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LONG PERIOD PERTURBATIONS OF EARTH SATELLITE ORBITS

ANALYTICAL AND
COMPUTATIONAL
MATHEMATICS, INC.

LONG PERIOD PERTURBATIONS
OF
EARTH SATELLITE ORBITS

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1.0 INTRODUCTION

In reference 1, Scheifele and Graf introduced a complete first order solution for the orbital motion of a satellite perturbed by earth oblateness. This solution was expressed in the DSϕ elements. In reference 2, Bond and Scheifele expressed the first order short period and secular J_2 solution in the non-singular PSϕ elements. This theory was implemented in an operational computer program named ASOP described in reference 3. In references 4 and 5, the PSϕ analytical theory was updated to include the drag effects. In reference 6, the theory was developed to account for the time dependent gravitational harmonics. The drag and time dependent geopotential terms have also been included in ASOP .

Bond also extended the PSϕ theory to include the first order long period terms and second order secular perturbations due to J_2 , J_3 , J_4 and J_5 . However, no documentation of the equations was ever published. In reference 7, Mueller developed a recursive theory to include the first order long period terms and second order secular perturbations due to zonal harmonics of any order. Mueller's theory plus the second order J_2 theory developed by Bond have now been implemented in ASOP .

The purpose of this report is to document all the equations involved in extending the PSϕ solution to include the long periodic and second order secular effects of the zonal harmonics.

2.0 METHOD OF SOLUTION

2.1 Notation

The DSϕ elements are a set of eight variables which have the following description:

Angle Elements:

$\alpha_1 = \phi$ true anomaly

$\alpha_2 = g$ argument of pericenter

$\alpha_3 = h$ longitude of ascending node

$\alpha_4 = \ell$ time element

Action Elements:

$\beta_1 = \Phi$ related to two body energy

$\beta_2 = G$ total angular momentum

$\beta_3 = H$ z-component of the angular momentum

$\beta_4 = L$ total energy

These may be canonically transformed to the PSϕ elements by the following relations:

$$
\begin{aligned}
\sigma_1 &= \phi + g + h \\
\sigma_2 &= -\sqrt{2(\Phi - G)}\ \sin\,(g + h) \\
\sigma_3 &= -\sqrt{2(G - H)}\ \sin\,(h) \\
\sigma_4 &= \ell \\
\rho_1 &= \Phi \\
\rho_2 &= \sqrt{2(\Phi - G)}\ \cos\,(g + h) \\
\rho_3 &= \sqrt{2(G - H)}\ \cos\,(h) \\
\rho_4 &= L
\end{aligned}
\qquad (1)
$$

The DSϕ Hamiltonian for the zonal oblateness problem is given by:

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 \tag{2}$$

where

$$F_o = \phi - \frac{\mu}{\sqrt{2L}} \qquad \text{(two body contributions)}$$

$$F_1 = \frac{1}{qr}\left[\left(\frac{x_3}{r}\right)^2 - \frac{1}{3}\right] \quad (J_2 \text{ contribution})$$

$$F_2 = \frac{1}{q}\sum_{n=3}^{N} \hat{J}_n \frac{1}{r^{n-1}} P_n\left(\frac{x_3}{r}\right) \quad \text{(higher zonal harmonics)}$$

$$\varepsilon = \frac{3}{2} J_2 \mu R_e^2$$

P_n are the Legendre polynomials, R_e is the mean equatorial radius of earth, and J_2 and $\hat{J}_n$ are oblateness coefficients.

2.2 Solution Algorithm

Von Zeipel's method of elimination of the short and long periodic terms is used. The solution first requires the transformation to eliminate the short periodic terms due to J_2. The generating function is assumed to be of the form

$$S = S_o + \varepsilon S_1$$

S_o give the identity transformation, S_1 is so chosen that the new Hamiltonian is no longer a function of short period variable ϕ. The Hamiltonian has the form

$$F'(\beta', g') = F_0' + \varepsilon F_1' + \varepsilon^2 F_2'$$

A more thorough discussion of the elimination of short periodic terms can be found in reference 2.

