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EARTH-MOON SYSTEM: DYNAMICS AND PARAMETER ESTIMATION

By

W. J. Breedlove, Jr.

Final Report
For the period February 17, 1975 - January 31, 1979

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia

Under
Research Grant NSG 1152
R. H. Tolson, Technical Monitor
Atmospheric Environmental Sciences Division

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P. O. Box 6369
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By

W. J. Breedlove, Jr.[1]

INTRODUCTION AND SUMMARY

General

This report constitutes the final report required under grant NSG 1152 issued by the National Aeronautics and Space Administration entitled "Earth-Moon System: Dynamics and Parameter Estimation." A preliminary survey was prepared under Master Contract Agreement NAS1-11707, Task Authorization No. 41. The results of that survey are in reference 1.

Original Objectives

Objectives of the original grant were to

1. Prepare a dynamic model of the coupled translational/rotational motion of the moon for use in a data-reduction process at NASA/Langley Research Center (LaRC). The available Lunar Laser Ranging Experiment (LURE) data perhaps in conjunction with other data types were to be utilized;

2. Obtain an accurate numerical integration scheme capable of solving the equations of motion to a high degree of accuracy;

3. Develop observational and normal equations for the LURE data type; and

4. Participate in the data analysis and interface grant-developed software with existing LaRC software for use in a parameter-estimation process.

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[1] Associate Professor, Department of Mechanical Engineering and Mechanics, Old Dominion University, Norfolk, Virginia, 23508.

Objectives of Continuation Grant

Objectives in an extension to the original grant included

5. Conversion of grant-developed software from the CDC 6600/6700 computers to the Old Dominion University DEC 10 computer;

6. Incorporation of BIH data into the software for Earth rotation;

7. Analysis of LURE data to determine or study the possibility of determining various parameters; and

8. Study of potentialities of using LURE data with other data types for parameter estimation.

Accomplishments

The following provides an assessment of the status of work towards the above objectives.

1. The Unified Model of Lunar Translation/Rotation (UMLTR) has been coded and verified. The model is capable of integrating all planets in a coupled sense with the rotational motion of the Moon. Comparisons of the UMLTR with DE-96 show accuracies of $\approx$ 500 m for the Earth-Moon barycenter after 3 years and $\approx$ 100 m for the heliocentric position of the Moon for the same time span. Comparisons of the UMLTR with Eckhardt's 300 series for lunar librations show standard deviations of less than 0.15 seconds in the physical librations ρ, $I\sigma$, and τ.

2. A paper "A Unified Model of Lunar Translation and Rotation for Use in the Reduction of Lure Data" was presented to the Division of Dynamical Astronomy of the AAS.

3. An accurate integrator was obtained, verified, and is currently in use in the UMLTR. The method is "Everhart's Single Sequence Method" (ref. 2).

4. A paper, "A Numerical Study of the Effects of Fourth Degree Terms in the Earth-Moon Potential on Lunar Physical

Librations," was presented at the Scientific Applications of Lunar Laser Ranging Symposium and published in the proceedings (Appendix A).

5. The BIH data has been obtained from the Jet Propulsion Lab (JPL); and has been incorporated in the software.

6. A program, NØRM, has been obtained from NASA/LaRC. This program forms normal equations to be used in the data-reduction process. It essentially solves the light-time problem, calculates the observable quantity, and forms observational related partial derivatives; subroutine TØPØS and certain time conversion routines from NØRM have been verified.

7. Programming of the variational equations is about 75 percent complete. Numerical partial derivatives were utilized for the entire study.

8. LURE normal points (and filtered data) for a six-year period are available (ref. 3).

9. A paper "A Unified Model of Lunar Translation/Rotation" has been submitted to the Celestial Mechanics Journal (Appendix B).

10. A study of the coupling between lunar translational and rotational motions was performed using the UMLTR. It was found that fourth degree lunar harmonics produced orbital perturbations of 5 cm, 0.0018 cm, and 0.0025 cm, respectively in semimajor axis, perigee, and longitude after three years due to the coupling mechanism.

11. A paper "On Lunar Orbital-Rotational Coupling and Figure-Figure Interactions in the Earth-Moon System" has been accepted for the 17th Aerospace Sciences Meeting of the AIAA (Appendix C).

12. A study of the feasibility of estimating simultaneously the lunar inertias A/MR^2, B/MR^2, and C/MR^2 using actual LURE observation times (ref. 3), and numerical partials to form a normal matrix has been accomplished. The results of

this study show no major problem in estimating these quantities when a rather limited set of parameters is considered. The results of this study are presented in Appendix C.

THE UNIFIED MODEL OF LUNAR TRANSLATION/ROTATION (UMLTR)

A numerical model which integrates the heliocentric equations of motion for all planets, the Earth-Moon barycenter, the geocentric motion of the Moon, and the Moon's rotational motion in a coupled sense has been derived, coded, and verified. A description of the UMLTR with details on various comparisons with Eckhardt's analytical model for the lunar physical librations and the JPL DE-96 ephemeris is presented in Appendix B. Reference 4 contains the derivations of the basic equations used in the UMLTR. A description of the numerical integration package used in the UMLTR as well as a description of an auxilliary program ANEAMØ is provided in reference 5. Program ANEAMØ evaluates analytic results for the rotational and translational motion of the Moon and interpolates DE-96 for the position of the Moon and planets. Reference 6 provides a description of some modifications to the UMLTR. Also, that reference provides a data analysis framework and the complete set of variational equations. A description of the various data files, programs, and auxilliary programs necessary for use with the UMLTR is presented in reference 7. Reference 8 describes the normal equation formulation to be used with the UMLTR. Finally, the general status of the work on NASA Grant NSG-1152 is presented in reference 9.

THE EFFECT OF FIGURE-FIGURE INTERACTIONS ON LUNAR PHYSICAL LIBRATIONS

One of the main purposes for development of the UMLTR was to investigate certain small effects on lunar rotation that might be discernible in the LURE data. One such effect, the figure-figure interaction (FFI) between the Earth and the

Moon, stems from certain terms in the Earth-Moon mutual potential which have been neglected in most models to date.

A disadvantage of numerical methods is that long-term behavior cannot be evaluated due to roundoff error growth. Thus, the effect of FFI was made only for a three-year period. Some preliminary numerical results are presented in Appendix A. There it was concluded that the effect of FFI on lunar physical librations amounted to at least 0.01" which mapped into range errors of 3 to 4 cm. This effect should be modeled, therefore, since the LURE data accuracy is expected to approach the 2- to 3-cm level.

Yoder in reference 10 subsequently showed, using an approximate theoretical analysis, that the effect of the FFI was to produce an 18.6-year libration of magnitude 0.0018" in longitude and 0.08" in latitude. This effect is now being included in the JPL physical libration model.

THE EFFECT OF TRANSLATIONAL/ROTATIONAL COUPLING ON THE LUNAR ORBIT

Another small effect heretofore neglected in analytical and numerical models is the rotational/translational coupling of lunar rotation with the lunar orbit. This effect is usually handled in an iterative manner, i.e., an ephemeris is produced assuming Cassini's laws are valid or that a given analytical theory represents the lunar rotation. Then, a rotational integration is made providing better knowledge for the next ephemeris generation.

The coupling effect is difficult to evaluate numerically. To gain some insight into the nature of, and magnitude of, the coupling effects, some numerical experiments were made. These experiments are described in Appendix C.

It was found that fourth degree lunar harmonics produce perturbations of 5 cm, 0.0018 cm and 0.0025 cm respectively in semimajor axis, perigee and longitude, after three years, due to the coupling mechanism.

AN ERROR ANALYSIS FOR ESTIMATING LUNAR INERTIAS FROM LURE DATA

The feasibility of estimating lunar inertias A/MR^2, B/MR^2, and C/MR^2 using the UMLTR and a single data set was studied. The results of this study are presented in Appendix C. This study did not involve a complete parameter set but indicates no major correlations or problems in estimating the above parameters using a data set such as LURE.

APPENDIX A

A NUMERICAL STUDY OF THE EFFECTS OF FOURTH DEGREE TERMS IN THE EARTH-MOON MUTUAL POTENTIAL ON LUNAR PHYSICAL LIBRATIONS

(Article from Scientific Applications of Lunar Laser Ranging.)

A NUMERICAL STUDY OF THE EFFECTS OF FOURTH DEGREE TERMS IN THE EARTH-MOON MUTUAL POTENTIAL ON LUNAR PHYSICAL LIBRATIONS*

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ABSTRACT. Lunar laser Ranging Data (LURE) is currently available with a range accuracy of 10 cm. Accuracies of 2 to 3 cm are ultimately expected. Previous analyses have shown that for an Earth-Moon range precision of 10 cm, certain fourth degree lunar harmonic gravity coefficients must be included in the lunar physical libration model. An order-of-magnitude calculation shows that some torques arising from previously neglected fourth-degree "figure-figure interaction" terms of the Earth-Moon mutual potential could produce effects on the physical librations that should be modeled for range accuracies of 2 to 3 cm.

An investigation of the effect of these "interaction" terms has been undertaken utilizing the "Unified Model of Lunar Translation/Rotation" (UMLTR) currently being developed by the author for eventual use in the analysis of the Lunar Laser Ranging Experiment (LURE) data.

This paper first presents a description of the UMLTR. Secondly, numerical results of an investigation of the effects of the mutual potential "figure-figure interaction" terms on the physical librations are presented. It is shown that these terms should be included in a lunar physical libration model when accuracies of 0.007 to 0.01 arc seconds (2 cm to 3 cm range precision) are required.

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* This work has been supported by the National Aeronautics and Space Administration under grant NSG 1152.

° Present address.

J D Mulholland (ed), Scientific Applications of Lunar Laser Ranging, 63-77

1. INTRODUCTION

The ever-increasing accuracy of the Lunar Laser Ranging Experiment (LURE) data has necessitated a re-examination of the theoretical models used in calculating the lunar orbit and lunar rotational motion.

Previous studies have shown that numerical or "special perturbation" models are better suited to provide the required accuracy than available analytical models. Thus, numerically integrated lunar orbits such as described by Oesterwinter and Cohen (1972) and Mulholland and Shelus (1973) have replaced the use of various versions of Brown's Lunar Theory. Also, solution of the lunar rotational equations of motion by numerical integration as done by Williams and Slade (1973) and Papo (1973) supplemented the use of analytical theories such as those of Koziel (1948) and Eckhardt (1965, 1970, 1973).

The above numerical models have made the assumptions that (1) lunar rotational and translational motions are uncoupled, and (2) Earth and lunar radii are small compared to the mutual Earth-Moon distance, r.

Duboshin (1958) has shown that translational and rotational motions of two interacting rigid bodies are rigorously coupled if terms are retained in the differential equations proportional to the inverse fourth power (r^{-4}) of the mutual distance between the bodies. If terms proportional to r^{-4} (r^{-5}) are retained then second (second and third) degree gravitational harmonics of both bodies appear in the forces and second and third (second, third, and fourth) degree harmonics appear in the torques. Also, it can be shown (see section 3) that if terms proportional to r^{-5} are retained in the differential equations, then "figure-figure interaction" terms from the mutual potential appear in the torques exerted on both bodies.

Existing models, however, indirectly account for coupled lunar translational and rotational motions. This is usually done by assuming that Cassini's laws are satisfied or by using an analytical theory for the lunar rotational motions while calculating the orbit. This precomputed lunar orbit is then used as an input to the numerically integrated rotational equations. This orbit can then be used in a recalculation of the lunar librations, etc.

The second assumption--to the author's knowledge--has not been accounted for in existing models. The consequences of assumption (2) are that the Earth is treated as a particle in the field of a non-spherical Moon and the Moon is treated as a particle in the field of a non-spherical Earth in the same model.

In other words, the rigorous mutual potential, as discussed by Brouwer and Clemence (1961) of the Earth-Moon system is not modeled.

Eckhardt (1973), Kaula and Baxa (1973), and Pesek (1973) have shown that the effects of the Earth acting on the third and fourth degree lunar gravity harmonics is observable within the current LURE data accuracy (10 cm). It can also be shown that torques acting on the Moon due to the interaction of the Earth's oblateness with lunar second degree coefficients are roughly of the same magnitude as those produced by the Earth acting on certain fourth degree lunar harmonics.

The preceding considerations, viz., (1) the trend to numerical integration, (2) translation-rotational coupling, and (3) mutual potential terms, led to the development of the Unified Model of Lunar Translation/Rotation (UMLTR). This model is essentially a numerical integration of the coupled translational-rotational motion of the Moon simultaneously with the translational motion of the planets. The heliocentric motions of the Moon and planets are integrated in a manner similar to that of Oesterwinter and Cohen (1972).

This paper presents a brief description of the UMLTR. It also presents a numerical investigation of the effect of previously neglected mutual potential terms on the lunar physical librations using the UMLTR.

The rotational motion of the Earth is omitted from these considerations since the usual practice is to obtain its rotational motion from observational data such as the BIH data. This practice is to be followed at least in the initial use of the UMLTR.

