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CR 160129

PERTURBATIONS OF
NON-RESONANT SATELLITE
ORBITS DUE TO A ROTATING EARTH

Analytical and
Computational
Mathematics, INC.

PERTURBATIONS OF
NON-RESONANT SATELLITE
ORBITS DUE TO A ROTATING EARTH

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JUNE 1978

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PERTURBATIONS OF

NON-RESONANT SATELLITE

ORBITS DUE TO A ROTATING EARTH

by

Alan Mueller

1.0   INTRODUCTION

The dominant perturbations of the motion of a satellite near the earth are due to the non-symmetrical gravitational field and the atmospheric drag. The gravitational field may be divided in two classes: terms independent of time (zonal harmonics) and terms which depend explicitly on time. A complete first order solution[†] for satellite motion perturbed by the second zonal harmonic has been developed by Scheifele (reference 1). This solution has been rewritten in the non-singular PSϕ elements by Bond (reference 2). In references 3 and 4, the perturbations due to drag are developed in PSϕ elements and added to this J_2 theory. In reference 5, the long period perturbations due to the additional zonal harmonics are included in the theory. The tesseral harmonics have yet to be treated under this unified theory and are the topic of this report.

The perturbations due to the tesseral harmonics can be placed in four categories:

   1)   Short period perturbations with a magnitude of

[†]   A first order solution here implies that the solution has a periodic error of second order and a secular error of third order.

about J_2^2 .

2) Intermediate period perturbations with a magnitude of between J_2 and J_2^2 .

3) Along track secular perturbations induced by the periodic perturbations in the mean motion.

4) Resonant perturbations.

For near earth satellites, the atmospheric drag perturbation continually pulls the satellite in and out of the different long period resonant frequencies. The result is that the resonances never become apparent and may be neglected.

Since the J_2 theory has been developed only to first order, the second order short period tesseral perturbations may be neglected. For the same reason, the intermediate period perturbations should be included if first order accuracy is to be maintained. The tesseral harmonics have no true secular perturbation but the periodicities in the mean motion induce a secular perturbation in the mean anomaly. This secular perturbation may be determined by simply using the average mean motion instead of the osculating mean motion. Graf (reference 6) finds the average mean motion in a numerical manner. The numerical studies in that reference verify the assumption that use of the average mean motion does account for the apparent secular trend in the mean anomaly. Although the results are good, use of a numerical method would be inconsistent with the idea of having a completely analytical theory.

To complete the solution of the motion of a near earth satellite, the averaged mean motion and the intermediate period perturbations need to be found in a completely analytical manner.

Since the previous PSϕ theory has applied Von Zeipel's solution technique, it seems natural to return to this method for a solution of the tesseral perturbation. As in the previous

theory, the solution will be found first in the singular $DS\phi$ elements and then rewritten in the $PS\phi$ elements to remove singularities. The notation used in the development is described in Appendix A.

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2.0 FORMULATION OF THE PROBLEM

Only the gravitational field will be considered since its interaction with the drag effects is small. The DSϕ hamiltonian for the gravitational potential is assumed to have the form

$$F = F_0 + \epsilon F_1 + \epsilon^2 F_2 \qquad , \tag{1}$$

where

$$F_0 = \phi - \frac{\mu}{\sqrt{2L}} \qquad , \qquad \text{(two-body)} \tag{2}$$

$$F_1 = \frac{1}{qr}\left(\left(\frac{z}{r}\right)^2 - \frac{1}{3}\right) \qquad , \qquad (J_2) \tag{3}$$

$$F_2 = F_Z + F_T \qquad , \qquad \text{(Zonals and Tesserals)} \tag{4}$$

$$F_Z = \frac{\mu}{q\epsilon^2} \sum_{n=3} C_{n,o} \frac{R_e^n}{r^{n-1}} P_n^o\left(\frac{z}{r}\right) \qquad , \tag{5}$$

$$F_T = \frac{\mu}{qr^2} \sum_{n=2} \sum_{m=1}^{n} \frac{R_e^n}{r^{n-1}} P_n^m\left(\frac{z}{r}\right)\left[C_{n,m} \cos m(\lambda-\theta) + \right.$$
$$\left. + S_{n,m} \sin m(\lambda-\theta) \right] \tag{6}$$

$$\epsilon = \frac{3}{2} J_2 \mu R_e^2 \quad . \tag{7}$$

P_n^m are the associated Legendre polynomials; R_e is the mean equatorial radius; J_2 , $C_{n,m}$, $S_{n,m}$ are the geopotential coefficients; λ is the longitude of the satellite and θ is the Greenwich hour angle.

The F_2 hamiltonian may be divided in zonal and tesseral harmonics. The functional dependence of F_Z and F_T is

$$F_Z = F_Z(\phi, g, \beta) \qquad . \tag{8}$$

$$F_T = F_T(\phi, g, h, \omega_\oplus \ell^{\dagger}, \beta) \qquad . \tag{9}$$

---

† The canonical time element ℓ will always appear, in the problem, premultiplied by the rotation rate of the earth $\omega_\oplus$.