An additional transformation must be made to eliminate the long periodic terms from F'. This transformation is defined by the generating function

$$S^* = S_0^* + \varepsilon S_1^*$$

Again, S_o^* gives the identify transformation, S_1^* is chosen such that the long period variable g' is eliminated from the Hamiltonian. The new Hamiltonian has the form

$$F''(\beta'') = F_0'' + \varepsilon F_1'' + \varepsilon^2 F_2''$$

A more thorough discussion of the elimination of the J_2 and higher order zonal perturbation long periodic terms can be found in references 1 and 7.

The solution algorithm can be divided into three steps:

(1) Initialize the primed variables

$$\begin{aligned} \sigma'_{k,0} &= \sigma_{k,0} + \varepsilon\left(\frac{\partial S_1}{\partial \rho_{k,0}} + \frac{\partial S_1^*}{\partial \rho_{k,0}}\right) \\ \rho'_{k,0} &= \rho_{k,0} - \varepsilon\left(\frac{\partial S_1}{\partial \sigma_{k,0}} + \frac{\partial S_1^*}{\partial \sigma_{k,0}}\right) \qquad k = 1,2,3,4 \end{aligned} \tag{3}$$

(2) Analytical integration of primed variables

$$\begin{aligned} \sigma_1' &= \sigma_{1,0}' + A_1\tau \\ \sigma_2' &= \sigma_{2,0}' \cos(A_2\tau) - \rho'_{2,0} \sin(A_2\tau) \\ \sigma_3' &= \sigma_{3,0}' \cos(A_3\tau) - \rho'_{3,0} \sin(A_3\tau) \\ \sigma_4' &= \sigma_{4,0}' + A_4\tau \\ \rho_1' &= \rho_{1,0}' \\ \rho_2' &= \rho_{2,0}' \cos(A_2\tau) + \sigma_{2,0}' \sin(A_2\tau) \\ \rho_3' &= \rho_{3,0}' \cos(A_3\tau) + \sigma_{3,0}' \sin(A_3\tau) \\ \rho_4' &= \rho_{4,0}' \end{aligned} \tag{4}$$

The definitions of A_1, A_2, A_3, A_4 are given in section 3.0 of this report, The relation between time t and the new independent variable τ is given by $\frac{dt}{dt} = r^2/q$, the definition of q is also given in section 3.0

(3) Back transformation

$$
\begin{aligned}
\sigma_k &= \sigma_k' - \varepsilon\left(\frac{\partial S_1}{\partial \rho_k} + \frac{\partial S_1^*}{\partial \rho_k}\right) \\
\rho_k &= \rho_k' + \varepsilon\left(\frac{\partial S_1}{\partial \sigma_k} + \frac{\partial S_1^*}{\partial \sigma_k}\right) \qquad k = 1,2,3,4
\end{aligned}
\tag{5}
$$

3.0 EQUATIONS FOR ELIMINATION OF LONG PERIODIC TERMS AND ANALYTICAL INTEGRATION OF PRIMED VARIABLES

A detailed description of generating function S_1 and derivatives of S_1 with respect to the PSϕ elements can be found in Appendix F of Reference 3. In this section a detailed description of generating function S_1^* and derivatives of S_1^* with respect to the PSϕ elements will be given. The derivatives of F_2'' with respect to the DSϕ elements will also be given.

3.1 Generating Function S_1^*

From Reference 1 we have:

$$S_1^* = \frac{1}{\left(\frac{\partial F_1}{\partial G}\right)_q} \left[\hat{S} - \frac{1}{2}\,\frac{f^2}{48}\,(2 - 3b + 6qB)e^2 b\,\sin(2g)\right] \tag{6}$$

$\hat{S}$ are terms related to higher order zonal perturbations. A detailed description of $\hat{S}$ can be found in reference 7.