2. THE UNIFIED MODEL OF LUNAR TRANSLATION/ROTATION

The Unified Model of Lunar Translation/Rotation (UMLTR), as currently programmed treats the Sun (1), Mercury (2), Venus (3), Mars (6), Jupiter (7), Saturn (8), Uranus (9), Neptune (10), and Pluto (11) as spherical homogeneous particles. It treats the Earth (4) and Moon (5) as aspherical rigid bodies.

The equations of motion for the above system referred to a heliocentric frame whose axes $\{X_i'\}$ translate with respect to an inertial reference frame (mean equator and equinox of 1950.0) are:

$$
\begin{aligned}
\{\ddot{\vec{r}}_i\} + G\{(m_1 + m_i)\vec{r}_i r_{1i}^{-3}\} \\
= \{m_i^{-1}\,\overline{\nabla}_i\,U^{DP}\} - G\left\{\sum_{\substack{j=2\\ i\neq j}}^{11} m_j \vec{r}_j r_{1j}^{-3}\right\}
\end{aligned}
\qquad \begin{matrix}(1)\\ \text{(cont'd.)}\end{matrix}
$$

$$+ \{m_i^{-1} \vec{\nabla}_i U_{45}\} + \{m_i^{-1} \vec{F}_i - m_1^{-1} \vec{F}_1\}$$
$$+ \{m_i^{-1} \vec{\nabla}_i U_{14} - m_i^{-1} \vec{\nabla}_4 U_{14}\} \qquad (1)$$
$$+ \{m_i^{-1} \vec{\nabla}_i U_{15} + m_1^{-1} \vec{\nabla}_5 U_{15}\} , \quad (i = 2, 11) \qquad \text{(concl'd.)}$$

and

$$\{\ddot{\beta}_j\} = \frac{1}{2} \frac{d}{dt} \left\{ ([\beta_{kj}]\{\Omega_j\}) \right\} ,$$
$$(j,k = 0, 1, 2, 3). \qquad (2)$$

In the above equations:

$U^{DP} = G \sum_{i>j=2}^{11} \sum^{11} m_i \, m_j \, r_{ij}^{-1}$, the direct planetary terms,

U_{45} is the Earth-Moon system mutual potential in the form derived by Giacaglia and Jeffreys (1971),

U_{14} is the potential for the interaction of the Sun with the aspherical Earth,

U_{15} is the potential for the interaction of the Sun with the aspherical Moon,

$\vec{F}_i$ includes relativity perturbations on all planets using the Eddington/Robertson form as presented by Anderson (1974), and tidal accelerations on Earth and lunar orbits formulated similarly to that by Oesterwinter and Cohen (1972),

$[\beta_{k,j}]$ is an orthogonal matrix of Euler parameters $(k,j = 0, 1, 2, 3)$, and

$\{\Omega_j\}$ is the angular velocity of the Moon with respect to a defined reference frame $(j = 0, 1, 2, 3)$.

The rotational motion of the Earth is adopted from Woolard (1953) in the current version of the UMLTR.

Equations (1) are standard in form except for the mutual potential terms in U_{45} and the implicit coupling between equations (1) and (2).

Equations (2) are the differential equations for the motion of a set of selenocentric--"body fixed"--principal axes, $\{z_i\}$, with respect to a set of reference axes $\{Z_i\}$. The relative orientation of these frames is shown in figure 1.

The orientation of $\{Z_i\}$ with respect to inertial space can be completely specified once the relative position of the Earth and Moon are known with respect to $\{X_i'\}$. In figure 1, the axes $\{X_i'^T\}$ are geocentric axes translating with respect to the frame $\{X_i'\}$. Thus, r, λ, φ are the spherical polar coordinates of the Moon relative to the Earth referred to $\{X_i'^T\}$. Finally, the unit vectors $\{\vec{K}_i\}$ of the reference frame $\{Z_i\}$ are related to the spherical polar unit vectors $\hat{r}, \hat{\lambda}\ \hat{\phi}$ via

$$\begin{aligned} \vec{K}_1 &= -\hat{r} \\ \vec{K}_2 &= -\vec{\lambda} \\ \vec{K}_3 &= \hat{\varphi} \end{aligned} \qquad (3)$$

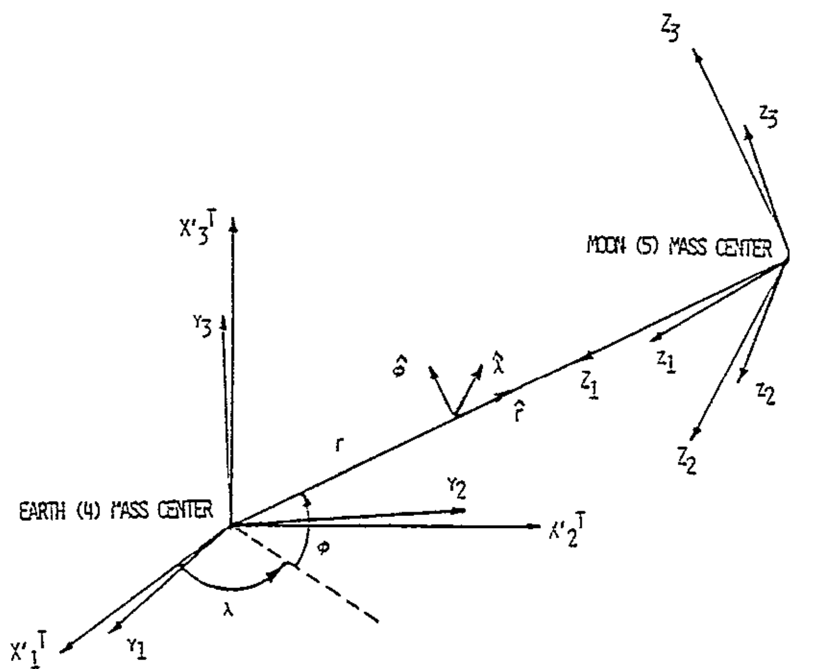

Figure 1. Coordinate frames for lunar rotational motion.

The relative orientation of $\{z_i\}$ and $\{Z_i\}$ is specified by the Euler parameters $\beta_0,\beta_1,\beta_2,\beta_3$ using the notation of Morton and Junkins (1973). Accordingly,

$$\{z_i\} = [c]\ \{Z_i\} \tag{4}$$

where

$$[c] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3-\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

Also, the matrix $[\beta_{jk}]$ of equation (2) is of the form

$$[\beta_{jk}] = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{5}$$

The angular velocity vector $\{\Omega_j\}$ is essentially the difference in absolute angular velocities of the Moon $\omega_1'\ \omega_2'\ \omega_3'$, and the reference axis system $-\dot{\lambda}\sin\phi,\ \dot{\phi},\ \dot{\lambda}\cos\phi$ expressed in components in the system $\{z_i\}$, viz.

$$\{\Omega_j\} = \begin{Bmatrix} 0 \\ \omega_1' \\ \omega_2' \\ \omega_3' \end{Bmatrix} - [c] \begin{Bmatrix} 0 \\ -\dot{\lambda}\sin\varphi \\ \dot{\varphi} \\ \dot{\lambda}\cos\phi \end{Bmatrix} \tag{6}$$

The time derivatives of $\{\Omega_j\}$ as required in equation (2) can be evaluated from Euler's equations for a rigid body, viz.

$$\begin{Bmatrix} \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} \doteq \begin{Bmatrix} T_1'/A' \\ T_2'/B' \\ T_3'/C' \end{Bmatrix} - \begin{Bmatrix} \alpha'\omega_2'\omega_3' \\ -\beta'\omega_1'\omega_3' \\ \gamma'\omega_1'\omega_2' \end{Bmatrix} \tag{7}$$

The torque T_i' $(i = 1, 2, 3)$ includes terms representing: (1) the action of a spherical Earth and Sun on lunar second degree harmonics, (2) the action of a spherical Earth on lunar third and fourth degree harmonics, and (3) the interaction of a triaxial Earth with lunar second degree harmonics as specified by the Earth-Moon mutual potential.

The set of second order equations (1) and (2) are integrated numerically--in the present version of the UMLTR--using an implicit single-sequence method developed by Everhart (1974). The method uses "Gauss-Radau spacings for substeps within each sequence," and integration orders of 7 to 31 are possible.

The following translational options are currently available in the program: (1) integrate all planets, (2) integrate Mercury, Venus, Earth, Moon, Mars, Jupiter, and (3) integrate Earth and Moon only.

In options (2) and (3) above, the remaining planetary ephemerides are read from JPL ephemeris tapes.

A comparison of the calculated values of the lunar physical librations ρ, $I\sigma$, and τ as defined by Eckhardt (1965, 1970), with the integrated values produced by the UMLTR is in progress. The results of the comparison are to be shown elsewhere, Breedlove (1976).

3. FOURTH DEGREE TORQUES ON THE MOON RESULTING FROM THE MUTUAL POTENTIAL

The mutual potential of two arbitrarily shaped rigid bodies may be put into the following standard form as first shown by Giacaglia and Jeffreys (1971):

$$U_{45} = \frac{Gm_4m_5}{r_{45}} \sum_{n=2}^{\infty} \sum_{m=0}^{n} Pnm(\sin \phi_{45}) \left(\frac{1}{r_{45}}\right)^n \Phi_{nm} , \qquad (8)$$

where

$$\Phi_{nm} = X_{nm} \cos m\lambda_{45} + Y_{nm} \sin m\lambda_{45} \quad .$$

The coordinate system implicit in equation (8) is centered at one of the bodies and $r_{45}, \phi_{45}, \lambda_{45}$ are the spherical polar coordinates of the mass center of the other body with respect to that coordinate system.

Representative values of the coefficients X_{nm} and Y_{nm} are listed here as presented in Giacaglia and Jeffreys (1971):

$$\begin{aligned} X_{2j} &= a^2C_{2j} + a'^2C'_{2j} \quad , \quad (j = 0,1,2) \\ Y_{2j} &= a^2S_{2j} + a'^2S_{2j} \quad , \quad (j = 1,2) \end{aligned} \qquad (9)$$

$$X_{40} = a^4C_{40} + a'^4C'_{40} + 6a^2a'^2(C_{20}C'_{20} + 2C_{22}C'_{22} + S_{22}S'_{22}) \quad , \tag{10}$$

etc.

The X_{nm} and Y_{nm} are composed of two types of terms, viz. (1) "particle-figure interaction" (PFI) terms such as a^2C_{21}, a^4C_{40}, a'^4C_{40}, and (2) "figure-figure interaction" (FFI) terms such as $a^2a'^2C_{20}C'_{20}$. In equation (10) a geocentric principal axis system is utilized, thus the C_{ij} and S_{ij} are constants. Since the C'_{ij} and S'_{ij} must be evaluated in the same frame, they become functions of the relative orientation of the two bodies.

Torque expressions may be derived from equation (8) by forming partial derivatives of U_{45} with respect to suitable rotational variables such as the direction cosines specifying the relative orientation of the two rigid bodies as shown by Beletskii (1966).

Torques due to "particle-figure interactions" and involving the third and fourth degree lunar harmonics may be found in the works of Kaula and Baxa (1973), Pesek (1973), and Eckhardt (1973).

Torques due to "figure-figure interactions" of a triaxial Earth with a Moon represented by second degree harmonics have been derived from equation (8) and are listed below:

$$\begin{aligned}
T'_1/A' = {} & Gm_4a^2\alpha'r^{-5}C_{20}[18P_{40}\gamma'\gamma'' \\
& + 3P_{41}\{(\alpha'\gamma'' + \gamma'\alpha'')c\lambda - (\gamma'\beta'' - \gamma''\beta')s\lambda\} \\
& + (P_{42}/2)\{(\alpha'\alpha'' - \beta'\beta'')c2\lambda - (\beta'\alpha'' + \alpha'\beta'')s2\lambda\}] \\
& + Gm_4a^2\alpha'r^{-5}C_{22}[-6P_{40}(\beta'\beta'' - \alpha'\alpha'') \\
& + 3P_{41}\{-(\alpha''\gamma' + \gamma''\alpha')c\lambda + (\gamma'\beta'' + \gamma''\beta')s\lambda\} \\
& + 3P_{42}\{\gamma'\gamma''c2\lambda\} \\
& + (P_{43}/2)\{(\alpha''\gamma' + \gamma''\alpha')c3\lambda + (\gamma'\beta'' + \gamma''\beta')s3\lambda\} \\
& + (P_{44}/4)\{(\alpha'\alpha'' - \beta'\beta'')c4\lambda \\
& + (\beta'\alpha'' + \alpha'\beta'')s4\lambda\}] \quad ,
\end{aligned} \tag{11}$$

$$\begin{aligned}
T'_2/B' = {} & Gm_4a^2\beta'r^{-5}C_{20}[-18P_{40}\gamma\gamma'' \\
& + 3P_{41}\{(\alpha\gamma'' + \gamma\alpha'')c\lambda - (\beta\gamma'' + \gamma\beta'')s\lambda\}
\end{aligned}$$

$$+ (P_{42}/2)\{(\beta\beta'' - \alpha\alpha'')c2\lambda - (\alpha\beta'' + \beta\alpha'')s2\lambda\}]$$