3.0 VON ZEIPEL'S SOLUTION TECHNIQUE

As was stated earlier, the Von Zeipel solution method is
to be used to find the average mean motion and to eliminate
the periodicities due to the tesseral harmonics. The genera-
ting function S is to be used for the elimination of the
short period function of ϕ and the intermediate period func-
tion of $\omega_e \ell$. The equations governing the transformation in
the $DS\phi$ space are

$$\alpha' = \frac{\partial S}{\partial \beta'} \quad , \qquad \qquad \beta = \frac{\partial S}{\partial \alpha} \quad . \qquad (10)$$

As in the hamiltonian, S is assumed to be of the form

$$S = S_0 + /\epsilon S_1 + \epsilon^2 S_2 \quad , \qquad (11)$$

$$S_2 = S_Z + S_T \quad , \qquad (12)$$

where S_T is a periodic function of ϕ and $\omega_e \ell$ and S_1
and S_Z are periodic functions of ϕ only. The development
of S_1 can be found in reference 1 and therefore the discus-
sion will be restricted to the development of S_2 .

The theory of references 1 and 2 neglected S_2 to main-
tain a first order accuracy. But this assumes that S_2 is
order $O(1)$. This is true for zonal perturbations but the
tesseral perturbations result in an S_2 which may become
larger than $O(1)$. This will be demonstrated in the following
analysis. The necessary Von Zeipel equations found from ref-
erence 1 are

$$\epsilon^0 \; : \; F_0' \; (\phi',L') = F_0 \; (\phi',L') \qquad (13)$$

$$\varepsilon^1 : \quad \frac{\partial F}{\partial \phi} \frac{\partial S_1}{\partial \phi} = - F_1 (\beta',\phi,g) + F_1' (\beta',g) \qquad (14)$$

$$\varepsilon^2 : \quad \frac{\partial F_0}{\partial \phi'} \frac{\partial S_2}{\partial \phi} + \frac{\partial F_0}{\partial L'} \frac{\partial S_2}{\partial \ell} = - \frac{1}{2} \sum_{k=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 F_0}{\partial \beta_k' \partial \beta_j'} \frac{\partial S_1}{\partial \alpha_k} \frac{\partial S_1}{\partial \alpha_j}$$

$$- \sum_{k=1}^{4} \frac{\partial F_1}{\partial \beta_k'} \frac{\partial S_1}{\partial \alpha_k} + \sum_{k=1}^{4} \frac{\partial F_1'}{\partial \alpha_k} \frac{\partial S_1}{\partial \beta_k'} - F_Z(\beta',\phi,g) - \qquad (15)$$

$$- F_T(\beta',\phi,g,h,\omega_\oplus \ell) + F_2'(\beta',g) \quad .$$

Assuming S_2 has the form given by 12 , then the expressions required to find S_Z and S_T are given by

$$\frac{\partial F_0}{\partial \phi'} \frac{\partial S_Z}{\partial \phi} = - \frac{1}{2} \sum_{k=1}^{4} \sum_{j=1}^{4} \frac{\partial^2 F_0}{\partial \beta_k' \partial \beta_j'} \frac{\partial S_1}{\partial \alpha_k} \frac{\partial S_1}{\partial \alpha_j} -$$

$$- \sum_{k=1}^{4} \frac{\partial F_1}{\partial \beta_k'} \frac{\partial S_1}{\partial \alpha_k} + \sum_{k=1}^{4} \frac{\partial F_1'}{\partial \alpha_k} \frac{\partial S_1}{\partial \beta_k'} - \qquad (16)$$

$$- F_Z(\beta',\phi,g) - F_2'(\beta',g) \quad .$$

$$\frac{\partial F_0}{\partial \phi'} \frac{\partial S_T}{\partial \phi} + \frac{\partial F_0}{\partial L'} \frac{\partial S_T}{\partial \ell} = - F_T \quad . \qquad (17)$$

If S_2 is to be used in eliminating the intermediate period terms then it is necessary to keep only those terms of order greater than $O(1)$. If S_2 is to be used in finding the

mean energy, only those terms which are functions of ℓ are considered. With these qualifications a much simpler set of equations can be used to find S_2 :

$$S_Z = 0 \quad , \tag{18}$$

$$\frac{\partial F_0}{\partial \phi'} \frac{\partial S_T}{\partial \phi} + \frac{\partial F_0}{\partial L'} \frac{\partial S_T}{\partial \ell} = - F_T(\beta', \phi, g, h, \omega_\oplus \ell) \quad . \tag{19}$$

S_Z is set to zero because the right hand sides of equation (16) are of order $O(1)$ and are independent of ℓ .

Thus the transformation due to the short and intermediate terms is given by

$$\alpha' = \alpha + \epsilon \frac{\partial S_1}{\partial \beta'} + \epsilon^2 \frac{\partial S_T}{\partial \beta'} \quad , \tag{20}$$

$$\beta = \beta' + \epsilon \frac{\partial S_1}{\partial \alpha} + \epsilon^2 \frac{\partial S_T}{\partial \alpha} \quad . \tag{21}$$

where

$$S_T > O(1) \quad .$$

The DSϕ element L is the total energy and can be related to the mean motion. The mean total energy L' can be then related to the instantaneous total energy by

$$L = L' + \epsilon^2 \frac{\partial S_T}{\partial \ell} \quad , \tag{22}$$

where

$$S_T \geq 0(1) \quad .$$

Remarks:

If the Von Zeipel's solution method had been applied to the classical Delaunay variables a much more complicated set of equations would appear. The average mean motion could also be found by differentiating the classical determining function S with respect to the mean anomaly. But since S_1 (or F_1) is a function of the mean anomaly, S_2 could not be considered zero since it would also be a function of the mean anomaly. Also the expression for the mean energy would be more complex and difficult to solve since terms of $0(\epsilon)$ would appear. All the usual complications of the second order theory would appear just to maintain a first order accuracy. But with the DSϕ elements the equations for finding the generating function remain concise and relatively easy to solve.

