DEVELOPMENT OF

BOOLEAN CALCULUS

AND

ITS APPLICATIONS

SEMI-ANNUAL STATUS REPORT NO. 3 — AUGUST 16, 1978
NASA GRANT NO . NSG 1436 — FEBRUARY 15, 1979

BY

MOIEZ A. TAPIA

PREPARED FOR
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
LANGLEY RESEARCH CENTER
HAMPTON, VIRGINIA 23665

(NASA-CR-158088) DEVELOPMENT OF BOOLEAN CALCULUS AND ITS APPLICATION Semi-annual Status Report, 16 Aug. 1978 - 15 Feb. 1979 (Miami Univ.) 78 p HC A05/MF A01 CSCL 12A

N79-16613

Unclas

G3/64 43630

ATTENTION:
DR. JERRY H. TUCKER, NASA TECHNICAL MONITOR
MR. FRANK S. KAWALKIEWICZ, GRANT ADMINISTRATOR

DEPARTMENT OF ELECTRICAL ENGINEERING
UNIVERSITY OF MIAMI
CORAL GABLES, FLORIDA 33124

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MOIEZ A. TAPIA

1. INTRODUCTION

The research reported here aims at developing Boolean calculus and finding its applications in digital design, fault location and detection. Most of the results obtained since the last Semi-Annual report (#2) are described in the papers in Appendices I & II. Results obtained in the area of noncombinational Boolean calculus are given in Section 4 of this report.

2. BOOLEAN CALCULUS

The mathematical system of Boolean Calculus was formally defined, described and submitted in the form of a paper entitled "Boolean Calculus for Digital Systems" given in Appendix I. The new results are decomposition of a Boolean function with respect to one of its arguments, x_i; a new interpretive definition of Boolean differential; the exact number of compatible integrals of a Boolean differential, if it is compatibly integrable; etc.

3. SYNTHESIS OF SEQUENTIAL SYSTEM USING BOOLEAN CALCULUS

The results obtained in the last few months in the area of synthesis of asynchronous sequential systems using edge-sensitive flipflops were put together "filling the gaps" and written in the form of a paper entitled "Synthesis of Asynchronous Sequential Systems Using Edge-Sensitive Flipflops". The latter is given in Appendix II and will be submitted shortly to IEEE Computer Transactions. Of most significance is the synthesis procedure outlined in it, which is a considerable improvement and refinement over the procedure given in Semi-Annual Report #1. Of course, this is a consequence of better understanding of the properties of DM table that we have achieved.

4. NONCOMBINATIONAL BOOLEAN CALCULUS

In Boolean calculus studied so far it was assumed that a function being studied is the output of a combinational system whose inputs are the arguments of the function. Also, while integrability of a differential expression was studied, it was tacitly assumed that an integral, if it exists would be realized with a combinational system.

An attempt was made to generalize the Boolean calculus that was developed with the limitations shown above. Such calculus, to be referred to, henceforth, as noncombinational Boolean calculus, will help us describe the output, after an input change, of a noncombinational system in terms of changes in the inputs to the system. Also, if the output, after an input change, is specified in terms of changes in inputs, realizability of such a specification using a noncombinational system will be studied. Some results obtained in this direction will be described in what follows. It will be assumed that only one variable can change at a time.

<u>Definition 3.1</u>: Δx_i, $1 \leq i \leq n$, denotes a change in x_i from 0 to 1.

(D3.1.1) $$\Delta x_i = \begin{cases} 1 \text{ when } x_i \text{ changes from 0 to 1} \\ 0 \text{ otherwise} \end{cases}$$

∇x_i, $1 \leq i \leq n$, denotes a change in x_i from 1 to 0.

(D3.1.2) $\nabla x_i = \begin{cases} 1, & \text{when } x_i \text{ changes from 1 to 0} \\ 0, & \text{otherwise} \end{cases}$

<u>Definition 3.2</u>: The terms $x_i \Delta x_i$, $\overline{x}_i \Delta x_i$, $x_i \nabla x_i$ and $\overline{x}_i \nabla x_i$ will be defined as follows:

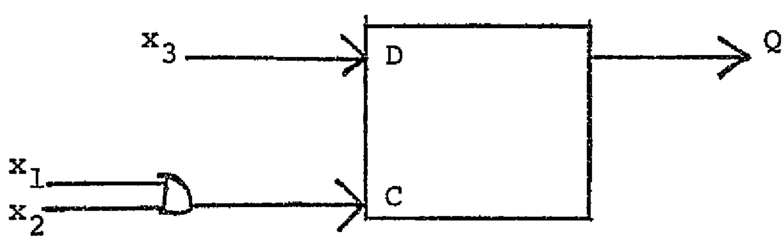
(D3.2.1) $x_i \Delta x_i = \Delta x_i$

(D3.2.2) $\overline{x}_i \Delta x_i = 0$

(D3.2.3) $x_i \nabla x_i = 0$

(D3.2.4) $\overline{x}_i \nabla x_i = \nabla x_i$

Consider a D-flipflop shown below:



Since the relationship between Q and x_1, x_2 and x_3 is not combinational, we cannot express Q in terms of a Boolean function of variables x_1, x_2 and x_3. However we could express the value of Q immediately following any transition of the clock.

Observe that

(3.1.1) $$C = x_1x_2$$

so that

(3.1.2) $$dC = x_1dx_2 + x_2dx_1$$

Since only the positive transitions of clock are of interest, we may describe the positive transitions of clock in terms of changes in x_1 and x_2

(3.1.3) $$\Delta c = x_1\Delta x_2 + x_2\Delta x_1$$

Let Q(T+) denote the value of Q immediately following any transition of clock. Then by definition

(3.1.4) $$Q(T+) = D\ .\Delta C$$

or

(3.1.5) $$Q(T+) = Dx_2\Delta x_1 + Dx_1\Delta x_2$$

If we let

(3.1.6) $D = x_3$, then

(3.1.7) $$Q(T+) = x_2x_3\Delta x_1 + x_1x_3\Delta x_2$$

Equation (3.1.7) points out that Q is 1 after the following transitions:

(1) $x_2 = x_3 = 1$ and x_1 changes from 0 to 1.

(2) $x_1 = x_3 = 1$ and x_2 changes from 0 to 1.

Observe that if $x_3 = 0$ and $x_1 = 1$ while x_2 changes from 0 to 1, then a transition does occur making Q to remain at or go to 0. This is not to be seen from equation (3.1.7) if the function D (i.e, x_3 in this case) is not kept separate from the transition terms. Hence a more desirable form for Q(T+) than that shown in equation (3.1.7) would be

(3.1.8) $$Q(T+) = D\cdot[x_2\Delta x_1 + x_1\Delta x_2].$$

Definition 3.3: The function Q(T+) as shown in equation (D3.3.1) below will be called next-value function.

(D3.3.1) $$Q(T+) = D.\left[\sum_{i=1}^{n}(\alpha_i\Delta x_i + \beta_i\nabla x_i)\right]$$ where D is a function of $\underline{x}$ and $\underline{y}$.

Obviously the function D outside the square bracket refers to the value that Q would assume if and when one of the transition terms inside the square brackets assumes value of 1.

Addressing ourselves to the reverse problem of synthesizing a network that would realize a next-value function Q(T+) of the form

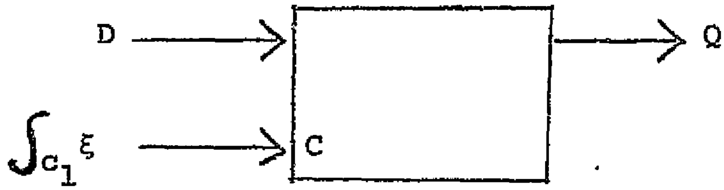
(3.1.9) $$Q(T+) = D.\left[\sum_{i=1}^{n}(\alpha_i\Delta x_i + \beta_i\nabla x_i)\right],$$

where α_i and β_i are assumed to be independent of x_i, for all i, without loss of generality (in view of Definition 3.2), all that we need to do is to find the exact integral, if it exists, of the differential expression

(3.1.10) $$d\xi = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i).$$

Of course, if the differential expression is not exactly integrable but compatibly integrable and if

$\int_{c1} d\xi$ is a compatible integral of the differential expression, then a realization of the form



will provide not only the transitions that are specified in equation (3.1.9) but, also, some additional transitions.

<u>Theorem 3.1</u>: If the next-value function of a system is given by

(T3.1.1) $$Q(T+) = D \cdot \left[\sum_{i=1}^{n} (\alpha_i \Delta x_i + \beta_i \nabla x_i)\right]$$

where

(T3.1.2) $$D = D_1 \cdot x_{i1}\, x_{i2} \cdot \dots \cdot x_{ik}, \quad 1 \leq k \leq n$$

(T3.1.3) $$F(\underline{x}) = \int_{\xi} \left(\sum_{i=1}^{n} \alpha_i dx_i + \beta_i d\overline{x}_i\right) \text{ and}$$

(T3.1.4) $$F(\underline{x}) = x_{i1} \cdot x_{i2} \cdot \dots \cdot x_{ik} \cdot F(\underline{x}),$$

then $Q_1(T+)$ described as

(T3.1.5) $$Q_1(T+) = D_1 \cdot \left[\sum_{i=1}^{n} (\alpha_i \Delta x_i + \beta_i \nabla x_i)\right]$$

realizes the same next-value function as $Q(T+)$ in equation (T3.1.1) does.

<u>Proof</u>: Suppose due to a transition described by $\partial m_p(\underline{x} - x_q)$, $Q(T+) = 1$ if $\underline{y} = S_0$. This implies that

$$D = 1 \text{ when } \underline{x} = b_p \;\&\; \underline{y} = S_0$$

Hence when $\underline{y} = S_0$ and $\underline{x} = b_p$

$$1 = D = D_1 \cdot x_{i1} \cdots x_{ik}$$

so that $D_1 = 1$ for $\underline{x} = b_p$ & $\underline{y} = S_0$

so that $Q_1(T+) = 1$. On the other hand if $Q_1(T+) = 1$ due to a transition $m_u(\underline{x} - x_v)$ when $\underline{y} = S_1$, then $D_1 = 1$ and $F(\underline{x}) = 1$ for $\underline{x} = b_u$ and equation (T3.1.6) implies that

$F(\underline{x}) = 1 = F(\underline{x}) \cdot x_{i1} \quad x_{ik}$ when $\underline{x} = b_u$,

which implies that $(x_{i1}.\text{-----}.x_{ik}) = 1$ when $\underline{x} = b_u$.

Hence when $\underline{x} = b_u$ & $y = S_1$, $D = D_1.(x_{i1}.\text{-----}.x_{ik})$

$$= 1 \cdot (1)$$

$$= 1$$

Hence $Q(T+)$ and $Q_1(T+)$ have identical values immediately after any transition.

Q.E.D.

<u>Theorem 3.2</u>: Theorem 3.1 is valid if equations (T3.1.2) & (T3.1.4) are replaced by equations (T3.2.2) and (T3.2.4) respectively as given below:

(T3.2.2) $\quad D = D_1 \cdot \overline{x}_{i1} \cdot \overline{x}_{i2}.\text{-----}.\overline{x}_{ik}$

(T3.2.4) $\quad F(\underline{x}) = \overline{x}_{i1}.\overline{x}_{i2}.\text{----}.\overline{x}_{ik}F(\underline{x})$.

The next-value functions for different types of flipflops (other than D-type) are currently under study. The results will be reported when the study is completed.

5. CONCLUSION

Procedure for synthesis of an asynchronous sequential system using clock-triggered flipflops and Boolean calculus has been revised to reflect the newly obtained results.

New results in the area of Boolean calculus are reported in Appendix I.

The concept of noncombinational Boolean calculus has been introduced. The next-value functions for flipflops are defined in terms of changes in the inputs. The reverse problem of synthesis is also considered.

Establisment of conditions for exact integrability, composition of differential functions, multi-variable-change calculus, methods of augmenting non-realizable DM tables so as to make them realizable, application to fault location and detection, etc. are among the many problems that remain to be solved.

APPENDIX I

BOOLEAN CALCULUS FOR DIGITAL SYSTEMS

ABSTRACT

A Boolean calculus is proposed for analysis and synthesis of digital systems, which describes how a Boolean function changes when one of its arguments changes. When the changes in a desired function are specified in terms of changes in its arguments, then ways of "integrating" (i.e. realizing) such a function, if it exists, are developed. Properties of various newly defined differential and integral operators are studied. Boolean calculus has applications in design of logic circuits and in fault analysis. In the former case, it leads to circuits which utilize less flipflops and logic gates than conventional methods do.

INDEX TERMS: Boolean algebra, Boolean calculus, Boolean transitions, direct and partial derivatives, Boolean differential decomposition of function, Boolean set, base of a Boolean function, function based on a set, differential expression, Boolean integration, compatible integral, exact integral, integration by parts, integral of zeroth order, integral of first order, edge-sensitive flipflops, fault analysis.

1. INTRODUCTION

The traditional methods of the analysis and the design of logic circuits are based on Boolean algebra, in which the functional relationships between the output values (or levels) and input values are of direct interest rather than changes in the output function in terms of changes in the input arguments. While designing an asynchronous sequential circuit, if Boolean calculus is employed and edge sensitive flipflops are used, then in many cases it leads to a simpler circuit utilizing fewer components than a circuit arrived at by using conventional techniques [5, 34, 35].

Throughout the paper, unless stated otherwise, a Boolean function $F(X_1, X_2, ---, X_n)$ of n Boolean variables $X_1, X_2, ---, X_n$ will be assumed. Also, it will be assumed that only one variable X_i $1 \leq i \leq n$, can change at a time.

2. REVIEW OF THE LITERATURE

This work is not the first to recognize the feasibility and desirability of establishing a mathematical system for Boolean functions analogous to ordinary calculus [1 - 5]. Based on earlier work by Reed [6], Akers [7] in 1959 obtained the mathematical properties of the Boolean difference. In 1962 Calingaert [8] made limited use of the Boolean difference. Hartman [9] employing the Boolean difference, developed a Boolean differential calculus, introducing in it a number of new concepts. Amar and Condulmari [10], and Sellers, Hsiao, and Bearnson [11] applied the Boolean difference to the problem of fault diagnosis. In recent years considerable work [12-21] has been done using the

Boolean difference for fault detection and diagnosis; moreover, Thayse, Davio, Deschamps, and Bioul [22-27] have shown that the Boolean difference is applicable to a number of areas other than fault diagnosis.

Talantsev [28] in Russia introduced special logical operators. One of Talantsev's operators, the d operator, has an important advantage over the Boolean difference. It not only gives the conditions under which a function will change due to changes in its arguments, but also describes the manner in which the function itself will change. His algorithm for integration, for a function of n variables, involves solving $nx2^n$ simultaneous Boolean equations. Obviously this is a tedious process. Lazarev and Piil have used Talantsev's operator for sequential circuit synthesis [29-31], and in one short paper [32] they present a method of integration. Even though there are situations where their method of integration is easier than Talantsev's, in general, it too, is quite awkward. In the area of differentiation there is only modest overlap between the work reported here and that of Brown and Young [33]. Furthermore, the type of integration considered by Brown and Young requires considerably more information about the function than just knowing its differential. The Boolean integration developed here can be used to determine the function from its differential alone, and the ease with which this can be done makes Boolean integration applicable to practical design problems.

