to mean the part of  $B$  that has connectivity  $a$ . We note that  $[B]_a = 0$  if  $B$  has no part with connectivity  $a$ . We also use the notations:

$$\begin{aligned} B_a &= \sum_{b(\subseteq a)} [B]_b \\ B^a &= \sum_{b(\not\subseteq a)} [B]_b \\ B_b^a &= \sum_{\substack{c(\subseteq b) \\ c(\not\subseteq a)}} [B]_c \end{aligned} \quad (1)$$

Using this notation we define the partition interactions

$$\begin{aligned} V_a &= \sum_{b(\subseteq a)} [V]_b \\ V^a &= \sum_{b(\not\subseteq a)} [V]_b \\ V_c^a &= \sum_{\substack{b(\subseteq a) \\ b(\not\subseteq c)}} [V]_b \end{aligned} \quad (2)$$

As an example consider the case of four particles:

$$\begin{aligned} V_{(123)(4)} &= V_{13} + V_{23} + V_{31} + V_{123} \\ V_{(14)(2)} &= V_{14} + V_{34} \end{aligned}$$

where  $V_{123}$  is a three-body force acting among particles 1, 2 and 3. The operator  $V_a$  is the sum of all interactions internal to the clusters of  $a$ ,  $V^a$  is the sum of all interactions external to the clusters of  $a$ , and  $V_b^a$  is the sum

of all interactions both external to the clusters of  $a$  and internal to the clusters of  $b$ .

In the same way we define the partition Hamiltonians:

$$H_a = \sum_{b(\subseteq a)} [H]_b = K + V_a. \quad (3)$$

We note  $K$  has connectivity 0. We observe

$$\begin{aligned} H_b^a &= V_b^a \\ H^a &= V^a. \end{aligned} \quad (4)$$

We now define the operators describing the full dynamics. The resolvent operator is

$$G(Z) = (Z - H)^{-1}.$$

It contains all of the N-body physics. In dynamic content  $G(Z)$  is equivalent to the transition operators [12]:

$$\begin{aligned} T_+^{ab}(Z) &= V^a + V^a G(Z) V^b \\ T_-^{ab}(Z) &= V^b + V^a G(Z) V^b. \end{aligned}$$

The matrix elements of either of these operators between asymptotic states gives the physical transition probabilities. Their on-shell matrix elements are identical but they differ off shell. We also define the various partition (subsystem) resolvents:

$$G_a(Z) = (Z - H_a)^{-1}.$$

In general, we suppress the Z-dependence of  $G$ ,  $G_a$  and  $T_i^{ab}$ . Since we use the partition operators  $G_a$ ,  $V_b^a$ , etc., to construct  $G$ ,  $T_i^{ab}$ , we must first understand the partition operators. We discuss these in the next subsection.

### C. Subsystem Spectral Properties

The basic philosophy behind the N-body problem is inductive: one solves the N-body problem using input from fewer body problems. The fewer body solutions are used to construct the partition resolvents,  $G_a$ , or their equivalents. Although  $G_a$  involves all N particles, when  $a \neq 1$ ,  $a$  has 2 or more noninteracting clusters each having fewer than N particles. Thus  $G_a$  describes the dynamics of several noninteracting fewer body systems, rather than a single N-body system. We define

$$H_i = K_i + V_i \quad i = 1, \dots, n_a$$

where  $V_i$  is the sum of all interactions internal to the  $i^{\text{th}}$  cluster of  $a$ , and  $K_i$  is the  $i^{\text{th}}$  clusters kinetic energy. The operator  $G_i = (Z - H_i)^{-1}$  is trivially related to the true fewer body resolvent by a simple imbedding. Hugenholtz [13] has shown that

$$G_a(Z) = G_1(Z) * G_2(Z) * \dots * G_{n_a}(Z)$$

where  $*$  is the convolution in  $Z$ . This determines the dynamics of  $H_a$  from fewer body systems.

Although  $H_a$  involves fewer body dynamics, it is an operator on the N particle Hilbert space. Because of the translational invariance of the relative positions of the clusters of  $a$ ,  $H_a$  has only a continuous spectrum. We distinguish between two types of eigenstates<sup>3</sup> of the partition Hamiltonians,  $H_a$ . The first class of partition Hamiltonian states will be denoted as maximally connected eigenstates. A maximally connected eigenstate

is a state where the particles of each cluster are in a bound state. These states are in 1-1 correspondence with the set of physically preparable initial states from which the full scattering states evolve, provided we let  $a$  vary over all partitions of N particles with  $n_a \geq 2$ . The complementary class of partition Hamiltonian eigenstates are those where the particles of at least one

cluster are in a scattering state. We call these scattering states of the partition Hamiltonian, and note that they also evolve from maximally connected states of another partition Hamiltonian,  $H_b$ , with  $b \subseteq a$ .

We define a channel  $\alpha_a$  by a partition,  $a$ , describing an asymptotic configuration, and a set,  $\alpha$ , of quantum numbers describing the internal state of each of the bound clusters of a maximally connected eigenstate of  $H_a$ . We denote the eigenstates of  $H_a$  by:

$$\begin{aligned} |\psi_a(\alpha_a)\rangle &= \text{maximally connected eigenstate of } H_a \text{ with} \\ &\quad \text{channel } \alpha_a. \\ |\psi_a(\beta_b)\rangle &= |\psi_a^+(\beta_b)\rangle = \text{scattering eigenstate of } H_b \text{ that} \\ &\quad \text{evolves from a maximally connected eigenstate} \\ &\quad \text{of } H_b \text{ with channel } \beta_b. \end{aligned} \quad (5)$$

For notational convenience we suppress the continuous parameters describing the momenta of the clusters of  $a$  (resp.  $b$ ). We define  $A_a$  by:

$$A_a = \{\alpha_a | a \neq 1, |\psi_a(\alpha_a)\rangle \text{ is a maximally connected eigenstate of } H_a\}. \quad (6)$$

We note  $A_0$  is just the set of scattering channels.

For each partition Hamiltonian  $H_a$  we let  $S_a(\beta_b)$  be the closure of the subspace of N-body Hilbert space generated by  $\{|\psi_a(\beta_b)\rangle\}$ . We assume that each partition Hamiltonian satisfies asymptotic completeness. In particular, if  $\mathcal{H}$  is our N-body Hilbert space, we assume asymptotic completeness [15]:

$$\forall a \neq 1 \quad \mathcal{H} = \bigoplus_{\substack{\beta_b \in A_0 \\ b \subseteq a}} S_a(\beta_b).$$

Asymptotic completeness allows us to define:

$$P_a(\beta_b) = (\text{projector on } S_a(\beta_b))^\dagger.$$

In terms of (5),  $P_a(\beta_b)$  admits the representation

$$P_a(\beta_b) = \int d\vec{p}_1 \dots d\vec{p}_{n_b-1} |\psi_a(\beta_b)\rangle \langle \psi_a(\beta_b)|. \quad (7)$$

It also follows that  $C_a(Z)$  admit. the spectral resolution

$$C_a(Z) = \sum_{\substack{b \in A_0 \\ b \in a}} \int d\vec{p}_1 \dots d\vec{p}_{n_b-1} \frac{[A_a^{(b)} - A_a^{(b)}]}{Z - \sum_{i=1}^{n_b-1} p_i^2/2\mu_i + E_{a,b}} \quad (3)$$

where the  $\vec{p}_i$  are a suitable set of momentum coordinates for the relative momentum of the clusters of  $b$ , and  $\mu_i$  are the corresponding reduced masses. These operators are the building blocks from which we construct our theory. In (5), (7), (8) and all subsequent work we assume the overall CM degrees of freedom have been removed.

### III. Hamiltonian Decomposition

In this section we define formally what we mean by a reaction mechanism (RM). Qualitatively, this is meant to describe the important processes occurring when a scattering or reaction process takes place. The choice of RM must rely on physical intuition, but for any choice the formalism provides a technique for obtaining mathematically appropriate equations to describe it and to obtain systematic corrections for all other processes.

The section is divided into four parts. We begin by considering some useful combinatoric relations. Next we use these results to write the full Hamiltonian as a sum of partition Hamiltonians. This yields decomposition of  $H$  in terms of asymptotic channels. Finally we define reaction mechanisms in general terms and construct the associated operators.

#### A. Combinatorics

The Stirling number of the second kind,  $S_N^m$ , is defined as the number of ways  $N$  distinguishable particles can be grouped into  $m$  non-empty clusters. We also define the coefficients  $C_a = C_{n_a} = (-)^a (n_a - 1)!$ . The main results required are <sup>4</sup> [16]:

Lemma 1:

$$\sum_{\substack{a \in \mathcal{P}(b) \\ n_a \geq 2}} C_a = \sum_{a \in \mathcal{P}(b)}' C_a = \sum_{n=2}^{n_b} S_{n_b}^n C_n = 1, \quad n_b \geq 2. \quad (9)$$

The prime on the sum means that the partition  $J$  is excluded. We use this notation throughout our paper. Lemma 1 is the key combinatoric result. The geometric importance is the following: when  $B_a$  is defined as in (1), one can show  $\sum_a C_a B_a$  is the disconnected part of  $B$ .

Another set of numbers that are of interest arise from the inversion of the relation

$$B_a = \sum_{b \in a} [B]_b = \sum_b \delta_{a \supseteq b} [B]_b.$$

The anticluster coefficients [17],  $\eta_b^a$ , are defined by

$$[B]_a = \sum_{b \in a} \eta_b^a B_b. \quad (10)$$

This allows us to extract  $[B]_a$  from  $B_a$ . The existence of the anticluster coefficients depends on the invertibility of the matrix  $\delta_{a \supseteq b}$ . We show that  $\delta_{a \supseteq b}$  not only has an inverse, but that the inverse is an integer matrix.

Put the partitions in ascending order by number of clusters. Since  $a \supseteq b \Rightarrow n_a \leq n_b$ ,  $n_a \leq n_b$  implies  $b$  is to the right of  $a$  and the entry 1 is above the diagonal. If  $n_a = n_b$ ,  $a \supseteq b \Rightarrow a = b$  so one obtains 1's along the diagonal. In this representation  $\delta_{a \supseteq b}$  has entries which are always 0 or 1 with 1's along the diagonal, and 0's below the diagonal. These three facts are enough to prove that the  $\eta_b^a$  exist and are integers. These coefficients can be easily constructed by row reduction. A specific example for three particles is given in Fig. 1. Explicit expressions for  $\eta_b^a$  can be obtained [18].

### B. Decomposition in Partition Hamiltonians

We now use Lemma 1 to express the full N-body Hamiltonian as a sum of partition Hamiltonians. Using (9) with  $b = \emptyset$  it follows that

$$K = \sum_a C_a K. \quad (11)$$

We also have the following:

Lemma 2:

$$\sum_a C_a V_a = V. \quad (12)$$

To prove this we note

$$\sum_a C_a V_a = \sum_a C_a \sum_{b \in a} [V]_b = \sum_b [V]_b \sum_{a \supseteq b} C_a.$$

By Lemma 1 we have

$$\sum_{a \supseteq b} C_a = 1 - \delta_{n_b, 1}.$$

It follows that

$$\sum_a C_a V_a = \sum_b [V]_b \sum_{a \supseteq b} C_a = V - [V]_1,$$

where we have used (2) with  $V = V_1$ . If there are no N-body forces  $[V]_1 = 0$ , and the result follows.

Results (11) and (12) lead to the important theorem:

Theorem 1:

$$H = \sum_a C_a H_a. \quad (13)$$

This is a trivial consequence of (11) and (12). It says that the full N-body Hamiltonian can be expressed as a sum of partition Hamiltonians. We recall that the partition Hamiltonians are fewer body operators.

Using the projection operators defined in (8) we may write

$$H_a = \sum_{b \in a} \sum_{c \in \Lambda_0} P_a(c_b) H_c. \quad (14)$$

Substituting (14) in (13) and changing to order of the summations we find

$$H = \sum_b \sum_{c \in \Lambda_0} \sum_{a \supseteq b} C_a P_a(c_b) H_c. \quad (15)$$

We note

$$\sum_b \sum_{c \in \Lambda_0} = \sum_b \sum_{c \in \Lambda_0}.$$

Defining

$$H(c_b) = \sum_{a \supseteq b} C_a P_a(c_b) H_a.$$

(15) becomes<sup>5</sup>

$$H = \sum_{c \in \Lambda_0} H(c). \quad (16)$$

This expresses  $H$  as a sum of operators constructed from fewer body operators, each of which corresponds to a different asymptotic channel. We do not need to know full  $N$ -body solutions to make this division.