$$+ Gm_4 a^2\beta' r^{-5}C_{22}[-6P_{40}(\beta\beta'' - \alpha\alpha'')$$

$$+ 3P_{41}\{(\alpha\gamma'' - \gamma\alpha'')c\lambda - (\beta\gamma'' + \gamma\beta'')s\lambda\}$$

$$- 3P_{42}\{\gamma\gamma'' c2\lambda\}$$

$$- (P_{43}/2)\{(\alpha\gamma'' + \gamma\alpha'')c3\lambda + (\beta\gamma'' + \gamma\beta'')s3\lambda\}$$

$$+ (P_{44}/4)\{(\beta\beta'' - \alpha\alpha'')c4\lambda - (\alpha\beta'' + \beta\alpha'')s4\lambda\}] \tag{12}$$

$$T_3'/B' = Gm_4 a^2\gamma' r^{-5}C_{20}[18P_{40}\gamma\gamma'$$

$$+ 3P_{41}\{(\alpha\gamma' + \alpha'\gamma)c\lambda + (\beta\gamma' + \gamma\beta')s\lambda\}$$

$$+ (P_{42}/2)\{(\alpha\alpha' - \beta\beta')c\lambda + (\alpha\beta' + \beta\alpha')s\lambda\}]$$

$$+ Gm_4 a^2\gamma' r^{-5}C_{22}[6P_{40}(\alpha\alpha' - \beta\beta')$$

$$+ 3P_{41}\{-(\alpha\gamma' + \gamma\alpha')c\lambda + (\beta\gamma' + \gamma\beta')s\lambda\}$$

$$+ 3P_{42}\{\gamma\gamma' c2\lambda\}$$

$$+ (P_{43}/2)\{(\alpha\gamma' + \gamma\alpha')c3\lambda + (\beta\gamma' + \gamma\beta')s3\lambda\}$$

$$+ (P_{44}/4)\{(\alpha\alpha' - \beta\beta')c4\lambda$$

$$+ (\alpha\beta' + \beta\alpha')s4\lambda\}] \tag{13}$$

where the relative orientation of the geocentric principal axis frame $\{y_i\}$ and the selenocentric principal axis frame $\{z_i\}$ is specified by

$$\{y_i\} = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix} \{z_i\} \tag{14}$$

using the notation similar to Beletskii (1966). In equations (11) to (13) the subscripts 45 have been omitted for clarity and the abbreviations $c \equiv \cos$ and $s \equiv \sin$ have been used.

The torque expressions in equations (11) to (13) contain the direction cosines $\alpha, \gamma', \gamma''$ of the rotation axis of the Earth with respect to the selenocentric frame $\{z_i\}$. It can be shown that these quantities exhibit periodic variations with the lunar nodal period (18.6 years), among others.

Lure (1963) presented a vector-dyadic form of equations (11) to (13) when the interaction of the Earth's oblateness with a triaxial Moon was considered.

4. THE EFFECT OF FIGURE-FIGURE INTERACTION TERMS OF THE EARTH-MOON MUTUAL POTENTIAL ON LUNAR PHYSICAL LIBRATIONS

A qualitative analysis of the effect of the "figure-figure interaction" (FFI) torques on the lunar physical librations was undertaken in the following manner. The UMLTR was used to produce a simulated set of physical librations over a three-year period beginning at J.D. 2441200.5. This set ("observations") was calculated including the effect of the usual second degree terms in the lunar torques as well as the FFI torques due to the Earth's oblateness. A "calculated" set of librations constituting the model was next produced in a similar manner but with the FFI torques omitted. The model was then fitted to the observations in a least squares sense by adjusting the lunar inertia ratios α', β', and γ' and the rotational initial conditions $(\beta_i)_0$ and $(\dot{\beta}_i)_0$. The results of this fit are shown in figure 2.

The nature of the residuals indicates that the FFI torques produce monthly variations in the latitude librations, ρ and $I\sigma$, with amplitudes of about .002". Also, the residuals in ρ, $I\sigma$, and τ indicate a long period effect that could not be fit over the adopted time interval (3 years). This long period effect is due to the 18.6 year variation in the direction cosines $\gamma, \gamma', \gamma''$ of the Earth's spin axis with respect to the lunar frame $\{z_i\}$ as discussed in section 3. Physical librations in ρ, $I\sigma$, and τ with amplitudes of 0.01" are apparent from figure 2 due to the presence of the FFI torques.

The above results were obtained with nominal values of $\beta = 0.6293 \times 10^{-3}$, $\gamma = 0.227 \times 10^{-3}$, and $C_{20} = -1.08261 \times 10^{-3}$. Data was obtained in 3-day increments and the 11th order version of Everhart's (1974) integrator was used. Planetary and lunar initial conditions were adopted from Oesterwinter and Cohen (1972)

5. CONCLUSIONS

The Unified Model of Lunar Translation and Rotation has been developed for studying lunar translational/rotational coupling and the effect of terms arising in the Earth-Moon mutual potential. It is being verified by comparisons with Eckhardt (1965, 1970, 1973, 1976) and Oesterwinter and Cohen (1972) for eventual use in the analysis of LURE data.

Figure-Figure Interaction torques produce effects on the lunar physical librations comparable in magnitude to those produced by

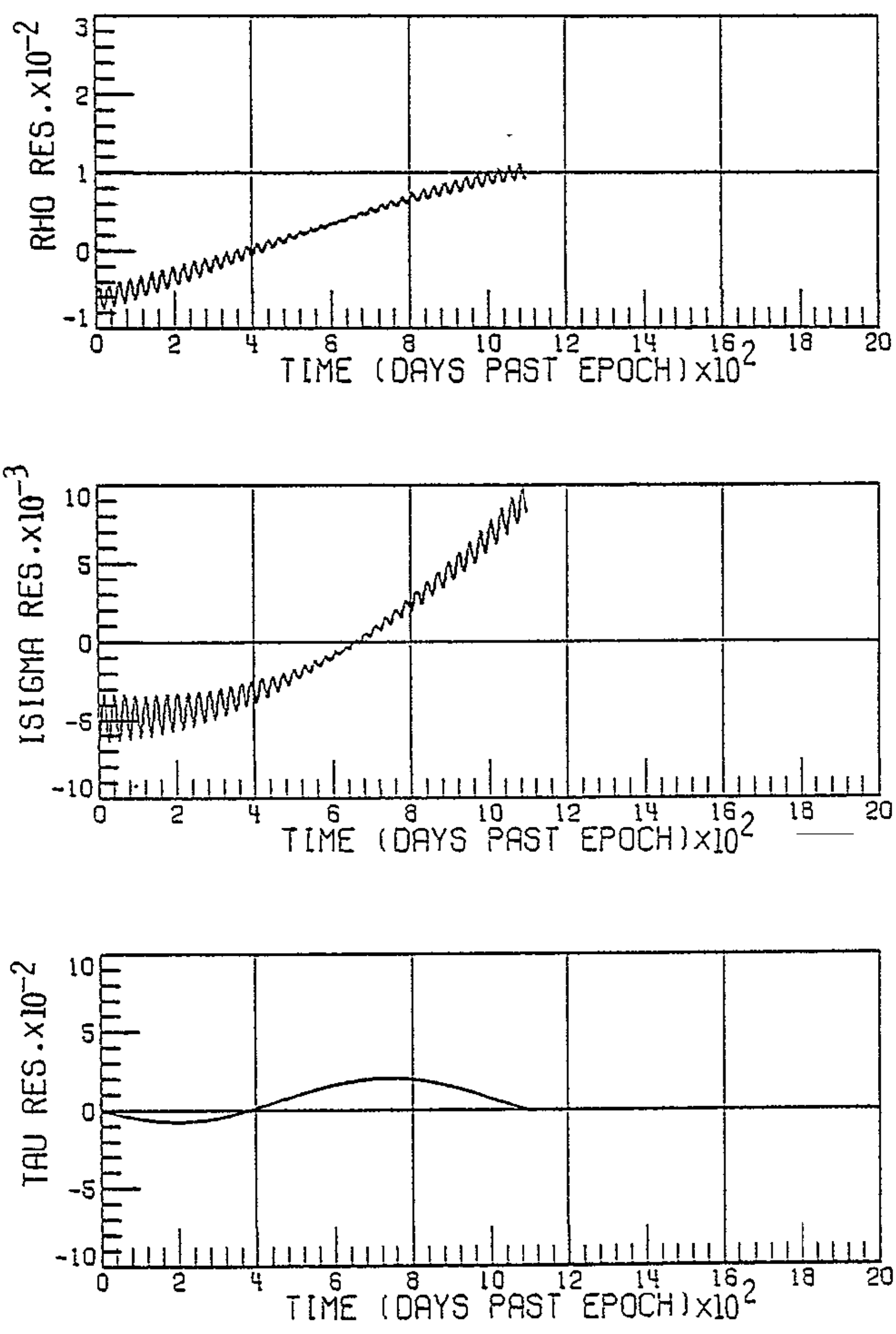

Figure 2. Residuals (arc sec) from fit of UMLTR (with no FFI torques) to UMLTR (with FFI torques).

certain fourth degree terms in the lunar gravity field. In particular, monthly variations of amplitude .002" were found in the latitude librations, ρ and $I\sigma$. Also, long period variations in ρ and $I\sigma$ and the longitude librations, τ, with amplitudes of ~0.01" and periods that are fractions and multiples of the lunar nodal period (18.6 years) were using a numerical experiment. The larger of the above effects causes range variations of ~10 cm. Thus, the FFI torques should be included in a lunar libration model accurate to ~0.01".

6. SYMBOLS

All primed quantities refer to the Moon, unprimed quantities refer to the Earth. The following symbols have not been defined previously:

$\vec{r}_i$	heliocentric position vector to planet i
m_i	mass of planet i
α'	$= (C' - B')/A'$
β'	$= (C' - A')/B'$
γ'	$= (B' - A')/C'$
a	reference radius
C_{jk}	gravity harmonic coefficients
S_{jk}	gravity harmonic coefficients
ρ, $I\sigma$	physical librations in latitude
τ	physical libration in longitude
P_{ij}	Associated Legendre polynomials
G	universal gravitational constant

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DISCUSSION

Counselman: Have you compared your theory with lunar laser observations yet?

Breedlove: No, we have not reached that stage, although we have the data.

Mulholland: I think that the reason for your large residuals in the comparison of libration models is probably due to having chosen a rather poor analytical model against which to compare. The errors in the comparison standard will give spurious corrections to the numerical starting conditions. Since the computer "Moon" has an infinite Q, this will stimulate artificial free librations in the numerical integration. You must use a better standard.

Breedlove: I have recently received the improved Eckhardt series 300, which I will use in the future.

King: Eckhardt now has an even newer theory, designated series 400.

APPENDIX B

A UNIFIED MODEL OF LUNAR TRANSLATION/ROTATION

(Article submitted to _Celestial Mechanics Journal_.)

A Unified Model of Lunar Translation/Rotation

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ABSTRACT:

A unified model of lunar translation/rotation (UMLTR) is described in which the heliocentric motions of the planets, the geocentric motion of the Moon, and the rotational motion of the Moon are integrated simultaneously. Novel features of the UMLTR include 1) inclusion of translational/rotational coupling, 2) inclusion of figure-figure interactions (FFI) between the Earth and Moon, 3) the use of Euler parameters for Lunar orientation, and 4) the use of a relatively new integrator developed by E. Everhart. At present the UMLTR fits Eckhardt's 300-301 series with standard deviations of 0.1", 0.15", and 0.1" in ρ, I_σ, and τ. The UMLTR lunar orbit differs from that in DE96 by 80 meters in 3 years and 200 meters in 6 years. Coupling effect residuals of 5 cm., 0.007", and 0.001" after 3 years arise in lunar semi-major axis, perigee, and longitude respectively if fourth order lunar harmonics are included over and above a third order model.

1. INTRODUCTION AND SUMMARY

The Lunar Laser Ranging Experiment (LURE) has placed three lunar ranging retro-reflectors on the Moon as described by Bender (1973). A series of measurements of the distance to these retro-reflectors has been inferred from measured round trip travel times of pulsed laser beams. These measurements have been made regularly since August 1969 and allow determination of the distance to about $\pm$ 8 cm. as reported by Williams (1976b). This same report indicates that a resolution of $\pm$ 2-3 cm. is expected eventually.

According to the LURE team, the LURE data can be used to obtain improved knowledge of 1) retro-reflector and earth-based telescope coordinates, 2) lunar center of mass motion, 3) lunar librational motion, 4) Earth rotational motion, 5) Earth continental drift, 6) gravitational theories, and 7) the internal composition of the Earth and Moon. Indeed, a number of major announcements have recently been made by the LURE team including 1) a confirmation of the equivalence principle of general relativity reported by Williams (1976a), and 2) confirmation that lunar laser ranging is capable of accurately determining the Earth's rotation as reported by Stolz (1976).

The ultimate accuracy of the LURE data has led a number of investigators to look into previously neglected physical and dynamical effects in existing models for the translational/rotational motions of the Moon. Thus Gurevich (1976) analyzed the tidal effect in the theory of lunar rotation; Calame (1976) has studied the presence of small free librations; Lambeck (1975) has investigated the effect of tidal dissipation in the oceans on the Moon's orbit; and Melosh (1975) has studied the effect of Mascons on the Moon's orientation.