3. Boolean Differentiation

In order to study the effect of change in a variable

X_i, on a function $F(\underline{x}) = F(x_1, x_2, ---, x_n)$ we will decompose it with respect to x_i, $1 \leq i \leq n$ as sum of 3 functions as

$$F = P_i X_i + Q_i \overline{X}_i + R_i \tag{3.01}$$

such that P_i, Q_i and R_i are independent of X_i and

$$P_i Q_i = P_i R_i = Q_i R_i = 0 \tag{3.02}$$

<u>Definition 3.1</u>: A function F that is decomposed as stated above, is said <u>to be the decomposition of F with respect to X_i</u>, $1 \leq i \leq n$.

<u>Definition 3.2</u>: Two minterms $m_a(\underline{x})$ and $m_b(\underline{x})$, $a \neq b$ are said to be <u>x_i - adjacent to each other</u> if every variable x_j, $j \neq i$ is in the same (true or complemented) form in both the minterms.

In order to decompose $F(\underline{x})$ with respect to x_i, we expand $F(\underline{x})$ as canonical sum of minterms and construct R_i by taking sum of all possible pairs of minterms in $F(\underline{x})$ which are X_i- adjacent to each other. Then delete these minterms from $F(\underline{x})$ so that it now becomes $(F(\underline{x}) \cdot \overline{R}_i)$. The sum of all the minterms in $F(\underline{x}) \cdot \overline{R}_i$ that have x_i in them in true form constitutes $P_i x_i$ and the sum of remaining minterms gives $Q_i \overline{x}_i$. This construction procedure brings out an interesting and useful property of the decomposition stated in Theorem 3.1.

<u>Theorem 3.1</u>: The decomposition of a function $F(\underline{x})$ with respect to X_i, $1 \leq i \leq n$, is unique.

<u>Definition 3.3</u>: The set of 2^n binary vectors or points $(X_1, X_2, ---, X_n)$ where $X_i = 0$ or 1 $1 \leq i \leq n$ such that X_i and X_j may or may not be equal if $i \neq j$, will be called <u>Boolean set of variables $X_1, X_2, ---, X_n$, denoted by $B(n)$</u>. <u>The Boolean set of</u>

<u>(n-1) variables</u> $x_1, x_2, ---, x_{i-1}, x_{i+1}, ---, x_n$, denoted by $\underline{x}/x_i$, will be denoted by <u>$B(n/i)$</u>.

<u>Definition 3.4</u>: The set SP_i of points $(b_1, b_2, ---, b_{i-1}, b_{i+1}, ---, b_n)$ denoted by $\underline{b}/b_i$, where b_j's are Boolean constants is the set of all possible points such that for every point in this set the value of Boolean function P_i in equation (3.0.1) is 1. Sets SQ_i and SR_i are defined similarly.

<u>Definition 3.5</u>: In relation to decomposition of $F(\underline{x})$ with respect to x_i, a function T_i, $1 \leq i \leq n$, will be defined as a Boolean function thusly:

$$T_i = \overline{P_i + Q_i + R_i} \tag{D3.5.1}$$

T_i is independent of x_i, since P_i, Q_i and R_i are independent of x_i. ST_i is defined as a set of points corresponding to function T_i in a way similar to Definition 3.4 for SP_i.

It can be seen that

$$SP_i \cap SQ_i = SP_i \cap SR_i = SP_i \cap ST_i = 0 \tag{D3.5.2}$$

$$SP_i \cap SR_i = SQ_i \cap ST_i = SR_i \cap ST_i = 0 \tag{D3.5.3}$$

$$SP_i \cup SQ_i \cup SR_i \cup ST_i = B(n/i) \tag{D3.5.4}$$

for all i, $1 \leq i \leq n$.

<u>Definition 3.6</u>: Given a set $S, \phi \subseteq S \subseteq B(n)$, a function $F(\underline{x})$ is said <u>to be based on the set</u> S if

$$F(\underline{x})\Big|_{\underline{x}=\underline{b}_o} = 1 \tag{D3.6.1}$$

if and only if $b_o \in S$. (D3.6.2)

On the other hand, if a function $F(\underline{x})$ is given, then set

$$S = \{\underline{b} \mid \underline{b} \in B(n) \text{ and } F(\underline{b}) = 1\} \tag{D3.6.3}$$

is called <u>the base of the function</u> F(x).

Observe that by Definition 3.6 the functions P_i, Q_i, R_i and T_i are based on sets SP_i, SQ_i SR_i and ST_i respectively. Also the sets SP_i, SQ_i, SR_i and ST_i are bases for the functions P_i, Q_i, R_i and T_i respectively.

<u>Theorem 3.2</u>: Given a Boolean function $F(\underline{x})$ and any i, $1 \leq i \leq n$, the Boolean set B(n/i) can be uniquely partitioned into sets SP_i, SQ_i, SR_i and ST_i such that

(1) for any point $\underline{x}$ with $\underline{x}/x_i = \underline{b}/b_i \in SP_i$, $F(\underline{x}) = x_i$ (T3.2.1)

so that on the set SP_i, F takes on the same value as x_i,

(2) for any point $\underline{x}$ with $\underline{x}/x_i = \underline{b}/b_i \in SQ_i$, $F(\underline{x}) = \overline{x}_i$ (T3.2.2)

so that on SQ_i, F takes on the value which is complement of the value of x_i;

(3) for any point $\underline{x}$ with $\underline{x}/x_i = \underline{b}/b_i \in SR_i$, F is a constant, in particular F = 1; (T3.2.3)

(4) for any point $\underline{x}$ with $\underline{x}/x_i = \underline{b}/b_i \in ST_i$, F is a constant, in particular F = 0. (T3.2.4)

<u>Proof:</u> The proof follows from Theorem 3.1.

<u>Definition 3.7</u>: The <u>direct (or inverse) partial derivative</u> of $F(\underline{x})$ with respect to x_i, $1 \leq i \leq n$, denoted by $\frac{\partial F}{\partial x_i}$ (or $\frac{\partial F}{\partial \overline{x}_i}$) is defined as a function of (n-1) variables $x_1, x_2, ---, x_{i-1}, x_{i+1}, ---, x_n$ such that whenever for a point $\underline{x} = \underline{b}/b_i$ the value of $\frac{\partial F}{\partial x_i}$ (or $\frac{\partial F}{\partial \overline{x}_i}$) is 1, then at that point F changes its value the

same (or opposite) way as x_i does. Observe that $\frac{\partial F}{\partial x_i}$ (or $\frac{\partial F}{\partial \overline{x}_i}$) is independent of x_i, $1 \le i \le n$.

<u>Theorem 3.3</u>: The direct and inverse partial derivatives of a function $F(\underline{x})$ with respect to x_i, $1 \le i \le n$ are given by the following equations:

$$\frac{\partial F}{\partial x_i} = P_i = (F(\underline{x})\,|_{x_i = 1}) \cdot \overline{(F(\underline{x})\,|_{x_i = 0})} \tag{T3.3.1}$$

$$\frac{\partial F}{\partial \overline{x}_i} = Q_i = (F(\underline{x})\,|_{x_i = 0}) \cdot \overline{(F(\underline{x})\,|_{x_i = 1})} \tag{T3.3.2}$$

$$\text{Also,} \quad \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial \overline{x}_i} = 0 \tag{T3.3.3}$$

<u>Proof:</u> From Definition 3.5 and Theorems 3.1 and 3.2, we have

$$\frac{\partial F}{\partial x_i} = P_i \tag{T3.3.4}$$

$$\frac{\partial F}{\partial \overline{x}_i} = Q_i \tag{T3.3.5}$$

From equation (3.0.1) we have

$$F\,|_{x_i=1} = P_i + R_i \tag{T3.3.6}$$

$$\text{and} \quad F\,|_{x_i = 0} = Q_i + R_i \tag{T3.3.7}$$

so that

$$(F\,|_{x_i=1}) \cdot \overline{(F\,|_{x_i=0})}$$

$$= (P_i + R_i) \cdot \overline{(Q_i + R_i)} \tag{T3.3.8}$$

In view of the fact that P_i, Q_i, R_i and T_i are functions based on sets SP_i, SQ_i SR_i and ST_i which are mutually disjoint and $SP_i \cup SQ_i \cup SR_i \cup ST_i = B(n/i)$, we have

$$\overline{Q_i + R_i} = T_i + P_i \qquad \text{(T3.3.9)}$$

Hence from equations (T3.3.6) and T3.3.7)

$$(F|_{x_i=1}) \cdot \overline{(F|_{x_i=0})}$$

$$= \quad (P_i + R_i) \quad (P_i + T_i)$$

$$= \quad P_i + R_i \; T_i$$

$$= \quad P_i = \frac{\partial F}{\partial x_i} \qquad \text{(T3.3.10)}$$

Similarly $(F|_{x_i=0}) \cdot \overline{(F|_{x_i=1})} = Q_i = \frac{\partial F}{\partial \overline{x}}$ (T 3.3.11)

Also, $\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial \overline{x_i}}$

$$= P_i \cdot Q_i$$

$$= 0 \quad \text{(from equation (3.0.2)} \qquad \text{(T3.3.12)}$$

Q.E.D.

Example 3.1

Consider the function

$$F = x_1x_2 + \overline{x}_1\, x_3 + x_2\, \overline{x}_3 \tag{E3.1.1}$$

Expanding F as sum of minterms

$$F = x_1x_2\overline{x}_3 + x_1x_2x_3 + \overline{x}_1\overline{x}_2x_3 + \overline{x}_1x_2x_3 + \overline{x}_1x_2\overline{x}_3 \tag{E3.1.2}$$

The decomposition of F with respect to x_1 is

$$F = (0)x_1 + (\overline{x}_2x_3)\, \overline{x}_1 + (x_2\overline{x}_3 + x_2x_3)$$

$$= (0)x_1 + (\overline{x}_2x_3)\, \overline{x}_1 + (x_2) \tag{E3.1.3}$$

$$\therefore \quad \frac{\partial F}{\partial x_1} = 0 \tag{E3.1.4}$$

$$\text{and} \quad \frac{\partial F}{\partial \overline{x}_1} = \overline{x}_2x_3 \tag{E3.1.5}$$

The decomposition of F with respect to x_2 is

$$F = (x_1\overline{x}_3 + x_1x_3 + \overline{x}_1\overline{x}_3)\, x_2 + (0)\overline{x}_2 + (\overline{x}_1x_3)$$

$$= (x_1 + \overline{x}_3)\, x_2 + (0)\overline{x}_2 + (\overline{x}_1x_3) \tag{E3.1.6}$$

$$\therefore \quad \frac{\partial F}{\partial x_2} = x_1 + \overline{x}_3 \tag{E3.1.7}$$

$$\text{and} \quad \frac{\partial F}{\partial \overline{x}_2} = 0 \tag{E3.1.8}$$

The decomposition of F with respect to x_3 is

$$F = (\overline{x}_1\overline{x}_2)\, x_3 + (0)\overline{x}_3 + (x_1x_2 + \overline{x}_1x_2)$$

$$= (\overline{x}_1\overline{x}_2)\, x_3 + (0)\overline{x}_3 + (x_2) \tag{E3.1.9}$$

$$\therefore \quad \frac{\partial F}{\partial x_3} = \overline{x}_1\overline{x}_2 \tag{E3.1.10}$$

and $\dfrac{\partial F}{\partial \overline{x}_3} = 0$ (E3.1.11)

<u>Lemma 3.1</u>: The decomposition of $\overline{F}$ with respect to x_i, $1 \leq i \leq n$, is given by

$$\overline{F} = Q_i \; x_i + P_i \; \overline{x}_i + T_i \tag{L3.1.1}$$

$$= Q_i \; x_i + P_i \; \overline{x}_i + \overline{P}_i \; \overline{Q}_I \; \overline{R}_i \tag{L3.1.2}$$

<u>Proof</u>: From Definition 3.1, we have

$$F = P_i x_i + Q_i \; \overline{x}_i + R_i \tag{L3.1.3}$$

so that

$$\overline{F} = \overline{(P_i x_i + Q_i \; \overline{x}_i + R_i)}$$

$$= \overline{(P_i + R_i)} \; x_i + \overline{(Q_i + R_i)} \; \overline{x}_i + \overline{P_i + Q_i + R_i}$$

$$= (T_i + Q_i) \; x_i + (T_i + P_i) \; \overline{x}_i + T_i$$

(last step follows from equations (D3.5.1) - (D3.5.4)).

$$= Q_i \; x_i + P_i \; \overline{x}_i + T_i \; (x_i + \overline{x}_i) + T_i$$

$$= Q_i \; x_i + P_i \; \overline{x}_i + T_i$$

$$= Q_i \; x_i + P_i \; \overline{x}_i + \overline{P}_i \; \overline{Q}_i \; \overline{R}_i \tag{L3.1.4}$$

Q.E.D.

Lemma 3.1 and Theorem 3.4 lead to the following theorem:

<u>Theorem 3.4</u>: For a function $F(\underline{x})$,

$$\frac{\partial \overline{F}}{\partial x_i} = Q_i = \frac{\partial F}{\partial \overline{x}_i} \qquad \text{(T3.4.1)}$$

$$\frac{\partial \overline{F}}{\partial \overline{x}_i} = P_i = \frac{\partial F}{\partial x_i} \qquad \text{(T3.4.2)}$$

We will now establish some properties of the derivatives of a Boolean function which will be needed later.

<u>Theorem 3.5</u>: Let G be a Boolean function that is independent of x_i, for some i, $1 \leq i \leq n$. Then

(1) $$\frac{\partial G}{\partial x_i} = 0 \qquad \text{(T3.5.1)}$$

(2) $$\frac{\partial G}{\partial \overline{x}_i} = 0 \qquad \text{(T3.5.2)}$$

(3) $$\frac{\partial}{\partial x_i}(GF) = G \cdot \frac{\partial F}{\partial x_i} \qquad \text{(T3.5.3)}$$

(4) $$\frac{\partial}{\partial \overline{x}_i}(GF) = G \cdot \frac{\partial F}{\partial \overline{x}_i} \qquad \text{(T3.5.4)}$$

(5) $$\frac{\partial}{\partial x_i}(G+F) = \overline{G} \cdot \frac{\partial F}{\partial x_i} \qquad \text{(T3.5.5)}$$

(6) $$\frac{\partial}{\partial \overline{x}_i}(G+F) = \overline{G} \cdot \frac{\partial F}{\partial \overline{x}_i} \qquad \text{(T3.5.6)}$$

<u>Proof</u>: Since G is not a function of x_i, the decomposition of G with respect to x_i will be given by

$$G = P_i' x_i + Q_i' \overline{x}_i + R_i'$$

$$= (0)x_i + (0)\overline{x}_i + R_i' \qquad \text{(T3.5.7)}$$

Hence by Theorem 3.4,

$$\frac{\partial G}{\partial x_i} = P_i' = 0 \tag{T3.5.1}$$

and $$\frac{\partial G}{\partial \overline{x}_i} = Q_i' = 0 \tag{T3.5.2}$$

Next, let the decomposition of F with respect to x_i be given by

$$F = P_i X_i + Q_i \overline{X}_i + R_i \tag{T3.5.8}$$

Since G is independent of X_i, the decomposition of (GF) with respect to X_i is

$$GF = (GP_i)\ X_i + (GQ_i)\ \overline{X}_i + (GR_i) \tag{T3.5.9}$$

Hence by Theorem 3.4,

$$\frac{\partial(GF)}{\partial\ x_i} = G\ P_i = G\ \frac{\partial F}{\partial x_i} \tag{T3.5.3}$$

and $$\frac{\partial(GF)}{\partial \overline{x}_i} = G\ Q_i = G\ .\ \frac{\partial F}{\partial \overline{x}_i} \tag{T3.5.4}$$

now $\frac{\partial}{\partial x_i}\ (G+F)$

$= \frac{\partial(\overline{\overline{G}+\overline{F}})}{\partial x_i}$ (by de Morgan's law)

$= \frac{\partial(\overline{G}\ \overline{F})}{\partial\ \overline{x}_i}$ (by Theorem 3.4)

$= \overline{G}\ .\ \frac{\partial(\overline{F})}{\partial \overline{x}_i}$

(since G is independent of x_i, relation (T3.5.4) can be used)

$$= \overline{G}\ .\ \frac{\partial F}{\partial x_i} \quad \text{(by relation T3.4.2)} \tag{T3.5.5}$$

Similarly we can prove that

$$\frac{\partial}{\partial \overline{x}_i}(G+F) = \overline{G} \cdot \frac{\partial F}{\partial \overline{x}_i} \qquad \text{(T3.5.6)}$$

Q.E.D.