The mean energy is found through a determining function which is defined by a partial differential equation with the anomaly of the satellite ϕ and the changing hour angle $\omega_\oplus \ell$. This expression is very similar in form to the part al differential equation found by Graf (reference 6) by applying the "Method of Averages" to the KS differential equations. Maybe it is not so peculiar that the Method of Averages and Von Zeipel's equations result in similar formulas.

If F_T can be written in a fourier series of the form

$$F_T = \sum_k \sum_j \left\{ C_{kj}(\beta',h,g) \cos (k\phi - j\omega_\oplus \ell) + \right.$$
$$\left. + S_{kj}(\beta',h,g) \sin (k\phi - j\omega_\oplus \ell) \right\} \quad , \tag{23}$$

then the solution for S_T in equation (19) can be easily shown

to be

$$S_T = -\sum_k \sum_j \frac{1}{k-\nu j} \left\{ C_{kj} \sin (k\phi - j\omega_\oplus \ell) - \right.$$
$$\left. - S_{kj} \cos (k\phi - j\omega_\oplus \ell) \right\} \quad , \tag{24}$$

where ν is the ratio of the frequency of the earth's rotation to the revolution frequency of the satellite.

$$\nu = \omega_\oplus \cdot \frac{\partial F_0}{\partial L} = \omega_\oplus \frac{\mu}{(2L)^{3/2}} \quad . \tag{25}$$

For near earth satellites this ratio is about one to sixteen,

$$\nu \approx \frac{1}{16} \quad . \tag{26}$$

Therefore when $k = 0$ and for $j < \frac{1}{\nu}$, the factor $\frac{1}{k-\nu j}$ becomes on the order of $O(\frac{1}{\nu})$. These terms result in the intermediate period perturbations which are the order of $O(\frac{\varepsilon^2}{\nu})$.

In order to complete the solution, the tesseral hamiltonian F_T must be expressed as a fourier series of the form given by equation (23). As before the solution will then be expressed in the non-singular PSϕ elements.

4.0 EXPANSION OF THE TIME DEPENDENT GEOPOTENTIAL

The definition of F_T as given by equation (6) is

$$F_T = \frac{\mu}{q\varepsilon^2} \sum_{n=2}^{\infty} \sum_{m=1}^{n} r^2 V_{n,m} \qquad , \qquad (27)$$

where

$$r^2 V_{n,m} = \frac{R_e^n}{r^{n-1}} P_n^m \left(\frac{z}{r}\right) \left[C_{n,m} \cos m (\lambda-\theta) + S_{n,m} \sin m (\lambda-\theta) \right] \qquad . \qquad (28)$$

We desire an expansion of the above expression in a fourier series of the canonical DSφ angular variables φ, g, h, and ℓ . The expansion will, in many respects, be similar to Kaula's expansion of the disturbing function in the classical angular variables (reference 7). The important difference is that expansions will now be carried out in the true anomaly instead of the mean anomaly.

As in Kaula, the potential may be expanded using the inclination F_{nmp}[†] to a form

$$r^2 V_{n,m} = \frac{R_e^n}{r^{n-1}} \sum_{p=0}^{n} F_{nmp} \left\{ A_{nm} \cos \psi_{nmp} + B_{nm} \sin \psi_{nmp} \right\} \qquad . \qquad (29)$$

[†] To maintain the notation of previous authors, this author has decided to keep the notation p as an index in the inclination function. For distinction, the semi-latus rectum will now appear as an italic "p" .

where

$$\psi_{nmp} = (n-2p)\,(\phi+g) + m\,(h-\theta) \quad , \qquad (30)$$

$$A_{nm} = \begin{cases} C_{nm} & n-m \quad \text{even} \\ \\ -S_{nm} & n-m \quad \text{odd} \end{cases} \quad , \qquad (31)$$

$$B_{nm} = \begin{cases} S_{nm} & n-m \quad \text{even} \\ \\ C_{nm} & n-m \quad \text{odd} \end{cases} \quad . \qquad (32)$$

The recursive relations for the inclination function have been derived by Giacaglia (reference 8). These relations and others appear in Appendix B.

Another transformation is necessary to replace the powers of r as a fourier series in ϕ :

$$\frac{1}{r^{n-1}} \begin{bmatrix} cos \\ sin \end{bmatrix} \psi_{nmp} = \frac{1}{p^{n-1}} \sum_{j=1-n}^{n-1} G_j^{n-1} \begin{bmatrix} cos \\ sin \end{bmatrix} \psi_{nmpj} \quad , \qquad (33)$$

$$\psi_{nmpj} = (n-2p+j)\,\phi + (n-2p)\,g + m\,(h-\theta) \quad . \qquad (34)$$

Note that this is a finite expansion with an eccentricity function $G_j^{n-1}(e)$. The development of G_j^{n-1} is given in Appendix C.