Existing models for the translational and rotational motion of the Moon generally contain the assumptions that (1) lunar rotational and translational motion are uncoupled, and (2) Earth and lunar radii are small compared to the Earth-Moon distance. A special perturbation model not incorporating the above two assumptions is currently being developed by the author. This model, the Unified Model of Lunar Translation/Rotation (UMLTR), is being used to investigate the above assumptions. This paper documents the UMLTR, provides comparisons with existing models, and gives the results of certain numerical experiments.

Preliminary results using an early version of the UMLTR were presented by Breedlove (1976a) showing that figure-figure interactions (FFI) of the Earth and Moon have at least a 2-3 cm. effect on the distance measured to the retro-reflector. These FFI can be investigated if the above assumptions are not made.

Novel features of the UMLTR include 1) the ability to integrate the coupled rotational-translational motion of the Moon, 2) inclusion of FFI torques based on the Earth-Moon mutual potential, 3) the use of Euler parameters for lunar orientation, 4) the use of a relatively new integrator developed by Everhart (1974); and 5) the option to integrate all planets, or only the inner planets or only the Earth-Moon barycenter together with the lunar translational-rotational motion. In the latter two options, DE96 provides the remaining planetary positions and velocities.

Section 2 provides a summary of the equations of motion with a brief description of the numerical integration scheme used. Section 3 provides the results of certain verification tests of the UMLTR including a comparison with DE96 for 6 years for translational motion; a closure test for 6.5 years of integration as a check on numerical roundoff errors and a 3 year comparison of the UMLTR with Eckhardt's (1976) 300-301 series for the rotational motion. Section 4 provides the results of some numerical experiments involving coupling and the effects of FFI.

At present, the UMLTR fits Eckhardts 300-301 series with standard deviations of 0.1", 0.15", 0.1" in ρ, I_{σ}, and τ. The UMLTR lunar orbit and that of DE96 differ by about 80 m after 3 years of integration and 200 m after 6 years of integration.

The effect of coupling is shown indirectly by plotting the lunar orbital residuals between UMLTR runs which differed only by the assumed lunar mass distribution. After 3 years, differences in semi-major axis, perigee, and longitude are 5 cm., 0.007", and 0.001" respectively when a run containing 4th order lunar harmonics is differenced with one containing 3rd order harmonics.

2. EQUATIONS OF MOTION

2.1 General Equations

The equations of motion for the Moon and planets used in the present version of the UMLTR are summarized here. The heliocentric motions of the planets and the Earth-Moon barycenter are integrated simultaneously with the geocentric translational motion of the Moon and its rotational motion.

The equations actually programmed are based on the more general equations:

$$m_i \ddot{\underline{r}}_i = \underline{\nabla}_i U + \underline{F}_i, \tag{1}$$

$$\{\ddot{\beta}_j\} = \frac{1}{2} \frac{d}{dt} \{[\beta_{kj}]\{\Omega_j\}\}, \tag{2}$$

with $i = 1,\ldots,11$; $j, k = 0,1,2,3$.

In equations (1) and (2), $\underline{r}_i$ is the position vector of the ith body with respect to the solar system barycenter and $[\beta_0 \beta_1 \beta_2 \beta_3]$ are Euler parameters providing the orientation of the Moon's principal axes with respect to a reference axis system defined later (section 2.6).

The potential function U in equation (1) is of the form

$$U = G \sum_{i=2}^{11} \sum_{\substack{j=2 \\ i \neq j}}^{11} m_i m_j r_{ij}^{-1} + G m_1 \sum_{j=2}^{11} m_j r_{1j}^{-1}$$

$$+ U^{45} + \sum_{\rho=2}^{11} U^{1\rho} \tag{3}$$

$$= \bar{U} + U^{45} + \sum_{\rho=2}^{11} U^{1\rho}$$

where the function $U^{\sigma\rho}$ appearing in this equation represent the mutual potential between bodies σ and ρ given in the form established by Giacaglia and Jeffreys (1971) viz.,

$$U^{\sigma\rho} = Gm_\sigma m_\rho \left[\sum_{n=2}^{\infty} \sum_{m=0}^{n} P_{nm}(\sin \phi_{\sigma\rho}) r_{\sigma\rho}^{-n} \Phi_{\sigma\rho} \right], \tag{4}$$

where

$$\Phi_{\sigma\rho} = X_{nm}^{\sigma\rho} \cos m \lambda_{\sigma\rho} + Y_{nm}^{\sigma\rho} \sin m \lambda_{\sigma\rho}.$$

The quantities $X_{nm}^{\sigma\rho}$ and $Y_{nm}^{\sigma\rho}$ are functions of the harmonic coefficients of bodies σ and ρ and $r_{\sigma\rho}$, $\phi_{\sigma\rho}$, $\lambda_{\sigma\rho}$ are the spherical polar coordinates of one body with respect to a coordinate system at the first body. The notation used for the "N" bodies is Sun = 1, Mercury = 2, Venus = 3, Earth = 4, Moon = 5, Mars = 6, Jupiter = 7, Saturn = 8, Uranus = 9, Neptune = 10, and Pluto = 11.

The force vector $\underline{F}_i$ is given by

$$\underline{F}_i = \underline{F}_i^r + \underline{F}_i^a + \underline{F}_i^t \tag{5}$$

where $\underline{F}_i^r$ represents the relativistic perturbations based on an isotropic, Parameterized Post-Newtonian N-body metric and the vector $\underline{F}_i^a$ represents the perturbations produced by the asteroids. Expressions for $\underline{F}_i^r$ and $\underline{F}_i^a$ were taken from the discussion of the JPL Development Ephemeris Number 96 by Standish et. al. (1976). The vector $\underline{F}_i^t$ represents the tidal coupling perturbations and was modeled as suggested by Oesterwinter and Cohen (1972). The vector, $\{\beta_i\}$, and matrix, $[B_{kj}]$ in equations (2) are composed of the Euler parameters β_0, β_1, β_2, β_3.

The angular velocity vector $\{\Omega_j\}$ in equations (2) is essentially the difference between the absolute angular velocity components of the moon and a reference axis system as described later in section 2.6.

2.2 UMLTR Model for Translational Motion of Planets

In the UMLTR, the heliocentric motions of the planets except the Earth are integrated in the form

$$\ddot{\underline{\rho}}_i = \ddot{\underline{r}}_i - \ddot{\underline{r}}_1 \qquad (i = 2,\ldots, 11,\ i \neq 4,5) \qquad (6)$$

where

$$\ddot{\underline{r}}_1 = m_1^{-1}\underline{\nabla}_1(\bar{U} + U^{14} + U^{15})$$

and

$$\ddot{\underline{r}}_i = m_i^{-1}\underline{\nabla}_i(\bar{U}) + m_i^{-1}\underline{F}_i^r + m_i^{-1}\underline{F}_i^a .$$

In equations (6), a simplified version of $m_i^{-1}\ \underline{F}_i^r$ is used. The version programmed is the Eddington-Robertson form as given by Anderson (1974), viz.,

$$m_i\underline{F}_i^r = Gm_1c^{-2}\rho_i^{-3}\{[2(\beta+\gamma)\phi_i - \gamma V_i^2]\underline{\rho}_i + 2(1+\gamma)(\rho_i\dot{\rho}_i)\dot{\underline{\rho}}_i\} \tag{7}$$

where

$$V_i^2 = \underline{\rho}_i \cdot \underline{\rho}_i$$

$$\phi_i = Gm_1\rho_i^{-1}$$

and β and γ are relativity parameters. Note that the term $m_1^{-1}\underline{F}_1^r$ is not included in equations (6) which, according to Moyer (1968) is a reasonable omission for accuracies of 1 m for the inner planets and 1 km for the outer planets.

The effect of the asteroids was included only as they perturbed Mars orbit. Standard orbits for the asteroids was adopted as given in Standish (1976).

In equations (6), the Sun is treated as spherical; C_{20} and C_{30} are retained for the Earth; and C_{20}', C_{22}', and S_{22}' are retained for the Moon.

2.3 UMLTR Model for Motion of Earth-Moon Barycenter

The heliocentric motion of the Earth-Moon barycenter in the UMLTR is integrated in the form

$$\ddot{\underline{\rho}}_B = m_5(m_4 + m_5)^{-1}\ \ddot{\underline{\rho}}_5 + m_4(m_4 + m_5)^{-1}\ \ddot{\underline{\rho}}_4 \tag{8}$$

where

$$\underline{\rho}_k = \underline{r}_k - \underline{r}_1 .$$

Using equations (1), $\ddot{\underline{\rho}}_B$ can be written as

$$\begin{aligned}\ddot{\underline{\rho}}_B = &-G(m_1+m_4+m_5)[m_4 M^{-1}\underline{\rho}_4\rho_{14}^{-3} + m_5 M^{-1}\underline{\rho}_5\rho_{15}^{-3}] \\ &+ M^{-1} \sum_{\substack{j=2 \\ j\neq 4,5}}^{11} \mu_j[m_5(\underline{\rho}_j-\underline{\rho}_5)\rho_{5j}^{-3} + m_4(\underline{\rho}_j-\underline{\rho}_4)\rho_{4j}^{-3} \\ &- M\underline{\rho}_j\rho_{1j}^{-3}] + M^{-1}[\underline{F}_4^r + \underline{F}_5^r] - Mm_1^{-1}\underline{F}_1^r \\ &+ (m_1 + m_4 + m_5)m_1^{-1}M^{-1}[\underline{\nabla}_5 U^{15} + \underline{\nabla}_4 U^{14}]\end{aligned} \tag{9}$$

where $\mu_j = Gm_j$

and

$$M = m_4 + m_5 .$$

In equation (9), $m_1^{-1}\ M\underline{F}_1^r$ is omitted and the remaining relativity terms come from equations (7) with

$$\phi_4 = Gm_1\rho_{14}^{-1} + Gm_5\rho_{45}^{-1} \tag{10a}$$

and

$$\phi_5 = Gm_1\rho_{15}^{-1} + Gm_4\rho_{45}^{-1}. \qquad (10b)$$

2.4 UMLTR Model for Geocentric Lunar Motion

The geocentric lunar motion of the Moon in the UMLTR is integrated in the form

$$\ddot{\underline{\rho}}_G = \ddot{\underline{\rho}}_5 - \ddot{\underline{\rho}}_4 \, . \qquad (11)$$

Again using equations (1), $\underline{\rho}_G$ can be written as

$$\begin{aligned}\ddot{\underline{\rho}}_G = {} & -G\underline{\rho}\, M\rho^{-3} - Gm_1[\underline{\rho}_5\rho_5^{-3} - \underline{\rho}_4\rho_4^{-3}] \\ & - G \sum_{\substack{j=2 \\ j\neq 4,5}}^{11} m_j[(\underline{\rho}_5 - \underline{\rho}_j)\rho_{5j}^{-3} - (\underline{\rho}_4 - \underline{\rho}_j)\rho_{4j}^{-3}] \\ & + m_5^{-1}\,[\underline{\nabla}_5 U^{15} + \underline{F}_5^r + \underline{F}_5^t] \\ & - m_4^{-1}\,[\underline{\nabla}_4 U^{14} + \underline{F}_4^r + \underline{F}_4^t] \\ & + Mm_4^{-1}m_5^{-1}[\underline{\nabla}_5 U^{45} - \underline{\nabla}_4 U^{45}]. \end{aligned} \qquad (12)$$

In equations (12), U^{45} retains C_{20} and C_{30} for the Earth and C'_{20}, C'_{22}, and S'_{22} for the Moon. The tidal coupling terms are computed as suggested by Oesterwinter and Cohen (1972) viz.,

$$m_i^{-1}\underline{F}_i^t = \frac{C}{(\mu_4 + \mu_5)} (\mu_4 \delta_{i5} - \mu_5 \delta_{i4}) \frac{(\underline{\rho}_G \times \dot{\underline{\rho}}_G) \times \underline{\rho}_G}{|\underline{\rho}_G \times \dot{\underline{\rho}}_G||\underline{\rho}_G|} \qquad (13)$$

where δ_{ij} is the delta function and C is the tidal coupling coefficient. The relativistic terms in equations (12) are computed as suggested by Moyer (1968) as

$$m_5^{-1}\underline{F}_5^r - m_4^{-1}\underline{F}_4^r = \ddot{\underline{r}}_M(S) - \ddot{\underline{r}}_E(S)$$

$$+ \ddot{\underline{r}}_M(E) - \ddot{\underline{r}}_E(M) \qquad (14)$$

where $\ddot{\underline{r}}_M(S)$ means the relativistic acceleration of the Moon due to the Sun, etc. In particular $r_M(S)$ and $\underline{r}_E(S)$ are calculated from equations (7) and (10). The terms $\underline{r}_M(E)$ and $\underline{r}_E(M)$ are calculated using the "N-body" metric in the form given by Standish (1976) viz.,

$$m_i^{-1}\underline{F}_i^r = \sum_{j\neq i} \mu_j(\underline{r}_j - \underline{r}_i)r_{ij}^{-3}\{-2(\beta+\gamma)c^{-2} \sum_{k\neq i} \mu_k r_{ik}^{-1}$$

$$- (2\beta-1)\, c^{-2} \sum_{k\neq j} \mu_k r_{jk}^{-1} + \gamma(V_i/c)^2 + (1+\gamma)(V_j/c)^2$$

$$- 2(1+\gamma)c^{-2}\, \dot{\underline{r}}_i \cdot \dot{\underline{r}}_j - (3/2)c^{-2}[(\underline{r}_i-\underline{r}_j) \cdot \dot{\underline{r}}_j r_{ij}^{-1}]^2$$

$$+ (1/2)c^{-2}(\underline{r}_j-\underline{r}_i) \cdot \ddot{\underline{r}}_j\} + c^{-2} \sum_{j\neq i} \mu_j r_{ij}^{-3}\{[\underline{r}_i-\underline{r}_j] \cdot$$

$$\cdot\, [(2 + 2\gamma)\dot{\underline{r}}_i - (1 + 2\gamma)\dot{\underline{r}}_j]\}(\dot{\underline{r}}_i - \dot{\underline{r}}_j)$$

$$+ (3/2 + 2\gamma)c^{-2} \sum_{j\neq i} \mu_j \underline{\ddot{r}}_j r_{ij}^{-1} \tag{15}$$

where instead of summing over "j", "j" is set equal to 4 for $\underline{r}_M(E)$ and "j" is set equal to 5 for $\underline{\ddot{r}}(M)$.