4. Boolean Differential

Analogous to calculus of real variables in which we express an incremental change in a function in terms of incremental changes in its arguments, we will define Boolean differential of a function.

Definition 4.1: dF will denote a change in the value of F (from "0" to "1" or "1" to "0"). dx_i (or $d\overline{x}_i$) is defined likewise. When we write $dF=dx_i$, then we mean that a "positive (or "negative") change in x_i causes a "positive" (or "negative") change in F. In order to relate dF, dx_i and the function F we will need to define dF and dx_i as among entities in Boolean algebraic system i.e. as Boolean variables. When "dF" (or "dx_i" or "$d\overline{x}_i$") is treated as such, dF has Boolean values as defined below:

$$dF = \begin{cases} 1, & \text{implies change in value of } F \\ 0, & \text{implies no change occurring in value of } F. \end{cases} \qquad \text{(D.4.1.1)}$$

Consider dF, the change in F, in terms of x_1, x_2 and dx_3, the change in x_3 as given below:

$$dF = x_1x_2\, dx_3 \qquad \text{(D4.1.2)}$$

When $x_1 = x_2 = 1,$ (D4.1.3)

then $dF = (1 \cdot 1)\, dx_3 = dx_3$ (D4.1.4)

Equation (D4.1.4) by Definition 4.1 can be interpreted to mean that a change in F is the same as the change in x_3 when $x_1x_2 = 1$.

On the other hand, when

$$x_1x_2 = 0 \tag{D4.1.5}$$

then
$$dF = 0.dx_3$$
$$= 0 \tag{D4.1.6}$$

meaning thereby that there is no change in F when $x_1x_2 = 0$ and x_3 changes.

Definition 4.2: The <u>Boolean differential of F with respect to x_i</u> $1 \leq i \leq n$, denoted by d_iF is defined as

$$d_iF = \frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial \overline{x}_i} d\overline{x}_i \tag{D4.2.1}$$

Definition 4.3: The <u>Boolean differential of F with respect to all variables $x_1, x_2, ---, x_n$</u> or <u>simply Boolean differential of F</u>, denoted by dF, is defined as

$$dF = \sum_{i=1}^{n} d_iF = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial \overline{x}_i} d\overline{x}_i \right) \tag{D4.3.1}$$

<u>Example 4.1</u>

Given
$$F = x_1 + \overline{x}_2 \tag{E4.1.1}$$
$$= x_1x_2 + \overline{x}_1\overline{x}_2 + x_1\overline{x}_2, \tag{E4.1.2}$$

find the Boolean differential of F.

Observe that F can be written as

$$F = (x_2)x_1 + (0)\overline{x}_1 + (x_2)$$
$$= P_1X_1 + Q_1\overline{X}_1 + R_1 \tag{E4.1.3}$$

so that

$$\frac{\partial F}{\partial x_1} = x_2 \qquad \text{(E4.1.4)}$$

and $$\frac{\partial F}{\partial \bar{x}_1} = 0 \qquad \text{(E4.1.5)}$$

Rewriting F as

$$F = (0)x_2 + (\bar{x}_1)\bar{x}_2 + x_1 \qquad \text{(E4.1.6)}$$

one sees that

$$P_2 = \frac{\partial F}{\partial x_2} = 0 \qquad \text{(E4.1.7)}$$

and $$Q_2 = \frac{\partial F}{\partial \bar{x}_2} = \bar{x}_1 \qquad \text{(E4.1.8)}$$

Hence

$$dF = x_2 dx_1 + (0)d\bar{x}_1 + (0)dx_2 + \bar{x}_1 d\bar{x}_2$$

$$= x_2 dx_1 + \bar{x}_1 d\bar{x}_2 \qquad \text{(E4.1.9)}$$

In view of the fact that we are allowing only one variable to change at a time, both dx_1 and $d\bar{x}_2$ cannot be "1" at the same time. It is clear from equation (E4.1.9) that when

$$x_2 = 1, \quad dF = dx_1 \qquad \text{(E4.1.10)}$$

and $$dF = d\bar{x}_2 \qquad \text{(E4.1.11)}$$

when $\bar{x}_1 = 1$.

<u>Definition 4.4</u>: A <u>differential expression</u>, denoted by $d\zeta$, is a Boolean expression of the form

$$d\zeta = \sum_{i=1}^{n} (\alpha_i \, dx_i + \beta_i \, d\bar{x}_i) \qquad \text{(D4.4.1)}$$

where in general α_i and β_i are functions of the (n-1) variables $x_1, x_2, ---, x_{i-1}, x_{i+1}, ---, x_n$ and α_i and β_i are independent of x_i for all i, $1 \leq i \leq n$. Observe that since by Definition 3.5, $\frac{\partial F}{\partial x_i}$ and $\frac{\partial F}{\partial \bar{x}_i}$ are independent of x_i, $1 \leq i \leq n$, Boolean differential of a function $F(\underline{x})$ as given in equation (D4.3.1) is a differential expression; however the converse is not true. For a differential expression to be a differential, there must exist a function such that its differential is the same as the given differential expression. For the expression $d\zeta$ in equation (D4.4.1) to be a differential, there must exist a function $\zeta(\underline{x})$ such that

$$\alpha_i = \frac{\partial \zeta}{\partial x_i} \qquad \text{(D4.4.2)}$$

and $$\beta_i = \frac{\partial \zeta}{\partial \bar{x}_i} \qquad \text{(D4.4.3)}$$

for all i, $1 \leq i \leq n$.

The following definition pertaining to relationship between two differential expressions will be needed in the latter part of the paper.

Definition 4.5: Given two differential expressions $d\zeta_1$ and $d\zeta_2$ where

$$d\zeta_1 = \sum_{i=1}^{n} (\alpha_{1i} dx_i + \beta_{1i} d\bar{x}_i) \qquad \text{(D4.5.1)}$$

and $$d\zeta_2 = \sum_{i=1}^{n} (\alpha_{2i} dx_i + \beta_{2i} d\bar{x}_i) \qquad \text{(D4.5.2)}$$

(a) $d\zeta_1$ and $d\zeta_2$ are said to be equal, that is

$$d\zeta_1 = d\zeta_2 \qquad \text{(D4.5.3)}$$

if $\alpha_{1i} = \alpha_{2i}$ (D4.5.4)

and $\beta_{1i} = \beta_{2i}$ (D4.5.5)

for all i, $1 \leq i \leq n$;

(b) $d\zeta_1$ is said to include or cover $d\zeta_2$, denoted by

$$d\zeta_1 \supseteq d\zeta_2 \tag{D4.5.6}$$

if $\alpha_{1i} \supseteq \alpha_{2i}$ (D4.5.7)

and $\beta_{1i} \supseteq \beta_{2i}$ (D4.5.8)

for all i, $1 \leq i \leq n$;

(c) $d\zeta$, the Boolean product of $d\zeta_1$ and $d\zeta_2$, denoted by $d\zeta_1 \cdot d\zeta_2$, is given by $d\zeta = d\zeta_1 \cdot d\zeta_2$

$$= \sum_{i=1}^{n} (\alpha_{1i}\, \alpha_{2i}\, dx_i + \beta_{1i}\beta_{2i} d\overline{x}_i), \tag{D4.5.9}$$

and

(d) $d\zeta$, the Boolean sum of $d\zeta_1$ and $d\zeta_2$, denoted by $d\zeta_1 + d\zeta_2$, is given by

$$d\zeta = d\zeta_1 + d\zeta_2$$

$$= \sum_{i=1}^{n} (\alpha_{1i} + \alpha_{2i})\, dx_i + (\beta_{1i} + \beta_{2i})\, d\overline{x}_i \,. \tag{D4.5.10}$$

5. Boolean Integration

In the preceding sections we saw that given a Boolean function $F(\underline{x})$ and a variable x_i, $1 \leq i \leq n$ we can identify the points in

B(n/i) where (a) F changes the same way as x_i,

or (b) F changes the way opposite to the way x_i changes,

or (c) F is insensitive to changes in x_i and F is constant at 1,

or (d) F is constant at 0 and insensitive to changes in x_i.

The differential expression thus describes changes in F that may occur due to change in any one of the variables $x_1, x_2, ---, x_n$, only one of them changing at a time. This is, indeed, useful in analysis. In synthesis, it is of interest to address ourselves to the question: "Is it possible to find a function F such that changes in F in terms of changes in its arguments are as prescribed?" Mathematically, if a differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i) \tag{5.0.1}$$

is given, can we find a function, F (say) such that

$$\frac{\partial F}{\partial x_i} = \alpha_i \tag{5.0.2}$$

and $$\frac{\partial F}{\partial \overline{x}_i} = \beta_i \tag{5.0.3}$$

for all i, $1 \leq i \leq n$?

We will need the following definition to pursue the answers to the question just raised.

<u>Definition 5.1</u>: F is said to be the <u>exact integral of $d\zeta$</u>, denoted by $\int_E d\zeta$ and $d\zeta$ is said <u>to be exactly integrable</u> if

$$\frac{\partial F}{\partial x_i} = \alpha_i \qquad \text{(D5.1.1)}$$

$$\frac{\partial F}{\partial \bar{x}_i} = \beta_i \qquad \text{(D5.1.2)}$$

for all i, $1 \leq i \leq n$

$$\text{and} \quad d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \qquad \text{(D5.1.3)}$$

<u>Definition 5.2</u>: Given a differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \qquad \text{(D5.2.1)}$$

if there exists a function F such that

$$\frac{\partial F}{\partial x_i} \supseteq \alpha_i \qquad \text{(D5.2.2)}$$

$$\text{and} \quad \frac{\partial F}{\partial x_i} \supseteq \beta_i \qquad \text{(D5.2.3)}$$

for all i, $1 \leq i \leq n$,

then F is said to be <u>a compatible integral of $d\zeta$</u> and is denoted by $\int_C d\zeta$ and $d\zeta$ is said to be <u>compatibly integrable</u>.

<u>Example 5.1</u>

Consider $d\zeta = x_2 dx_1 + \bar{x}_1 d\bar{x}_2$ (E5.1.1)

then $\int_E d\zeta = x_1 + \bar{x}_2$

$= F$ (say), (E5.1.2)

since $dF = x_2 dx_1 + \bar{x}_1 d\bar{x}_2$

$= d\zeta$ (E5.1.3)

which satisfies equations (D5.1.1) and (D5.1.2). Consider next

$$F_1 = x_1x_2 + \bar{x}_1\bar{x}_2 \tag{E5.1.4}$$

so that $$dF_1 = x_2dx_1 + \bar{x}_2d\bar{x}_1 + x_1dx_2 + \bar{x}_1d\bar{x}_2 \tag{E5.1.5}$$

Hence $$x_2 = \frac{\partial F_1}{\partial x_1} \supseteq \alpha_1 = x_2 \tag{E5.1.6}$$

$$\bar{x}_2 = \frac{\partial F_1}{\partial \bar{x}_1} \supseteq \beta_1 = 0 \tag{E5.1.7}$$

$$x_1 = \frac{\partial F_1}{\partial x_2} \supseteq \alpha_2 = 0 \tag{E5.1.8}$$

and $$\bar{x}_1 = \frac{\partial F_1}{\partial \bar{x}_2} \supseteq \beta_2 = \bar{x}_1 \tag{E5.1.9}$$

Therefore by Definition 5.2, a compatible integral of $d\xi$ is

$$\int_c d\zeta = F_1 = x_1x_2 + \bar{x}_1\bar{x}_2 \tag{E5.1.10}$$

In what follows we will obtain ways of finding all possible compatible integrals of $d\zeta$, if $d\zeta$ is compatibly integrable. To accomplish this we need the following integral operators:

<u>Definition 5.3</u>: The <u>zeroth order integral of $d\zeta$,</u> denoted by $\int_o d\zeta$, is defined as

$$\int_o d\zeta = \sum_{i=1}^{n} (\alpha_i\bar{x}_i + \beta_i x_i) \tag{D5.3.1}$$

where $$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \tag{D5.3.2}$$

Also, the <u>first order integral of $d\zeta$,</u> denoted by $\int_1 d\zeta$, is defined as

$$\int_1 d = \sum_{i=1}^{n} (\alpha_i x_i + \beta_i \bar{x}_i) \tag{D5.3.3}$$

<u>Definition 5.4</u>: A binary point $\underline{b}_o \in B(n)$ is said to be a <u>"one" of a function $F(x)$</u> if

$$F(\underline{b_o}) = 1. \tag{D5.4.1}$$

It is said to be a "zero" of $F(x)$ if

$$F(\underline{b_o}) = 0 \tag{D5.4.2}$$

Lemma 5.1: If the differential

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i) \tag{L5.1.1}$$

is compatibly integrable and F_1 is a compatible integral of $d\zeta$, then every one of $\int_1 d\zeta$ is, also, a one of F_1.

Proof: Since F_1 is a compatible integral, by Definition 5.2,

$$\frac{\partial F_1}{\partial x_i} \supseteq \alpha_i \tag{L5.1.2}$$

and
$$\frac{\partial F_1}{\partial \overline{x}_i} \supseteq \beta_i \tag{L5.1.3}$$

also
$$dF_1 = \sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial x_1} dx_i + \frac{\partial F_1}{\partial \overline{x}_1} d\overline{x}_i\right) \tag{L5.1.4}$$

so that the ones of

$$\frac{\partial F_1}{\partial x_1} x_i \ \left(\text{or } \frac{\partial F_1}{\partial \overline{x}_1} \overline{x}_1\right), \quad 1 \leq i \leq n, \quad \text{are,}$$

also, the ones of F_1.

From equation (L5.1.2) (or (L5.1.3)) the ones of $\alpha_i x_i$ (or $\beta_i \overline{x}_i$), $1 \leq i \leq n$, are the ones of $\frac{\partial F_1}{\partial x_i} x_i$ (or $\frac{\partial F_1}{\partial \overline{x}_1} \overline{x}_i$) for all $1 \leq i \leq n$.

Hence the ones of $(\alpha_i x_i + \beta_i \overline{x}_i)$, $1 \leq i \leq n$, are, also, the ones of F_1. Hence the ones of $\int_1 d\zeta$ are the ones of F_1.

Q.E.D.

On arguing on a similar basis, we can establish the following lemma.

Lemma 5.2: If the differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \tag{L5.2.1}$$

is compatibly integrable and $F_1(\underline{x})$ is a compatible integral of $d\zeta$, then the ones of $\int_o d\zeta$ are zeroes of F_1.

Theorem 5.1: A necessary condition for compatible integrability of the differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \tag{T5.1.1}$$

is that

$$(\int_o d\zeta) \cdot (\int_1 d\zeta) = 0 \tag{T5.1.2}$$

for all $\underline{x} \in B(n)$.