A final transformation is required to place the disturbing function in the form of equation (23). In the expression of the angle ψ_{nmpj} , the hour angle may be written as

$$\theta = \theta_o + \omega_{\oplus} t \quad , \qquad (35)$$

where θ_o is the hour angle at time $t = 0$. Time is now
a dependent variable in the $DS\phi$ system given by

$$t = \ell + \frac{\mu}{(2L)^{3/2}} \left[M(\phi) - \phi \right] , \tag{36}$$

where

$$M(\phi) = E - e \sin E . \tag{37}$$

Replacing equations (35) and (36) into equation (34) one finds

$$\psi_{nmpj} = (n-2p+j) \phi + m\nu (\phi - M) + (n-2p) g +$$
$$+ m (h - \omega_\oplus \ell - \theta_o) . \tag{38}$$

Thus we define a new function $N_k^m (\nu, e)$ which gives

$$\begin{bmatrix} cos \\ sin \end{bmatrix} \psi_{nmpj} = \sum_k N_k^m \begin{bmatrix} cos \\ sin \end{bmatrix} \psi_{nmpjk} , \tag{39}$$

where

$$\psi_{nmpjk} = (n-2p+j+k) \phi + (n-2p) g + m (h - \omega_\oplus \ell - \theta_o) . \tag{40}$$

The development of N_k^m can be found in Appendix D.

Accumulating the results we arrive at the final expres-
sion for the disturbing function F_T :

$$F_T = \frac{\mu}{\varepsilon^2 q} \sum_{n=2}^{n} \sum_{m=1}^{n} \sum_{p=0}^{n-1} \sum_{j=1-n}^{\infty} \sum_{k=-\infty}^{\infty} V_{nmpjk} \quad , \qquad (41)$$

where one element is given by

$$V_{nmpjk} = J_{nmpjk} \left\{ A_{nm} \cos \psi_{nmpjk} + B_{nm} \sin \psi_{nmpjk} \right\} \qquad (42)$$

and

$$J_{nmpjk} = \frac{R_e^n}{p^{n-1}} F_{nmp} \, G_j^{n-1} \, N_k^m \quad . \qquad (43)$$

Thus one element of the determining function S_T defined by equation (19) becomes

$$S_{Tnmpjk} = \frac{-J_{nmpjk}}{(n-2p+j+k-\nu m)} \left\{ A_{nm} \sin \psi_{nmpjk} - \right.$$

$$\left. - B_{nm} \cos \psi_{nmpjk} \right\} \quad . \qquad (44)$$

The mean energy can then be found from equations (44) and (22):

$$L' = L + \frac{\mu}{q} \sum_n \sum_m \sum_p \sum_j \sum_k \Delta L_{nmpjk} \quad , \qquad (45)$$

where ΔL_{nmpjk} is expressed as

$$\Delta L_{nmpjk} = m\omega_\oplus \frac{J_{nmpjk}}{(n-2p+j+k-\nu m)} \left\{ A_{nm} \cos \psi_{nmpjk} + \right.$$

$$\left. + B_{nm} \sin \psi_{nmpjk} \right\} \quad . \tag{46}$$

The elimination of the intermediate period terms may also be found by determining the partial derivatives of each element of S_{Tnmpjk} in which

$$k = 2p - n - j \quad . \tag{47}$$

5.0 EXPRESSING THE GEOPOTENTIAL IN PSϕ ELEMENTS

In the expansion of the geopotential, the angle ψ_{nmpjk} becomes undefined for small eccentricities and inclinations. The fourier expansion of equations (41), (42) and (43) should be transformed into a fourier series in the true longitude σ_1 and the angle $\omega_\oplus \ell + \theta_o$, and polynomials of the well defined functions

$$\eta = e \sin (g+h) \qquad \delta = s \cdot \sin h$$
$$\zeta = e \cos (g+h) \qquad \beta = s \cdot \cos h \qquad . \qquad (48)$$

With the definition of the true longitude $\sigma_1 = \phi + g + h$, the angle ψ_{nmpjk} becomes

$$\psi_{nmpjk} = \theta_{nmpjk} - (j+k)(g+h) + (m-n+2p) h \ , \qquad (49)$$

$$\theta_{nmpjk} = (n-2p+j+k) \ \sigma_1 - m(\omega_\oplus \ell + \theta_o) \qquad . \qquad (50)$$

Keeping the notation of Giacaglia (reference 9) we make new definitions

$$q^\dagger = j + k \qquad , \qquad (51)$$

$$\alpha = m - n + 2p \qquad , \qquad (52)$$

so that ψ_{nmpjk} becomes

$$\psi_{nmpjk} = \theta_{nmpjk} - q(g+h) + \alpha h \qquad . \qquad (53)$$

$\dagger$
To maintain the notation of previous authors, this author has decided to keep the notation of q as an index in the expansion. Please observe the definition of q given in Appendix A.