2.5 UMLTR Model for Lunar Rotational Motion

The rotational motion of a rigid Moon satisfies Liouville's equations. A special form of these equations are referred to as Euler's equations, viz.,

$$\{\dot{\omega}_i\} = \{M_i/I_i'\} - \begin{Bmatrix} \alpha\omega_2\omega_3 \\ -\beta\omega_1\omega_3 \\ \gamma\omega_1\omega_2 \end{Bmatrix} . \tag{16}$$

In the UMLTR, the orientation of the Moon is specified by a set of Euler parameters

$$\{\beta_j\} = \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \tag{17}$$

which represent a rigid rotation from a set of orbitally defined reference axes $\{Z_i\}$ to the lunar principal axes $\{z_i\}$. These axes are described in section 2.6.

The relative orientation of $\{Z_i\}$ and $\{z_i\}$ is given by

$$\{z_i\} = [C]\{Z_i\} \tag{18}$$

where

$$[C] = \begin{bmatrix} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{bmatrix}$$

using the notation of Morton, et al. (1974).

The absolute angular velocity of the Moon, $[\omega_1\omega_2\omega_3]$ is given by

$$\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = 2[\beta_{jk}]^{-1}\{\dot{\beta}_j\} + [\tilde{C}]\begin{Bmatrix} 0 \\ -\dot{\lambda} \sin \phi \\ \dot{\phi} \\ \dot{\lambda} \cos \phi \end{Bmatrix} \tag{19}$$

where $[\beta_{jk}]$ is an orthogonal matrix of Euler parameters viz.,

$$[\beta_{jk}] = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} .$$

In equations (19), $[\tilde{C}]$ is matrix $[C]$ augmented by a row and column of zeroes - the zeroes forming the first row and column of $[\tilde{C}]$. Also, in these equations the angular velocity of the reference frame, $\{z_i\}$, is given by $\lfloor -\dot{\lambda}\sin\phi, \dot{\phi}, \dot{\lambda}\cos\phi \rfloor$. Finally if equations (19) are solved for $\{\dot{\beta}_j\}$ and put in a second order form, then the lunar rotational equations assume the form given in equations (2) viz.,

$$\{\ddot{\beta}_j\} = \frac{1}{2}\frac{d}{dt}\{[\beta_{kj}]\{\Omega_j\}\}$$

where

$$\{\Omega_j\} = \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} - [\tilde{C}] \begin{Bmatrix} 0 \\ -\dot{\lambda}\sin\phi \\ \dot{\phi} \\ \dot{\lambda}\cos\phi \end{Bmatrix} .$$

The time derivatives of $\{\Omega_j\}$ as required in equations (2) are evaluated from Euler's equations, (16).

2.6 Coordinate Frames

The basic inertial reference frame, X_i $(i=1,2,3)$, is located at the Earth's mass center and is the traditional mean equator and equinox of 1950.0 system. Axes x_i, and y_i, with unit vectors $\underline{i}_i$ and $\underline{j}_i$, are Earth and Moon principal axis systems. Axes Z_i, with unit vectors $\underline{K}_i$, are the reference axes for the lunar rotational motion.

The unit vectors $\underline{K}_i$ have a defined relation to the unit vectors $\underline{i}_r$, $\underline{i}_\lambda$, $\underline{i}_\phi$ of a spherical polar system locating the Moon with respect to X_i, viz.,

$$\underline{K}_1 = -\underline{i}_r,\ \underline{K}_2 = -\underline{i}_\lambda,\ \underline{K}_3 = \underline{i}_\phi.$$

2.7 UMLTR Model for Earth Rotational Equations of Motion

Although the rotational equations of motion for the Earth have been derived and coded in a manner similar to that done for the Moon (Section 2.4) those equations are not integrated in the current version of the UMLTR. Instead the nutation, precession, and spin of the Earth are defined by the formulae given by Lundquist and Veis (1966).

It was found necessary to use the rigorous rotation matrix for nutation however, rather than the linearized version incorporated there. The UMLTR can also read the JPL DE96 Ephemeris tape for nutation and nutation rates.

2.8 Forces and Torques

The forces utilized in the UMLTR have been outlined in preceding sections. Detailed derivations are presented in Breedlove (1975, 1976b, 1977). The lunar torques coded in the UMLTR are listed in Table 1. Pesek (1973) did not list all of the terms in the 4th order particle-figure interaction (PFI) torques and made some approximations in the remaining terms. The full fourth order PFI torques were coded in the UMLTR, however. A convenient vector form for the FFI torques when only C_{20} is considered is given by Chobotov (1964) in the form

$$\underline{T} = -(15/2)\mu_4 C_{20} R^2 \rho^{-5} [(1 - 7\cos^2\theta)\underline{i}_r \cdot \bar{\bar{I}} \times \underline{i}_r \qquad (20)$$

$$+ 2\cos\theta(\underline{j}_3 \cdot \bar{\bar{I}} \times \underline{i}_r + \underline{i}_r \cdot \bar{\bar{I}} \times \underline{j}_3) - \frac{2}{5}\underline{j}_3 \cdot \bar{\bar{I}} \times \underline{j}_3]$$

where $\theta = \underline{j}_3 \cdot \underline{i}_r$ and $\bar{\bar{I}}$ is the lunar inertia dyadic.

LUNAR GRAVITY COEFFICIENTS INVOLVED	TYPE OF INTER-ACTION*	BODIES INVOLVED	REFERENCE
2nd Order	PFI	Earth, Sun Moon	Eckhardt (1970)
3rd Order	PFI	Earth, Moon	Pešek (1973)
4th Order	PFI	Earth, Moon	Pešek (1973)
4th Order	FFI	Earth (C_{20}, C_{22}) Moon (C'_{20}, C'_{22})	Breedlove (1976a)

*PFI = Particle - Figure Interaction

*FFI = Figure - Figure Interaction

TABLE 1. LUNAR TORQUES

2.9 Numerical Integration Method

A relatively new integrator, as described by Everhart (1974a, 1974b), is being utilized in the UMLTR. This integrator uses an implicit single sequence algorithm. Substeps within each sequence are taken using Gauss-Radau spacings for increased efficiency and accuracy. The integrator has the form of an empirical time series method and has been shown to be related to the implicit Runge-Kutta method. Orders of 7,11,15,19,23,27 and 31 may be chosen and the 11th order option is being utilized on all UMLTR runs.

3. NUMERICAL VERIFICATION TESTS

Results from numerical experiments and from several comparisons of the UMLTR with existing numerical/analytical models are presented here. All runs described here were made on the Old Dominion University DEC-10 computer in double precision. This computer maintains a precision of 18 decimal digits in double precision with 72 bits per double precision word.

3.1 Comparison with DE96

Various comparison runs have been made with DE96 for the purpose of validating the UMLTR for integration of the Moon and the planets. In the run described here, planetary initial conditions were obtained from DE96 for Julian Date 2441200.5 and all constants used in the DE96 run were duplicated in the UMLTR. The initial conditions for β_i, $\dot{\beta}_i$ $(i = 0,1,2,3)$ were obtained from Eckhardt's (1976) 300 series. All rotational physical parameters were set to those values specified by the 300 series. A program, EMAN, was developed to (in part) convert the physical librations of the 300 series into the Euler parameters β_i. Initial rates $\dot{\beta}_i$ are calculated by numerical differentiation of a cubic spline fit to β_i in EMAN.

The basic differences between DE96 and the UMLTR in terms of the planetary integrations are that 1) the full relativistic equations as given by Standish (1976) have not been programed in the UMLTR, (see Section 2.2) and 2) the UMLTR integrations assume the Sun's mass center is the origin of an inertial reference frame. With the above initial conditions, a run was made for six years. The results of the comparison between the UMLTR and DE96 are presented in Table 2. The quantity tabulated there is

$$\delta_r = \sqrt{(X_{DE96} - X_{UMLTR})^2 + (Y_{DE96} - Y_{UMLTR})^2 + (Z_{DE96} - Z_{UMLTR})^2}$$

in meters. The differences, δ_r, are periodic with Jupiter having the largest values probably due to the omission of the Sun's acceleration in the UMLTR. An earlier comparison run was made without modeling the asteroids Juno, Pallas, and Vesta resulting in δ_r's of 398 m and 4608 m for Mars after 400 and 800 days respectively. Inclusion of the effect of these asteroids on Mars orbit in the UMLTR produces the δ_r's quoted in Table 2. The run described here used the 11th order integrator with a constant sequence size of 1 day. CPU time on the DEC10 for the six year run was approximately 102 minutes.

PLANET / TIME PAST EPOCH	400 d	800 d	1200 d	1600 d	2000 d	2400 d
MERCURY	56	115	54	179	190	245
VENUS	102	377	184	342	263	669
EARTH-MOON BARYCENTER	306	458	493	391	152	196
MOON (GEOCENTRIC)	37	82	35	120	199	88
MARS	162	873	82	641	752	268
JUPITER	61	327	1,278	5,202	7,847	12,400
SATURN	45	161	338	579	923	1,421
URANUS	9	38	144	137	229	329
NEPTUNE	3	11	26	35	63	59
PLUTO	150	13	13	50	79	144

TABLE 2. COMPARISON OF UMLTR AND DE96 FOR SIX YEARS (δ_r in meters).

The primary goal of this work however, is to produce an accurate integration of lunar translational/rotational motion. The accuracy of comparison of the Lunar orbit with DE96 is encouraging although no detailed comparison of the UMLTR and the lunar ephemeris in DE96 (LE44) has been attempted. Also it should be noted that options exist in the UMLTR to integrate only the Earth and Moon or only the inner planets and Moon while reading all remaining planetary positions from DE96.

3.2 Closure Test

A closure test was performed using the UMLTR as a rough check on roundoff error. A run was initiated at Julian Date (JD) 2441200.5 and terminated at JD 2442400.5. The terminal conditions were then used to perform a backward integration terminating at JD = 2441200.5. The same initial conditions used in the previous test were used here. The RMS position vector error for the Moon was 18 cm. and the RMS velocity vector error was 7 cm./day. In terms of the physical librations, the error in ρ was 0.004", the error in I_σ was 0.07", and the error in τ was 0.07". These are an indication of the roundoff errors after about 6.5 years of integration.

3.3 Comparison with Eckhardt's 300 Series

An iterative weighted least squares technique was used to fit the lunar physical librations ρ, I_σ, and τ calculated by the

UMLTR with those available from Eckhardts (1976) 300 series. Eckhardt's 300 solution was supplemented by a set of "additive" and "planetary" terms supplied by Williams (1976c) and by the second order term described by Williams, et. al (1973). Also Eckhardt's 301 series terms were utilized to provide the effects of 4th degree terms in the lunar gravitational potential.

Differences of the UMLTR and Eckhardt's solution for ρ, I_σ, and τ were calculated at 3 day increments for 1200 days. The quantity

$$\sum_i (\rho_{300} - \rho_{UMLTR})_i^2 + (I\sigma_{300} - I\sigma_{UMLTR})_i^2$$

$$+ (\tau_{300} - \tau_{UMLTR})_i^2$$

was chosen as the quantity to be minimized where the sum is over all "observations". The fit was made by adjusting the initial conditions, $(\beta_i)_0$ and $(\dot{\beta}_i)_0$ (i=0,1,2,3) and 9 bias parameters using numerical partial derivatives.

The quantities $(\beta_i)_0$ and $(\dot{\beta}_i)_0$ are not all independent so that in performing the estimation and in calculating the partial derivatives the following constraints are maintained

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1, \tag{20}$$

$$\beta_0\dot{\beta}_0 + \beta_1\dot{\beta}_1 + \beta_2\dot{\beta}_2 + \beta_3\dot{\beta}_3 = 0.$$

In practice only β_0, β_1, β_2, $\dot{\beta}_0$ $\dot{\beta}$, $\dot{\beta}_2$ were estimated with β_3 and $\dot{\beta}_3$ being calculated from the constraints by the method given in Moyer (1971).