Proof: Suppose $d\zeta$ is compatibly integrable so that there exists $F_1(\underline{x})$ such that

$$F_1 = \int_c d\,\zeta. \tag{T5.1.3}$$

Also, suppose that there exists $\underline{b_o}$ such that

$$[(\int_o d\,\zeta) \cdot (\int_1 d\zeta)]\Big|_{\underline{x} = \underline{b_o}} = 1 \tag{T5.1.4}$$

which implies that

$$(\int_o d\zeta)\Big|_{\underline{x} = \underline{b_o}} = 1 \qquad \text{(T5.1.5)}$$

and $$(\int_1 d\zeta)\Big|_{\underline{x} = \underline{b_o}} = 1 \qquad \text{(T5.1.6)}$$

From Lemma 5.1 and equation (T5.1.6) $\underline{b_o}$ is a one of F_1. (T5.1.7)

From Lemma 5.2 and equation (T5.1.5) $\underline{b_o}$ is a zero of F_1. (T5.1.8)

Statements (T5.1.7) and (T5.1.8) contradict each other. Hence there exists no $\underline{b_o} \in B(n)$ that satisfies equation (T5.1.4). Hence equations (T5.1.2) is a necessary condition for $d\zeta$ to be compatibly integrable.

Q.E.D.

<u>Lemma 5.3</u>: If the differential expression

$$d\zeta = \Sigma(\alpha_i\, dx_i + \beta_i d\bar{x}_i) \qquad \text{(L5.3.1)}$$

and $$(\int_o d\zeta) \cdot (\int_1 d\zeta) = 0 \qquad \text{(L5.3.2)}$$

for $\underline{x} \notin B(n)$,

then (a) $\alpha_i \int_1 d\zeta = \alpha_i x_i$, (L5.3.3)

(b) $\alpha_i \underline{\int_o d\zeta} = \alpha_i \bar{x}_i$ (L5.3.4)

and (c) $\alpha_i\ (\int_o d\zeta) = \alpha_i x_i$ (L5.3.5)

<u>Proof</u>: From Definition 5.3 and equation (L5.3.2) we have

$$0 = (\Sigma\alpha_j x_j + \beta_j \bar{x}_j) \cdot (\Sigma\alpha_i \bar{x}_i + \beta_i x_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i\alpha_j\, \bar{x}_i x_j + \beta_i\alpha_j x_i x_j + \alpha_i\beta_j \bar{x}_i \bar{x}_j \qquad \text{(L5.3.6)}$$

$$+ \beta_i\beta_j\, x_i \bar{x}_j)$$

Hence for all i,j, $1 \leq i \leq n$, $1 \leq j \leq n$

$$\alpha_i\alpha_j\,\bar{x}_i x_j \;+\; \beta_i\alpha_j x_i x_j \;+\; \alpha_i\beta_j\,\bar{x}_i\bar{x}_j \;+\; \beta_i\beta_j x_i\bar{x}_j \;=\; 0 \qquad \text{(L5.3.7)}$$

so that $\alpha_i\alpha_j\bar{x}_i x_j = \beta_i\alpha_j x_i x_j = \alpha_i\beta_j\bar{x}_i\bar{x}_j = \beta_i\beta_j x_i\bar{x}_j = 0$ (L5.3.8)

Now

$$\alpha_i \int_1 d\zeta = \alpha_i \left(\sum_{j=1}^{n} \alpha_j x_j + \beta_j\bar{x}_j\right)$$

$$= \alpha_i x_i + \alpha_i\beta_i x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} (\alpha_i\alpha_j x_j + \alpha_i\beta_j\bar{x}_j)$$

$$= \alpha_i x_i + \alpha_i\beta_i\bar{X}_i$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{n} (\alpha_i\alpha_j x_i x_j + \alpha_i\alpha_j\bar{x}_i x_j + \alpha_i\beta_j x_i\bar{x}_j$$

$$+ \alpha_i\beta_j\bar{x}_i\bar{x}_j) \qquad \text{(L5.3.9)}$$

In equation (L5.3.8), setting $i=j$ yields

$$\alpha_i\beta_i = 0 \qquad \text{(L5.3.10)}$$

for all i, $1 \leq i \leq n$.

Hence using equations (L5.3.8) - (L5.3.10), we get

$$\alpha_i \int_1 d\zeta = \alpha_i x_i + \alpha_i x_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n} (\alpha_j x_j + \beta_j\bar{x}_j)\right) = \alpha_i x_i \qquad \text{(L5.3.3)}$$

By interchanging i and j in equation (L5.3.9), we get

$$\alpha_i \int_0 d\zeta = \alpha_i\,\bar{x}_i \qquad \text{(L5.3.4)}$$

Now

$$\alpha_i\,\overline{(\int_0 d\zeta)}$$

$$= \alpha_i (1 \oplus \int_0 d\zeta)$$

$$= \alpha_i \oplus \alpha_i \int_0 d\zeta$$

$$= \alpha_i \oplus \alpha_i \overline{x}_i \quad \text{(from equation (L5.3.4))}$$

$$= \alpha_i x_i \tag{L5.3.5}$$

Q.E.D.

<u>Theorem 5.2:</u> If the differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i) \tag{T5.2.1}$$

and $$(\textstyle\int_0 d\zeta) \cdot (\int_1 d\zeta) = 0 \tag{T5.2.2}$$

for all $\underline{x} \in B\,(n)$, then F given by

$$F = \textstyle\int_1 d\zeta + \psi(\int_0 d\zeta) \tag{T5.2.3}$$

is a compatible integral of $d\zeta$, where ψ is an arbitrary function of $\underline{x}$.

<u>Proof:</u> ANDing both the sides of equation (T5.2.3) by α_i, we have

$$\alpha_i F = \alpha_i \textstyle\int_1 d\zeta + \psi x_i \overline{(\int_0 d\zeta)}$$

$$= \alpha_i x_i + \psi \alpha_i x_i \quad \text{(from Lemma 5.3)}$$

$$= \alpha_i x_i \;(1 + \psi)$$

$$= \alpha_i x_i \tag{T5.2.4}$$

By Theorem 3.5, since α_i is independent of x_i,

$$\alpha_i \frac{\partial F}{\partial x_i} = \frac{\partial (\alpha_i F)}{\partial x_i}$$

$$= \frac{\partial (\alpha_i x_i)}{\partial x_i} \quad \text{(from equation (T5.2.3))}$$

$$= \alpha_i \frac{\partial x_i}{\partial x_i}$$

$$= \alpha_i \tag{T5.2.5}$$

$$\therefore \quad \frac{\partial F}{\partial x_i} \supseteq \alpha_i . \tag{T5.2.6}$$

Similarly $\quad \dfrac{\partial F}{\partial \overline{x}_i} \supseteq \beta_i \qquad$ (T5.2.6)

Hence by Definition 5.2, F is a compatible integral of $d\zeta$.

Q.E.D.

Theorem 5.3: A differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i) \tag{T5.3.1}$$

is compatibly integrable if and only if

$$(\int_0 d\zeta) \cdot (\int_1 d\zeta) = 0 \tag{T5.3.2}$$

for all $\underline{x} \in B(n)$.

Proof: The proof follows from Theorems 5.1 and 5.2.

A word regarding the arbitrary function $\psi(\underline{x})$ in equation (T.5.2.2) is in order. If sets D_0 and D_1, $0 \subseteq D_i \subseteq B(n)$, i=0 and 1, are bases (Definition 3.6) of functions $\int_0 d\zeta$ and $\int_1 d\zeta$, then every distinct ψ would give rise to a distinct compatible integral if ψ is based on a subset (not necessarily proper) of $D=\overline{\{D_0 \cup D_1\}}$. In fact if ψ is based on a subset of D, then the factor $(\int_0 d\zeta)$ that is ANDed with ψ in equation (T5.2.2) may be dropped since

$\overline{D_o} \supset \{\overline{D_o \cup D_1}\} = D$. Hence we can modify Theorem 5.2 as shown in the next theorem.

<u>Theorem 5.4</u>: If the differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i) \qquad \text{(T5.4.1)}$$

and

$$(\int_0 d\zeta) \cdot (\int_1 d\zeta) = 0 \qquad \text{(T5.4.2)}$$

$$\text{for all } \underline{x} \in B(n),$$

then F given by

$$F = \int_1 d\zeta + \Theta(\underline{x}) \qquad \text{(T5.4.3)}$$

is a compatible integral of $d\zeta$, where $\Theta(\underline{x})$ is based on a subset of D, where

$$D = \{\overline{D_o \cup D_1}\}, \qquad \text{(T5.4.4)}$$

D_o and D_1 being the bases of $\int_o d\zeta$ and $\int_1 d\zeta$, respectively. Moreover if the number of points in D is m, then there are 2^m distinct compatible integrals of $d\zeta$.

<u>Proof</u>: The essence of the proof is outlined in the discussion preceding the theorem. A formal proof can be given using Tapia-Tucker method [36,37] for obtaining complete solution for Boolean equations.

<u>Example 5.2</u>

A clock function $C(x_1,x_2,x_3)$ is to be realized which goes through, at least, the transitions specified in the differential expression

$$dC = (x_2\overline{x}_3 + \overline{x}_2 x_3)\, dx_1 + (x_1\overline{x}_3)\, d\, x_2$$

$$+ (x_1 x_3)\, d\, \overline{x}_2 + (x_1\overline{x}_2) dx_3 + (x_1 x_2)\, d\overline{x}_3. \qquad \text{(E5.2.1)}$$

Find C, if it exists.

We have

$$\int_0 dc = (\overline{x}_1 x_2 \overline{x}_3 + \overline{x}_1\overline{x}_2 x_3) + x_1\overline{x}_2\overline{x}_3$$

$$+ x_1 x_2 x_3 + x_1\overline{x}_2\overline{x}_3 + x_1 x_2 x_3 \qquad \text{(E5.2.2)}$$

$$= \overline{x}_1 x_2 \overline{x}_3 + \overline{x}_1\overline{x}_2 x_3 + x_1\overline{x}_2\overline{x}_3 + x_1 x_2 x_3$$

and

$$\int_1 dC = x_1 x_2\overline{x}_3 + x_1\overline{x}_2 x_3 + x_1 x_2\overline{x}_3 + x_1\overline{x}_2 x_3 + x_1\overline{x}_2 x_3 + x_1 x_2\overline{x}_3$$

$$= x_1 x_2 \overline{x}_3 + x_1\overline{x}_2 x_3. \qquad \text{(E5.2.3)}$$

Obviously

$$(\int_0 dC) \cdot (\int_1 dC) = 0 \qquad \text{(E5.2.4)}$$

Hence by Theorem 5.3, a compatible integral does exist.

Also, D referred to in equation (T5.4.4) is

$$D = \overline{\{D_0 \cup D_1\}}$$

$$= \overline{\{(0,0,1),\ (0,1,0),\ (1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)\}}$$

$$= \{(0,0,0),\ (0,1,1)\} \qquad \text{(E5.2.5)}$$

Hence $\theta(\underline{x})$ has 4 possible values,

$$\theta_1(\underline{x}) = 0 \qquad \text{(E5.2.6)}$$

$$\theta_2(\underline{x}) = \overline{x}_1\overline{x}_2\overline{x}_3 \qquad \text{(E5.2.7)}$$

$$\theta_3(\underline{x}) = \overline{x}_1 x_2 x_3 \qquad \text{(E5.2.8)}$$

$$\theta_4(\underline{x}) = \overline{x}_1\overline{x}_2\overline{x}_3 + \overline{x}_1 x_2 x_3 \qquad \text{(E5.2.9)}$$

Hence there are four solutions by Theorem 5.4.

$$C_1 = \int_1 dC + \theta_1 = x_1x_2\bar{x}_3 + x_1\bar{x}_2x_3 \qquad \text{(E5.2.10)}$$

$$C_2 = x_1x_2\bar{x}_3 + x_1\bar{x}_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3 \qquad \text{(E5.2.11)}$$

$$C_3 = x_1x_2\bar{x}_3 + x_1\bar{x}_2x_3 + \bar{x}_1x_2x_3 \qquad \text{(E5.2.12)}$$

and

$$C_4 = x_1x_2\bar{x}_3 + x_1\bar{x}_2x_3 + \bar{x}_1\bar{x}_2\bar{x}_3 + \bar{x}_1x_2x_3 . \qquad \text{(E5.2.13)}$$

Observe that

$$dC_1 = (x_2 \oplus x_3)\ dx_1 + (x_1\bar{x}_3)\ dx_2 + (x_1x_3)\ d\bar{x}_2$$

$$+ (x_1\bar{x}_2)\ dx_3 + (x_1x_2)\ d\bar{x}_3 \qquad \text{(E5.2.14)}$$

$$= dC \qquad \text{(E5.2.15)}$$

Hence C_1 realizes those transitions which are specified in dC and no transitions which are not specified in dC. In fact by Definition 5.1, C_1 is also the exact integral of dC in equation (E5.2.1). Let us now examine C_2.

$$dC_2 = (x_2 \oplus x_3)\ dx_1 + \underline{(\bar{x}_2\bar{x}_3)\ d\bar{x}_1} + (x_1\bar{x}_3)\ dx_2$$

$$+ (x_1x_3 + \underline{\bar{x}_1\bar{x}_3})\ d\bar{x}_2 + (x_1\bar{x}_2)\ dx_3$$

$$= (x_1x_2 + \underline{\bar{x}_1\bar{x}_2})\ dx_3 \qquad \text{(E5.2.16)}$$

Observe that C_2 realizes the transitions represented by differential terms $\bar{x}_2\bar{x}_3d\bar{x}_1$, $\bar{x}_1\bar{x}_3d\bar{x}_2$ & $\bar{x}_1\bar{x}_2dx_3$ which are not specified in dC in equation (E5.2.1). However it does realize all the transitions specified in dC.

As shown above, a differential expression that is exactly integrable is, also, compatibly integrable. Hence the necessary condition that

$$(\int_c d\ )\ .\ (\int_1 d\ ) = 0$$

for all $\underline{x} \in B(n)$

$$\text{for} \qquad d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i)$$

to be compatibly integrable is also necessary for $d\zeta$ to be exactly integrable. However, unlike in the case of compatible integrability, the condition is not sufficient as can be seen from the example that follows:

<u>Example 5.3</u>

Consider the differential expression

$$d\zeta = x_1 dx_2 \qquad \text{(E5.3,1)}$$

We have

$$\int_o d\zeta = \alpha_1\bar{x}_1 + \beta_1 x_1 + \alpha_2\bar{x}_2 + \beta_2 x_2$$

$$= 0 + 0 + x_1\bar{x}_2 + 0 \qquad \text{(E5.3.2)}$$

and $\int_1 d\zeta = \alpha_1 x_1 + \beta_1\bar{x}_1 + \alpha_2 x_2 + \beta_2\bar{x}_2$

$$= 0 + 0 + x_1 x_2 + 0 \qquad \text{(E5.3.3)}$$

Obviously $(\int_o d\zeta)\ .\ (\int_1 d\zeta) = 0$ (E5.3.4)

for all $\underline{X} \in B(n)$.

And the expression is integrable - at least in the compatible sense - by Theorem 5.3.