### C. Reaction Mechanisms

In this part we define a reaction mechanism (RM). Qualitatively an RM is a phenomenological picture of how a reaction proceeds, usually motivated by some observed property of the system. For scattering experiments the most important properties are cross sections. In some cases they give enough useful information to isolate a dominant RM, but often must be supplemented by physical intuition. Given any set of important asymptotic channels, we have a natural way of isolating the dynamics that propagate through these channels. This leads us to define a reaction mechanism as follows.

#### Definition:

A reaction mechanism (RM) is a collection

$$A \in A_0 \text{ of asymptotic channels.} \quad (17)$$

In any specific example one still has to decide on physical grounds what states are important. Clearly, states responsible for large portions of the cross section should be important. On the other hand, if we are below the threshold for a certain channel, it may still be important through off-shell effects. Our phenomenological pictures will be important guides in picking out an RM, especially with regard to these off-shell effects.

Analyticity is a very powerful tool in scattering theory; given the discontinuities across the scattering cuts, residues of the bound state poles, and the asymptotic behavior at  $Z = \infty$  we may use analytic considerations to recover all  $N$ -body physics. Through analyticity and unitarity constraints the discontinuities across the scattering cuts are related to the physical cross sections. This relation is expressed in the optical theorem [18] which in our notation has the form:

$$\text{Im} \langle \psi_A | T | \psi_A \rangle = \frac{1}{2} \langle \psi_A | T | \psi_A \rangle \langle \psi_A | T | \psi_A \rangle = \frac{1}{2} \langle \psi_A | T | \psi_A \rangle \langle \psi_A | T | \psi_A \rangle \quad (18)$$

where  $\langle \psi_A | T | \psi_A \rangle$  is the incident current, and  $\langle \psi_A | T | \psi_A \rangle$  is the partial cross section for scattering from  $\psi_A$  to  $\psi_A$ . Through the optical theorem one tries to isolate the parts of the dynamics responsible for the various final states. One of the aims of this paper is to prove the following:

#### Theorem 2:

The decomposition  $H = \sum_{A \in A_0} H(A)$  gives a decomposition of the full  $N$ -body Hamiltonian into terms corresponding to each final channel in the following sense. Given any RM,  $A \in A_0$ , we define

$$H(A) = \sum_{a \in A} H(a). \quad (19)$$

$H(A)$  defines a unitary theory whose final state spectrum consists only of the channel(s)  $A$ . If  $A = A_0$  we get our full dynamic theory. We postpone the proof of this theorem until Sec. V.

We define some special RM's. If  $A$  is an RM, we define the complementary RM,  $A'$ , by  $A' = A_0 \setminus A$ . A trivial RM,  $A$ , is an RM where

$$A = \{a \mid b \geq c, a \in A\}.$$

These RM's are termed trivial because the Hamiltonian  $H(A)$  has no interactions between the clusters of  $a$ , so the dynamics are derivable directly from fewer body problems. Few-body RM's are those where each of the partitions  $\{a \mid a \in A\}$  involve no more than four clusters. These are important because in a connected kernel approach, their dynamic complexity is comparable to that of a few-body problem.

In the remainder of this section we use the operator  $H(A)$  to define dynamic RM operators. In the next two sections we develop connected kernel equations for the RM operators and their complements. We show how these may be combined in another connected kernel equation to get the full reaction dynamics.

#### D. RM Operators

In a full reaction theory one begins with a Hamiltonian; one then constructs resolvents, transition operators and other dynamic quantities. We follow this procedure for the RM Hamiltonian,  $H(A)$ , defined by (19). First we construct effective RM interactions. Using (15) we see  $H(A)$  admits the representation

$$H(A) = \sum_b \sum_{a_b \in A} \sum_{a(\supseteq b)} C_a P_a(v_b) H_a = \sum_a C_a \sum_{b(\subseteq a)} \sum_{a_b \in A} P_a(v_b) H_a. \quad (20)$$

We define

$$P_a(A) = \sum_{b(\subseteq a)} \sum_{a_b \in A} P_a(v_b) \quad (21)$$

and the RM partition Hamiltonians by

$$H_a(A) = P_a(A) H_a = P_a(A) H_a P_a(A). \quad (22)$$

For  $A = A_0$  this reduces to  $P_{A_0}(A_0) = 1$  and  $H_{A_0}(A_0) = H_{A_0}$ . With this definition

(20) becomes

$$H(A) = \sum_a C_a H_a(A).$$

allowing us to construct  $H(A)$  from the projection operators  $P_a(A)$ . Note  $H(A_0) = H$ .

Next we show

**Lemma 3:**

$$\sum_{a(\subseteq b)} [H(A)]_a = H_b(A). \quad (23)$$

Lemma 3 is an important compatibility condition between the analytic properties described by  $P_a(A)$  and the connectivity properties described by  $[H(A)]_b$ . In order to prove this result we need some additional machinery.

We define wave operators

$$\begin{aligned} u_a^b(z) &= 1 + G_a(z) V_a^b \\ q_a^{b+}(z^2) &= 1 + V_a^b G_a(z^2) \end{aligned}$$

and the operator

$$E_b(v_b; E) = \int [E_b(v_b) - E - E'(p_1, \dots, p_{n_b-1}, 0) - v_b(v_b)] dp_1 \dots dp_{n_b-1}$$

where  $E'$  is the energy of the state  $(p_1, \dots, p_{n_b-1}, 0)$ .

Using these operators, the states  $|\psi_a(v_b)\rangle$  admit the representation

$$|\psi_a(v_b)\rangle = u_a^b(E^+); \psi_b(v_b) \rangle. \quad (24)$$

This allows us to relate partition scattering states with maximally connected partition states. From (7), (22), and (24) it follows that

$$H_a(A) = \sum_{b(\subseteq a)} \sum_{a_b \in A} \int dE u_a^b(E^+) E_b(v_b; E) u_a^{b+}(E^-) \quad (25)$$

where  $E^+ = E + i0^+$ . We observe that  $E_b$  has connectivity  $b$ , while  $u_a^b$  and  $u_a^{b+}$  have connectivities contained in  $a$ . Since  $b \subseteq a$ ,  $u_a^b(E^+) E_b(v_b; E) u_a^{b+}(E^-)$  is a sum of terms with connectivities  $c$  such that  $a \supseteq c \supseteq b$ . It is easy to show

**Lemma 4:**

$$[u_a^b(E^+) E_b(v_b; E) u_a^{b+}(E^-)]_c = [u_c^b(E^+) E_b(v_b; E) u_c^{b+}(E^-)]_c \quad a \supseteq c \supseteq b. \quad (26)$$

We prove (26) by expanding  $u_a^b(E^+)$  and  $u_a^{b+}(E^-)$  as follows:

$$u_a^b(E) = 1 + (G_c + G_c V_a^c G_a)(V_c^b + V_a^{bc}) = (1 + G_c V_c^b) + (\text{terms with connectivity external to } c),$$

similarly with  $u_a^{b+}(E^-)$ . Since  $1 + G_c V_c^b = u_c^b(E^+)$ , and since terms with connectivities external to  $c$  give no contribution to the  $c$  connected part of an expression, (26) becomes

$$[u_a^b(E^+) E_b(v_b; E) u_a^{b+}(E^-)]_c = [u_c^b(E^+) E_b(v_b; E) u_c^{b+}(E^-)]_c + (\text{terms with connectivity external to } c).$$

Since the last term is necessarily 0, (26) follows.

We are now in a position to prove (23). The first step in this proof is to extract  $[H(A)]_a$ . Using (25), (20), (21), and (22) we have

$$H(A) = \sum_a C_a \sum_{b(\subseteq a)} \sum_{a_b \in A} \int dE u_a^b(E^+) E_b(v_b; E) u_a^{b+}(E^-).$$

Making a connectivity expansion of this using (26) we obtain

$$H(A) = \sum_a c_a \sum_{b(\subseteq a)} \sum_{c \in A} \sum_{(a \supseteq c) \supseteq b} \int dE [E_c^b(E^+) E_b(c; E) E_c^{b+}(E^-)]_c.$$

Changing the order of the sums gives

$$H(A) = \sum_c \sum_{b(\subseteq c)} \sum_{a \in A} \sum_{(a \supseteq c) \supseteq b} c_a \int dE [E_c^b(E^+) E_b(c; E) E_c^{b+}(E^-)]_c. \quad (27)$$

By (9)  $\sum_{a(\subseteq c)} c_a = 1 - \delta_{nc} \cdot 1$ . Using this in (27) we obtain

$$H(A) = \sum_c \sum_{b(\subseteq c)} \sum_{a \in A} \int dE [E_c^b(E^+) E_b(c; E) E_c^{b+}(E^-)]_c.$$

This allows us to make the identification:

$$[H(A)]_a = \sum_{b(\subseteq a)} \sum_{c \in A} \int dE [E_c^b(E^+) E_b(c; E) E_c^{b+}(E^-)]_a. \quad (28)$$

Using (28) in the left-hand side of (23) we obtain

$$\sum_{a(\subseteq b)} [H(A)]_a = \sum_{a(\subseteq b)} \sum_{c \in A} \sum_{a \in A} \int dE [E_c^a(E^+) E_a(c; E) E_c^{a+}(E^-)]_a.$$

Using (26) we obtain

$$\sum_{a(\subseteq b)} [H(A)]_a = \sum_{a(\subseteq b)} \sum_{c \in A} \sum_{a \in A} \int dE [E_c^a(E^+) E_a(c; E) E_c^{a+}(E^-)]_a.$$

Changing the order of the sums, using (25) gives

$$\begin{aligned} \sum_{a(\subseteq b)} [H(A)]_a &= \sum_{c(\subseteq b)} \sum_{a \in A} \sum_{(b \supseteq a) \supseteq c} \int dE [E_c^a(E^+) E_a(c; E) E_c^{a+}(E^-)]_a \\ &= \sum_{c(\subseteq b)} \sum_{a \in A} \int dE [E_c^a(E^+) E_a(c; E) E_c^{a+}(E^-)]_a \\ &= H_b(A). \end{aligned}$$

This proves (23).

Since the  $P_a(A)$  are fewer body operators (projectors on eigenstates of partition Hamiltonians), the operators  $H_a(A) = P_a(A) H_A$  can be constructed from solving fewer body problems. From (10) it follows that (23) may be inverted to give the  $[H(A)]_a$ 's in terms of these fewer body solutions via:

$$[H(A)]_a = \sum_b \tau_{ab} H_b(A). \quad (29)$$

We can now define the RM partition interactions. Following (2), (3) and

(4) we define

$$\begin{aligned} V^a(A) &= \sum_{b(\not\supseteq a)} [H(A)]_b \\ V_b^a(A) &= \sum_{c(\not\supseteq a)} [H(A)]_c \end{aligned} \quad (30)$$

These correspond to interactions in a full reaction theory through the correspond-

$$\begin{aligned} V^a + H^a &= \sum_{b(\not\supseteq a)} [H]_b = \sum_{b(\not\supseteq a)} [H(A_0)]_b = V^a(A_0) \\ V_b^a + H_b^a &= \sum_{c(\not\supseteq a)} [H]_c = \sum_{c(\not\supseteq a)} [H(A_0)]_c = V_b^a(A_0). \end{aligned}$$

From the representation (30) it is an easy matter to show

$$\begin{aligned} V^a(A) &= V_b^a(A) + V^{ab}(A) \\ H_a(A) &= V_a^b(A) + H_{ab}(A) \\ H(A) &= V^a(A) + H_a(A), \text{ etc.} \end{aligned} \quad (31)$$

Next we consider RM resolvent operators. By analogy with the full N-body reaction theory we define

$$\begin{aligned} G(A; Z) &= (Z - H(A))^{-1} \\ G_a(A; Z) &= (Z - H_a(A))^{-1}. \end{aligned} \quad (32)$$

Since  $H(A_0) = H$ ;  $H_a(A_0) = H_a$ , it follows that  $G(A_0; Z) = G(Z)$  and  $G_a(A_0; Z) = G_a(Z)$  in the full reaction theory. (16) implies that the RM partition resolvents can be constructed from solutions of fewer body problems. We define

$$Q_a(A) = 1 - P_a(A).$$

It follows from (22) and (32) that

$$G_a(A; Z) = P_a(A) G_a(Z) + \frac{Q_a(A)}{Z}. \quad (33)$$

This gives an explicit representation of  $G_a(A; Z)$  in terms of solutions of fewer body problems. Because of relations (31) and (32) it follows that one has the resolvent equations



$$G(A) - G_a(A) = G_a(A) V^a(A) G(A) = G(A) V^a(A) G_a(A) \quad (34)$$

$$\begin{aligned} G_a(A) - G_b(A) &= G_a(A) (H_a(A) - H_b(A)) G_b(A) = G_b(A) (H_a(A) - H_b(A)) G_a(A) \quad (35) \\ &= G_a(A) (V^b(A) - V^a(A)) G_b(A) = G_b(A) (V^b(A) - V^a(A)) G_a(A) \\ &= G_a(A) (V_a^b(A) - V_b^a(A)) G_b(A) = G_b(A) (V_a^b(A) - V_b^a(A)) G_a(A). \end{aligned}$$

We have suppressed the  $Z$ . If we set  $A = A_0$  these become the standard resolvent relations.