In calculating the numerical partial derivatives with respect to $(\beta_i)_0$, the usual normalization procedure used when integrating Euler parameters must be suspended. Usually, these parameters are normalized by a factor, QN, where

$$QN = \sqrt{\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2}$$

during the numerical integration.

The bias parameters, b_i $(i = 1,\ldots,6)$ are of the form

$$\rho_{UMLTR} = \rho_{CALCULATED} + b_1 + b_2 t$$

$$I\sigma_{UMLTR} = I\sigma_{UMLTR} + b_3 + b_4 t$$

$$\tau_{UMLTR} = \tau_{UMLTR} + b_5 + b_6 t,$$

while the remaining bias parameters b_7, b_8, b_9 represent arbitrary constant rigid body Euler rotations about axes $\{z_i\}$.

The epoch for the fit was 2441200.5 and all initial conditions and parameters are consistent with DE96 and Eckhardt's 300 series for that date. Again initial rotational conditions were calculated by program EMAN.

The results of this fit are shown in Figure 1, where residuals (O-C) in ρ, $I\sigma$, and τ are shown in arc seconds plotted against days past 2441200.5. The means and standard deviations of these data are given in Table 3. The fit changed the nominal initial conditions at most in the 13th decimal place. The estimated values of the bias parameters are given in Table 4. All bias values were given nominal values of zero.

LIBRATION	MEAN (Arc sec)	STANDARD DEVIATION (Arc sec)
ρ	-0.1011D-4	0.093
$I\sigma$	0.1850D-4	0.148
τ	0.3602D-3	0.053

TABLE 3. STATISTICS OF UMLTR FIT TO ECKHARDT 300-301 SERIES

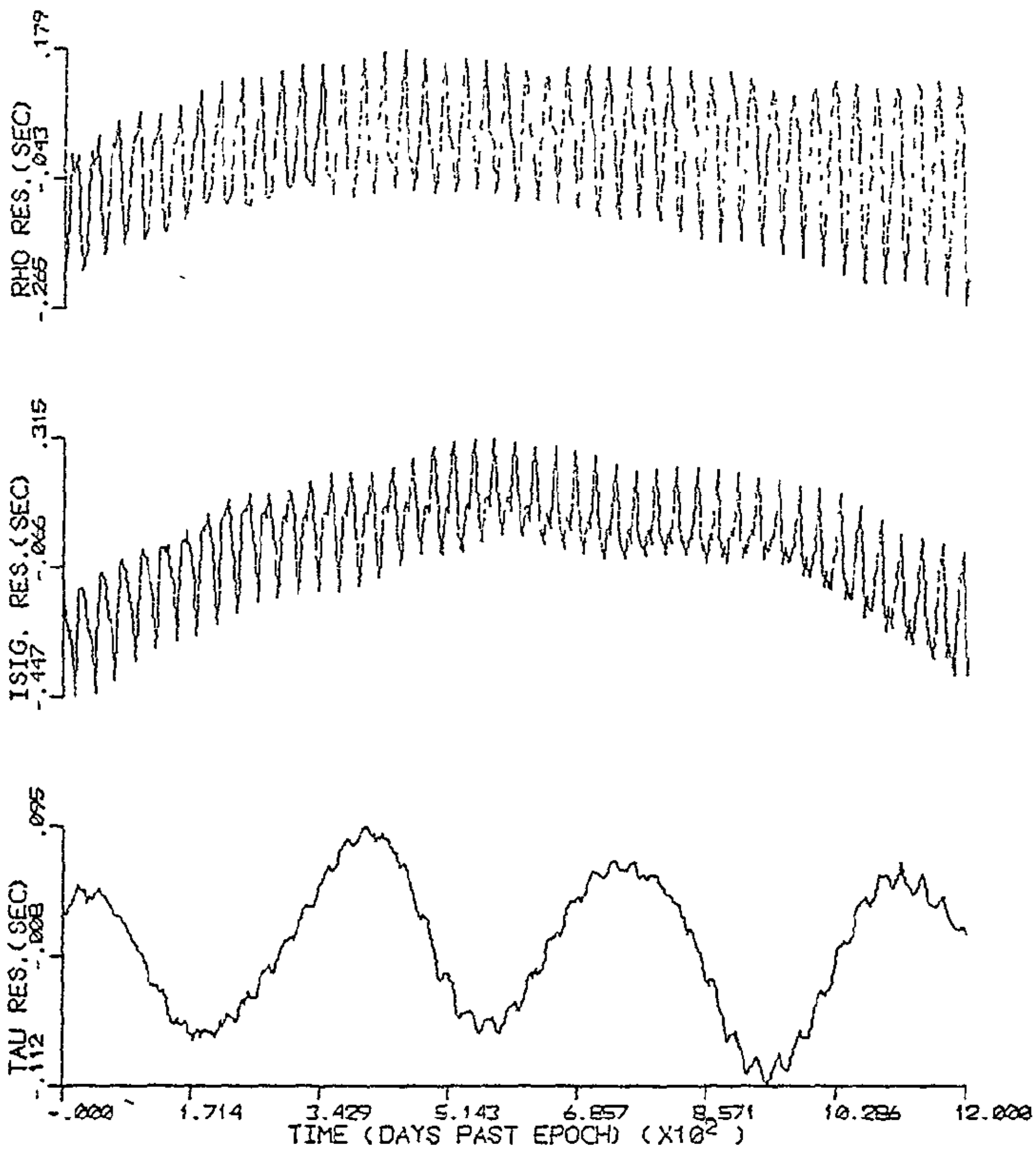

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Figure 1.- UMLTR fit to Eckhardt 300-301 series.

Parameter	Estimated Value
b_1	0.199"
b_2	-0.004"/d
b_3	6.042"
b_4	-0.011"/d
b_5	-0.035"
b_6	0.292×10^{-4}"/d
b_7	0.010"
b_8	0.013"
b_9	0.018"

TABLE 4. ESTIMATED BIAS PARAMETERS FOR UMLTR FIT TO 300-301 SERIES.

The residual patterns of Figure 1 show the existence of long period free librations in ρ and $I\sigma$ that could not be fit over this data arc. The residuals in τ show that the solutions differ by annual terms of about 0.1 arc second and that the free libration in τ has been suppressed. The estimated values of the bias parameters b_1-b_9 indicates that the coordinate reference frames of the 300 series and the UMLTR don't coincide precisely.

4. TRANSLATIONAL/ROTATIONAL COUPLING AND FIGURE-FIGURE INTERACTIONS

Several runs of the UMLTR have been made to provide an indication of the effect of the earth-Moon mutual potential on lunar physical librations and the effect of considering translational/rotational coupling for the Moon.

4.1 Coupling of Lunar Orbital/Rotational Motions

Duboshin (1958) has shown that if terms factored by ρ_G^{-4} are retained in the differential equations of motion that coupling is present and that the force expressions contain terms involving second degree gravity harmonics while the torque expressions contain terms involving second and third degree gravity harmonics. The UMLTR models terms factored by ρ_G^{-4} in the forces and by ρ_G^{-5} in the torques.

A nominal run of the UMLTR was made modeling second and third order lunar gravity harmonics. Translational initial conditions came from DE96 at JD = 2441200.5. Rotational initial conditions were determined from a fit of the UMLTR to Eckhardt's 300 series to eliminate large free librations in τ.

Two additional runs were made using the same translational initial conditions but changing the lunar gravity model and the rotational initial conditions. In the first case lunar second, third, and fourth degree harmonics were retained while in the second case only second degree harmonics were retained. Rotational initial conditions, in each case, were found from fits of the UMLTR to Eckhardt's series.

Figure 2 shows the residuals between the nominal and the first case run. Residuals in semi-major axis in meters, eccentricity, inclination, argument of perigee, node, and true longitude are shown. All angular residuals are in seconds of arc. The elements

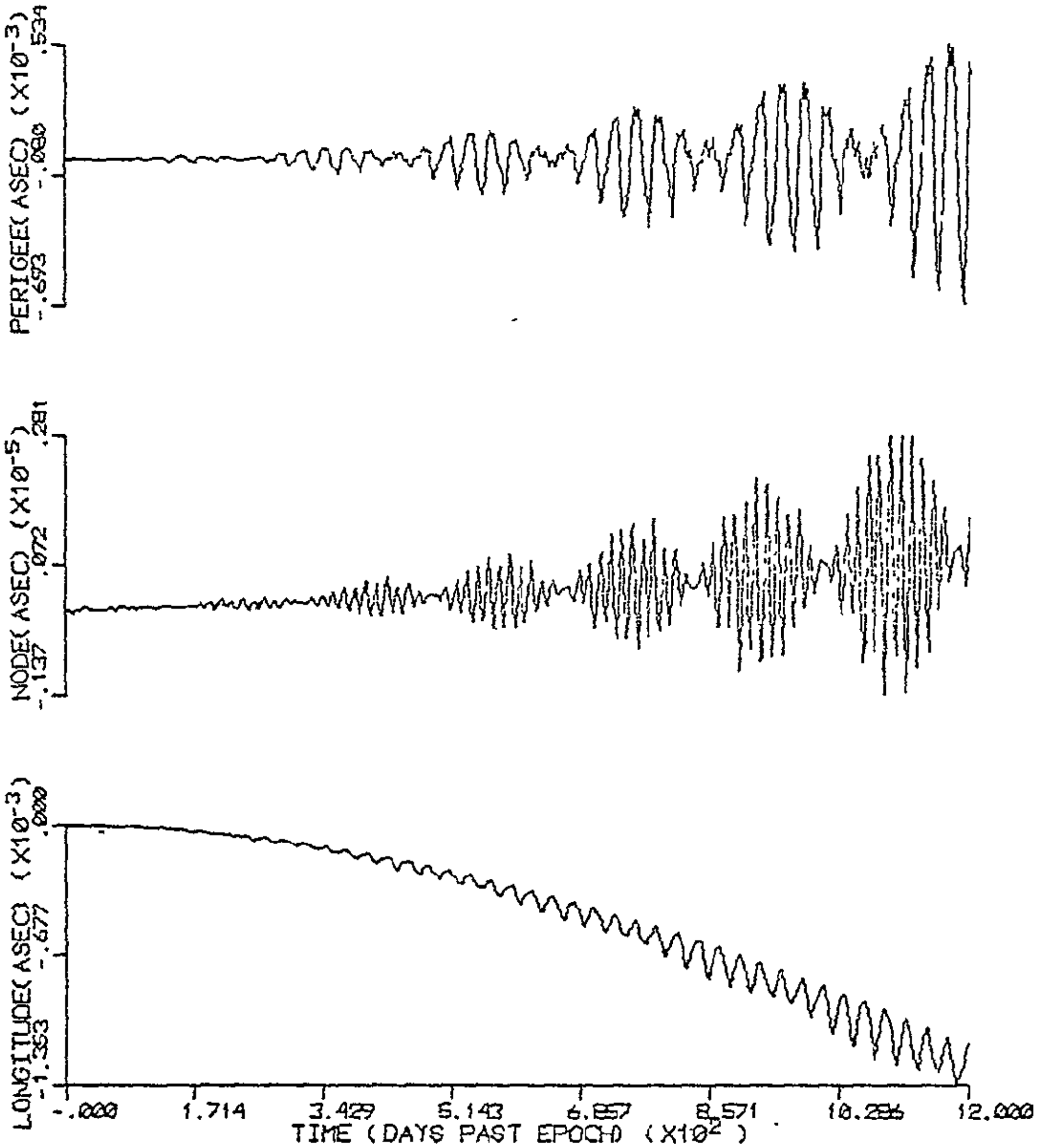

Figure 2.- (Cont.) Lunar orbital residuals between 4th and 3rd degree torque models.

are with respect to the fixed 1950.0 system. The major effects are the growth in longitude amounting to 0.001" in 3 years, the periodic growth in perigee amounting to 0.007" in 3 years, and the growth in semimajor axis of 5 cm. in 3 years. Identical residual patterns were obtained for the case two run but the magnitudes were much greater. After 3 years, the residuals in semimajor axis, perigee, and longitude for the latter case are 6.6 meters, .09", and .17" respectively. Apparent secular motions may also be noted in inclination, node, and longitude. The ratio of the effects noted above between cases one and two are the same as the ratio of the constant longitude bias between the two cases.

The model in the UMLTR directly includes the effects of 2nd order lunar harmonics in the forces governing translational motion. The inclusion of 3rd and 4th degree terms in the torques can only affect the translational motion through the coupling mechanism. That is, the presence of these torques affects the lunar orientation which, in turn, affects the forces.

4.2 The Effect of FFI on Lunar Physical Librations

The effect of FFI interactions on physical librations was shown in a preliminary study in Breedlove (1976a). There it was shown that these terms have at least an 0.01" effect on lunar physical librations and thus should be included in any model analysing the LURE data. Yoder (1977) has subsequently derived approximate analytical expressions for the FFI effects on lunar physical librations. That study indicates "that the largest terms produce an

18.6 year libration of magnitude .007" in longitude and 0.08" in latitude as seen from an inertial frame."