Also, D_o, D_1 and D referred to in equation (T5.4.4) are given by

$$
\begin{aligned}
D &= \overline{\{D_o \cup D_1\}} \\
&= \overline{\{(1,0),\ (1,1)\}} \\
&= \{(0,0),\ (0,1)\} \qquad \text{(E5.3.5)}
\end{aligned}
$$

Hence by Theorem 5.4 we have 4 compatible integrals of $d\xi$ given by F_o, F_1, F_2 and F_3 as given below:

$$
\begin{aligned}
F_o &= \int_1 d\zeta + \theta_o \\
&= x_1x_2 + 0 \qquad \text{(E5.3.6)}
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= \int_1 d\zeta + \theta_1 \\
&= x_1x_2 + \bar{x}_1\bar{x}_2 \qquad \text{(E5.3.7)}
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= \int_1 d\zeta + \theta_2 \\
&= x_1x_2 + \bar{x}_1x_2 \\
&= x_2 \qquad \text{(E5.3.8)}
\end{aligned}
$$

$$
\begin{aligned}
F_3 &= \int_1 d\zeta + \theta_3 \\
&= x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 \\
&= x_2 + \bar{x}_1 \qquad \text{(E5.3.9)}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
dF_o &= x_2dx_1 + x_1dx_2 \\
&\neq d\zeta \qquad \text{(E5.3.10)}
\end{aligned}
$$

$$dF_1 = x_2dx_1 + \bar{x}_2d\bar{x}_1 + x_1dx_2 + \bar{x}_1d\bar{x}_2$$
$$\neq d\zeta \qquad \text{(E5.3.11)}$$

$$dF_2 = dx_2$$
$$\neq d\zeta \qquad \text{(E5.3.12)}$$

$$dF_3 = \bar{x}_2d\bar{x}_1 + x_1dx_2$$
$$\neq d\zeta \qquad \text{(E5.3.13)}$$

Hence none of the compatible integrals F_o, F_1, F_2 and F_3 is exact by Definition 5.1, and no more compatible integral exists which is distinct from these four integrals by Theorem 5.5 that follows. Hence $d\xi$ is not exactly integrable. Thus the condition that $(\int_o d\zeta) \cdot (\int_1 d\zeta)$ be 0 for all $\underline{x} \in B(n)$ is necessary but not sufficient for $d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_2 d\bar{x}_i)$ to be exactly integrable.

Theorem 5.5: Let D_o, $D_o \subsetneq B(n)$, and D_1, $D_1 \subseteq B(n)$, be Boolean sets, which are bases of functions $\int_o d\zeta$ and $\int_1 d\zeta$, respectively, $d\zeta$ being the differential expression given by

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\bar{x}_i). \qquad \text{(T5.5.1)}$$

Let D be another Boolean set such that

$$D = \overline{\{D_o \cup D_1\}} \qquad \text{(T5.5.2)}$$

and m = number of distinct points in D. (T5.5.3)

Also, let $\theta_i(\underline{x})$'s be functions based on distinct subsets of D, $\quad 1 \leq i \leq 2^m$ (T5.5.4)

If F is a compatible integral of $d\zeta$, then F must belong to the class of 2^m distinct functions defined by

$$F_i(x) = \int_1 d\zeta + \theta_i(\underline{x}) \qquad (T5.5.5)$$

$$1 \leq i \leq 2^m$$

Proof: Since $d\zeta$ is compatibly integrable by hypothesis, there exist, by Theorem 5.4, 2^m distinct ingetral F_i's given by equation (T5.5.5). We want to establish that these are all the compatible integrals of $d\zeta$ that could exist.

First observe that $F_i(\underline{x})$, $1 \leq i \leq 2^m$ in equation (T5.5.5) is based on set $\{D_1 \cup E_i\}$ where

$$E_i \subseteq D \qquad (T5.5.6)$$

From equations (T5.5.2) and (T5.5.6), D_1 and E_i are non-intersecting so that every distinct set E_i defines a distinct set

$$G_i = D_1 \cup E_i \qquad (T5.5.7)$$

and $$G_i \subseteq \overline{D}_o, \qquad (T5.5.8)$$

$$1 \leq i \leq 2^m$$

Hence $F_i(\underline{x})$ is based on G_i, where

$$D_1 \subseteq G_i \subseteq \overline{D}_o, \qquad (T5.5.9)$$

all i, $1 \leq i \leq n$.

Now we will show that given a function $F(\underline{x})$ based on a set H, $H \subseteq B(n)$, which is a compatible integral of $d\zeta$ in equation

(T5.5.1), H must satisfy the bounds

$$D_1 \subseteq H \subseteq \overline{D}_o \qquad \text{(T5.5.10)}$$

so that $F(\underline{x}) = F_i(\underline{x})$ for some i. By Lemmas 5.1 and 5.2 $F(\underline{x})$ must be based on a superset of D_1 and must not be based on a subset of D_o so that H does satisfy equation (T5.5.10).

Q.E.D.

Preliminary results pertaining to necessary and sufficient conditions for a differential expression to be exactly integrable are given in a recent publication [3]. These results in detailed analysis along with the method of finding exact integral of a given function, if it exists, and other useful and interesting properties of integrals of higher order (not defined in this paper) will be published in the near future.

Given a differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i)$$

if $(\int_o d\zeta) \cdot (\int_1 d\zeta) \neq 0$ for some $b \in B(n)$, then the expression cannot be integrated exactly nor compatibly. However it could be decomposed as sum of several differential expressions, each one of which may be integrable separately as defined below.

<u>Definition 5.5</u>: A differential expression is said to be <u>integrable by parts</u> if $d\zeta$ can be written as

$$d\zeta = \sum_{k=1}^{m} d\zeta_k \qquad m \geq 1 \qquad \text{(D5.5.1)}$$

where $d\zeta_k$, is compatibly integrable for all k, $1 \leq k \leq m$. Any compatible integral of $d\zeta_k$, $1 \leq k \leq m$ will be called a <u>partial integral</u> of $d\zeta_k$. A complete set of partial integrals <u>of differential expression</u> is a set of functions, $\{F_1, F_2, ---, F_m\}$ where for each $d\zeta_k$ in equation (D5.5.1), there is a function F_k in the set such that

$$dF_k \supseteqq d\zeta_k \tag{D5.5.2}$$

It will now be shown that any differential expression is integrable by parts.

<u>Theorem 5.6</u>: Any differential expression

$$d\zeta = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i) \tag{T5.6.1}$$

is integrable by parts.

<u>Proof</u>: Observe that for any i, $1 \leq i \leq n$,

$$(\textstyle\int_1 \alpha_i dx_i) \cdot (\textstyle\int_0 \alpha_i x_i dx_i)$$

$$= (\alpha_i x_i) \cdot (\alpha_i \overline{x}_i)$$

$$= 0 \tag{T5.6.2}$$

and $$(\textstyle\int_1 \beta_i d\overline{x}_i) \cdot (\textstyle\int_0 \beta_i d\overline{x}_i) \tag{T5.6.2}$$

$$= 0 \tag{T5.6.3}$$

so that by Theorem 5.3 differential terms $\alpha_i dx_i$ and $\beta_i d\overline{x}_i$ are compatibly integrable. Hence each and every term on the right hand side of equation (T5.6.1) is compatibly integrable. Hence by Definition 5.5, $d\zeta$ is integrable by parts.

Q.E.D.

Example 5.4

Consider the differential expression

$$d\zeta = \bar{x}_3 dx_1 + \bar{x}_3 dx_2 + \bar{x}_2 dx_3 \tag{E5.4.1}$$

For this expression

$$\int_1 d\zeta = \bar{x}_3 x_1 + \bar{x}_3 x_2 + \bar{x}_2 x_3 \tag{E5.4.2}$$

and

$$\begin{aligned} \int_0 d\zeta &= \bar{x}_3\bar{x}_1 + \bar{x}_3\bar{x}_2 + \bar{x}_2\bar{x}_3 \\ &= \bar{x}_1\bar{x}_3 + \bar{x}_2\bar{x}_3 \end{aligned} \tag{E5.4.3}$$

$$\begin{aligned} \therefore\quad & (\textstyle\int_1 d\zeta) \cdot (\textstyle\int_0 d\zeta) \\ &= x_1\bar{x}_2\bar{x}_3 + \bar{x}_1 x_2\bar{x}_3 \\ &\neq 0 \end{aligned} \tag{E5.4.4}$$

Hence by Theorem 5.4 $d\zeta$ is not compatibly integrable. Let us, therefore, decompose $d\zeta$ as

$$d\zeta = d\zeta_1 + d\zeta_2 + d\zeta_3 \tag{E5.4.5}$$

where

$$d\zeta_1 = \bar{x}_3 dx_1 \tag{E5.4.6}$$

$$d\zeta_2 = \bar{x}_3 dx_2 \tag{E5.4.7}$$

and

$$d\zeta_3 = \bar{x}_2 dx_3 \tag{E5.4.8}$$

Hence $\{x_1\bar{x}_3,\ x_2\bar{x}_3,\ \bar{x}_2 x_3\}$ is a complete set of partial integrals of $d\zeta$.

Also, $d\zeta$ can be separated as

$$d\zeta = (\bar{x}_3 dx_1) + (\bar{x}_3 dx_2 + \bar{x}_2 dx_3) \tag{E5.4.9}$$

which leads to

$\{x_1\bar{x}_3, x_2 + x_3\}$ as another complete set of partial integrals of $d\zeta$.

6. Conclusion

Boolean calculus as proposed here is a powerful tool for analysis as well as synthesis of logic circuits. The use of Boolean integration in synthesis of asynchronous circuits using clock-triggered flipflops has led to circuits which require less flipflops and logic gates than circuits synthesized using conventional methos [5], thus reducing complexity, cost and size and improving reliability.

All the differential operators used in the references can be expressed in terms of partial derivatives defined in the paper.

Earlier methods to realize a function from the specified changes in its value in terms of changes in its arguments do not possess the simplicity and ease that the method presented here does.

The concept of the exact integral was introduced for this purpose. In virtue of the fact that in real-life situations we do have don't-care conditions and/or transitions, the concept of a compatible integral was introduced in order to generalize the concept of the exact integral. Moreover, if the exact integral did not exist for a specified differential but a compatible integral did, then the undesired transitions (changes) in the integral may be inhibited using a simple logic circuit. Integration by parts is further generalization of compatible integration, which has possible applications in logic circuits.

7. Acknowledgements

The authors are pleased to acknowledge gratefully the helpful suggestions offered by Professor F.G. Gray and Mr. R.D. Hofler for the research reported here. Special mention must be made of Dr. W.G. Batte, who suggested the possibility of developing the calculus of Boolean variables analogous to the calculus of real variables,

REFERENCES

(1) J.H. Tucker, "A Transition Calculus for Boolean Functions," Ph.D. Dissertation, Dept. of Electrical Engineering, Virginia Polytechnic Institute and State University, Blacksburgh, Virginia, May, 1974.

(2) J.H. Tucker and A.W. Bennett, "A Transition Calculus for Boolean Functions," Proceedings of IEEE Southeast-Con., Orlando Florida, April 29-May 1,1974.

(3) J.H. Tucker, M.A. Tapia, and A.W. Bennett, "Boolean Integration," Proceedings of IEEE Southeast-Conference, Clemson, S.C., April 5-7, 1976

(4) J.H. Tucker, M.A. Tapia and A.W. Bennett, "Boolean Differentiation of Integration Using Karnaugh Maps," Proceedings of IEEE Southeast-Conference Williamsburg, Va., April 1977

(5) M.A. Tapia, "Development of Boolean Calculus and Its Applications," NASA Langley Basic Research Review, Hampton Va., April 1978.

(6) I.S. Reed, "A Class of Multiple-Error-Correcting Codes and the Decoding Scheme," Trans. Inst. Radio Engrs., IT-4, 1954, pp.38-49.

(7) S.B. Akers, "On a Theory of Boolean Functions," J. Soc. Ind. Appl. Math., Vol. 7, Dec. 1959, pp. 487-498.

(8) P. Calingaert, "Switching Function Canonical Forms Based on Commutative and Associative Binary Operations," AIEE Trans. Communication and Electronics, Vol. 79, January 1961, pp. 808-814.

(9) F.B. Hartman, "Boolean Differential Calculus," IBM TR 22.526, Dec.20, 1967.

(10) V. Amar and N. Condulmari, "Diagnosis of Large Combinational Networks," IEEE Trans. Electronic Computers (Correspondence), Vol. EC-16, October 1967, pp 675-680.

(11) F.F. Sellers, M.Y. Hsiao, and L.W. Bearnson, "Analyzing Errors With The Boolean Difference," First Annual IEEE Computer Conf., September 6-8, 1967, Chicago, Ill.

(12) F.F. Sellers, M.Y. Hsiao, and L.W. Bearnson, "Analyzing Errors With the Boolean Difference," IEEE Trans. Computers, Vol. C-17, July 1968, pp. 676-683.

(13) F.F. Sellers, M.Y. Hsiao, and L.W. Bearnson, "Error Detecting Logic for Digital Computers, New York: McGraw-Hill, 1968.

(14) M. Davio and P. Piret, "Les Derivees Booleennes et Leur Application au Diagnostic," Rev. MBLE, Vol. 12, July 1969, pp. 63-76.

(15) P.M. Marinos, "Boolean Differences as a Means for Deriving Complete Sets of Input Test Sequences for Fault Diagnosis in Switching Systems," Bell Telephone Laboratories, Greensboro, N.C. Tech. Memo MM-69-6313-1, August 1969.

(16) P.N. Marinos, "A Method of Deriving Minimal Complete Sets of Test Input Sequences Using Boolean Difference," IEEE Computer Group Repository, R-70-22.

(17) P.N. Marinos, "A Method of Deriving Minimal Complete Sets of Test Input Sequences Using Boolean Difference," IEEE Computer Group Repository, R-70-22.

(18) G.E. Whitney, "Algebraic Fault Analysis for Constrained Combinational Networks, "IEEE Trans. Computers, Vol. C-20, Feb. 1971, pp. 141-148.

(19) L.W. Bearnson and C.C Carroll, "On the Design of Minimal Length Fault Test for Combinational Circuits," IEEE Trans. Computers, Vol. C-20, Nov. 1971, pp. 1353-1356.

(20) M.Y. Hsiao, D.K. Chia, "Boolean Difference for Fault-Detection in Asynchronous Sequential Machines," IEEE Trans. Computers, Vol. C-20, Nov. 1971, pp. 1356-1361.

(21) A.D. Friedman and P.R. Menon, "Fault Detection in Digital Circuits," Prentice-Hall, New Jersey, 1971.

(22) A. Thayse, "Transient Analysis of Logical Networks Applied to Hazard Detection," Phillips Res. Rep., Vol. 25, October 1970, pp. 261-336.

(23) A. Thayse, "Boolean Differential Calculus," Phillips Res. Rep., Vol.26 June 1971, pp. 229-246.

(24) G. Bioul and M. Davio, "Taylor Expansions of Boolean Functions and of Their Derivatives," Phillips Res. Rep., Vol. 27, Feb. 1972, pp. 1-7.

(25) A. Thayse, "Testing of Asynchronous Sequential Switching Circuits," Phillips Res. Rep., Vol. 27, Feb. 1972, pp. 99-107.

(26) A. Thayse, "On Some Iteration Properties of Boolean Functions," Philips Res. Rep., Vol. 28, Jan. 1973, pp. 107-119.

(27) A. Thayse and M. Davio, "Boolean Differential Calculus and Its Application to Switching Theory," IEEE Trans. Computers, Vo. C-22, April 1973, pp. 409-420.

(28) A.D. Talantsev, "On the Analysis and Synthesis of Certain Electrical Circuits by Means of Special Logical Operators," Avt. 1 Telem., Vol.20, No. 7, pp. 989-907.

(29) V.G. Lazarev and E.I. Piil, "Synthesis Method for Finite Automata," Avt. 1 Telem., Vol. 22, No. 9, September 1961, pp. 1194-1201.

(30) V.G. Lazarev and E.I. Piil, "A Method for Synthesizing Finite Automata with Voltage-Pulse Feedback Elements," Avt. 1 Telem., Vol. 23, No. 8, August 1962, pp. 1037-1043.