Next we define the RM transition operators

$$\begin{aligned} T_+^{ab}(A; Z) &\equiv V^a(A) + V^a(A) G(A; Z) V^b(A) \\ T_-^{ab}(A; Z) &\equiv V^b(A) + V^a(A) G(A; Z) V^b(A) \end{aligned} \quad (36)$$

Again these satisfy the property that

$$T_{\pm}^{ab}(A_0; Z) = T_{\pm}^{ab}(Z).$$

The dynamic content of the  $T_{\pm}^{ab}(A; Z)$  is the same as that of  $G(A; Z)$ . This follows from (36) and the relations

$$\begin{aligned} G(A) &= G_a(A) + G_a(A) T_+^{ab}(A) G_b(A) \\ G(A) &= G_b(A) + G_a(A) T_-^{ab}(A) G_b(A). \end{aligned}$$

We have succeeded in defining dynamic operators that describe any RM and which have a natural correspondence with operators in the full reaction theory. Theorem 2 ties these operators to the specific reaction mechanism.

In this section we defined the concept of reaction mechanism. For any reaction mechanism this definition allows us to isolate dynamic operators having the same structure as the corresponding operators in the full reaction theory. Since the algebraic structure relating the various resolvents and interactions are all preserved, the dynamic RM operators,  $G(A)$  or  $T_{\pm}^{ab}(A)$ , can be calculated using any first principles reaction theory. For a suitable choice of RM, this may result in simpler practical calculations. Furthermore the structure of the equations should help us understand the dynamic processes involved in the RM. In the next section we will use N-body techniques to derive mathematically well-behaved equations for the RM operators.

#### IV. Dynamic Equations

In this section we write connected kernel equations for the RM resolvents and transition operators. We also show the relationship between the full dynamic operators and the RM dynamic operators.

##### A. RM Operators

In constructing dynamic equations for the operators  $G(A)$  and  $T_{\pm}^{ab}(A)$  we first dispose of the case of trivial reaction mechanisms. Recall that  $A$  is trivial if

$$b = \bigcup_{\{a \supseteq c, a_c \in A\}} a \neq 1.$$

By (31) we have  $H(A) = H_a(A) + V^b(A)$ . Using (28) and (30) we see

$$\begin{aligned} V^b(A) &= \sum_{a(\not\supseteq b)} \sum_{c(\subseteq a)} \sum_{a_c \in A} \int [\Omega_a^c(E^+) E_c(a_c; E) \Omega_a^{+c}(E^-)]_a dE \\ V^b(A) &= \int \sum_{c \supseteq a} \sum_{a(\supseteq c)} \int [\Omega_a^c(E^+) E_c(a_c; E) \Omega_a^{+c}(E^-)]_a dE. \end{aligned} \quad (37)$$

We note  $a_c \in A$ ,  $a \supseteq c \Rightarrow a \subseteq b$  by triviality. It follows that all non-zero terms in the sum (37) have  $a \subseteq b$ . The sum must vanish because of the restriction  $a \not\supseteq b$ . Thus  $V^b(A) = 0$ . This means  $H(A) = H_b(A)$ , and consequently, that  $G(A) = G_b(A)$ . Since  $G_b(A)$  is determined from solutions of fewer body problems, both  $G(A)$  and  $T_{\pm}^{ab}(A)$  are known. We see that there is no new dynamical content in trivial RM's. The corresponding operators can be constructed directly from fewer body solutions.

The situation for nontrivial RM's is not so simple. We begin by giving connected kernel equations for the RM resolvent  $G(A)$ . Assume  $A$  is nontrivial. We define

$$\Omega_c^a(A) = G_c V_c^a(A).$$



We also define the operators  $\tilde{G}_{ab}(A)$  by the equation:

$$\tilde{G}_{ab}(A) \equiv G_a(A) \delta_{ab} + \sum_c G_a(A) v_c^a(A) \tilde{G}_{cb}(A). \quad (38)$$

In the next section we show that the first iterated kernel of this equation is connected. Assuming our RM interactions  $v_c^a(A)$  are consistent with the operating F.C.A., the first iterate of (38) will be a compact kernel equation. This condition guarantees a unique solution<sup>7</sup> [4]. We show that the operator

$$\tilde{G}(A) \equiv \sum_c \tilde{G}_{ac}(A) \quad (39)$$

may be identified with the RM resolvent  $G(A)$ . We have

**Theorem 3:**<sup>8</sup>

$$G(A) = \tilde{G}(A). \quad (40)$$

To prove this we first show  $\tilde{G}(A)$  is independent of  $a$ , then that it satisfies all of the resolvent identities (34).

To show  $\tilde{G}(A)$  is well defined we use the perturbation expansion generated by (38). We need the following:

**Lemma 5:**

$$G_a(A) - G_b(A) = \sum_{c_1} G_b(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A) + \sum_{c_1 c_2} G_b(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A) (v_{c_2}^{c_1}(A) - v_{c_2}^a(A)) G_{c_2}(A) + \dots \quad (41)$$

To prove this lemma we first note

$$\sum_b v_b^a(A) = \sum_b C_b v_b^a(A) = \sum_b C_b \sum_{c \in \{b\}} [H(A)]_c = \sum_{c \in \{a\}} [H(A)]_c \sum_{b \in \{c\}} C_b. \quad (42)$$

By Lemma 1 we have

$$\sum_{b \in \{c\}} C_b = 1 - \delta_{n_c, 1}.$$

Since

$$H(A) = \sum_a P_a(A) H_a.$$

it has no fully connected pieces, so (42) becomes

$$\sum_b v_b^a(A) = \sum_{c \in \{a\}} [H(A)]_c = v_c^a(A) \quad (43)$$

by (30). Using (43) in (35) we find

$$G_a(A) - G_b(A) = \sum_{c_1} G_b(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A). \quad (44)$$

Iterating (44) with  $b = c_1$ ,  $c_1 = c_2$  gives

$$G_a(A) - G_b(A) = \sum_{c_1} G_b(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A) + \sum_{c_1 c_2} G_a(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A) (v_{c_2}^{c_1}(A) - v_{c_2}^a(A)) G_{c_2}(A).$$

Repeating this iteration an infinite number of times gives (41).

We now show that the definition (39) is independent of  $a$ , i.e., for any  $a$  and  $b$

$$\sum_c (\tilde{G}_{ac}(A) - \tilde{G}_{bc}(A)) = 0$$

to each order in  $v(A)$ . Equation (38) generates the expansion:

$$\tilde{G}_{ac}(A) = G_a(A) \delta_{ac} + G_a(A) v_c^a(A) G_c(A) + \sum_{c_1} v_{c_1}^a(A) G_{c_1}(A) v_c^{c_1}(A) G_c(A) + \dots$$

Summing over  $c$  gives

$$\sum_c \tilde{G}_{ac}(A) = G_a(A) + \sum_c G_a(A) v_c^a(A) G_c(A) + \sum_{c_1 c_2} G_a(A) v_{c_1}^a(A) G_{c_1}(A) v_{c_2}^{c_1}(A) G_{c_2}(A) + \dots \quad (45)$$

Using (44) in (45) gives

$$\begin{aligned} \sum_c \tilde{G}_{ac}(A) &= (G_b(A) + \sum_{c_1} G_b(A) (v_{c_1}^b(A) - v_{c_1}^a(A)) G_{c_1}(A) + \dots) \\ &\quad + (1 + \sum_d v_d^a(A) G_d(A) + \sum_{d_1 d_2} v_{d_1}^a(A) G_{d_1}(A) v_{d_2}^{d_1}(A) G_{d_2}(A) + \dots). \end{aligned} \quad (46)$$

In order to show that this is the expansion for  $\sum_c \tilde{G}_{bc}(A)$  we consider the  $n^{\text{th}}$  order term of (46) in  $v$ . This term is (suppressing the  $A$ 's):

$$\begin{aligned}
& \sum_b G_b (v_{c_1}^b - v_{c_1}^a) G_{c_1} (v_{c_2}^{c_1} - v_{c_2}^a) \dots (v_{c_{n-1}}^{c_{n-2}} - v_{c_{n-1}}^a) G_{c_{n-1}} (v_{c_n}^{c_{n-1}} - v_{c_n}^a) G_{c_n} \\
& + \sum_b G_b (v_{c_1}^b - v_{c_1}^a) G_{c_1} (v_{c_2}^{c_1} - v_{c_2}^a) \dots (v_{c_{n-1}}^{c_{n-2}} - v_{c_{n-1}}^a) G_{c_{n-1}} v_{c_n}^a G_{c_n} \\
& + \sum_b G_b (v_{c_1}^b - v_{c_1}^a) G_{c_1} (v_{c_2}^{c_1} - v_{c_2}^a) \dots (v_{c_{n-2}}^{c_{n-3}} - v_{c_{n-2}}^a) G_{c_{n-2}} v_{c_{n-1}}^a G_{c_{n-1}} v_{c_n}^{c_{n-1}} G_{c_n} \dots \\
& + \sum_b G_b v_{c_1}^a G_{c_1} v_{c_2}^{c_1} \dots v_{c_n}^{c_{n-1}} G_{c_n} = \sum_b G_b v_{c_1}^b G_{c_1} v_{c_2}^{c_1} \dots v_{c_n}^{c_{n-1}} G_{c_n}. \quad (47)
\end{aligned}$$

In order to observe the cancellations in (47) one must start from the right of the top line. The term with the single underline and all factors to its left cancel the second line. Moving to the left, the term with the double underline, everything to its left and the terms,  $-G_{c_{n-1}} v_{c_n}^a G_{c_n}$ , to the right cancel with line 3. We continue this process until we get to the term with the triple underline. This term, along with what remains on the right cancels the last  $n+1^{\text{st}}$  line. What remains is the expression on the right-hand side of the equality. This is the  $n^{\text{th}}$  order term in the sum. Putting these terms together each such  $n$  gives:

$$\sum_c \tilde{G}_{ac}^{(n)}(A) = G_b(A) + \sum_c G_b(A) v_{c_n}^b(A) G_c(A) + \dots$$

which is just the perturbation expansion for  $\sum_c \tilde{G}_{bc}^{(n)}(A)$ . This proves that  $\tilde{G}(A)$  is well defined.

Now we proceed to demonstrate that we can identify  $\tilde{G}(A)$  with  $G(A)$ . Using (38) we have

$$\begin{aligned}
\tilde{G}(A) &= \sum_b G_a(A) \delta_{ab} + \sum_c G_a(A) v_c^a(A) \sum_b G_{cb}(A) \\
\tilde{G}(A) &= G_a(A) + G_a(A) \left[ \sum_c v_c^a(A) \right] \tilde{G}(A).
\end{aligned}$$

From (41) we have

$$\tilde{G}(A) = G_a(A) + G_a(A) v^a(A) \tilde{G}(A).$$

This must hold for any  $a$ . Therefore  $\tilde{G}(A)$  satisfies all relevant resolvent identities and may be identified with the RM resolvent operator  $G(A)$ , since satisfying all Lippmann-Schwinger equations is enough to yield uniqueness, even though a single one does not.<sup>8</sup> This concludes the proof of (40).

It follows that for nontrivial reaction mechanisms, the RM resolvent may be obtained by equations (31) and (38). The kernel and driving terms of (38) are determined from solutions of  $i$ -body problems using (22), (24), (30), and (33).