NOTATION

$''$	Seconds of arc
m_j	Mass of jth particle
c	Speed of Light
C	Tidal coupling coefficient
α, β, γ	Lunar Principle Inertia Ratios
C_{ij}, S_{ij}	Earth Gravity Harmonic Coefficients
C'_{ij}, S'_{ij}	Moon Gravity Harmonic Coefficients

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APPENDIX C

ON LUNAR ORBITAL-ROTATIONAL COUPLING AND FIGURE-FIGURE INTERACTIONS IN THE EARTH-MOON SYSTEM

(Paper prepared for AIAA 17th Aerospace Sciences Meeting.)

ON LUNAR ORBITAL-ROTATIONAL COUPLING AND FIGURE-FIGURE INTERACTIONS IN THE EARTH-MOON SYSTEM

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Abstract

A numerical model - the Unified Model of Lunar Translation/Rotation (UMLTR) - has been developed by the author. The model numerically integrates the heliocentric motion of all planets, the Earth-Moon barycenter, and the geocentric motion of the Moon coupled with its rotational motion. Results are presented of numerical experiments performed with the UMLTR which show coupling - effect orbital residuals of 5cm., .0007", and 0.001" in semi-major axis, perigee and longitude - after three years - between a third-order lunar gravity model and a fourth-order model. Figure-figure interactions provide perturbations of at least 0.01" in the lunar physical librations. Finally, the possibility of estimating simultaneously lunar inertia ratios, α, β, γ with lunar coefficients C_{20} and C_{22} is discussed based on the UMLTR and the Lunar Laser Ranging Experiment (LURE) observable.

Nomenclature

A,B,C	Lunar principal moments of inertia
C_{20}, C_{22}, S_{22}	Lunar Harmonic Gravity Coefficients
$C'_{20}, C'_{22}, C'_{30}$	Earth Harmonic Gravity Coefficients
M	Lunar mass
R	Mean lunar radius

1. Introduction and Summary

Consideration of the coupling between lunar orbital and rotational motions and the effect of the mutual potential between the Earth and Moon have been - for the most part - ignored in many numerical and analytical models developed in the past. The complexity of any model incorporating these effects and the level of observational accuracy available in the past did not warrant their treatment. Recently, however, the observational accuracy has improved to such an extent that inclusion of the above effects may be important. The Lunar Laser Ranging Experiment (LURE) is currently providing measurements of the equivalent distance from an Earth-bound telescope to the lunar retro-reflectors accurate to 13cm (over the period 1969-1977). The anticipated accuracy level for this data is 2-3cm.

The author has developed a numerical model for the coupled translational/rotational motion of the Moon referred to as the Unified Model of Lunar Translation/Rotation (UMLTR). This model consists of the direct numerical integration of the heliocentric orbits of all planets, the Earth-Moon barycenter, and the geocentric orbit of the Moon coupled with its rotational motion. The integrator has the option of integrating the planetary motions or reading their ephemerides from DE96. A relatively new integration method is utilized in the UMLTR as developed by E. Everhart.[1,2] This method is an implicit single-sequence method using "Gauss-Radau spacings for substeps within each sequence". Integration orders of 7 to 31 are possible. Other novel features of the UMLTR are that Euler parameters are used to model the lunar rotational motion and the figure-figure interactions (FFI) of the Earth and Moon are modelled from the mutual potential function.

The effect of FFI interactions and coupling on the Moon's motions has had little discussion in the literature. A preliminary numerical study of the effect of figure-figure interactions on the lunar physical librations was given by Breedlove.[3] Yoder[4] investigated the effect of spin-spin interactions on the lunar physical librations from a theoretical standpoint. Langrange[5] investigated the effects of lunar libration on the lunar orbit. W. J. Eckert[6] and Spencer Jones[7] attributed discrepancies between the observed and modelled motions of the lunar node and perigee to the "lunar figure, inaccuracies in observed and theoretical values, and possible effects of unknown forces". Based on these discrepancies, Eckert estimated values for certain lunar inertias and their ratios.

In a more general sense, a number of authors have discussed the problem of translational/rotational motion of mutually attracting rigid bodies. Duboshin[8] showed that if terms factored by r^{-3} - where r is the mutual distance between two rigid bodies - are retained in the differential equations then one has the case of semi-independent motions. That is, the translational motion can be solved independently and then used as input to the rotational solution. If terms factored by r^{-4} are retained then translational/rotational motions are coupled and if terms factored by r^{-5} are retained the FFI terms appear in the torques. Finally, the FFI terms appear in the force expressions if terms factored by r^{-6} are retained. Beletskii[9] mentions that V. T. Kondurar "observed that the perturbations introduced in the lunar motion by its "shape effect" are of the same order of magnitude as some of the effects considered in the modern theory of the Moon". Several special-case solutions are available in the literature based on assumed mass distributions in both bodies. Examples of these are given by Lanzano[10] and Kinoshita[11].

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More practically, a vigorous treatment of the orbital-rotational coupling may provide for improved accuracy in the determination of the lunar moments of inertia from observational data. This will provide more confidence in the model of the lunar interior.

The determination of the lunar inertias A/MR^2, B/MR^2, and C/MR^2 has always been accomplished by combining data obtained from two fundamentally different sources. For example, the motion of the lunar perigee and node provide a measure of K and L where

$$L=(3/2)(C-A)/MR^2=3C_{22}-(3/2)C_{20},$$

and

$$K=(3-2)(B-A)/MR^2=6C_{22},$$

and the lunar rotational motion provides a means of determining

$$\beta=(C-A)/B$$

and

$$\gamma=(B-A)/C.$$

Using individually determined L and K values or β and γ values leaves the inertias undetermined. But, if e.g., L, β, K, and γ are algebraically combined, estimates of the inertias can be obtained. Michael[12] summarizes various solutions for the inertias using the above approach. Eckert[6] also utilized this approach.

Recently, lunar orbiter, Explorer, Luna and Apollo orbital data have been utilized to estimate

$$C_{20}=[(A+B)/2-C]/MR^2$$

which can be algebraically combined with β and γ obtained from the LURE data in order to solve for the lunar inertias. Gapcynski, et al[13] found $C/MR^2=0.392\pm0.003$ using this approach. Sinclair, et al[14] proposed a simultaneous estimation of all six lunar inertia tensor components using LURE data and Lunar Orbiter IV data.

Typically, β has always been better determined than C_{20} or γ and Williams[15] reports that the uncertainty in C/MR^2 is dominated by the error in C_{20} which means that orbiting spacecraft data must be improved.

Since the UMLTR models the coupled orbital-rotational motions of the Moon, the rotational parameters $\alpha=(C-A)/B, \beta$, and γ and the orbital parameters C_{20} and $C_{22}=(B-A)/4MR^2$ appear in the equations of motion. These five parameters are not independent however, but are connected by the constraints

$$C_{22}=\gamma(1+\beta)C_{20}/2[\gamma(1-\beta)-2\beta] \tag{1}$$

and

$$\alpha=(\beta-\gamma)/(1-\beta\gamma). \tag{2}$$

Thus, the use of the UMLTR in the data reduction process offers the possibility of solving for the above parameters simultaneously using a single data set such as the LURE data set or the VLBI (Very Long Base Interferometry) data set with the attendant accuracy of such data.

The purpose of this paper is to present the results of some numerical experiments made using the UMLTR in an investigation of FFI effects, coupling effects and the feasibility of estimating the parameters of α, β, γ, C_{20} and C_{22} using the LURE data set.

Section 2 provides a brief summary of the UMLTR. Section 3 presents the results of a numerical study using the UMLTR which gives some indication of the magnitude of orbital-rotational coupling effects on the lunar orbit. Section 4 summarizes previously obtained results concerning FFI effects. Section 5 presents the results of a preliminary covariance study which indicates that the lunar inertias A/MR^2, B/MR^2 and C/MR^2 can be estimated simultaneously using a single accurate data set such as that available from the LURE.

2. The Unified Model of Lunar Translation/Rotation (UMLTR)

The equations of motion utilized in the UMLTR are based on the general equations,

$$m_i\ddot{\vec r}_i=\vec\nabla_i U+\vec F_i \qquad (i=1, \ldots, 11) \tag{3}$$

$$\left\{\dot\beta_j\right\}=\frac{1}{2}\frac{d}{dt}\left(\left[B\,(\beta_j)\right]\left\{\Omega_j\right\}\right) \qquad (j=0, 1, 2, 3) \tag{4}$$

where $\vec r_i$ is the inertial position vector of the ith body (Sun, 9 planets, Moon), U is the gravitational potential function, $\vec F$ is a vector of forces, $\{\beta_j\}$ is a set of Euler parameters for lunar orientation, and $\{\Omega_j\}$ is an angular velocity vector.

More explicitly, in the UMLTR, the heliocentric orbits of the planets and the Earth-Moon barycenter are integrated as subjected to n - body effects, relativistic effects modeled using the Eddington-Robertson equations as given by Anderson[16], and indirect effects due to the interaction of the Sun with the Earth figure (C'_{20} and C'_{30}) and Moon figure (C_{20}, C_{22}, S_{22}).

The effect of Ceres, Pallas, Juno and Vesta on Mars is also included.

The geocentric motion of the Moon is integrated in the UMLTR, as subjected to n - body forces, tidal coupling modeled according to Oesterwinter and Cohen[17], Earth figure - Sun interaction effects, and relativistic effects modeled using the full "N - Body" metric suggested by Moyer[18] in the form given by Standish[19].

The orientation of the Moon is specified by locating its principal axes, $\{z_i\}$, with respect to a set of reference axes $\{Z_i\}$ using the Euler parameters $\lfloor\beta_0, \beta_1, \beta_2, \beta_3\rfloor$ in the form given by Morton, et al[20]. Axes $\{Z_i\}$ with unit vectors $\{\vec K_i\}$ are defined by $\vec K_1=-\vec i_r$, $\vec K_2=-\vec i_\lambda$, $\vec K_3=\vec i_\phi$

where $\vec{i}_r$, $\vec{i}_\lambda$, and $\vec{i}_\phi$ are unit vectors of a spherical polar coordinate system locating the lunar mass center with respect to a geocentric mean equator and ecliptic of 1950.0 system. Matrix $[B\{\beta_j\}]$ of equation (4) is an orthogonal matrix of Euler parameters and angular velocity vector $\{\Omega_j\}$ of that equation is the angular velocity of the Moon with respect to $\{z_i\}$.

The torques modeled in the UMLTR involve those due to particle-figure interactions between the Earth, Sun, and lunar second degree harmonics, particle-figure interactions between the Earth and lunar third and fourth degree harmonics and figure-figure interactions between the Earth (C'_{20}, C'_{22}) and the Moon (C_{20}, C_{22}).

An associated program, EMAN, has been developed that provides lunar orientation from Eckhardt's[21] 300 series libration theory and lunar position from DE96.

3. Translational/Rotational Coupling

If terms factored by r^{-4} are retained in the differential equations, coupling is present and the forces depend on lunar orientation. Equivalently, lunar second degree harmonics appear in the forces and lunar third degree harmonics appear in the torques. Existing developments, e.g., that of Oesterwinter and Conen[17] contain the second degree harmonic terms in the forces. Some adopted model for lunar rotation must be used in those developments however, to provide the lunar orientation. The adopted model could range from one based on Cassini's laws to one based on analytical or numerical solutions for lunar rotation made using an adopted lunar orbit. This process is not necessary in the UMLTR however, since the lunar orientation is automatically provided.

The effects of coupling are numerically investigated using the UMLTR by determining the effect of various lunar gravity models on the lunar orbital elements a, e, i, ω, Ω and true longitude, L. Here, each run has identical translational initial conditions but the lunar rotational initial conditions and the lunar gravity model differ for each run. Three models are used viz., (i) a lunar second degree harmonic model (II), (ii) a lunar second and third harmonic model (III), and (iii) a lunar second, third, and fourth degree harmonic model (IV). In each run, rotational initial conditions are determined from a least squares fit of the UMLTR to Eckhardt's[21] 300-301 series for the lunar physical librations ρ, $I\sigma$, τ. These fits are made over a 3-year period to supress large free librations in the longitudinal libration τ that appear with an arbitrary choice of initial conditions. Translational initial conditions come from DE96 and all parameters are consistent with DE96 and Eckhardt's 300-301 series.

Figures 1 and 2 show the residuals for a 3-year period from the Julian day epoch 2441200.5 between models III and IV Of prime significance are the $\sim1.5''/cy^2$ quadratic term in longitude which corresponds to an $\dot{a}$=0.27cm./cy. The quadratic terms in node and inclination are $\sim9.7\times10^{-4}''/cy^2$ and $\sim4.7\times10^{-4}''/cy^2$ respectively. Also; small bi-monthly terms are present in the node and inclination modulated by the 173-day solar nodal passage terms. The eccentricity and perigee exhibit a purely periodic character with small monthly terms modulated by the 205-day solar apsidal passage terms. Finally, a periodic growth in semi-major axis of ~5cm. is noted as well as maximum growth in longitude residuals of .001" and in perigee of .0007" in 3 years.

The apparent growth in all of those curves are in reality, a portion of a longer period effect than the 3-year integration time.