(31) V.G. Lazarev and E.I. Piil, "The Simplification of Pulse-Potential Forms," Avt. 1 Telem., Vol. 24, No. 2, February 1963, pp. 271-276

(32) V.G. Lazarev and E.I. Piil, "On the Integration of Potential-Pulse Forms," Dokl. AN SSSR, Vol. 139, July 1961, pp. 556-559.

(33) A. Brown and H. Young, "Toward an Algebraic Theory of the Analysis and Testing of Digital Networks," AAS & ORS Annual Meeting, Denver, Colorado, June 17-20, 1969, AAS Paper 69-236.

(34) J.R. Smith, Jr. and C.H. Roth, Jr., "Differential Mode Analysis and Synthesis of Sequential Switching Networks," Elec. Res. Cen., Univ. of Tex., Austin, Tech. Rep. 63, 1969.

(35) J.R. Smith, Jr. and C.H. Roth, Jr., "Analysis and Synthesis of Asynchronous Sequential Networks Using Edge-Sensitive Flip-Flops," IEEE Trans. Computers, Vol. C-20, Aug. 1971, pp. 847-855.

(36) M.A. Tapia, "Solution of Boolean Equations and Their Applications," Proc. 6th Annual Southeastern Symposium on System Theory, Baton Rouge, La., Feb. 21-22, 1974.

(37) M.A. Tapia and Jerry H. Tucker, "Complete Solution of Boolean Equations," revised and resubmitted to IEEE Transactions on Computers, October 1978.

APPENDIX II

SYNTHESIS OF ASYNCHRONOUS SEQUENTIAL SYSTEMS USING EDGE-SENSITIVE FLIPFLOPS

ABSTRACT

Conventional methods for synthesizing asynchronous sequential systems do not use clock-triggered flipflops. It has been shown [5,6,7] that synthesis techniques for such systems which utilize edge-sensitive (clock-triggered) flipflops lead to networks which require less flipflops and logic gates and which are less expensive and more reliable. The proposed paper aims at developing formal procedures for synthesis of asynchronous sequential systems using commercially available edge-sensitive flipflops.

INDEX TERMS: Asynchronous sequential systems, clock-triggered flipflops, commercial flipflops, edge-sensitive flipflops, Boolean differential, Boolean differential expression, Boolean integration, compatible and exact integrals, fundamental mode asynchronous system, differential mode system, realizability criteria.

I. INTRODUCTION

In conventional asynchronous sequential system design, direct emphasis is placed on relationship between outputs and inputs in terms of their levels only, and the possibility of using edge sensitiveness property of logic elements is not utilized. Smith and Roth have shown [6,7] that if edge-sensitive flipflops are used in the design of an asynchronous system and the edge-sensitiveness property is judiciously taken advantage of, then in many cases it leads to a realization that requires less flipflops and logic gates than conventional method does for a given system. Smith and Roth technique [6,7] utilizes a "general model of edge-sensitive flipflop" in their approach. The method proposed this paper is applicable to any commercially available clock-triggered flipflop that responds to a clock transition (positive or negative).

2. DIFFERENTIAL MODE MODEL

Definition 2.1: A Fundamental Mode Asynchronous system

(D2.1.1) FMAS $\triangleq$ (I, S, O, f, g) where

(D2.1.2) $I \triangleq$ set of p distinct input conditions = $\{I_j\}$

(D2.1.3) $S \triangleq$ set of q states of the system = $\{S_j\}$

(D2.1.4) $O \triangleq$ set of outputs = $\{O_j\}$

(D2.1.5) $f \triangleq$ output function $\triangleq f(S_k, I_j)$, $\forall j$ & k and

(D2.1.6) $g \triangleq$ next state function $\triangleq g(S_k, I_j)$, $\forall j$ & k

will be assumed [10,12,13]. It will further be assumed that only one state variable and only one input variable is allowed to change at a time.

In order to make it convenient to express the next state and output in terms of the change in the input and the present state, we will transform the FMA system to a Differential Mode model defined below. This is comparable to the DM Machine of Smith and Roth [6,7] but really different than that.

Definition 2.2: Given a fundamental mode asynchronous system FMAS, a Differenial Mode System, DMS will be defined as a 6-tuple as given below:

(D2.2.1) $DMS = (I', I^{*\prime}, S', O', f', g')$ where

(D2.2.2) $I' = I$,

(D2.2.3) $I^{*\prime} \triangleq \{(I_j, I_k) \mid \forall_{j,k}\}$

(D2.2.4) $S' = S$

(D2.2.5) $f' \triangleq$ output function of DMS

(D2 2.6) $g' \triangleq$ next state function of DMS

The function g' is related to the function g of the FMA system as shown below:

(D2.2.9) $g'(S_h, I_j, I_k)$

$$\triangleq \begin{cases} S_i, & \text{if } g(S_h, I_j) = S_h,\ g(S_h, I_k) = S_i \\ & \quad \text{and } g(S_i, I_k) = S_i \\ S_i, & \text{if } g(S_h, I_j) = S_h \text{ and there exist } S_{i1}, S_{i2}, ---, S_{in} \ \& \ S_i \\ & \quad \text{such that } g(S_h, I_k) = S_{i1}, \\ & \quad g(S_{i1}, I_k) = S_{i2}, ---, g(S_{in}, I_k) = S_i \\ & \quad \text{and } g(S_i, I_k) = S_i. \\ \underline{\quad\quad}, & \text{if } g(S_h, I_j) = S_h \ \& \ g(S_h, I_k) = \underline{\quad\quad} \\ \underline{\quad\quad}, & \text{if } g(S_h, I_j) = S_h \text{ and there exist} \\ & S_{i1}, S_{i2}, ---, S_{in} \text{ such that} \\ & g(S_h, I_k) = S_{i1},\ g(S_{i1}, I_k) = S_{i2} \\ & g(S_{i2}, I_k) = S_{i3}, ---, g(S_{in}, I_k) = \underline{\quad\quad} \\ \underline{\quad\quad}, & \text{if } g(S_h, I_j) \neq S_h \end{cases}$$

The function $f'(S_h, I_j, I_k)$ is related to the function $f(S_k, I_j)$ of the FMA system as shown below:

(D2.2.10) $$f'(S_h, I_j, I_k) \triangleq \begin{cases} f(S_i, I_k), \text{ if } g'(S_h, I_j, I_k) = S_i \\ \text{---}, \text{ if } g'(S_h, I_j, I_k) \text{ is unspecified} \end{cases}$$

Before we develop procedure for synthesizing the asynchronous sequential system described by the equations (D2.2.1) through (D2.2.10), we will assume that the FMA system (and hence DM system) is amenable to single variable - change state assignment. Let us further assume that the system has n input variables $X_1, X_2 \text{---}, X_n$ m state variables $Y_1, Y_2 \text{---}, Y_m$ and hence m clock-triggered flipflops that respond to positive transitions. We will, therefore, need to realize m clock functions, say C_j's such that whenever an input change occurs then one (and only one) of the clock functions goes through a positive transition providing a proper state transition. Towards this end, we will define differential operators and differential expression to describe changes in a given function $F(X_1, X_2, \text{---}, X_n)$ in terms of changes in its arguments [1, 2, 3, 4, 11].

Definition 2.3:

(D2.3.1) $$dx_i \triangleq \begin{cases} 1, \text{ when } X_i,\ 1 \leq i \leq n, \text{ changes from 0 to 1 or from 1 to 0} \\ 0, \text{ when } X_i \text{ does not change at all} \end{cases}$$

dF will be defined similarly

(D2.3.2) $dF = dX_i$ by definition implies that when X_i changes from 0 to 1 (or 1 to 0), so does F change from 0 to 1 (or 1 to 0).

In order to relate changes in F due to changes in X_i under different conditions we will treat dF and dX_i, $1 \leq i \leq n$ as entities in Boolean algebra having values of 0 or 1 as defined in equation (D2.3.1).

Consider the equation

(D2.3.3) $$dF = (X_2 \cdot X_3) dx_1 + (X_1 \cdot X_2) d\bar{X}_3$$

When $X_2 = X_3 = 1$ and X_1 is changing, then $d\bar{X}_3 = 0$ and $dF = dx_1$ so that F changes the same way as X_1 changes. Similarly when $X_1 = X_2 = 1$ and X_3 changes, then $dF = d\bar{X}_3$ and F changes the same way as $\bar{X}_3$ changes.

Differential expression, denoted by $d\xi$, will be defined as

(D2.3.4) $$d\xi = \sum_{i=1}^{n} \left(\alpha_i dX_i + \beta_i d\bar{X}_i \right)$$

where α_i and β_i, $1 \leq i \leq n$ are functions of $X_1, X_2, ---, X_{i-1}, X_{i+1}, ---, X_n$ (and independent of X_i) and only one of the variables $X_1, X_2, ---, X_n$ is allowed to change at a time.

Differential expression as given in (D2.3.4) will be used to describe changes in clock functions in terms of changes in input and state variables.

The following definitions, relationships and theorems have been reported earlier [1,2,3,4,11] and will be presented here briefly for the sake of completeness and convenience of reference.

<u>Definition 2.4</u>: For a Boolean function $F(X_1, X_2, ---, X_n)$, of n variables $X_1, X_2, ---, X_n$ <u>Boolean differential of</u> F, denoted by dF is defined as

(D2.4.1) $$dF = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial X_i} dX_i + \frac{\partial F}{\partial \overline{X_i}} d\overline{X_i} \right)$$

The summation in Equation (D2.4.1) is with respect to the inclusive OR, and the partial derivatives are defined by

(D2.4.2) $$\frac{\partial F}{\partial X_i} = \left(F(\underline{x}) |_{X_i=1} \right) \cdot \overline{\left(F(\underline{x}) |_{X_i=0} \right)} \text{ and}$$

(D2.4.3) $$\frac{\partial F}{\partial \overline{X_i}} = \left(F(\underline{x}) |_{X_i=0} \right) \cdot \overline{\left(F(\underline{x}) |_{X_i=1} \right)}$$

With the interpretation given in Definition D2.3, equation (D2.4.1) completely describes changes in F due to change in variable X_i, $1 \le i \le n$.

<u>Definition 2.5</u>: <u>The integral of zeroth order</u>, written as $\int_0 d\xi$, of the Boolean expression

(D2.5.1) $$d\xi = \sum_{i=1}^{n} (\alpha_i dx_i + \beta_i d\overline{x}_i)$$

is given by

(D2.5.2) $$\int_o d\xi = \sum_{i=1}^{n}(\alpha_i \bar{X}_i + \beta_i X_i)$$

and the integral of first order, written as $\int_1 d\xi$, of the expression $\alpha\xi$ in equation (D2-5-1) is given by

(D2.5.3) $$\int_1 d\xi = \sum_{i=1}^{n}(\alpha_i X_i + \beta_i \bar{X}_i).$$

Definition 2.6: A given differential expression $d\xi$ given in (D2.5.1) is said to be compatibly integrable if there exists a function F such that

(D2.6.1) $$\frac{\partial F}{\partial X_i} \supseteq \alpha_i \text{ and } \frac{\partial F}{\partial \bar{X}_i} \supseteq \beta_i$$

for all i, $1 \leq i \leq n$. If F satisfying equation (D2.6.1) does exist then F is called a compatible integral of $d\xi$. The differential expression is said to be exactly integrable if there exists function F such that

(D2.6.2) $$dF = d\xi$$

If F satisfying the above equation does exist, then F is called the exact integral of $d\xi$.

Theorem 2.1; The necessary and sufficient condition for compatible integrability of a given differential expression

(T2.1.1) $$d\xi = \sum_{i=1}^{n} (\alpha_i dX_i + \beta_i d\bar{X}_i)$$ is that

(T2.1.2) $$\left(\int_0 d\xi\right) \cdot \left(\int_1 d\xi\right) = 0$$

Theorem 2.2: If a given differential expression $d\xi$ is integrable, then a compatible integrable of $d\xi$ is given by

(T2.2.1) $$\int_0 d\xi = \int_1 d\xi + K$$ where

(T2.2.2) $$\phi \subseteq K \subseteq \left(\int_0 d\xi + \int_1 d\xi\right)$$

3. A DIFFERENTIAL MODE SYSTEM

EXAMPLE 3.1

Consider the FMA system described by the reduced flow table below:

	X_1X_2	00	01	11	10
(A,B)	1	1,0	1,0	3,-	2,-
(C,F)	2	2,1	1,-	4,-	2,1
(D,G)	3	2,-	3,1	3,1	2,-
(E,H)	4	1,-	1,-	4,0	4,0

Fig. 3.1

Since the minimal sytem shown in the table has 4 states, 2 flipflops will be required to realize the system if conventional techniques are used.

On transforming the systemin Figure 3.1 to DM system, we get DM table given in Figure 3.2.

X_1X_2	00 10	00 01	01 11	01 00	11 01	11 10	10 00	10 11
1	2,1	1,0	3,1	1,0	-	-	-	-
2	2,1	1,0	-	-	-	-	2,1	4,0
3	-	-	3,1	2,1	3,1	2,1	-	-
4	-	-	-	-	1,0	4,0	1,0	4,0

Figure 3.2

Using the conventional methods to reduce flow tables for FMA systems, the DM table can reduced as shown in Figure 3.3 Observe that (1,4) and (2,3) are compatible pairs.

X_1X_2	00 10	00 01	01 11	01 00	11 01	11 10	10 00	10 11
(1,4) A	B,1	A,0	B,1	A,0	A,0	A,0	A,0	A,0
(2,3) B	B,1	A,0	B,1	B,1	B,1	B,1	B,1	A,0

Figure 3.3

The reduced DM system has only two states. Let y=0 and y=1 be the assignments for states A and B respectively.

Observe that if y=0, the flipflop must change its state when (1) X_2 = 0 and X_1 changes from 0 to 1

or (2) X_2 = 1 and X_1 changes from 0 to 1.

If y = 1, the flipflop must change when (1) X_1 = 0 and X_2 changes from 0 to 1 or (2) X_1 = 1 and X_2 changes from 0 to 1.

This tells us when the clock should go through a positive transition. The desired changes in clock function in terms of changes in X_1 and X_2 can be described by the differential expression (E3.1.1) below:

$$\text{(E3.1.1)} \qquad dc = \bar{y}\left(\bar{X}_2 dx_1 + X_2 dx_1\right) + y\left(\bar{X}_1 dx_2 + X_1 dx_2\right)$$
$$= \bar{y}\left(dx_1\right) + y\left(dx_2\right)$$

Observe that

$$\text{(E3.1.2)} \qquad \left(\int_0 dc\right) \cdot \left(\int_1 dc\right) = \left(\bar{y}\bar{x}_1 + y\bar{x}_2\right) \cdot \left(\bar{y}x_1 + yx_2\right) = 0$$

so that by Theorem 2.1, dc is compatibly integrable and by Theorem 2.2 a compatible integral of dc is

$$\text{(E3.1.3)} \qquad \int_c dc = \bar{y}x_1 + yx_2$$

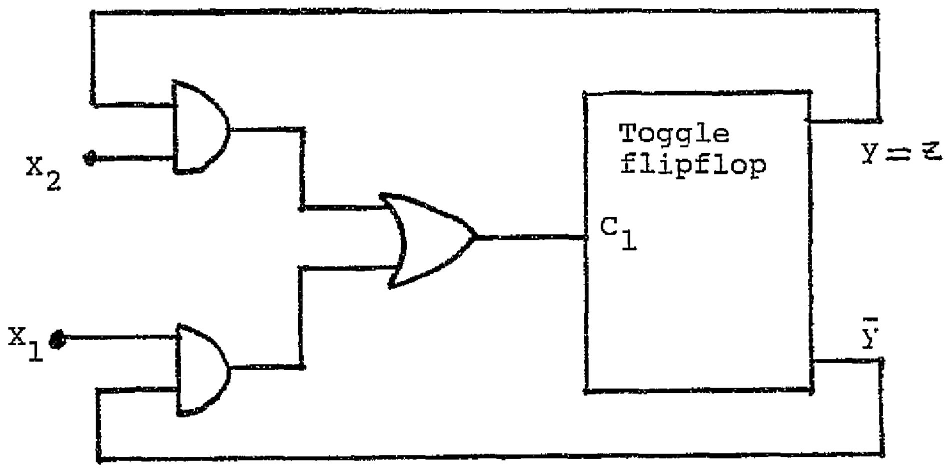
Let us, then, try

(E3.1.4) $\qquad C_1 = \bar{y}x_1 + yx_2$ as the input to the clock pin of a flipflop to be used. We will examine whether a toggle flipflop can be used so that the least amount of combinational logic will be required.