By defining

$$\eta_q = e^{|q|} \; sin \; q \; (g+h) \qquad \delta_\alpha = s^{|\alpha|} \; sin \; \alpha \; h \qquad ,$$
$$\zeta_q = e^{|q|} \; cos \; q \; (g+h) \qquad \beta_\alpha = s^{|\alpha|} \; cos \; \alpha \; h \qquad , \tag{54}$$

$$R_{\alpha q} = \zeta_q \beta_\alpha + \eta_q \delta_\alpha \qquad , \tag{55}$$

$$I_{\alpha q} = \zeta_q \delta_\alpha - \eta_q \beta_\alpha \qquad , \tag{56}$$

the expressions for the functions of ψ_{nmpjk} become

$$e^{|q|} s^{|\alpha|} \; cos \; \psi_{nmpjk} = R_{\alpha q} \; cos \; \theta_{nmpjk} +$$
$$+ I_{\alpha q} \; sin \; \theta_{nmpjk} \qquad . \tag{57}$$

$$e^{|q|} s^{|\alpha|} \; sin \; \psi_{nmpjk} = R_{\alpha q} \; sin \; \theta_{nmpjk} +$$
$$- I_{\alpha q} \; cos \; \theta_{nmpjk} \qquad . \tag{58}$$

Recursive relations exist for the definitions of equation (54). They are

$$\eta_q = \eta_{q-1} \; \zeta_1 + \zeta_{q-1} \; \eta_1 \qquad , \tag{59}$$

$$\zeta_q = \zeta_{q-1} \; \zeta_1 - \eta_{q-1} \; \eta_1 \qquad , \tag{60}$$

$$\eta_{-q} = - \eta_q \qquad , \qquad \zeta_{-q} = \zeta_q \qquad , \tag{61}$$

$$\zeta_1 = Q \; \rho_2 \qquad , \qquad \eta_1 = - Q \; \alpha_2 \; , \tag{62}$$

$$\delta_\alpha = \delta_{\alpha-1}\ \beta_1 + \beta_{\alpha-1}\ \delta_1 \qquad . \tag{63}$$

$$\beta_\alpha = \beta_{\alpha-1}\ \beta_1 - \delta_{\alpha-1}\ \delta_1 \qquad , \tag{64}$$

$$\delta_{-\alpha} = -\delta_\alpha \qquad , \qquad \beta_{-\alpha} = \beta_\alpha \qquad , \tag{65}$$

$$\beta_1 = P\rho_3 \qquad , \qquad \delta_1 = -P\sigma_3 \qquad . \tag{66}$$

With the expressions (57) and (58), one is now able to write each element of the disturbing function given by equation (42) entirely in non-singular coordinates:

$$V_{nmpjk} = \bar{J}_{nmpjk} \left\{ (A_{nm}\ R_{\alpha q} - B_{nm}\ I_{\alpha q})\ cos\ \theta_{nmpjk} + \right.$$
$$\left. + (A_{nm}\ I_{\alpha q} + B_{nm}\ R_{\alpha q})\ sin\ \theta_{nmpjk} \right\} \qquad , \tag{67}$$

$$\bar{J}_{nmpjk} = \frac{R_e^n}{p^{n-1}}\ \bar{F}_{nmp}\ \bar{G}_j^{n-1}\ \bar{N}_k^m\ e^{(|j|+|k|-|j\cdot k|)} \qquad . \tag{68}$$

$$\bar{F}_{nmp} = \frac{F_{nmp}}{s^{|q|}} \qquad . \tag{69}$$

$$\bar{G}_j^{n-1} = \frac{G_j^{n-1}}{e^{|j|}} \qquad . \tag{70}$$

$$\bar{N}_k^m = \frac{N_k^m}{e^{|k|}} \qquad . \tag{71}$$

The expressions of the barred values are found from the relations of the unbarred values. These barred expressions

are free of singularities. The relations for $\overline{F}_{nmp}$, $\overline{G}_j^{n-1}$ and $\overline{N}_k^m$ may be found in Appendices B, C and D. respectively.

6.0 CONCLUSIONS

Orbit perturbations due to the time dependent harmonics
of the gravitational field of the earth have been studied for
the case of near-earth satellites. Since for these orbits, the
atmospheric drag perturbations never permits the satellite to
remain long in a resonant frequency, the resonant perturbations
have been neglected. Because the short period perturbations
due to the tesseral harmonics are of second order, these terms
have also been neglected.

The solution for the satellite motion which includes the
intermediate period perturbations of between first and second
order have been found by using the Von Zeipel solution method
with a general, recursive expansion of the geopotential in the
non-singular PSϕ elements. Also, the along track secular
perturbations induced by the periodic perturbations in the
mean motion are eliminated by the computation of the average
energy using the same theory.

This theory has been implemented in an operational com-
puter program ASOP (reference 12) and numerical experiments
verify the expected accuracy.