#### B. Full N-Body Equations

In part A we derived connected kernel equations for  $G(A) = (Z - H(A))^{-1}$ . Given a reaction mechanism  $A$  there is a complementary reaction mechanism  $A'$ . Since  $H = \sum_{a \in A_0} H(a)$ ,  $A' = A_0 \setminus A$  implies that

$$H = H(A) + H(A'). \quad (48)$$

From (48) follow the relations

$$\begin{aligned}
G &= G(A) + G(A) H(A') G \\
G &= G(A') + G(A') H(A) G.
\end{aligned}$$

Iterating these equations gives<sup>9</sup>

$$G = G(A) + G(A) H(A') G(A') + G(A) H(A') G(A') H(A) G. \quad (49)$$

In the next section we prove that (49) is a connected kernel equation for any  $c$  of  $A$ . At this stage we have a full N-body theory. We note that we actually have a family of N-body theories— one for each complementary pair of reaction mechanisms. Since our main interest is in scattering phenomena, it is advantageous to reformulate these equations in terms of transition operators. We do this in the last part of this section.

### C. Scattering Operators

Recall the definitions of our RM scattering operators (Eq. (36)). These can be calculated from (36), (38), and (39). Direct calculation of the  $T(A)$ 's will clearly save us a quadrature or two. Another advantage in calculating the RM transition operators directly is that for any  $A$ ,  $G_A(A)$  is never zero. It follows that (38) is a system of coupled equations with  $B_N - 1$  coupled equations [19] where  $B_N$  is the  $N^{\text{th}}$  Bell number, i.e., the number of distinct partitions of  $N$  particles. The  $-1$  is because we have no terms for the one-cluster partition  $\underline{1}$ . For sufficiently simple RM's transition operator equations may couple together far fewer amplitudes.

Transition operator equations for the  $T_+^{ab}(A)$  may be obtained in a straightforward way by rewriting (36) (using (34)) as

$$T_+^{ab}(A) = V^a(A) G(A) G_b^{-1}(A).$$

Using (43) and (34) in (48) we obtain

$$T_+^{ab}(A) = \sum_c V_c^a(A) G_c(A) G_b^{-1}(A) + \sum_c V_c^a(A) G_c(A) T_+^{cb}(A). \quad (50)$$

We note for  $A = A_0$  these are the fully-off-shell BRS equations in precursor form [8]. The same proof that shows the first iterated kernel of (38) is connected, also shows that the first iterated kernel of (50) is connected. The advantage of (50) is that  $V_c^a(A) = 0$  when  $c$  is such that  $c \not\supset d$  for at least one  $a_d \in A$ , reducing the number of coupled equations. Generally the largest reduction will be when  $A$  is a few-body RM. We note that (38) and (50) will both have vanishing iterated kernels when  $A$  is trivial. An equation which can yield corrections to  $T_+^{ab}(A)$  from the complementary RM, can be obtained by rewriting (49) as a scattering operator equation. One obtains

$$T_+^{ab} = -G_b^{-1} + G_a^{-1} G(A) G_b^{-1} + G_a^{-1} G(A) \hat{T} G(A) G_b^{-1} \quad (51)$$

$$\hat{T} = T_+^{00}(A') + H(A') G(A) H(A) G(A') \hat{T}. \quad (52)$$

Equation (52) is the transition operator version of (49). It also has a connected kernel. Equation (51) can be simply characterized when it is evaluated between on-shell states of the form  $\langle \phi_a(\alpha_a) |, | \phi_b(\beta_b) \rangle$  with  $\alpha_a, \beta_b \in A$ . In this case (51) becomes

$$\langle \phi_a(\alpha_a) | T_+^{ab} | \phi_b(\beta_b) \rangle = \langle \phi_a(\alpha_a) | (T_+^{ab}(A) + \Omega^{+a}(A) \hat{T} \Omega^{+b}(A)) | \phi_b(\beta_b) \rangle \quad (53)$$

where  $\Omega^{+a}(A) = G_a^{-1}(A) G(A)$ ,  $\Omega^{+b}(A) = G(A) G_b^{-1}(A)$ . Each term in (53) has a simple physical interpretation. The first term  $T_+^{ab}(A)$  describes the scattering that occurs purely through the primary RM,  $A$ . The wave operator  $\Omega^{+b}(A)$  distorts the incoming wave through the primary RM. The operator  $\hat{T}$  scatters the distorted wave through the secondary RM,  $A'$ , at least once. The wave operator  $\Omega^{+a}(A)$  distorts the outgoing wave through the primary RM. The treatment is particularly appropriate when the primary RM describes most of the physics, and the  $A'$  corrections can be included perturbatively. When  $a \notin A$  one must use (51) directly.

Once we show that (38) and (49) or (50) and (52) are connected kernel equations the  $N$ -body aspects of our theory are complete. We can construct the  $N$ -body scattering states by applying  $G G_b^{-1}$  to the states  $|\phi_b(\alpha_b)\rangle$  (the maximally connected partition eigenstates). Fredholm theory guarantees that the residues of the poles of  $G$  are finite rank projection operators on the  $N$ -body bound states. Since any degeneracy is always finite, the  $N$ -body bound states can all be isolated. If we were to continue by solving the  $N+1$  body problem, these solutions would then be needed to construct the  $P_a(A)$ 's for  $N$ -body systems.

The other remaining problem is to prove Theorem 2 which is needed to show that the operator  $T^{ab}(A)$  is related to the RM- $A$  in the sense discussed in (19). This problem and the connectivity problem are considered in the next section.

### V. Properties of Equations

In this section we discuss properties of the dynamic equations introduced in Sect. IV. We show that (38), (49), (50), and (52) are connected kernel equations for any choice of nontrivial A. Assuming that all of the operators used as input are consistent with the operating F.C.A., all of these equations will have unique solutions by Fredholm theory. We also show the RM transition operators satisfy an optical theorem of the form (18) involving only those partial cross sections associated with the RM-A.

#### A. Connectivity of the RM Operator Equations

We show that the first iterates of (38) and (50) are connected. For (38) the iterated kernel is  $\sum_c' G_a(A) v_c^a(A) G_c(A) v_b^c(A)$ , while for (50) it is  $\sum_c' v_c^a(A) G_c(A) v_b^c(A) G_b(A)$ . To show both of these are connected it is sufficient to show<sup>10</sup>

##### Theorem 4:

$$\sum_c' v_c^a(A) G_c(A) v_b^c(A) \text{ is connected for any nontrivial A.} \quad (54)$$

By definition of  $v$  we have

$$\sum_c' v_c^a(A) G_c(A) v_b^c(A) = \sum_c' C_c v_c^a(A) G_c(A) v_b^c(A) C_b. \quad (55)$$

We can drop the  $C_b$  on the right of (55) without changing any results. Making a connectivity expansion of  $v_c^a(A) G_c(A)$  we have

$$\begin{aligned} \sum_c' C_c \sum_{d(\subseteq c)} [v_d^a(A) G_d(A)]_d v_b^c(A) \\ = \sum_c' C_c \sum_{d(\subseteq c)} [(v_d^a(A) + v_c^{ad}(A))(G_d(A) + G_d(A) v_c^d(A) G_c(A))]_d v_b^c(A). \end{aligned} \quad (56)$$

We have used (31) and (35) in the above. Note  $v_d^c(A) = 0$  in (35) for  $d \subseteq c$ .

Since terms with connectivities external to  $d$  (upper  $d$ 's) do not contribute to the  $d$  connected part of an operator, (56) becomes

$$\sum_c' C_c \sum_{d(\subseteq c)} [v_d^a(A) G_d(A)]_d v_b^c(A).$$

Changing the order of the sums gives

$$\sum_d' [v_d^a(A) G_d(A)]_d \sum_{c(\supseteq d)}' C_c v_b^c(A).$$

Using the representation (30) for  $v_b^c(A)$  gives

$$\begin{aligned} \sum_d' [v_d^a(A) G_d(A)]_d \sum_{c(\supseteq d)}' C_c \sum_{\substack{e(\subseteq b) \\ e(\supseteq d)}} [H(A)]_e \\ = \sum_d' \sum_{e(\subseteq b)} [v_d^a(A) G_d(A)]_d [H(A)]_e \left[ \sum_{c(\supseteq d)}' (1 - \delta_{e \subseteq c}) C_c \right]. \end{aligned} \quad (57)$$

Now from Lemma 1 we have

$$\sum_{c(\supseteq d)}' C_c = 1, \quad \sum_{c(\supseteq d)}' \delta_{e \subseteq c} C_c = \sum_{c(\supseteq d \cup e)}' C_c = 1 - \delta_{n_d \cup e = 1}. \quad (58)$$

Using (58) in (57) gives

$$\sum_c' v_c^a(A) G_c(A) v_b^c(A) = \sum_d' \sum_{e(\subseteq b)} [v_d^a(A) G_d(A)]_d [H(A)]_e \delta_{n_d \cup e = 1}.$$

The condition  $n_d \cup e = 1$  means that  $d \cup e = 1$  which implies that  $[1_d \cdot 1_e]$  is connected. Thus  $\sum_c' v_c^a(A) G_c(A) v_b^c(A)$  is completely connected for any nontrivial A. The connectivity of (38) and (50) follow immediately. Nontriviality must be assumed because for trivial A the iterated kernel is zero.

#### B. Connectivity of Full Dynamic Equations

Now we show that Eqs. (49) and (52) have connected kernels. Here we deal with kernels of the form  $G(A) H(A') G(A') H(A)$  and  $H(A') G(A) H(A) G(A')$ . It suffices to show<sup>10</sup>

**Theorem 5:**

$H(A') G(A') H(A)$  and  $H(A') G(A) H(A)$  are connected.

**Proof:**

Since the proofs of these results are similar we only consider the  $H(A') G(A') H(A)$  case. Using (20), (21), (22), (31) and (34) we have

$$H(A') G(A') H(A) = \sum_a C_a H_a(A') [G_a(A') + G_a(A') V^a(A') G(A')] [H_a(A) + V^a(A)]$$

$$= \sum_a C_a H_a(A') G_a(A') H_a(A) + \sum_a C_a H_a(A') G_a(A') V^a(A') G(A') H(A)$$

$$+ \sum_a C_a H_a(A') G_a(A') V^a(A).$$

The second and third terms on the right contain  $\sum_a C_a H_a(A') G_a(A') V^a(A')$  and  $\sum_a C_a H_a(A') G_a(A') V^a(A)$ . These can be shown to be connected using the same methods employed in proving (54). The remaining term is  $\sum_a C_a H_a(A') G_a(A') H_a(A)$ .

Using (22) and (33) this term becomes

$$\sum_a C_a H_a P_a(A') \left( \frac{Q_a(A')}{Z} + P_a(A') G_a P_a(A') \right) P_a(A) H_a. \quad (59)$$

For complementary reaction mechanisms we have by (8), (21)

$$P_a(A) = Q_a(A')$$

$$P_a(A') P_a(A) = 0 \quad (60)$$

$$P_a(A') Q_a(A') = 0.$$

Using (60) in (59) gives

$$\sum_a C_a H_a(A') G_a(A') H_a(A) = 0.$$

It then follows that  $H(A') G(A') H(A)$  is connected.

We remark that  $A$  and  $A'$  do not actually have to be complementary; they only need by disjoint. This means that several such disjoint reaction mechanisms may be joined in this manner. We discuss this in Appendix A. We also

note that unlike (54), Theorem 5 holds for both trivial and nontrivial RM's.

**C. Spectral Properties of RM Operators**

In order to understand unitarity, we must first understand the spectral properties of the operators that serve as input in the equations for the RM operators.

We begin by discussing the eigenstates of  $H_a(A)$ . (See, e.g., (22).) If  $a_b \in A$  we have

$$P_a(A) |z_a(a_b)\rangle = |z_a(a_b)\rangle$$

$$Q_a(A) |z_a(a_b)\rangle = 0. \quad (61)$$

By (5), (22) and (61) we have

$$0 = (E_{a_b} - H_a) |z_a(a_b)\rangle = (E_{a_b} - H_a(A)) |z_a(a_b)\rangle. \quad (62)$$

If  $a_b \notin A$  it follows that

$$H_a(A) |z_a(a_b)\rangle = H_a P_a(A) Q_a(A) |z_a(a_b)\rangle = 0.$$

We see that  $H_a(A)$  will have three kinds of eigenstates: maximally connected eigenstates of  $H_a$  with  $a_b \in A$ ; scattering eigenstates of  $H_a$  that evolve from maximally connected eigenstates of  $H_b$  with  $a_b \in A$ ; and eigenstates of  $H_a$  with  $a_b \notin A$ . We will refer to this last class as zero energy states.