Now we introduce $\sin(g)$ and $\cos(g)$

$$\sin(g) = \frac{1}{CD}\,(\sigma_6\sigma_3 - \sigma_7\sigma_2)$$

$$\cos(g) = \frac{1}{CD}\,(\sigma_6\sigma_7 + \sigma_2\sigma_3) \tag{7}$$

where

$$C = \sqrt{2(\Phi - G)}$$

$$D = \sqrt{2(G - H)}$$

To write S_1^* in terms of PSϕ elements we introduce the following abbreviations

$$Q = \left\{\frac{\sigma_8}{\mu^2}\left[\frac{2\mu}{\sqrt{2\sigma_8}} - \frac{1}{2}\,(\sigma_2^2 + \sigma_6^2)\right]\right\}^{\frac{1}{2}} \tag{8}$$

$$p = \frac{1}{\mu}\left[-\frac{1}{2}(\sigma_2^2 + \sigma_6^2) + \frac{\mu}{\sqrt{2\sigma_8}}\right]^2 \tag{9}$$

$$e = (1 - \frac{2\sigma_8}{\mu}\,p) = QD \tag{10}$$

$$b = 1 - \frac{G^2}{H^2} \tag{11}$$

$$\chi = eb^{\frac{1}{2}} sin(g) = \frac{1}{2}\frac{Q}{G}\sqrt{2(G + H)}\,(\sigma_6\sigma_3 - \sigma_7\sigma_2) \tag{12}$$

$$\psi = eb^{\frac{1}{2}} cos(g) = \frac{1}{2}\frac{Q}{G}\sqrt{2(G + H)}\,(\sigma_6\sigma_7 + \sigma_2\sigma_3) \tag{13}$$

$$\Theta = e^2 b sin(2g) = 2\chi\psi \quad . \tag{14}$$

Now we have

$$S_1^* = \frac{1}{\left(\frac{\partial F_1}{\partial G}\right) q}\left[\hat{S} - \frac{1}{2}\frac{f^2}{48}(2 - 3b + 6qB)\Theta\right] \tag{15}$$

where

$$q = -\frac{1}{2}(\sigma_6^2 + \sigma_2^2 - \sigma_5) + \frac{\mu}{\sqrt{2\sigma_8}} \tag{16}$$

$$f = \frac{1}{pq} \tag{17}$$

$$B = \frac{2H^2}{G^3} \tag{18}$$

$$\frac{\partial F_1}{\partial G} = \frac{1}{2} f\left[f\left(\frac{2}{\mu} qd + p\right)\left(\frac{2}{3} - b\right) + B\right] \tag{19}$$

and

$$d = (p\mu)^{\frac{1}{2}} \quad . \tag{20}$$

Let

$$\begin{aligned} T_a &= -\frac{1}{2}\frac{f^2}{48} \\ T_b &= (2 - 3b + 6qB)T_a \\ T_c &= \Theta T_b \\ T &= \hat{S} + T_c \end{aligned} \tag{21}$$

then

$$S_1^* = \frac{T}{\left(\frac{\partial F_1}{\partial G}\right) q} \tag{22}$$

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3.2 Derivatives of S_1^*

Let

$$S_{1k}^* = \frac{\partial S_1^*}{\partial \sigma_k}\ , \qquad k = 1,2,3.\dots,8$$

From now on the subscript k represents partial derivatives with respect to the 8 PSϕ elements, unless otherwise specified.

- $$S_{1k}^* = -\frac{1}{\left(\frac{\partial F_1}{\partial G}\right) q}\left\{ T\left[\frac{\left(\frac{\partial F_1}{\partial G}\right)_k}{\left(\frac{\partial F_1}{\partial G}\right)} + \frac{q_k}{q}\right] - T_k \right\} \tag{23}$$

- $$\left(\frac{\partial F_1}{\partial G}\right)_k = \frac{f_k}{2}\left[f\left(\frac{2qd}{\mu} + p\right)\left(\frac{2}{3} - b\right) + B\right] + \frac{f}{2}\left[B_k - f\left(\frac{2qd}{\mu} + p\right) b_k + f\left(\frac{2}{3} - b\right)p_k + f\left(\frac{2}{3} - b\right)\frac{2}{\mu}\left(dq_k + qd_f\right)\right] \tag{24}$$