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APPENDIX A - NOTATION

APPENDIX A

NOTATION

The definitions of the DSϕ elements are as follows:

The angular variables

$$
\begin{aligned}
\alpha_1 &= \phi && \text{true anomaly} \\
\alpha_2 &= g && \text{argument of pericenter} \\
\alpha_3 &= h && \text{ascending node} \\
\alpha_4 &= \ell && \text{time element}
\end{aligned}
\tag{A1}
$$

The action variables

$$
\begin{aligned}
\beta_1 &= \Phi && \text{related to two-body energy} \\
\beta_2 &= G && \text{angular momentum magnitude} \\
\beta_3 &= H && \text{Z component of angular momentum} \\
\beta_4 &= L && \text{the total energy}
\end{aligned}
\tag{A2}
$$

These may be transformed to the canonical PSϕ elements:

$$
\begin{aligned}
\sigma_1 &= \phi + g + h & \rho_1 &= \phi \\
\sigma_2 &= -\sqrt{2(\phi-G)}\,\sin\,(g+h) & \rho_2 &= \sqrt{2(\phi-G)}\,\cos\,(g+h) \\
\sigma_3 &= -\sqrt{2(G-H)}\,\sin\,h & \rho_3 &= \sqrt{2(G-H)}\,\cos\,h \\
\sigma_4 &= \ell & \rho_4 &= L
\end{aligned}
\tag{A3}
$$

Abbreviations used in the text are

$$
p = \frac{1}{\mu}\left(G-\phi + \frac{\mu}{\sqrt{2L}}\right)^2 \qquad \text{(semi-latus rectum)}
$$

$$
q = G - \frac{1}{2}\left(\phi - \frac{\mu}{\sqrt{2L}}\right)
$$

$$e = \sqrt{1 - \frac{2L}{\mu} p} \qquad \text{(eccentricity)}$$

$$s = \sin I = \sqrt{1 - \frac{H^2}{G^2}} \qquad \text{(I is the inclination)}$$

$$c = \cos I = \frac{H}{G}$$

$$Q = \frac{\sqrt{\rho_4}}{\mu} \left[\frac{2\mu}{\sqrt{2\rho_4}} + G - \phi \right]^{\frac{1}{2}}$$

$$P = \frac{\sqrt{2(G+H)}}{2G}$$

APPENDIX B - INCLINATION FUNCTION

APPENDIX B

INCLINATION FUNCTION

Recursive relations for the Inclination Function F_{nmp} have been derived by Giacaglia (reference 8). The first recurrence relation given in that reference is simple and valid for all n, m, p except when $n = m$:

$$F_{nmp} = (-1)^{n-m+1} \frac{(2n-1)s}{2(n-m)} \left[F_{n-1,m,p} - F_{n-1,m,p-1} \right] -$$

$$- \left(\frac{n-1+m}{n-m} \right) F_{n-2,m,p} \quad . \tag{B1}$$

The other relations derived in that reference were not suitable because they had singularities for zero inclinations. A new recurrence relation derived through induction by this author completes the definition of the inclination function:

$$F_{nnp} = \frac{2n-1}{2} \left\{ s^2 (1+c)^{-1} F_{n-1,n-1,p-1} + \right.$$

$$\left. + (1+c) F_{n-1,n-1,p} \right\} \quad . \tag{B2}$$

In all the recurrence relations

$$F_{nmp} = 0 \qquad \text{for } p < 0 \text{ and } p > n$$
$$m < 0 \qquad m > n \quad . \tag{B3}$$

The definition of the barred inclination function is

$$\bar{F}_{nmp} = \frac{F_{nmp}}{s^{|\alpha|}} \qquad \alpha = m - n + 2p \qquad . \qquad (B4)$$

From equation (B1) one finds the relation for $\bar{F}_{nmp}$ is

$$\bar{F}_{nmp} = (-1)^{n-m+1} \frac{2n-1}{2(n-m)} \left[a\, \bar{F}_{n-1,m,p} - b\, \bar{F}_{n-1,m,p-1} \right] -$$

$$(B5)$$

$$- \frac{(n-1+m)}{(n-m)} \bar{F}_{n-2,m,p-1} \qquad .$$

The values of a and b are found from the conditions

	n >m+2p	n<m+2p	n=m+2p
a =	1	s^2	s^2
b =	s^2	1	s^2

From equation (B2) the definition of $\bar{F}_{nnp}$ becomes

$$\bar{F}_{nnp} = \frac{2n-1}{2} (1+c)^{-1} \left\{ \bar{F}_{n-1,n-1,p-1} + \right.$$

$$(B6)$$

$$\left. + (1+c)^2\, \bar{F}_{n-1,n-1,p} \right\} \qquad .$$

As before the following applies

$$\bar{F}_{nmp} = 0 \qquad \text{for} \quad p < 0 \quad \text{and} \quad p > n \qquad (B7)$$
$$m > 0 \quad \text{and} \quad m > n \quad .$$

The starting value of the recurrence relation is

$$\bar{F}_{000} = 1 \qquad . \qquad\qquad\qquad (B8)$$

APPENDIX C - ECCENTRICITY FUNCTION

APPENDIX C

ECCENTRICITY FUNCTION

The eccentricity function G_j^n is defined by the relation

$$\left(\frac{p}{r}\right)^n \begin{bmatrix} sin \\ cos \end{bmatrix} i\phi = \sum_{j=-n}^{n} G_j^n \begin{bmatrix} sin \\ cos \end{bmatrix} (i + j)\phi \qquad (C1)$$

where $\dfrac{p}{r} = 1 + e \, cos \, \phi$.