Observe that

$$\text{(E3.1.5)} \qquad dc_1 = \bar{y}dx_1 + ydx_2 + \bar{x}_1x_2dy + x_1\bar{x}_2d\bar{y}$$

so that transitions $(\bar{x}_1x_2dy)$ and $(x_1\bar{x}_2d\bar{y})$ which are not specified by equation (E3·1·1) may be present. However a close examination at Figure 3.3 reveals the fact that when $y=0$ and input changes to $x_1x_2 = 01$, then y does not change to 1 so that $(\bar{x}_1x_2dy)$ is a transition that cannot occur. Similarly $(x_1\bar{x}_2d\bar{y})$ cannot occur either. Hence the clock function C_1 will provide exactly those transitions which are specified by equation (E3.1.1). Hence the following realization.



(A possible hazard can be eliminated by adding X_1X_2 to the right hand side of equation (E3.1.4).)

4. PROPERTIES OF DIFFERENTIAL MODE SYSTEM

Given a DMS table obtained by transforming an FMA table, that has the same number of states as the latter and whose state transition graph [10,12,13] admits of a single-variable-change state assignment, it will be shown later that it is realizable using clock-triggered flipflops. However realizability of a DMS table that is further reduced, if reducible, after transformation depends on several characteristics of the table, that are described in what follows:

<u>Definition 4.1.</u>: A DM system table is said to be <u>level-wise output-unambiguous</u>, if there exist no input conditions I_i, I_j, I_k and states S_a and S_b, I_i and I_k being adjacent, $I_j \neq I_k$, S_a and S_b not necessarily distinct, such that $g'(S_a, I_j, I_i)$, $g'(S_b, I_k, I_i)$, $f'(S_a, I_j, I_i)$ and $f'(S_b, I_k, I_i)$ are defined and

(D4.1.1) $\quad g'(S_a, I_j, I_i) = g'(S_b, I_k, I_i) = S_c \quad$ (say)

(D4.1.2) $\quad f'(S_a, I_j, I_i) = O_{jc} \neq O_{kc} = f'(S_b, I_k, I_i)$.

	I_j I_i	I_k I_i
S_a (or S_b)	S_c, O_{jc}	S_c, O_{kc}

	I_j I_i	I_k I_i
S_a	S_c, O_{jc}	
S_b		S_c, O_{kc}

A DM system table which is not level-wise output-unambiguous will be called <u>level-wise output-ambiguous</u>.

<u>Definition 4.2</u>: A DM system table is said to be <u>level-wise next-state-ambiguous,</u> if there exist inputs I_i, I_j and I_k and states S_a, S_b and S_c such that

(D4.2.1) $\qquad S_b \neq S_c$

(D4.2.2) $\qquad g'(S_a, I_j, I_i) = S_b$ and

(D4.2.3) $\qquad g'(S_a{'}I_k, I_i) = S_c$

	I_j I_i		I_k I_i	
S_a	S_b,	O_{ai}	S_c,	O_{ai}

A DM table which is not level-wise next-state ambiguous is said to be <u>level-wise next-state-unambiguous</u>.

<u>Definition 4.3</u>: If a change in the value of state variable y_j resulting from a change in input causes another state variable y_i, for some $i \neq j$, to change its value, then a <u>secondary transition</u> or <u>ripple</u> is said to occur in the flipflop that is associated with the state variable y_i. If in a DM system a ripple cannot occur, the system is called <u>ripple-free</u>.

In the following definitions, we will assume that the DM system has n input variables $x_1, x_2, \ldots, x_n$, m state variables $y_1, y_2, \ldots, y_m$, $q(=2^m)$ distinct states $S_0, \ldots S_{q-1}$ k output variables

$O_1, O_2, \ldots, O_k$ & $p(=2^n)$ distinct input conditions.

<u>Definition 4.4</u>: I_k represents the binary vector $(b_1, b_2, \ldots, b_n)$ such that b_i's are 0's or 1's and k is the numerical value of $(b_1, b_2, \ldots, b_n)$, when the latter is interpreted as a binary number.

Observe that $m_k(\underline{x})\Big|_{\underline{x}=I_k} = 1$, where $m_k(\underline{x})$ is the $k\underline{th}$ minterm of $x_1, x_2, \ldots, x_n$.

<u>Definition 4.5</u>: S_k represents the binary vector $(b_1, b_2, \ldots, b_m)$ such that b_j's are 0's or 1's and k is the numerical value of $(b_1, b_2, \ldots, b_m)$, when the latter is interpreted as a binary number.

Observe that $m_k(\underline{y})\Big|_{\underline{y}=S_k} = 1$, where $m_k(\underline{y})$ is the $k\underline{th}$ minterm of $\underline{y}$

<u>Definition 4.6</u>: S_{i1} and S_{i2} are said to be y_j-adjacent to each other, if their representations as defined in Definition 4.5 agree in every bit except the $j\underline{th}$ one.

<u>Definition 4.7</u>: $m_j(\underline{x} - x_i)$ denotes the product term obtained by deleting the variable x_i from the jth minterm of variables $x_1, x_2, \ldots, x_n$.

Definition 4.8: $\partial m_j(\underline{x} - x_i)$ denotes transitions as defined below:

(D4.8.1) $$\partial(m_j(\underline{x}-x_i)) \triangleq \begin{cases} m_j(\underline{x}-x_i)\ dx_i, & \text{if } x_i \text{ in minterm } m_j(\underline{x}) \text{ is in true form} \\ m_j(\underline{x}-x_i)\ d\bar{x}_i, & \text{if } x_i \text{ in } m_j(\underline{x}) \text{ is in complemented form.} \end{cases}$$

Theorem 4.1: Consider a DM system table whose realization exists. Then corresponding to every row (or state) S_{i1} and input change from I_{j1}, to I_{j2}, $0 \leq i1 \leq q-1, 0 \leq j1 \leq p-1, 0 \leq j2 \leq p-1$, I_{j1} and I_{j2} being x_i-adjacent, if the next state function $g'(S_{i1}, I_{j1}, I_{j2})$ is defined and

(T4.1.1) $$g'(S_{i1}, I_{j1}, I_{j2}) = S_{i2} \quad \text{where}$$

(T4.1.2) S_{i1} and S_{i2} are y_k-adjacent, then

(T4.1.3) $$dC_k \supseteq m_{i1}(\underline{y}) \cdot \partial(m_{j2}(\underline{x} - x_i)),\ 1 \leq k \leq m$$

Proof: Equations (T4.1.1) and (T4.1.2) imply that y_k must change its value when the system is in state S_{i1} and input variable x_i (say) changes its value so that the final input condition is I_{j2}. Hence at the time of this change C_k

must go through a positive transition described by

$m_{i1}(\underline{y}) \cdot \partial\left(m_{j2}(\underline{x}) - x_i\right)$

Hence

$$\text{(T4.1.4)} \qquad dC_k \supseteq m_{i1}(\underline{y}) \cdot \partial\left(m_{j2}(\underline{x} - x_i)\right)$$

Q.E.D.

Theorem 4.1 presents a way of constructing differential expression dC_k, for $\forall k$, by using each row (i.e. 'present' state) an each column (i.e. input change) such that the (next state) entry corresponding to them is defined and is different than the present state corresponding to the row in which it lies. Construction of a differential expression has already been shown in Example 3.1.

Lemma 4.1: Consider states S_a and S_b, $S_a \neq S_b$ and a change of input from I_i to I_j in a DM system whose table is specified. Let S_a and S_b be y_k-adjacent and

$$\text{(L4.1.1.)} \qquad S_b = g'(S_a, I_i, I_j).$$

Then $m_a(\underline{y}) \cdot m_i(\underline{x})$ and $m_a(\underline{y})\, m_j(\underline{x})$ and the clock function C_k satisfy the following relations if the system is realizable.

$$\text{(L4.1.2)} \qquad C_k \not\supseteq m_a(\underline{y}) \cdot m_i(\underline{x}) \text{ and}$$

$$\text{(L4.1.3)} \qquad C_k \supseteq m_a(\underline{y}) \cdot m_j(\underline{x}).$$

Proof: When the system is in state S_a and input changes from I_i to I_j, the system goes to state S_b requiring that the state of flipflop k change.This necessitates that the clock function C_k go through positive transition. This implies that the value of C_k is 0 when the system is in state S_a and the input is I_k. Hence

(L4.1.2) $\quad C_k \not\supset m_a(\underline{y}) \cdot m_i(x)$.

When the input changes to I_k, the value of C_k must become 1 so that

(L4.1.3) $\quad C_k \supset m_a(\underline{y}) \cdot m_j(\underline{x})$.

Hence the Lemma. Q.E.D.

<u>Theorem 4.2</u>: Differential expression corresponding to every clock function for a specified DM system is integrable if and only if there exist no input conditions I_i, I_{j1} and I_{j2} (I_{j1} and I_{j2} not necessarily distinct) and no states S_a and S_b , $S_a \neq S_b$ such that

(T4.2.4) $\quad S_b = g'(S_a, I_i, I_{j1})$

(T4.2.5) $\quad S_b = g'(S_a, I_{j2}, I_i)$

<u>Proof</u>: Let S_a and S_b be y_k-adjacent. Assume that I_i, I_{j1}, I_{j2}, S_a and S_b exist that satisfy both the equations (T4.2.4) and (T4.2.5). Hence from Lemma 4.1 and these equations, we have simultaneously

(T4.2.6) $\quad C_k \supset m_a(\underline{y}) \cdot m_i(\underline{x})$ and

(T4.2.7) $\quad C_k \not\supset m_a(\underline{y}) m_i(\underline{x})$.

Obviously no function can satisfy relationships (T4.2.6) and (T4.2.7) simultaneously. Hence the differential expression for the clock C_k is not integrable.

If no I_i, I_{j1}, I_{j2}, S_a and S_b exist which satisfy equations (T4.2.4) and (T4.2.5), then $m_a(\underline{y}) \cdot m_i(\underline{x})$ is either "zero" or "one", but not both, of C_k for $\forall a$, i and k. Hence the terms in $\int_1 dC_k$ and $\int_0 dC_k$ are non-intersecting

$$\therefore \int_0 dC_k \cdot \int_1 dC_k = 0, \text{ for } \forall_k$$

Hence differential expression dC_k is integrable for $\forall_k$.

Q.E.D.

<u>Theorem 4.3</u>: A necessary condition for a given DM system table to be realizable is that there exist no input conditions I_i, I_{j1}, and I_{j2} (I_{j1} and I_{j2} may or may not be distinct) and no states S_a and S_b, $S_a \neq S_b$, such that

(T4.3.1) $\qquad S_b = g'(S_a, I_i, I_{j1})$ and

(T4.3.2) $\qquad S_b = g'(S_a, I_{j2}, I_i)$

<u>Proof</u>: The proof follows from Theorem 4.2.

<u>Lemma 4.2</u>: Given a DMS table derived from an FMA system that has the same number of states as the former, the differential expressions for clock functions for the DM system are integrable if a state assignment exists such that exactly one variable changes during any state transition.

<u>Proof</u>: Suppose at least one of the differential expressions is not integrable. Then by Theorem 4.2 there exist inputs I_i, I_{j1} and I_{j2} and states S_a, S_b, such that

(L4.2.1) $\quad S_a \neq S_b$

(L4.2.2) $\quad S_b = g'(S_a, I_i, I_{j1})$ and

(L4.2.3) $\quad S_b = g'(S_a, I_{j2}, I_i)$.

From equation (L4.2.3) and FMAS-to-DMS transfer equations in Definition 2.2 we have

(L4.2.4) $\quad g(S_a, I_{j2}) = S_a$

Also from equation (L4 2.2) and FMAS-to-DMS transfer equations in Definition 2.2 we have

(L4.2.5) $\quad g(S_a, I_i) = S_a$

Equations (L4.2.4) and (L4.2.5) clearly show that we must have in the DM system

(L4.2.6) $\quad g'(S_a, I_{j2}, I_i) = S_a$

From equations (L4.2.6) and (L4.2.3) we have

(L4.2.7) $S_a = S_b$

which contradicts our assumption (L4.2.1).

Hence the Lemma

Q.E.D.

Lemma 4.3: Given a DMS table which is directly derived from an FMA system which has the same number of states as the former, the DMS table is level-wise output-unambiguous.

Proof: Suppose the DMS table is output-ambiguous. Then by Definition 4.1 there exist inputs I_i, I_j and I_k and states S_a and S_b such that

(L4.3.1) $g'(S_a\ I_j,\ I_i) = g'(S_b,\ I_k,\ I_i) = S_c$ (say)

(L4.3.2) $f'(S_a,\ I_j,\ I_i) = O_{jc}$ (say)

(L4.3.3) $f'(S_b,\ I_k,\ I_i) = O_{kc}$ (say) and

(L4.3.4) $O_{jc} \neq O_{kc}$

From Definition 2.2. of DMS table and equation (L4·3·1)

(L4·3·5) $$f'(S_a, I_j, I_i) = f(g'(S_a, I_j, I_i), I_i) = f(S_c, I_i)$$ and

(L4·3·6) $$f'(S_a, I_k, I_i) = f(g'(S_a, I_k, I_i), I_i) = f(S_c, I_i)$$

Hence

(L4·3·7) $$f'(S_a, I_j, I_i) = f'(S_a, I_k, I_i)$$

or

(L4·3·8) $$O_{jc} = O_{kc}$$

which contradicts relation (L4.3.4)

Hence the Lemma.

Q.E.D.

<u>Lemma 4.4</u>: If a DMS table derived from an FMA system has the same number of states as the latter, then the table is level-wise next-state-unambiguous.

<u>Proof</u>: Suppose the table is level-wise-next-state-ambiguous. Then by Definition 4.2 there exists inputs I_i, I_j and I_k and states S_a, S_b and S_c such that

(L4.4.1) $\quad S_b = g'(S_a, I_j, I_i)$

(L4.4.2) $\quad S_c = g'(S_a, I_k, I_i)$ and

(L4.4.3) $\quad S_b \neq S_c$.