The most interesting eigenstates of  $H_a(A)$  are the scattering eigenstates. From the above we know that these are also scattering eigenstates of  $H_a$ . Because  $a_b \in A$  we note that  $|z_b(a_b)\rangle$  is a maximally connected eigenstate of  $H_b(A)$ . Considered as eigenstates of  $H_a$ , the scattering states of  $H_a(A)$  satisfy:

$$|z_a^+(a_b)\rangle = |z_b(a_b)\rangle + C_b^+ V_a^b |z_b^+(a_b)\rangle. \quad (63)$$

Alternatively we may ask what is the scattering eigenstate of  $H_a(A)$  that evolves from the eigenstate  $|z_b(a_b)\rangle$  of  $H_b(A)$ . These boundary conditions give

$$|\phi_a^+(a_b); A\rangle = |\phi_b(a_b)\rangle + G_b^+(A) V_a^b(A) |\phi_a^+(a_b); A\rangle. \quad (64)$$

To determine the relationship between (63) and (64) we consider  $|\phi_a^+(a_b)\rangle$  as an eigenstate of  $H_a(A)$  rather than  $H_A$ . Equation (62) allows us to write

$$(E_{a_b}^+ - H_b(A)) |\phi_a^+(a_b)\rangle = V_a^b(A) |\phi_a^+(a_b)\rangle.$$

This gives

$$|\phi_a^+(a_b)\rangle = |\epsilon_b(a_b)\rangle + G_b^+(A) V_a^b(A) |\phi_a^+(a_b)\rangle \quad (65)$$

where  $|\epsilon_b(a_b)\rangle$  satisfies  $(E_{a_b} - H_b(A)) |\epsilon_b(a_b)\rangle = 0$ . If we write (63) and (65) in a coordinate representation, as the clusters of  $b$  get far apart, the contributions from  $V_a^b$  and  $V_A^b$  vanish. Since the function  $\langle \vec{r} | \phi_a^+(a_b) \rangle$  in (63) and (65) is the same, we must have

$$|\epsilon_b(a_b)\rangle = |\phi_b(a_b)\rangle.$$

Using (35) and (65) we can express (65) as

$$|\phi_a^+(a_b)\rangle = (1 + G_a^+(A) V_a^b(A)) |\phi_b(a_b)\rangle. \quad (66)$$

Using (35) we may express (64) as

$$|\phi_a^+(a_b); A\rangle = (1 + G_a^+(A) V_a^b(A)) |\phi_b(a_b)\rangle. \quad (67)$$

Comparing (66) and (67) we have

$$|\phi_a^+(a_b)\rangle = |\phi_a^+(a_b); A\rangle.$$

This identification allows us to make the identifications

$$G_a^+ G_b^{+1} |\phi_b(a_b)\rangle = (1 + G_a^+ V_a^b) |\phi_b(a_b)\rangle = (1 + G_a^+(A) V_a^b(A)) |\phi_b(a_b)\rangle = G_a^+(A) G_b^{+1}(A) |\phi_b(a_b)\rangle \quad (68)$$

for  $b \leq a$ , when operating on maximally connected states  $|\phi_b(a_b)\rangle$  with  $a_b \in A$ .

This result will be important in deriving unitarity relations for the  $T_{+}^{ab}(A)$ .

#### D. RM Unitarity

We now prove Theorem 2 by showing that the RM transition operators (36) satisfy a unitarity relation consistent with the underlying RM. This means

Theorem 6: (RM Optical Theorem)

$$\text{Im} \langle \phi_a(a_a); E | T_{+}^{aa}(A; E) | \phi_a(a_a); E \rangle = -\frac{1}{2} \text{Im} \langle \sum_{b \in A} \langle \phi_a(a_a); E | \sigma_{a_a} \cdot \sigma_b (A; E) \rangle \rangle. \quad (69)$$

Proof:

Throughout this proof we use the following notation:

$$\Delta_{\epsilon} A(E) = A(E + i\epsilon) - A(E - i\epsilon)$$

$$\Delta A(E) = \lim_{\epsilon \rightarrow 0} [A(E + i\epsilon) - A(E - i\epsilon)]$$

$$A(E^+) = A(E + i0^+)$$

$$Z = E + i\epsilon.$$

To show our unitarity relation we calculate  $\Delta_{\epsilon} T_{+}^{ab}(A; E)$ . In order to evaluate this we need some additional results. We observe (36) implies

$$T_{+}^{ab}(A) = T_{+}^{ab}(A) + V^b(A) - V^a(A). \quad (70)$$

Using (50) and (70) it is easy to show

$$T_{+}^{ab}(A) = V^a(A) + \sum_c T_{+}^{ac}(A) G_c(A) V^b(A) \quad (71)$$

$$T_{-}^{ab}(A) = V^b(A) + \sum_c T_{-}^{ac}(A) G_c(A) V^b(A) \quad (72)$$

where the operators  $\hat{T}^{ac}(A)$  are defined

$$\hat{T}^{ab}(A) = V_b^a(A) + \sum_c V_c^a(A) G_c(A) \hat{T}^{cb}(A) = V_b^a(A) + \sum_c \hat{T}^{ac}(A) G_c(A) V_b^c(A). \quad (73)$$

Using (71) we find

$$\begin{aligned} \Delta_{\epsilon} T_{+}^{ab}(A; E) &= \sum_c (\hat{T}^{ac}(A) G_c(A) - \hat{T}^{ac}(A)^* G_c(A)^*) V^b(A) \\ &= \sum_c (\Delta_{\epsilon} \hat{T}^{ac}(A; E) G_c(A) + \hat{T}^{ac}(A) \Delta_{\epsilon} G_c(A; E)) V^b(A). \end{aligned} \quad (74)$$

We now use



**Lemma 6:**

$$\Delta_c \hat{T}^{ac}(A;E) = \sum_d \hat{T}^{ad}(A) \Delta_c G_d(A;E) \hat{T}^{dc}(A)^*. \quad (75)$$

This and the subsequent lemmas in this proof are proven in Appendices B, C,

and D. Using (75) in (74) we find

$$\Delta_c \hat{T}_+^{ab}(A;E) = \sum_c \sum_d \{ \hat{T}^{ad}(A) \Delta_c G_d(A;E) \hat{T}^{dc}(A)^* G_c(A) + \delta_{cd} \} V^b(A).$$

Using (72) this becomes

$$\Delta_c \hat{T}_+^{ab}(A;E) = \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A;E) \hat{T}_-^{cb}(A)^*.$$

Taking the limit as  $\epsilon \rightarrow 0$  gives

**Lemma 7:**

$$\Delta \hat{T}_+^{ab}(A;E) = \sum_c \hat{T}^{ac}(A;E^+) \Delta G_c(A;E) \hat{T}_-^{cb}(A;E^-). \quad (76)$$

From (7), (8), (21) and (33) it follows that

$$\Delta G_c(A;E) = -2\pi i \sum_{d(\subseteq c)} \sum_{\alpha_d \in A} \int d\vec{p}_1 \dots d\vec{p}_{n-1} |\phi_c(\alpha_d)\rangle \delta(E - E'(\alpha_d, \vec{p})) \langle \phi_c(\alpha_d)|. \quad (77)$$

Since the  $\delta$ -function keeps the states  $|\phi_c(\alpha_d)\rangle$  on shell we may use (66) and

(68) to express (77) as

$$\begin{aligned} \Delta G_c(A;E) &= \sum_{d(\subseteq c)} \sum_{\alpha_d \in A} G_c(A;E^+) G_d^{-1}(A;E^+) \\ &\times [-2\pi i \int d\vec{p}_1 \dots d\vec{p}_{n-1} |\phi_d(\alpha_d)\rangle \delta(E - E(\alpha_d; \vec{p})) \langle \phi_d(\alpha_d)|] G_d^{-1}(A;E^-) G_c(A;E^-). \end{aligned} \quad (78)$$

To simplify our expressions we define

$$D(\alpha_d;E) = -2\pi i \int d\vec{p}_1 \dots d\vec{p}_{n-1} |\phi_d(\alpha_d)\rangle \delta(E - E(\alpha_d; \vec{p})) \langle \phi_d(\alpha_d)|. \quad (79)$$

Using (78) and (79) in (76) gives

$$\begin{aligned} \Delta \hat{T}_+^{ab}(A;E) &= \sum_c \hat{T}^{ac}(A;E^+) \sum_{d(\subseteq c)} \sum_{\alpha_d \in A} G_c(A;E^+) G_d^{-1}(A;E^+) D(\alpha_d;E) G_d^{-1}(A;E^-) \\ &\times G_c(A;E^-) \hat{T}_-^{cb}(A;E^-). \end{aligned} \quad (80)$$

Since we already have good equations for  $\hat{T}_+^{ab}(A;E)$  we may use the solved form

(36) to write

$$G_d^{-1}(A;E^-) G_c(A;E^-) \hat{T}_-^{cb}(A;E^-) = G_d^{-1}(A;E^-) G_c(A;E^-) [1 + V^c(A) G(A;E^-)] V^b(A).$$

Using (34) this becomes

$$\begin{aligned} G_d^{-1}(A;E^-) G_c(A;E^-) \hat{T}_-^{cb}(A;E^-) &= G_d^{-1}(A;E^-) G_c(A;E^-) V^b(A) + V^b(A) + V^d(A) G(A;E^-) V^b(A) \\ &= \hat{T}_-^{db}(A;E^-). \end{aligned} \quad (81)$$

We remark that one does not need the solved form of  $\hat{T}_-^{ab}(A)$  to prove (81). One can show it directly from the dynamic equations using arguments similar to those in Sect. IV.A. Using (81) in (80) gives

$$\Delta \hat{T}_+^{ab}(A;E) = \sum_c \sum_{d(\subseteq c)} \sum_{\alpha_d \in A} \hat{T}^{ac}(A;E^+) G_c(A;E^+) G_d^{-1}(A;E^+) D(\alpha_d;E) \hat{T}_-^{db}(A;E^-). \quad (82)$$

Changing the order of the summation in (82) gives

$$\Delta \hat{T}_+^{ab}(A;E) = \sum_d \sum_{\alpha_d \in A} \left[ \sum_{c(\supseteq d)} \hat{T}^{ac}(A;E^+) G_c(A;E^+) G_d^{-1}(A;E^+) \right] D(\alpha_d;E) \hat{T}_-^{db}(A;E^-). \quad (83)$$

We now note

**Lemma 8:**

$$\hat{T}^{ac}(A;E^+) G_c(A;E^+) G_d^{-1}(A;E^+) D(\alpha_d;E) = 0 \text{ if } c \not\supseteq d$$

This lemma means that we can extend the  $c$  sum in (83):  $\sum_{c(\supseteq d)} \rightarrow \sum_c$ . Using

(73) we get

$$\begin{aligned} \sum_c \hat{T}^{ac}(A;E^+) G_c(A;E^+) G_d^{-1}(A;E^+) &= \sum_c V_c^a(A) G_c(A;E^+) G_d^{-1}(A;E^+) + \sum_c V_c^a(A) G_c(A;E^+) \\ &\times \left[ \sum_e \hat{T}^{ce}(A;E^+) G_e(A;E^+) G_d^{-1}(A;E^+) \right]. \end{aligned} \quad (85)$$

Comparing (85) to (50) and noting that the solution must be unique it follows that

$$\sum_c \hat{T}^{ac}(A;E^+) G_c(A;E^+) G_d^{-1}(A;E^+) = \hat{T}_+^{ab}(A;E^+). \quad (86)$$

Using (86) in (83) after extending the  $c$  sum gives

$$\Delta T_+^{ab}(A;E) = \sum_{d \in A} T_+^{ad}(A;E^+) D_d(\alpha_d;E) T_-^{db}(A;E^-) \quad (87)$$

where we have used  $\sum_d \sum_{a \in A} = \sum_{a \in A}$ . From (36) we have  $\{T_+^{ab}(A;E^+)\}^+ = T_-^{ba}(A;E^-)$ . It follows that

$$\Delta T_+^{ab}(A;E) = \sum_{d \in A} T_+^{ad}(A;E^+) D_d(\alpha_d;E) \{T_+^{bd}(A;E^+)\}^+ \quad (88)$$

Equations (87) or (88) are the basic operator unitarity relations for our RM transition operators. To put them in optical theorem form (69) we take on-shell matrix elements of (88) between states  $|\phi_a(\alpha_a)\rangle$  for  $\alpha_a \in A$ . Using this with (79) gives

$$2i \operatorname{Im} \langle \phi_a(\alpha_a) | T_+^{aa}(A;E^+) | \phi_a(\alpha_a) \rangle = -2\pi \sum_{b \in A} \int d\vec{p}_1 \dots d\vec{p}_{n_b-1} \delta(E - E(d_b; \vec{p})) \times |\langle \phi_a(\alpha_a) | T_+^{ab}(A;E^+) | \phi_b(\beta_b) \rangle|^2 \quad (89)$$

$$= -1[2\pi \sum_{b \in A} \int |\langle \phi_a(\alpha_a) | T_+^{ab}(A;E^+) | \phi_b(\beta_b) \rangle|^2 \frac{d\rho_f(\alpha_a, \vec{p}, E)}{d\vec{p}} d\vec{p}_1 \dots d\vec{p}_{n_b-1}] \quad (90)$$

where  $\rho_f$  is the density of final states. Using the standard quantum mechanical relation between the cross section and the transition operators [21], we can identify (90) with

$$= -1 |J_{inc}| \sum_{b \in A} \sigma_{\alpha_a \rightarrow \beta_b}(E; A). \quad (91)$$

Combining (89) and (91) we get our RM optical theorem (69). As stated before this unitarity result proves Theorem 2. This justifies the interpretation of the operators  $T^{ab}(A)$  as transition operators describing the reaction mechanism A in the sense of the definition (17).