The binomial expansion of $\left(\dfrac{p}{r}\right)^n$ becomes

$$\left(\frac{p}{r}\right)^n = \sum_{s=0}^{n} \binom{n}{s} \left(\frac{e}{2}\right)^s \sum_{t=0}^{s} \binom{s}{t} cos \, (s-2t) \qquad . \qquad (C2)$$

Using equation (C2) and the definition of equation (C1) the expression for G_j^n becomes

$$G_j^n = e^{|j|} \sum_{s=|j|}^{n} \binom{n}{s} e^{s-j} \binom{s}{\frac{s-j}{2}} \left(\frac{1}{2}\right)^s \qquad \text{for} \quad j > 0$$

$$ \qquad (C3)$$

$$G_{-j}^n = G_j^n \qquad .$$

Recurrence relations exist for the binomial coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad . \tag{C4}$$

The expression for $\bar{G}^n_j$ becomes

$$\bar{G}^n_j = \frac{G^n_j}{e^{|j|}} = \sum_{k=|j|}^{n} \binom{n}{k} e^{k-j} \binom{k}{\frac{k-j}{2}} \left(\frac{1}{2}\right)^k$$

$$\tag{C5}$$

$$\bar{G}^n_{-j} = \bar{G}^n_j \quad .$$

APPENDIX D - THE TIME EXPANSION

APPENDIX D

THE TIME EXPANSION

It is desired to make the following expansion

$$
\begin{bmatrix} cos \\ sin \end{bmatrix} m\nu \ (\phi - M) = \sum_k N_k^m \begin{bmatrix} cos \\ sin \end{bmatrix} k\phi \qquad . \tag{D1}
$$

To develop this expansion, a complex notation is adopted which has the same meaning

$$
\exp \left[im\nu \ (\phi - M) \right] = \sum_k N_k^m \exp \ (ik\phi) \quad , \tag{D2}
$$

where N_k^m is defined by

$$
N_k^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left[im\nu \ (\phi - M) - ik\phi \right] d\phi \quad . \tag{D3}
$$

As in the development of Hansen coefficients (reference 10) some abbreviations are useful

$$
\frac{\sqrt{1+e}}{\sqrt{1-e}} = \frac{1+\beta}{1-\beta} \qquad \text{or} \qquad \beta = \frac{e}{1 + \sqrt{1-e^2}} \quad , \tag{D4}
$$

$$
z = \exp iE \quad . \tag{D5}
$$

The relation between the true and eccentric anomaly

$$\tan \frac{1}{2} \phi = \frac{\sqrt{1+e}}{\sqrt{1-e}} \tan \frac{1}{2} E \qquad (D6)$$

can be written in another form by equations (D4) and D5) as

$$\exp i\phi = \frac{z-\beta}{1-\beta z} = z \left(1-\frac{\beta}{z}\right)(1-\beta z)^{-1} \qquad . \qquad (D7)$$

Similarly the relation between mean and eccentric anomalies

$$M = E - e \sin E \qquad (D8)$$

may be written as

$$\exp iM = z \cdot \exp\left[\frac{e}{2}\left(z - \frac{1}{z}\right)\right] \qquad . \qquad (D9)$$

By differentiating equation (D7), the expression for $d\phi$ becomes

$$d\phi = \frac{(1-\beta^2)}{i} (1-\beta z)^{-1} z^{-1} \left(1-\frac{\beta}{z}\right)^{-1} dz \qquad . \qquad (D10)$$

Thus the integrand in equation (D3) becomes

$$\exp i\left[(m\nu-k)\phi - m\nu M\right] = \frac{1-\beta^2}{i}(1-\beta z)^{k-m\nu-1}\left(1-\frac{\beta}{z}\right)^{-(k+1-m\nu)} z^{-1-k} \cdot$$

(D11)

$$\cdot \exp\left[\frac{m\nu e}{2}\left(z-\frac{1}{z}\right)\right] dz \qquad .$$

The powers of $(1-\beta z)$ and $1-\frac{\beta}{z}$ can be expanded with small parameter β

$$f_s = (1-\beta z)^{k-m\nu-1}$$

$$\frac{df_s}{dz} = -\beta(1-\beta z)^{k-m\nu-2}(k-m\nu-1)$$

$$\frac{d^2 f_s}{dz^2} = (-\beta)^2(1-\beta z)^{k-m\nu-3}(k-m\nu-1)(k+m\nu-2)$$

(D12)

$$\frac{d^5 f_s}{dz^5} = (-\beta)^s(1-\beta z)^{k-(m\nu+s+1)}\prod_{j=0}^{s-1}\left[k-(m\nu+j+1)\right] \qquad .$$

$$\frac{d^s f_s(z=0)}{dz^s} = -(\beta)^s\prod_{j=0}^{s-1}\left[k-(m\nu+j+1)\right]$$