From Definition 2.2 and equation (L4.4.1) we get

(L4.4.4) $\quad g(S_a, I_j) = S_a$

(L4.4.5-1) $\quad g(S_a, I_i) = S_{i_1} \neq S_b$ (say)

(L4.4.5-2) $\quad g(S_{i_2}, I_i) = S_{i_1}$

$\vdots$

(L4.4.5-n) $\quad g(S_{i_n}, I_i) = S_b,\ n \geq 1$

(L4.4.6) $\quad g(S_b, I_i) = S_b$

Also from equation (L4.4.2) and Definition 2.2,

(L4.4.7) $\quad g(S_a, I_k) = S_a$

Hence using relations (L4.4.5-1) through (L4.4.5-n), (L4.4.6) and Definition 2.2., we get

(L4.4.8) $\quad g'(S_a, I_k, I_i) = S_b$

From equations (L4.4.8) and (L4.4.2) we get

(L4.4.9) $S_b = S_c$

which contradicts equation (L4.4.3). On the other hand if

(L4.4.10) $S_{i1} = S_b$ in equation (L4.4.5), then also we get

(L4.4.8) $g'(S_a, I_k, I_i) = S_b$ and a contradiction.

Hence the Lemma.

Q.E.D.

Lemma 4.5: A DMS table derived from an FMA system has the same number of states as the latter. If a state S_a, $0 \leq a \leq (q-1)$ in the DM system is such that $g'(S_a, I_{i1}, I_i)$ is specified with

(L4.5.1) $g'(S_a, I_{i1}, I_i) \triangleq S_b$ (say),

then

(L4.5.2) $g'(S_b, I_k, I_i) = \begin{cases} S_b \text{ or} \\ \text{Unspecified} \end{cases}$

for any I_k adjacent to I_i.

Proof: From Definition 2.2 and equation (L4.5.1) for the FMA system we get

(L4.5.3) $g(S_b, I_i) = S_b$.

Also, from Definition 2.2, $g'(S_b, I_k, I_i)$ is defined (i.e., a specified entry) only if

(L4.5.4) $g(S_b, I_k) = S_b$

If equation (L4.5.4) is satisfied, then equation (L4.5.3) and Definition 2.2 imply that

(L4.5.5) $g'(S_b, I_k, I_i) = S_b$

If $g(S_b, I_k) \neq S_b$ or $g(S_b, I_{i1})$ is not defined (i.e., not specified), then by Definition 2.2, $g'(S_b, I_k, I_i)$ is unspecified). In any case, the relation (L4.5.2) is satisfied.

Q.E.D.

<u>Theorem 4.4</u>: If a DM system table has the same number of states as the FMA system from which it is derived and a state assignment exists for the system such that only one state variable changes during any state transition, then the DM system table can be realized using clock-triggered flipflops and logic gates.

<u>Proof</u>: For the sake of simplicity we will consider realization using D-flipflops, the i^{th} flipflop having "level" input D_i, clock input C_i and output Q_i, $1 \leq i \leq m$. Also, it will be assumed that these flipflops respond to positive transitions in their clocks. As indicated in proof of Theorem 4.1, differential expression dC_i, $1 \leq i \leq m$ can be written corresponding to each clock function C_i.

By hypotehsis and Lemma 4.2 these differential expressions are integrable (at least in the compatible integrability sense), and hence all the clock functions can be realized.

By Lemma 4.4 and the hypothesis, the next-state in the DM system table is uniquely specified (if specified) in terms of any input (at the end of any transition) and any present state. Hence the next-state variable functions D_i's can be realized using logic gates.

By Lemma 4.3 and the hypothesis, the output in the DM table is uniquely specified (if specified) in terms of any input and any state. Hence the output functions can be realized using logic gates.

It may be noted that the clock functions may include clock transitions which are not specified in the corresponding differential expressions. However these "undesired" transitions, even if positive, will cause no problem, since the next state that the system must enter into due to any one of these transitions is either uniquely specified or not specified at all.

The only thing now we have to consider is the effect of change in a state variable on the clock functions. If such a change causes a positive transition in one or more of the clock functions, there may be undesired state transition(s) causing malfunction. We will now show that no such malfunction can occur. Consider a transition from state S_a to state S_b, $S_a \neq S_b$, due to a change in input from I_{i1} to I_i. Assume S_a and S_b are y_j-adjacent.

When the input changes from I_{i1} to I_i while the DM system is in state S_a, the system will go to state S_b and the value of y_j will change. Suppose there exists C_p, for some p, $1 \leq p \leq m$ such that when the input is I_i and y_j changes as indicated above, the change in C_p causes a positive transition in C_p. This is possible only if

(T4.4.1) $C_p \supseteq \pi(y) \cdot \pi(X)$.

However by hypothesis we have

(L4.4.2) $S_b = g'(S_a, I_{i1}, I_i)$ with

(L4.4.2A) $S_a \neq S_b$ so that

these relations with Lemma 4.5 yield

(T4.4.3) $g'(S_b, I_k, I_i) = \begin{cases} S_b \\ \text{or} \\ \text{undefined} \end{cases}$

for $\forall I_k$.

In equation (T4.4.3) since there is no change in state S_b, by Theorem 4.1 we have no term of the form $m_b(\underline{y})\partial m_i(\underline{x} - x_k)$ in C_q, $\forall_q$ and $\forall_k$. This implies that

(T4.4.4) $C_q \not\supset m_b(\underline{y}) \cdot m_i(\underline{x})$ for $\forall_q$, $1 \leq q \leq m$.

Relations (T4.4.1) and (T4.4.4) contradict each other. Hence our assumption that a change in state variable could cause a further change in the state of itself or any other flipflop is incorrect. Hence the system realization is ripple-free.

Hence the DM system table can be realized using clock-triggered flipflops and logic gates.

Q.E.D.

Theorem 4.5: Any finite-state asynchronous sequential system can be realized using clock-triggered flipflops and logic gates.

Proof: Any finite-state asynchronous sequential system can be described as an FMA system [10,12,13]. Also the FMA system table could be augmented so that the augmented table admits of a single-state-variable-change assignment [10,12,13]. This table can, then, be transformed to a DM system with the same number of states by Definition 2.2, which can be realized using clock-triggered flipflops and logic gates by Theorem 4.4.

Hence every finite-state asynchronous sequential system can be realized using clock-triggered flipflops and logic gates.

Q.E.D.

Theorem 4.6: The complexity of a network realization of a finite-state asynchronous sequential system, consisting of clock-triggered flipflops and logic gates obtained as shown in Theorem 4.4 method is comparable to that of a network realization of the same system, consisting of S-R flipflops (without clock inputs) and logic gates obtained by conventional method for synthesis of an FMA system.

Proof: The method in Theorem 4.4 requires the same number of state variables and hence flipflops for a given finite-state asynchronous sequential system as the conventional method does for the same system. Hence (in the worst case) realizations obtained by the two distinct approaches have comparable complexity.

Q.E.D.

5. SYNTHESIS PROCEDURE

Realizability of a finite-state asynchronous sequential system using clock-triggered flipflops and logic gates has been established in Theorem 4.5. In this section we will outline a procedure for synthesizing a finite-state asynchronous sequential system using clock-triggered flipflops and logic gates. We will assume that an FMA model table for the given system is already obtained and that the table is already augmented, if necessary, so that it admits of a single-variable-change state assignment.

As shown in Example 3.1, even if the FMA system table is reduced, the DMS table derived from it by transformation equations in Definition 2.2 may be further reducible. If we do reduce it further, if possible, then realizability of the reduced DMS table is not guaranteed, since the reduced DMS table system may fail to satisfy one or more of the following conditions necessary for its realizability:

(1) level-wise output-unambiguity

(2) level-wise next-state-unambiguity

(3) the table admitting of a single-variable-change state assignment.

If the three conditions are satisfied we would have to look for a ripple-free (Definition 4.3) realization which may or may not exist. Quite often even if the possibility of ripples exists, the conditions under which they could occur may be don't care conditions thus making the sytem virtually ripple-free.

If DMS table is further reducible after transformation, the number of flipflops required is less than that if the system were realized directly from the FMA system table using conventional table and hence the complexity of the realization in the former case would be less than that in the latter case.

In the worst case if the reduced DMS table is not realizable, we can go back to the unreduced DMS table obtained from the FMAS table, which is guaranteed to be realizable with complexity comparable to that of a realization obtained by conventional method.

The next Theorem describes the conditions under which ripples can occur in the realization of a DMS table that is obtained by reducing the DMS table obtained from an FMA system.

Theorem 5.1: Consider a DMS table that is obtained by reducing the DMS table obtained from an FMA system. A ripple may occur in the k^{th} flipflop, $1 \leqslant k \leqslant m$, if and only if

(T5.1.1) $$\frac{\partial c_k}{\partial y_{j1}} + \frac{\partial c_k}{\bar{y}_{j1}} \neq 0$$

for some y_j, say y_{j1}, $1 \leqslant j1 \leqslant m$, $j1 \neq k$, where y_j is the state variable associated with flipflop j.

Proof: Assume $\frac{\partial c_k}{\partial y_{j1}} \geqslant m_{j2}(\underline{x}) \cdot m_{j3}(\underline{y} - y_{j1})$.

Then if the system could get into (total) state such that $m_{j2}(\underline{x}) \cdot m_{j3}(\underline{y} - y_{j1}) = 1$ and in such a state if y_{j1} changes from 0 to 1, then the clock will go through a positive transition and change the state of the flipflop , thus causing a ripple to occur. If $\frac{\partial c_k}{\partial \bar{y}_{j1}} \neq 0$, we could show in a similar way that a ripple may occur.

On the other hand, if

(T5.1.2) $$\frac{\partial c_k}{\partial y_{i1}} + \frac{\partial c_k}{\partial y_{j1}} = 0,$$

c_k is insensitive to changes in y_{j1}, $1 \leqslant j1 \leqslant m$, and no ripples could occur in flipflop k.

Q.E.D.

Procedure for synthesis of an asynchronous sequential system will, now, be described in Procedure 5.1.

5.1 PROCEDURE

(1) Transform the given FMAS table to DMS table using Definition 2.2.

(2) Employ conventional techniques of reducing an incompletely specified sequential system table [10,12,13] and reduce the DMS table obtained in step (1), if it is reducible. If the table did get reduced, go to step (3). If not, go to step (12).

(3) Determine if the reduced table is level-wise next-state unambiguous and output-unambiguous. If it is not so, go to step (4). If it is so, go to step (5).

(4) Obtain another reduced table, if it exists, of the DMS table obtained in step (1), that has not been tried yet. If no such table exists, go to step (12). If it does exist, go to step (3).

(5) Find a state assignment for the reduced DMS table being examined such that exactly one state variable changes during any state transition and go to step (6). If no such assignment exists, go to step (4).

(6) Employ Theorem 4.2 to determine if the differential expressions for the inputs to the clock pins of the flipflops are integrable. If the necessary and sufficient conditions for

integrability are not satisfied, go to step (12). If they are satisfied, go to (next) step (7).

(7) Consider every (next-state) entry in every row that is different than the (present) state that it represents. If the (present) state and the next state are y_k adjacent, then as shown in Theorem 4.1 a differential term is added to dC_k, the differential expression for clock input to flipflop k, $1 \leq k \leq m$. This is repeated for every next-state entry in the reduced DMS table that is adjacent to the state corresponding to the row in which it lies. When all the differential expressions, dC_k's are obtained go to step (7A).

(7A) Find a set of compatible integrals of differential expressions dC_k, $1 \leq k \leq m$.

(8) Compute $\frac{\partial C_k}{\partial y_i}$ and $\frac{\partial C_k}{\partial \bar{y}_i}$. for $\forall i$, $i \neq k$ $1 \leq i \leq n$ and $\forall k$, $1 \leq k \leq m$.

(9) If there exists at least one I, say I_1, one y_j, say y_{j1} and one k, say k_1 such that

(P5.1.1) $$\left(\frac{\partial C_{k1}}{y_{j1}} + \frac{\partial C_{k1}}{\bar{y}_{j1}}\right) \neq 0 \;, \quad k_1 \neq j_1$$

and not all of the differential terms on the left side of the relation (P5.1.1) are don't-care transitions, then go to the next step. Otherwise go to step (11).

(10) Find another distinct set of compatible integrals of differential expressions obtained in step (6), that has not been tried yet and go step (8). If no such set exists go to step (4).

(11) Minimize the set of compatible integrals of differential expressions dC_k, $1 \leq k \leq m$ and obtain a combinational network for each compatible integral in the set. From the DMS table, find minimal Boolean expressions for the J-K- or S-R- or D-inputs (depending upon the type of flipflops used) for each flipflop and realize them using logic gates. Similarly realize output functions in terms of state variables and input variables that determine them. This completes the synthesis procedure.

(12) Take the (unreduced) DMS table in step (1) and obtain differential expression for clock input to flipflop k, $1 \leq k \leq m$ as indicated in Theorem 4.1.

(13) Find a set of compatible integrals of differential expressions dC_k, $1 \leq k \leq m$. Go to step (11).

6. CONCLUSION

Design of asynchronous sequential systems using clocked flipflops has been known for a long time. However, such design using simpler circuits has been essentially limited to those cases where the logic designer possesses sufficient experience and inspiration to intuitively obtain such an implementation. Smith and Roth [6,7] presented a formal approach to realize asynchronous sequential system using "general model of edge-sensitive flipflop" [6,7]. The formal synthesis procedure proposed here is applicable to synthesis of such systems using any commercially available clock-triggered flipflops.

We have shown that any asynchronous sequential system could be realized using the proposed approach. In many cases this approach leads to designs which are less complex, less costly, more reliable and smaller in size than those obtained using conventional design techniques [10,12,13]. In the worst case the complexity of the design obtained by the proposed approach are comparable to that obtained by conventional techniques.

REFERENCES

(1) J.H. Tucker, "A Transition Calculus for Boolean Functions," Ph.D. Dissertation, Dept. of Electrical Engineering, Virginia Polytechnic Institute and State University, Blacksburgh, Virginia, May, 1974.

(2) J.H. Tucker and A.W. Bennet, "A Transition Calculus for Boolean Functions," Proceedings of IEEE Southeast-Con., Orlando Florida, April 29-May 1, 1974.

(3) J.H. Tucker, M.A. Tapia, and A.W. Bennett, "Boolean Integration," Proceedings of IEEE Southeast-Conference, Clemson, S.C., April 5-7, 1976.

(4) J.H. Tucker, M.A. Tapia and A.W. Bennett, "Boolean Differentiation of Integration Using Karnaugh Maps," Proceedings of IEEE Southeast-Conference Williamsburg, Va., April 1977.

(5) M.A. Tapia, "Development of Boolean Calculus and Its Applications, "NASA Langley Basic Research Review, Hampton Va., April 1978.

(6) J.R. Smith, Jr. and C.H. Roth, Jr., "Differential Mode Analysis and Synthesis fo Sequential Switching Networks," Elec. Res. Cen., Univ. of Tex., Austin, Tech. Rep. 63, 1969.

(7) J.R. Smith, Jr. and C.H. Roth, Jr., "Analysis and Synthesis of Asynchronous Sequential Networks Using Edge-Sensitive Flip-Flops," IEEE Trans. Computers, Vol. C-20, Aug. 1971, pp. 847-855.

(8) A.D. Talantsev, "On the Analysis and Synthesis of Certain Electrical Circuits by Means of Special Logical Operators," Avt. l Telem., Vol. 20, No. 7, pp. 898-907.

(9) V.G. Lazarev and E.I. Piil, "Synthesis Method for Finite Automata," Avt. l Telem, Vol. 22, No. 9, September 1961, pp. 1194-1201.

(10) P.E. Wood, Switching Theory, McGraw-Hill, 1968.

(11) M.A. Tapia, J.H. Tucker and A.W. Bennett, "Boolem Calculus for Digital Systems," submitted to IEEE Computer Transactions.

(12) Nagle, Carroll and Irwin, "An Introduction to Computer Logic", Prentice-Hall, 1975.

(13) Zvi Kohavi, "Introduction to Switching and Finite Automata Theory," McGraw-Hill, 1970.