## VI. Summary And Conclusion

### A. Summary

We have presented a unified treatment of different reaction mechanisms in nonrelativistic N-body scattering. This theory is based on connected kernel integral equations that are expected to become compact given reasonable constraints on the potentials<sup>11</sup>. These equations can be formulated for any important set of asymptotic channels. The operators  $T_+^{ab}(A)$  are approximate transition operators that describe the scattering proceeding through an arbitrary reaction mechanism A. These operators are uniquely determined by a connected kernel equation and satisfy an optical theorem consistent with the choice of reaction mechanism. Connected kernel equations relating  $T_+^{ab}(A)$  to the full  $T^{ab}$  allow us to correct the approximate solutions for any ignored process to any order.

Although our theory is capable of treating any reaction mechanism A, it is only when the individual states of A involve a few clusters, that the equations for the  $T^{ab}(A)$ 's result in a substantial simplification of the exact theory. In this case of few-body reaction mechanisms, two basic simplifications occur. The most important is that the number of continuous vector variables in the kernel of the integral equations reduces to  $N_A - 1$ , where  $N_A = \max_{\alpha_a \in A} n_{\alpha_a}$ . Since the vector nature of these variables can be reduced by separable expansions and partial wave techniques, one can usually get the number of continuous degrees of freedom down to  $N_A - 1$ . This reduces the numerical complexity of the many-body problem to that of a few-body problem which may admit exact numerical treatments. The other simplification is in the number of equations involved. If one defines the set

$$P_A = \{a | a \geq b, \alpha_b \in A\},$$



the number of distinct partitions in this set is the number of coupled equations for  $T^{ab}(A)$ . For highly clustered states  $b$ , this number is considerably less than  $B_N - 1$ , which is the number of equations for the exact  $T^{ab}$  when every possible type of multiparticle force exists.

This theory represents a considerable improvement over model theoretic treatments of few-body reaction mechanisms. The approximate dynamics is dictated directly from the reaction mechanism and automatically includes all corrections (such as spectroscopic factors and overcounting corrections) needed to maintain the integrity of the underlying N-body nature of the system. It shows how all corrections to the approximate solution come in, including effects of particle identity. It gives a unified treatment of all few-body reaction mechanisms, with the same dynamic simplicity of the model calculation, but can include complicated reaction mechanisms involving overlapping configurations where it is difficult to formulate models.

### B. Applications

The most important consideration in applying these methods to physical systems is choosing a dominant reaction mechanism. It is only when the reaction mechanism  $A$  accounts for most of the dynamics that the approximation  $T_+^{ab} \rightarrow T_+^{ab}(A)$  can be expected to be good. In this same situation one expects the corrections should be amenable to perturbative treatments. Finding the most important set of asymptotic states  $A$  may not be simple. One expects that final states that are responsible for large reaction cross sections should be included in  $A$ . On the other hand, other states may have to be included even if they have negligible or no cross section. There can be several reasons for this. First, the available phase space on the energy

shell limits the size of various cross sections, but the associated processes could still be important off shell. A better measurement would be to consider the ratio of the actual cross section to the available phase space. For some processes we may be below the threshold for the opening of the channel. Again these states may introduce important off-shell effects. One can try to use a phenomenological picture of the reaction mechanism to decide if these states are important. Another possibility is to look at the size of the cross section above threshold. If substantial matrix elements are indicated, we might expect the state to be important. At this stage ingenuity and physical insight are important ingredients in choosing reaction mechanisms. Once this is done, however, the formalism presented here always yields a mathematically well-behaved framework.

The most important applications to our formalism is to few-body reaction mechanisms. It is here that the formalism should result in a sufficient simplification to allow for numerical calculation. The methods of Benze and Redish [22] for treating identical particles in N-body systems can be applied to our equations and result in an additional reduction of the number of coupled equations when identical particles are involved.

An example of the type of system that one could apply these methods to is nucleon-nucleus scattering above about 50 MeV. Here it is observed that the dominant processes are single nucleon knockout [23]. This suggests keeping those channels involving the incident nucleon, a nucleon from the nucleus and the residual core. Our formalism allows us to treat all possible configurations of two nucleons and a core simultaneously. When the BR symmetrization is applied we would get four coupled two-vector variable integral equations corresponding to the configurations  $(p,A)$ ,  $(n,A')$ ,  $(p,n,A-1)$ ,  $(p,p,A-1)$

The six continuous degrees of freedom could then be reduced to one or two using partial wave techniques and separable expansions. For comparison we note a three-body problem with local potentials and distinguishable particles involves three coupled two-vector variable integral equations. It follows that the dynamics for scattering of N-body systems with three-body dominant reaction mechanisms will have essentially the same complexity as a three-body system has. A calculation of this type is in preparation.

Another application concerns processes involving several reaction mechanisms. If these reaction mechanisms are amenable to different types of dynamic treatments, our formalism allows us to treat each one individually, using different methods for each one. It then shows us how they must be combined to give the full dynamics. Applications of this type are somewhat more ambitious than the previous types. We will not discuss them further.

The last application is with few-body systems. We note for any choice of A our equations give a reaction theory equivalent to N-body Schrödinger theory. Unlike some few-body theories our formalism incorporates all combinations of many-body forces. Thus it is a useful formalism for exploring the important questions concerning the existence of few-body forces in many-body systems.

In conclusion we constructed an exact unified formalism for truncating an exact N-body scattering theory to describe any set few-body reaction mechanisms in a natural and unitary manner. We are able to handle any type of reaction mechanism, and still maintain the connected structure of our theory.

## Appendix A

In this appendix we discuss extensions of the basic theory: the inclusion of N-body forces, and extensions to more than two reaction mechanisms.

### A. N-body Forces

In order for an N-body force to be compatible with our theory it must satisfy an appropriate F.C.A. This means that the N-body force must be compact as a Hilbert space operator (assuming we have removed the CM degrees of freedom). Since the decomposition (13) cannot support a fully connected operator, we must modify our formalism. There are two considerations: How the equations must be modified, and how the modifications affect unitarity.

The compactness of the N-body forces makes them easy to include. Different methods of treating these forces are suitable for different applications. We present one of many possible techniques for including them. We let  $V_1$  denote our N-body force, and assume that it can be split into a part corresponding to each RM,

$$V_1 = V_1(A) + V_1(A')$$

where  $V_1(A)$  and  $V_1(A')$  are compact, and may be zero. Using our previous notation (2), (3) we define

$$\begin{aligned} V^a(A) &= V^a(A) + V_1(A) \\ V_b^a(A) &= V_b^a(A) + V_1(A) \\ G(A) &= (Z - H(A) - V_1(A))^{-1}. \end{aligned} \quad (A1)$$

With these definitions our RM transition operator becomes

$$\begin{aligned} T_+^{ab}(A) &= V^a(A) + V^a(A) G(A) V_b^b(A) \\ &= V^a(A) (G_b(A) + G(A) V_b^b(A) G_b^{-1}(A)). \end{aligned} \quad (A2)$$

Similarly for  $T_-^{ab}(A)$ . From (A1) it follows that

$$G(\underline{A}) = G_b(\underline{A}) + G(\underline{A}) V^b(\underline{A}) G_b(\underline{A}).$$

Using this in (A2) gives

$$T_+^{ab}(\underline{A}) = V^a(\underline{A}) G(\underline{A}) G_b^{-1}(\underline{A}).$$

By Lemma 1

$$\sum_c C_c V_1(\underline{A}) = V_1(\underline{A})$$

which implies

$$\begin{aligned} T_+^{ab}(\underline{A}) &= \sum_c C_c V_c^a(\underline{A}) (G_c(\underline{A}) + G_c(\underline{A}) V_c^b(\underline{A}) G(\underline{A})) G_b(\underline{A}) \\ T_+^{ab}(\underline{A}) &= \sum_c C_c V_c^a(\underline{A}) G_c(\underline{A}) G_b^{-1}(\underline{A}) + \sum_c C_c V_c^a(\underline{A}) G_c(\underline{A}) T_+^{cb}(\underline{A}). \end{aligned}$$

We define

$$v_c^a(\underline{A}) \equiv C_c V_c^a(\underline{A})$$

giving

$$T_+^{ab}(\underline{A}) = \sum_c v_c^a(\underline{A}) G_c(\underline{A}) G_b^{-1}(\underline{A}) + \sum_c v_c^a(\underline{A}) G_c(\underline{A}) T_+^{cb}(\underline{A}). \quad (A3)$$

This generalizes Eq. (50) when N-body forces are included. Because the first iterated kernel of (A3) differs from that of (50) only by terms containing  $V_1(\underline{A})$ , it follows that the first iterated kernel is connected. We note it differs from (50) by the modification

$$v_c^a(\underline{A}) \rightarrow v_c^a(\underline{A}) + C_c V_1(\underline{A}).$$

We get a similar equation for the  $A'$ -RM.

We may now use our previous results on unitarity to show (A2) satisfies a proper unitarity relation. To do this we note

$$\Delta_c T_+^{ab}(\underline{A}) = V^a(\underline{A}) (\Delta_c G(\underline{A})) V^b(\underline{A}). \quad (A4)$$

We also note

$$G(\underline{A}) = G(\underline{A}) + G(\underline{A}) V_1(\underline{A}) G(\underline{A}) \rightarrow \quad (A5)$$

$$\begin{aligned} \Delta_c G(\underline{A}) &= G(\underline{A}) - G(\underline{A})^* + (1 + G(\underline{A}) V_1(\underline{A})) \Delta_c G(\underline{A}) (1 + V_1(\underline{A}) G(\underline{A})^*) \\ &\quad - (1 + G(\underline{A}) V_1(\underline{A})) \Delta_c G(\underline{A}) (1 + V_1(\underline{A}) G(\underline{A})^*) \\ &= G(\underline{A}) - G(\underline{A})^* - G(\underline{A}) + G(\underline{A})^* + G(\underline{A}) V_1(\underline{A}) G(\underline{A})^* - G(\underline{A}) V_1(\underline{A}) G(\underline{A}) \\ &\quad + G(\underline{A})^* V_1(\underline{A}) G(\underline{A})^* - G(\underline{A}) V_1(\underline{A}) G(\underline{A})^* - G(\underline{A}) V_1(\underline{A}) G(\underline{A}) V_1(\underline{A}) G(\underline{A})^* \\ &\quad + G(\underline{A}) V_1(\underline{A}) G(\underline{A})^* V_1(\underline{A}) G(\underline{A})^* + (1 + G(\underline{A}) V_1(\underline{A})) \Delta_c G(\underline{A}) (1 + V_1(\underline{A}) G(\underline{A})^*) \end{aligned}$$

Using (A5) this reduces to

$$\Delta_c G(\underline{A}) = (1 + G(\underline{A}) V_1(\underline{A})) \Delta_c G(\underline{A}) (1 + V_1(\underline{A}) G(\underline{A})^*). \quad (A6)$$

From (36) and (88) it follows that

$$\Delta G(\underline{A}) = \sum_{a \in A} (1 + G(\underline{A}) V^a(\underline{A})) D(\gamma_a) (1 + V^a(\underline{A}) G(\underline{A})^*).$$

We take the limit as  $\epsilon \rightarrow 0$  in (A6). By the discussion in Appendix C, the compactness of  $V_1(\underline{A})$  allows us to bring this limit inside the terms  $1 + G(\underline{A}) V_1(\underline{A})$ , etc., giving

$$\begin{aligned} \Delta G(\underline{A}) &= \sum_{a \in A} (1 + G(\underline{A}) V_1(\underline{A})) (1 + G(\underline{A}) V^a(\underline{A})) D(\gamma_a) (1 + V^a(\underline{A}) G(\underline{A})^*) (1 + V_1(\underline{A}) G(\underline{A})^*) \\ &= \sum_{a \in A} (1 + G(\underline{A}) (V_1(\underline{A}) + V^a(\underline{A}))) D(\gamma_a) (1 + (V_1(\underline{A}) + V^a(\underline{A})) G(\underline{A})^*). \end{aligned}$$

Using this in (A4) with (A2) gives

$$\Delta T^{ab}(\underline{A}) = \sum_{c \in A} T_+^{ac}(\underline{A}) D(\gamma_c) T_-^{cb}(\underline{A})^*. \quad (A7)$$

Comparing (A7) and (87) it follows that  $T_+^{ab}(\underline{A})$  also satisfies a proper unitarity relation. Thus the inclusion of N-body forces in the RM operator equations does not change the unitarity structure of the operators. To combine the RM's Eq. (49) must be modified so

$$G(\underline{A}) \rightarrow G(\underline{A})$$

$$H(\underline{A}) \rightarrow H(\underline{A}) + V_1(\underline{A}).$$

Similarly for  $A \rightarrow A'$ . Again the connectivity is maintained.