The residuals between models II and III exhibit the same behavior as shown in Figures 1 and 2. The magnitude of the effects are much larger, however. The largest effects are growths in semi-major axis longitude and perigee residuals of ~7m, ~0 17" and ~0.1", respectively, as well as a quadratic term $\sim160''/cy^2$ in the longitude. This latter value is consistent with an $\dot{a}$ of ~24m/cy The node and inclination exhibit quadratic terms of $\sim0.1''/cy^2$ and $\sim0.07''/cy^2$, respectively.

It is interesting to note that the ratios of the quadratic terms between the II-III residuals and the III-IV residuals is the same as the ratio of the constant offset in τ (longitude libration) predicted for those cases by Eckhardt's 300 series.

The model in the UMLTR directly includes the effects of second order lunar harmonics in the forces governing translational motion. The inclusion of third and fourth degree terms in the torque models can only affect the translational motion through the coupling mechanism. That is, the presence of these torques affects the lunar orientation which, in turn, affects the forces.

This analysis shows that even fourth degree lunar harmonics can have a measurable effect on the lunar orbit contributing at least 5 cm. effects in the geocentric lunar radius after 3 years. For consistency, if fourth degree lunar harmonics are included in the lunar torques, then third degree lunar harmonics should be retained in the forces.

4. Figure-Figure Interactions

FFI arise in the mutual potential of the Earth-Moon system when the usual assumption that Earth and lunar radii are small compared to their mutual distance is discarded

Breedlove[3] considered the interaction of C'_{20} and C'_{22} for the Earth with a triaxial Moon using the UMLTR. In that paper, a numerical study of the effects of the FFI was presented which showed that monthly variations of the order of ~0.002" were present in the latitude librations (ρ and $I\sigma$) while long period variations in ρ, $I\sigma$, and τ should be expected with amplitudes ~0 01" and periods that are multiples and fractions of the lunar nodal period (18.6 years). These effects cause range variations of ~10cm. Thus, it was suggested that these terms be modeled in any analysis of LURE data accurate to ~0.01".

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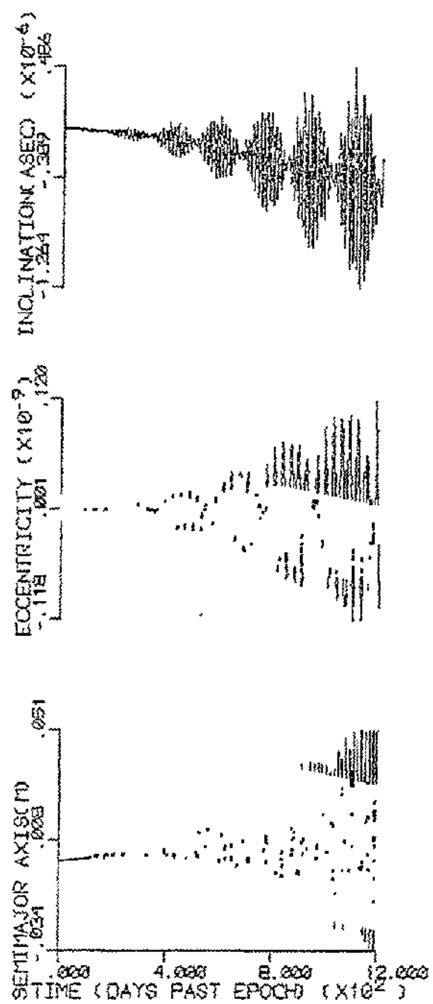

FIGURE 1. RESIDUALS IN SEMI-MAJOR AXIS, ECCENTRICITY, AND INCLINATION BETWEEN MODELS III AND IV

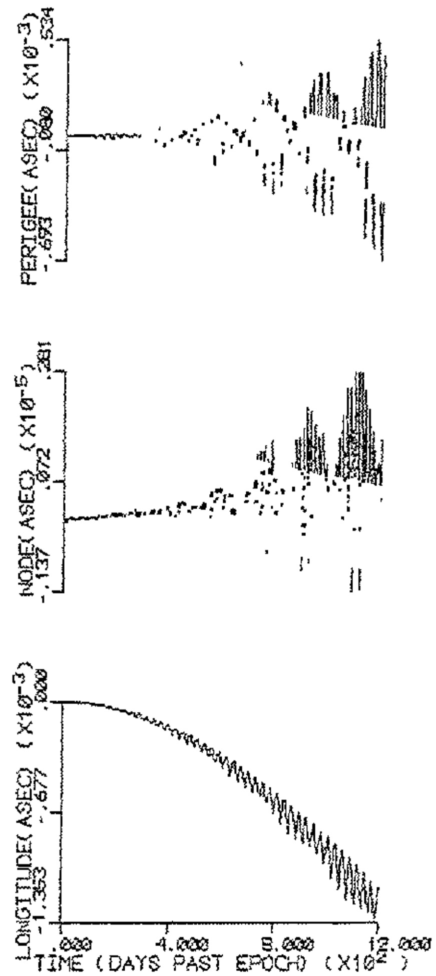

FIGURE 2. RESIDUALS IN LONGITUDE, NODE AND PERIGEE BETWEEN MODELS III AND IV

Subsequently, Yoder[4] used an approximate theoretical analysis to show that the major FFI effects produced the following perturbations in the latitude librations, $\delta\rho$, and the longitude librations, $\delta\tau$.

$$\delta\rho = -0''.074 \exp(-i\phi') - 0''.0026 \exp(-i\Omega) \qquad (5)$$
$$+ 0''.0014 \exp[-i(\ell+\phi')] - 0''.001 \exp[-i(2\lambda-\ell-\phi')]$$
$$\delta\tau = 0''.0067 \sin(\Omega-\phi') - 0''.001 \sin 2\omega. \qquad (6)$$

In the above, ϕ' is the node angle describing the Earth's spin axis with respect to the ecliptic, ℓ is the lunar mean anomaly, $\omega=(F-L)$ is the lunar argument of pericenter, and i is the imaginary unit.

5. A Covariance Analysis for the Simultaneous Estimation of Lunar β, γ, and C_{20}

The feasibility of simultaneously estimating lunar β, γ, and C_{20} using only LURE data was investigated by forming a normal matrix based on the LURE observable, τ^*, numerical partial derivatives and actual observation times over the period September 1967 to January 1973. This normal matrix was then inverted to give a covariance/correlation matrix providing a relative indication of the feasibility of estimating the above quantities.

Nineteen (19) parameters were estimated as follows:

$\vec{\rho}_{G_0}$ - lunar geocentric initial position vector,

$\dot{\vec{\rho}}_{G_0}$ - lunar geocentric initial velocity vector,

β_{i_0} $(i=0,1,2)$ - lunar Euler parameter initial values,

$\dot{\beta}_{i_0}$ $(i=0,1,2)$ - lunar Euler parameter initial rate values,

$\beta = (C-A)/B$ - lunar inertia ratio,

$\gamma = (B-A)/C$ - lunar inertia ratio,

$C_{20} = [(A+B)/2-C]/MR^2$ - lunar oblateness,

$\alpha = (C-B)/A$ - lunar inertia ratio,

$C_{22} = (B-A)/4Ma^2$ - lunar tesseral harmonic

β_{3_0} - lunar Euler parameter initial value, and

$\dot{\beta}_{3_0}$ - lunar Euler parameter initial rate value.

The first 15 of the above parameters were treated as independent and the remaining four were subject to the following exact constraints

$$S_1 = \alpha = (\beta-\gamma)/(1-\beta\gamma), \qquad (7)$$

$$S_2 = C_{22} = \gamma(1+\beta)C_{20}/2[\gamma(1-\beta)-2\beta], \qquad (8)$$

$$S_3 = \beta_3 = \pm\sqrt{1-\beta_0{}^2-\beta_1{}^2-\beta_2{}^2}, \quad \text{and} \qquad (9)$$

$$S_4 = \dot{\beta}_3 = -(1/\beta_3)(\beta_0\dot{\beta}_0+\beta_1\dot{\beta}_1+\beta_2\dot{\beta}_2). \qquad (10)$$

Initial values for $\vec{\rho}_{G_0}$, $\dot{\vec{\rho}}_{G_0}$ were taken from DE96 at the initial epoch JD2440468.9217100694(UT) and β_{i_0}, $\dot{\beta}_{i_0}$ were taken from program EMAN at that initial epoch. Nominal values for the other parameters were

$$\beta = 6.31\times10^{-4}$$
$$\gamma = 2.275\times10^{-4}$$
$$C_{20} = 2.075\times10^{-4}.$$

Following Moyer[22], the covariance matrix is

$$\Gamma = \left[J+\tilde{\Gamma}_x^{-1}\right]^{-1} \qquad (11)$$

where

$$J = \left(A_x+A_xS_x\right)^T W \left(A_x+A_sS_x\right), \qquad (12)$$

$$A = \left[\frac{\partial\tau^*}{\partial p}\right],$$

and

$$p = \left\{\frac{x}{s}\right\}.$$

In equation (11), p is the parameter set, τ^* is the LURE observable, x are the independent parameters, s are the constrained parameters, W is the data weighting matrix and $\tilde{\Gamma}_x$ is the a priori covariance matrix for x.

Following Mulholland[23], the elements of matrix A are computed from

$$\frac{\partial\tau^*}{\partial p_i} = \vec{D}^* \cdot \left[\frac{\partial\vec{\rho}_G}{\partial p_i} - \frac{\partial\vec{\rho}''}{\partial p_i} + \frac{\partial\vec{\kappa}''}{\partial p_i}\right]\left(\frac{2}{C}\right) \qquad (13)$$

where C is the speed of light, $\vec{\rho}''$ is the geocentric position vector of the laser transmission station and $\vec{\kappa}''$ is the selenocentric position vector of the retroflector both expressed in the 1950.0 mean Earth equator and equinox system. Unit vector $\vec{D}^* = \left(\vec{\rho}_G-\vec{\rho}''+\vec{\kappa}''\right)/\left|\vec{\rho}_G-\vec{\rho}''+\vec{\kappa}''\right|$. Tacitly assumed in the deviation of equation 13 is that the Earth transmission station and the lunar retroreflector do not move during the ~2.5 seconds roundtrip light travel time.

The derivatives $\partial\tau/\partial p_i$ were computed as follows:

1. LURE normal point observation times taken from Shelus[24] were converted from UTC to ET using the formulation given by Moyer[22] and JPL smoothed BIH data for the pole position, A1-UT1 and A1-UTC as supplied by Williams[25],

2. The vectors $\vec{\rho}_G$, $\vec{\rho}''$, $\vec{\kappa}''$, and $\vec{D}^*$ were calculated at the LURE observation times using the UMLTR, and finally,

3. Numerical partials for $\partial\vec{\rho}_G/\partial p_i$ and $\partial\vec{\kappa}''/\partial p_i$ were generated using the UMLTR.

At each observation time, the matrix J was incremented using a data weight corresponding to the typical LURE data accuracy of 13cm. or 4.3×10^{-10}sec. as given by Williams[15].

The covariance-correlation matrix was formed after 528 days, 887 days, and 1212 days.

There were no significant correlations among β, γ, C_{20} and the remaining parameters. The formal standard deviations in β, γ, and C_{20} were as shown in Table 1.

Standard Deviation	Parameter		
	β	γ	C_{20}
After 528 days	$.29 \times 10^{-13}$	$.21 \times 10^{-7}$	$.68 \times 10^{-7}$
After 887 days	$.37 \times 10^{-14}$	$.25 \times 10^{-8}$	$.27 \times 10^{-8}$
After 1212 days	$.24 \times 10^{-14}$	$.86 \times 10^{-9}$	$.16 \times 10^{-8}$

TABLE 1. STANDARD DEVIATIONS FOR β, γ, C_{20}.

The above figures are meant to provide a relative indication of the feasibility of simultaneously estimating β, γ, and C_{20} and hence, A/Ma^2, B/Ma^2, and C/Ma^2 using the LURE observable. Many other parameters should be included for a more definitive study.

The parameters β and γ are routinely estimated using LURE data, one of the latest results being presented by Williams[15] as
$\beta = (631.26 \pm 0.3) \times 10^{-6}$ and
$\gamma = (227.37 \pm 0.6) \times 10^{-6}$.

The LURE observable is more sensitive to β and γ as shown in Table 1 than to C_{20} but the magnitudes indicate that C_{20} should be determinable from LURE data in a simultaneous estimation of these parameters using the UMLTR.

6. Conclusions

Coupling effects have been shown to produce residuals of 5cm., .0007", and 001" after 3 years in the orbital parameters: semi-major axis, perigee and longitude, respectively. These effects stem from the inclusion of fourth order terms in the lunar potential which modify the lunar orientation and, therefore, the forces producing the lunar orbit.

FFI interactions have been shown to produce effects as great as 0.01" in the lunar physical librations.

Modeling, the dynamics of the Moon in a coupled translational-rotational sense as accomplished in the UMLTR offers the advantages of providing rigorously for these coupling and FFI effects. It also offers the possibility of estimating, for the first time, the lunar inertias A/MR^2, B/MR^2, and C/MR^2 from a single extremely accurate data set viz. the LURE data.

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