By Taylor's expansion

$$f_s = 1 + \sum_{s=1}^{\infty} \frac{d^s f(z=0)}{dz^s} \frac{1}{s!} z^s$$

or

$$f_s = \sum_{s=0}^{\infty} a_s \beta^s z^s \qquad\qquad (D13)$$

where

$$a_o = 1 \qquad a_s = \frac{(-1)^s}{s!} \prod_{j=0}^{s-1} \left[k - (m\nu + j + 1) \right]$$

or

$$a_s = - \left(\frac{k - m\nu - s}{s} \right) a_{s-1} \quad .$$

Similarly

$$f_t = (1 - \beta w)^{-(k+1-m\nu)} \qquad \text{where} \quad w = \frac{1}{z}$$

$$\frac{df_t}{dw} = \beta (1 - \beta w)^{-(k-m\nu+2)} (k - m\nu + 1)$$

$$\frac{d^2 f_t}{dw^2} = \beta^2 (1 - \beta w)^{-(k-m\nu+3)} (k - m\nu + 1)(k - m\nu + 2) \qquad (D14)$$

$$\frac{d^t f_t}{dw^t} = \beta^t (1-\beta w)^{-(k-m\nu+1+t)} \prod_{j=0}^{t-1} (k-m\nu+1+j)$$

$$\frac{d^t f_t (w=0)}{dw^t} = \beta^t \prod_{j=0}^{t-1} (k-m\nu+1+j) \quad .$$

Therefore:

$$f_t = \sum_{t=0}^{\infty} b_t \beta^t w^t \quad \text{or} \quad \sum_{t=0}^{\infty} b_t \beta^t z^{-t} \tag{D15}$$

where

$$b_0 = 1 \qquad b_t = \frac{1}{t} \prod_{j=0}^{t-1} (k-m\nu+1+j)$$

or

$$b_t = \left(\frac{k-m\nu+t}{t}\right) b_{t-1} \quad .$$

With equations (D13) and (D15) the integral becomes

$$N_k^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp i \left[(m\nu-k)\phi - m\nu M\right] d\phi$$

$$\tag{D16}$$

$$N_k^m = (1-\beta^2) \sum_{s=0} \sum_{t=0} a_s b_t \beta^{s+t} \frac{1}{2\pi i} \oint z^{-1-(k-s+t)} \exp\left[\frac{m\nu e}{2}\left(z-\frac{1}{z}\right)\right] dz \quad .$$

But the Bessel function is defined as

$$J_n(\alpha) = \frac{1}{2\pi i} \oint z^{-1-n} \exp\left[\frac{\alpha}{2}\left(z - \frac{1}{z}\right)\right] dz \qquad , \qquad (D17)$$

so that the final form becomes

$$N_k^m = (1-\beta^2) \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} a_s b_t \beta^{s+t} J_{k-s+t}(m\nu e) \qquad , \qquad (D18)$$

where

$$a_0 = 1 \qquad a_s = -\frac{(k-m\nu-s)}{s} a_{s-1} \qquad (D19)$$

$$b_0 = 1 \qquad b_t = \frac{(k-m\nu+t)}{t} b_{t-1} \qquad . \qquad (D20)$$

The barred function $\bar{N}_k^m$ can be found from its definition and equation (D18)

$$\bar{N}_k^m = \frac{N_k^m}{e^{|k|}} \qquad . \qquad (D21)$$

If one notes the definition of β in equation (D4), one finds

$$\bar{N}_k^m = (1-\beta^2) \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{a_s b_t}{(1+\sqrt{1-e^2})^{s+t}} e^{2\sigma(m\nu)} |k-s+t| \hat{J}_{k-s+t}(m\nu e) \qquad (D22)$$

The value of σ is determined by the following conditions

	$k \gtrless 0$	$k < 0$
$k-s+t \gtrless 0$	$\sigma = t$	$\sigma = t + k$
$k-s+t < 0$	$\sigma = s - k$	$\sigma = s$

(D23)

Note that σ can never be negative.

The function $\hat{J}_n(\alpha)$ is defined from the Bessel function as

$$\hat{J}_n(\alpha) = \frac{J_n(\alpha)}{\alpha^{|n|}} \qquad .$$

(D24)

It can be found from following recursive relation

$$\hat{J}_{n-1} = 2n\,\hat{J}_n - \alpha^2\,\hat{J}_{n+1} \qquad .$$

(D25)

with starting values

$$J_{n+1} = \left(\frac{1}{2}\right)^{n+1}\frac{1}{(n+1)!}$$

$$J_n = \left(\frac{1}{2}\right)^n \left\{\frac{1}{n!} - \frac{\frac{\alpha^2}{4}}{(n+1)!}\right\}$$

(D26)

The expressions for N_k^m were determined in another manner by Bond (reference 11) up to order e^3 . These expressions are given here in the barred form.

$$\bar{N}_o^m = 1 - (m\nu e)^2$$

$$\bar{N}_{-1}^m = -m\nu \left[1 - m\nu e^2 \left(\frac{3}{8} + \frac{1}{2} m\nu\right)\right]$$

$$\bar{N}_1^m = m\nu \left[1 + m\nu e^2 \left(\frac{3}{8} - \frac{1}{2} m\nu\right)\right]$$

$$\bar{N}_{-2}^m = \frac{m\nu}{2} \left(\frac{3}{4} + m\nu\right) \tag{D27}$$

$$\bar{N}_2^m = -\frac{m\nu}{2} \left(\frac{3}{4} - m\nu\right)$$

$$\bar{N}_{-3}^m = \frac{-m\nu}{2} \left(\frac{1}{3} + \frac{3}{4} m\nu + \frac{1}{3} (m\nu)^2\right)$$

$$\bar{N}_3^m = \frac{m\nu}{2} \left(\frac{1}{3} - \frac{3}{4} m\nu + \frac{1}{3} (m\nu)^2\right)$$