## B. Several RM's

In some applications one may find it desirable to deal with several disjoint reaction mechanisms,  $\{A_i, i=1 \dots n | A_i \cap A_j = \emptyset, \bigcup_{i=1}^n A_i = A_0\}$  as opposed to a pair complementary RM's. In this case the RM operator equations (38) or (50) remain unchanged. If we define  $G(A_i \cup A_j) = (Z - H(A_i) - H(A_j))^{-1}$ , (49) is replaced by the equations

$$G(A_1 \cup A_2) = G(A_1) + G(A_1) H(A_2) G(A_2) + G(A_1) H(A_2) G(A_2) H(A_1) G(A_1 \cup A_2)$$

$$G(A_1 \cup A_2 \cup A_3) = G(A_1 \cup A_2) + G(A_1 \cup A_2) H(A_3) G(A_3) + G(A_1 \cup A_2) H(A_3) G(A_3) (H(A_1) + H(A_2)) G(A_1 \cup A_2 \cup A_3) \quad (A8)$$

$$G = G\left(\bigcup_{i=1}^{n-1} A_i\right) + G\left(\bigcup_{i=1}^{n-1} A_i\right) H(A_n) G(A_n) + G\left(\bigcup_{i=1}^{n-1} A_i\right) H(A_n) G(A_n) \left(\sum_{i=1}^{n-1} H(A_i)\right) G.$$

The proof used to show (49) has a connected kernel also shows that each of the equations (A8) have connected kernels, the key point being that the disjointness of the  $A_i$ 's make the disconnected parts of the kernels into products of orthogonal operators.

## Appendix B

In this appendix we prove Lemma 6 which says

$$\Delta_c \hat{T}^{ab}(A) = \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A) \hat{T}^{cb}(A)^*$$

where

$$\hat{T}^{ab}(A) = v_b^a(A) + \sum_c v_c^a(A) G_c(A) \hat{T}^{cb}(A) = v_c^a(A) + \sum_c \hat{T}^{ac}(A) G_c(A) v_b^c(A). \quad (B1)$$

To prove this we note

$$\begin{aligned} \Delta_c \hat{T}^{ab}(A) &= \hat{T}^{ab}(A) - \hat{T}^{ab}(A)^* \\ &= \hat{T}^{ab}(A) - \hat{T}^{ab}(A)^* - \sum_c \hat{T}^{ac}(A) (G_c(A) - G_c(A)^*) \hat{T}^{cb}(A)^* + \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A) \hat{T}^{cb}(A)^* \\ &= \sum_c \hat{T}^{ac}(A) (\delta_{cb} + G_c(A) \hat{T}^{cb}(A)^*) - \sum_d (\delta_{ad} + \hat{T}^{ad}(A) G_d(A)) \hat{T}^{db}(A)^* \\ &\quad + \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A) \hat{T}^{cb}(A)^*. \quad (B2) \end{aligned}$$

Using (B1) in (B2) gives

$$\sum_{c,d} (\hat{T}^{ad}(A) G_d(A) + \delta_{ad}) (v_c^d(A) - v_c^d(A)^*) (\delta_{cb} + G_c(A) \hat{T}^{cb}(A)^*) + \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A) \hat{T}^{cb}(A)^* \quad (B3)$$

But  $v_c^d(A) = v_c^d(A)^*$  so (B3) becomes<sup>12</sup>

$$\Delta_c \hat{T}^{ab}(A) = \sum_c \hat{T}^{ac}(A) \Delta_c G_c(A) \hat{T}^{cb}(A)^*$$

which is the desired result.

## Appendix C

In this appendix we prove Lemma 7 of the text. The context of this lemma is that we can bring the limit in (76) through the  $\tilde{T}^{ac}(A)$  and  $T_-^{cb}(A)^*$ . The problem arises because the operators become unbounded as  $\epsilon \rightarrow 0$ . This means the limit only makes sense when the operators are restricted to a suitably dense set of vectors in our N-body Hilbert space, and furthermore, products of these operators make sense only for certain combinations. Thus there are two concerns: the operators themselves, and the structure of the equations determining how the operators are put together. In this appendix we only concern ourselves with the second of these considerations.

The type of problem that arises is when some  $G_d(A)$  in  $\tilde{T}^{ac}(A)$  and some  $G_d(A)^*$  in  $T_-^{cb}(A)^*$  are connected by  $\delta$  functions in some of their continuum variables. This can cause singularities that would not normally appear, through the mechanism

$$\int dp \frac{2ic}{(p^2 + E)^2 + c^2} \rightarrow \int 2\pi i \delta(E + p^2) dp$$

where the  $2ic$  comes from the  $\Delta_c(A)$ . Other combinations with  $\tilde{T}^{ac}$  and  $\Delta_c(A)$ ,  $T_-^{cb}(A)^*$  and  $\Delta_c(A)$  can also cause problems. All of these problems can be traced to the fact that two similar denominators are separated by  $\delta$  functions in the momentum variables. We will show that our equations are structured so that this will never happen in Eq. (76).

To do this we use (38), (72) and (73) to express (76) as

$$\begin{aligned} \Delta_c T_+^{ab}(A) &= \sum_c \tilde{T}^{ac}(A) \Delta_c G_c(A) T_-^{cb}(A)^* \\ &= \sum_c (v_c^a(A) + \sum_{d,c} v_d^a(A) \tilde{G}_{dc}(A) v_c^a(A)) \Delta_c G_c(A) (v^b(A) + \sum_{d,g} v_f^c(A) \tilde{G}_{fg}(A)^* v^b(A)). \end{aligned}$$

In this form the singularities of the  $T$ 's are all contained in the  $\tilde{G}$ 's. Since by Theorem 4  $\sum_c v_c^a(A) \Delta_c G_c(A) v_f^c(A)$  is a connected operator, it follows that

there are no nontrivial  $\delta$  functions separating the singularities of  $\tilde{T}^{ac}(A)$  and  $T_-^{cb}(A)^*$ . We need only check that no similar singularities of  $\tilde{G}_{de}(A)$  or  $\tilde{G}_{fg}(A)^*$  are separated from  $\Delta_c G_c(A)$  by  $\delta$  functions in their momentum variables. It suffices to consider the product

$$\sum_e \tilde{G}_{de}(A) \sum_c v_c^e(A) \Delta_c G_c(A) v_f^c(A).$$

We let  $\gamma_g \in A$  be a cut of  $G_c(A)$ , which requires  $g \subseteq (c)$ . Consider the first appearance of this cut in  $\tilde{G}_{de}$ . Using (38) we can write

$$\tilde{G}_{de}(A) = \int_a (\tilde{G}_{da}(A) v_e^a(A) + \delta^{de}) G_e(A). \quad (C1)$$

We first argue that there is no problem when this cut first occurs in  $G_e(A)$  because of the following: If the cut  $\gamma_g$  is in  $G_e(A)$  then  $g \subseteq (e)$ . When we consider  $G_e(A) v_c^e(A) \Delta_c G_c(A)$ , the fact that  $g \subseteq c \cap e$  means that  $v_c^e(A) = v_g^e(A) + v_c^{eg}(A) = v_c^{eg}(A)$ . This requires that  $v_c^e(A)$  either has an interaction external to  $g$ , or else it is 0. The presence of an interaction with connectivity external to  $g$  means that there cannot be a  $\delta$  function in all the relative momenta of the clusters of  $g$  separating  $G_e(A)$  and  $\Delta_c G_c(A)$ . Hence there is no problem if the cut  $\gamma_g$  first appears in  $G_e(A)$  in (C1). If the cut  $\gamma_g$  first appears in the  $\tilde{G}_{da}(A)$  in (C1), there is clearly no  $\delta$  function because it is separated from  $\Delta_c G_c(A)$  by the connected string  $\sum_e v_e^a(A) G_e(A) v_c^e(A)$ . Singularities arising from the product  $\Delta_c G_c(A) v_f^c(A) \tilde{G}_{fg}(A)^*$  can be treated analogously. These arguments justify Lemma 7 provided the expression is evaluated between a suitable class of state vectors.



## Appendix D

In this appendix we prove

$$\tilde{T}^{ac}(A) G_c(A) G_d^{-1}(A) D(\alpha_d) = 0 \quad \text{for } c \not\supset d \quad (D1)$$

To show this we evaluate the left-hand side of (D1). Equation (79) implies

$$\tilde{T}^{ac}(A) G_c(A) G_d^{-1}(A) D(\alpha_d) = 1_c \tilde{T}^{ac}(A) G_c(A) D(\alpha_d). \quad (D2)$$

This will vanish as  $\epsilon \rightarrow 0$  unless  $\tilde{T}^{ac}(A) G_c(A)$  is singular when applied to  $D(\alpha_d)$ . This will happen if we can factor a  $G_d(A)$  out of the expression  $\tilde{T}^{ac}(A) G_c(A)$ . To prove (D1) we show that  $c \not\supset d$  means that the first occurrence of  $G_d(A)$  is separated from  $D(\alpha_d)$  by an operator external to  $d$ . We show that this combination cannot give rise to a singularity. Since  $D(\alpha_d)$  looks like a maximally connected eigenstate,  $|\phi_d(\alpha_d)\rangle$ , of  $H_d$  to  $\tilde{T}^{ac}(A) G_c(A)$ , we consider

$$\lim_{\epsilon \rightarrow 0} 1_c \tilde{T}^{ac}(A) G_c(A) |\phi_d(\alpha_d)\rangle \quad c \not\supset d.$$

We note  $c \not\supset d$  means that there is at least one cluster of  $d$  that is not contained in any single cluster of  $c$ . Interactions internal to this cluster necessarily must occur in the perturbation expansion of  $G_d(A)$ . The fact that this cluster is external to  $c$  means that these same interactions cannot occur in the perturbation expansion of  $G_c(A)$ . It follows that  $G_d(A)$  cannot be factored out of  $G_c(A)$ , however that does not mean it can be factored out of  $\tilde{T}^{ac}(A) G_c(A)$ . To see that this cannot happen we note

$$\tilde{T}^{ac}(A) G_c(A) = \sum_b (\delta_{ab} + T^{ab}(A) G_b(A)) v_c^b(A) G_c(A). \quad (D3)$$

Since  $v_c^b(A)$  has only terms internal to  $c$ , the interactions external to  $c$  that are necessary to complete the perturbation expansion of  $G_d(A)$  can, at best, come from the  $G_b(A)$  in (D3). Fortunately  $v_c^b(A)$  consists of interactions external to  $b$ . If the required interaction is in  $G_b(A)$ , it must be internal to  $b$ . But for that  $b$ ,  $v_c^b(A)$  will contain terms external to this cluster.

If a term is external to a cluster of partition, it is necessary external to the whole partition. It follows that the first occurrence of  $G_d(A)$  will be separated from  $|\phi_d(\alpha_d)\rangle = D(\alpha_d)$  by at least one interaction external to  $d$ .

We show that under the above conditions the singularity due to  $G_d(A)$  is washed out. It then follows that (D2) vanishes as  $\epsilon \rightarrow 0$ . To show this it is sufficient to show

$$\lim_{\epsilon \rightarrow 0} 1_c G_d(A) V^d(A) |\phi_d(\alpha_d)\rangle = 0.$$

Using (71) and (33), the above becomes

$$\lim_{\epsilon \rightarrow 0} 1_c \left\{ \frac{Q_d(A)}{E + i\epsilon} + \sum_{d' \subset d} \sum_{\delta' \in A} \int d\vec{p}_1 \dots d\vec{p}_{n_{d'}-1} \frac{|\phi_{d'}(\alpha_{d'})\rangle \langle \phi_{d'}(\alpha_{d'})|}{E + i\epsilon + E_{\delta'} - \sum_{i=1}^{n_{d'}-1} p_i^2/2\mu_i} \right\} V^d(A) |\phi_d(\alpha_d)\rangle \quad (D4)$$

We note that  $I_f(Z) = \int_0^\infty \frac{p^2 f(p) dp}{(Z - p^2)}$  is bounded as  $Z \rightarrow \infty$  for any bounded integrable  $f(p)$ . If our state vectors and the interaction  $V^d(A)$  are suitably well behaved, there will be no singularity if any of the momenta in (D4) get integrated. To show that at least one of these  $\vec{p}_i$ 's gets integrated we must consider two cases. Case 1 is when  $d' \neq d$ . In this case the denominator in (D4) has at least one momenta involving particles in a single cluster of  $d$ . Maximal connectivity of  $|\phi_d(\alpha_d)\rangle$  means that  $D(\alpha_d)$  is an operator with connectivity  $d$ . This means that the presence of  $|\phi_d(\alpha_d)\rangle$  or  $D(\alpha_d)$  cause this momentum to get integrated over. In this case we have no singularity. Case 2 is when  $d' = d$ . In this case the momenta,  $\vec{p}_i$ , in (D4) are the Jacobi momenta of the clusters of  $d$ . Here  $V^d$ , consisting of terms external to  $d$ , causes at least one of these momenta to be integrated. This also washes out the singularity. We remark that the term  $\frac{Q_d(A)}{E + i\epsilon}$  causes a problem at  $E = 0$ . This will not affect our unitarity relations when  $E \neq 0$ . Since the singularities of concern do not appear, the  $i\epsilon$  in (D2) causes (D1) to vanish as  $\epsilon \rightarrow 0$ . This proves Lemma 8.

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1. We assume the center of mass motion has been factored out.

2. This assumption allows us to use the terms connected and compact almost interchangeably. In general our kernels are analytic operator valued functions of the complex energy  $Z$ . Compactness as a Hilbert space operator following from the F.C.A. is assumed to hold only for  $Z$  in the domain of analyticity of the kernel. Typically for  $Z$  on the scattering cut our operators are no longer compact. Generally solutions still exist on the scattering cuts, but only as unbounded operators. This means that the transition operators only make sense when they are applied to a certain dense set of initial state vectors. The physically interesting case of  $Z$  on the scattering cut can be treated by Fredholm methods if one can introduce a new norm that excludes the troublesome initial states. Often the kernels can be shown to be compact on the scattering cut in the topology of a suitably chosen norm. In [4] Steinberg shows that the analytic Fredholm theorem holds on Banach spaces.

3. None of the partition Hamiltonian eigenstates are state vectors of our  $N$ -body Hilbert space, but can be considered as generalized eigenvectors in a suitably rigged Hilbert space [14].

4. This lemma follows from substituting the recursion relation (24.1.411A) of ref. [16] into the check  $\sum_{m=0}^N (-)^{m-N} m! S_N^m = 1$  (24.1.411B).

5. It is interesting to consider the relation between (16) and the

spectral resolution of the full Hamiltonian. The full spectral resolution takes on the form

$$H = \int_{\alpha \in A_0} H(\alpha)_{full} + \int_{\beta \in B} H(\beta)_{full}$$

where  $B = \{N\text{-body bound states}\}$ . Assuming that  $H$  has no  $N$ -body forces we have

$$\int_{\beta \in B} H(\beta)_{full} + \int_{\alpha \in A_0} [H(\alpha)]_{\perp} = 0.$$

It follows that

$$H = \int_{\alpha \in A_0} (H(\alpha)_{full} - [H(\alpha)_{full}]_{\perp}).$$

One can show that the  $H(\alpha)$  of (16) is just  $H(\alpha)_{full} - [H(\alpha)_{full}]_{\perp}$ . Since the difference between  $H(\alpha)$  and  $H(\alpha)_{full}$  involves only terms where all  $N$  particles are simultaneously correlated, expansion (16) can be considered as the best approximation to the full spectral resolution that can algebraically be obtained purely from fewer body solutions.

6. One might be concerned that the  $(-)$  signs of the  $C_A$  which are buried in (19) might cause the RM Hamiltonian  $H(A)$  to grossly violate lower boundedness of the spectrum, in the sense that one of the cuts might get flipped to the negative direction. That this does not happen follows by applying the analytic Fredholm theorem [4] to eq. (18) iterated once and then extracting the analytic structure of the kernel using eqs. (33) and (8).

7. We note the analytic Fredholm theorem gives uniqueness provided a solution exists. To show the solution exists in the domain of analyticity one must show that it exists for at least one  $Z$  in this domain. Generally this is no problem because a  $Z$  can generally be found for which the perturbation series converges.

8. The proof of this relation is not the most satisfying. If one is willing to introduce the RM transition operators (36) first, it is possible to give a better proof. Taking (50) as the fundamental equation, using (38) and (73) as definitions one can show

$$T^{ab}(A)G_b(A) = \int_c T^{ac}(A)G_c(A) = \int_{c,d} v_c^a(A)G_{cd}(A) \quad (v_b \neq 1)$$

by showing all three expressions satisfy the same connected kernel equation, whose solution must be unique (see 7). Theorem 3 then follows using (38) and  $G(A) = G_a(A) + G_a(A)T_+^{ab}(A)G_b(A)$ .

9. We note  $G(A)$  and  $G(A')$  will have poles corresponding to both approximate N-body bound states, and to the  $\frac{Q_a(A)}{Z}$  terms in (33). Since the kernel of (49) involves products of operators containing these singularities with other operators that may vanish on these poles in some appropriate sense, it does not follow that the full solution  $G$  must contain any of these pole singularities.

10. Recall that the algebra of compact operators is a 2 sided ideal of the algebra of bounded operators [20]. This means that the product of a compact operator with a bounded operator is compact. Since our F.C.A. guarantees compactness (by connectivity) of

$$H(A')G(A)H(A) \quad \text{and} \quad \int_c v_c^a(A)G_c(A)v_b^c(A)$$

for  $Z$  not on the scattering cut (see 2), and since  $G(A)$  and  $G_a(A)$  are bounded for such  $Z$ , compactness of the appropriate kernel follows from Theorems 4,5.

11. In this formulation of the N-body problem we are not free to impose constraints on the effective interactions directly. Their properties are determined by solutions of fewer body problems. Generally the terms that appear involve products of potentials and projectors on fewer body bound states. Assuming the real interactions are well behaved, one expects the bound states to be elements of some fewer body Hilbert space. The bound state projectors imbedded in N-body Hilbert space will then be Hilbert-Schmidt (in fact finite rank) on the appropriate fiber (in the sense of Simon in [11]). This type of structure, combined with reasonable interactions should yield effective interactions that are sufficiently well behaved to obtain an F.C.A..

12. If one were to try to mock up particle creation effects, the effective interactions would become energy dependent. We would no longer have  $v_c^d(A) - v_c^d(A)^* = 0$ . Instead the discontinuity in  $v_c^d(A)$  would give rise to the various particle production cuts. The structure of our unitarity relation should then give some indication of the type of analytic structure that  $v_c^d(A)$  would require in order to get the right type of production cross sections. A serious treatment of this should also include relativistic kinematics.



## References

1. J. Goldstone, Proc. Roy. Soc. (London) A279 (1957), 267.  
N. M. Hugenholtz, Physica 23 (1957), 481.
2. B. H. Brandow, Rev. Mod. Phys. 39 (1967), 771.  
G. A. Baker, Rev. Mod. Phys. 43 (1971), 479.  
"Effective Interactions and Operators in Nuclei", (B. E. Barrett, Ed.)  
Springer Verlag (Berlin) 1975.
3. S. Weinberg, Phys. Rev. 133 (1964), B232.  
O. A. Yakubovskii, Sov. J. Nuc. Phys. 5 (1967), 937.  
Gy. Bencze, Nuc. Phys. A210 (1973), 568.  
E. F. Redish, Nuc. Phys. A225 (1974), 16.  
D. J. Kouri and F. S. Levin, Nuc. Phys. A253 (1975), 395.  
W. Tobocman, Phys. Rev. C9 (1974), 2966.
4. M. Reed and B. Simon, "Functional Analysis", Ch. VI.14-15, Academic Press, 1972.  
S. Steinberg, Arch. Rat. Mech. Anal. 31 (1968), 372.
5. K. E. Atkinson, "A Survey of Numerical Methods for the Solution of  
Fredholm Integral Equations of the Second Kind", S.I.A.M., 1976.  
S. Weinberg, Phys. Rev. 131 (1963), 440.  
S. K. Adhikari and I. H. Sloan, Nuc. Phys. A241 (1975), 429.
6. M. I. Haftel et. al., Phys. Rev. C16 (1977), 42.
7. D. F. Jackson, "Nuclear Reactions", Chapman and Hall, 1975.  
K. M. Watson, Phys. Rev. 89 (1953), 575.  
A. K. Kerman, H. McManus and R. M. Thaler, Ann. Phys. (N.Y.) 8 (1959), 551.
8. P. Benoist-Gueutal, M. L'Huillier, E. F. Redish and P. C. Tandy, to be  
published,  
K. L. Kowalski, Phys. Rev. C16 (1977), 7.

9. Gy. Bencze, unpublished.  
W. N. Polyzou, unpublished.
10. C. Lovelace, Phys. Rev. 135 (1964), B1225.
11. B. Simon, "Quantum Mechanics for Hamiltonians Defined as Quadratic  
Forms", p. 186, Princeton U. Press, 1971.  
W. Hunziker, Phys. Rev. 135 (1964), B800.
12. R. G. Newton, "Scattering Theory of Waves and Particles", Sec. 16.3.1,  
Mc Graw Hill, 1966.
13. N. M. Hugenholtz, Physica 23 (1957), 481.  
S. Weinberg, Phys. Rev. 133 (1964), B232.
14. I. M. Gel'fand and N. Ya. Vilenkin, "Generalized Functions", Vol. IV,  
Academic Press, 1964.
15. J. R. Taylor, "Scattering Theory", Sec. 16.c, Wiley, 1972.
16. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions",  
p. 824, Nat. Bur. Stds., 1964.
17. M. L'Huillier, unpublished.
18. A. Messiah, "Quantum Mechanics", Vol. II Ch. XIX.31, Wiley.
19. Gy. Bencze, Phys. Lett. 72B (1977), 155.
20. T. Kato, "Perturbation Theory for Linear Operators", Ch. III, Sec. 4,  
Springer Verlag, 1966.
21. See ref. 18., Ch. XIX.18.
22. Gy. Bencze and E. F. Redish, to be published (J.M.P.).
23. N. S. Wall and P. G. Roos, Phys. Rev. 150 (1969), 811.  
D. Corley et. al., Nuc. Phys. A184 (1972), 437.  
F. R. Kroll and N. S. Wall, Phys. Rev. C1 (1970), 138.

(123) 1(23) 2(13) 3(12) 123 (123) 1(23) 2(13) 3(12) 123

(123)	1	1	1	1	1	1	-1	-1	-1	2
1(23)	0	1	0	0	1	0	1	0	0	-1
2(13)	0	0	1	0	1	0	0	1	0	-1
3(12)	0	0	0	1	1	0	0	0	1	-1
123	0	0	0	0	1	0	0	0	0	-1

Fig. 1

Construction of anticluster coefficients ( $N = 3$ ).