- $$T_k = \left(\frac{\partial \hat{S}}{\partial p} - \frac{2}{p} T_c\right) p_k + \left(\frac{\partial \hat{S}}{\partial b} - 3T_a \Theta\right) b_k + \frac{\partial \hat{S}}{\partial e^2} e_k^2 + \left(\frac{\partial \hat{S}}{\partial \psi} + 2T_b \chi\right)\psi_k + \left(\frac{\partial \hat{S}}{\partial \chi} + 2T_b \psi\right)\chi_k - \left(\frac{2}{q} T_c - 6BT_a \Theta\right) q_k + 6T_a q\Theta B_k \tag{25}$$

- $p_k = 0 \qquad \text{for } k = 1,3,4,5,7$

$$p_2 = -2\,\frac{\sqrt{\mu p}}{\mu}\,\sigma_2$$

$$p_6 = -2\,\frac{\sqrt{\mu p}}{\mu}\,\sigma_6 \tag{26}$$

$$p_8 = -2\,\frac{\sqrt{\mu p}}{\left(2\sigma_8\right)^{3/2}}$$

- $q_k = 0 \qquad \text{for } k = 1,3,4,6,7$

$$q_2 = -\sigma_2$$

$$q_5 = \frac{1}{2} \tag{27}$$

$$q_8 = -\frac{\mu}{2}\,\frac{1}{\left(2\sigma_8\right)^{3/2}}$$

- $G_k = 0 \qquad \text{for } k = 1,3,4,7,8$ (28)

  $G_2 = -\sigma_2$

  $G_5 = 1$

  $G_6 = -\sigma_6$

- $H_k = G_k \qquad \text{for } k = 1,2,4,5,6,8$ (29)

  $H_3 = -\sigma_3$

  $H_7 = -\sigma_7$

- $f_k = -f\left(\frac{p_k}{p} + \frac{q_k}{q}\right)$ (30)

- $b_k = -\frac{2H}{G^2}\left(H_k - \frac{H}{G} G_k\right)$ (31)

- $d_k = \frac{1}{2}\left(\frac{\mu}{p}\right)^{\frac{1}{2}} p_k$ (32)

- $B_k = \frac{H}{G^3}\left(4H_k \quad \frac{H}{G} G_k\right)$ (33)

- $e_k^2 = -2\sigma_8 \mu p_k \qquad \text{for } k = 1,2,3,4,5,6,7$ (34)

  $e_8^2 = -2\sigma_8 \mu p - 2\mu p$

- $\chi_k = (\sigma_6\sigma_3 - \sigma_7\sigma_2)\frac{\sqrt{2(G+H)}}{2G}\left[Q_k - \frac{Q}{G} G_k + \frac{Q}{2(G+H)}(G_k + H_k)\right]$

  $k = \ , 4, 5, 8$ (35)

  $\chi_2 = (\sigma_6\sigma_3 - \sigma_7\sigma_2)\frac{\sqrt{2(G+H)}}{2G}\left[Q_2 - \frac{Q}{G} G_2 + \frac{Q}{2(G+H)}(G_2 + H_2)\right]$

  $- \sigma_7 \frac{\sqrt{2(G+H)}}{2G}$

  $\chi_3 = (\sigma_6\sigma_3 - \sigma_7\sigma_2)\frac{\sqrt{2(G+H)}}{2G}\left[Q_3 - \frac{Q}{G} G_3 + \frac{Q}{2(G+H)}(G_3 + H_3)\right]$

  $+ \sigma_6 \frac{Q\sqrt{2(G+H)}}{2G}$

$$\chi_6 = (\sigma_6\sigma_3 - \sigma_7\sigma_2)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_6 - \frac{Q}{G}G_6 + \frac{Q}{2(G+H)}(G_6 + H_6)\right] + \frac{\sigma_3 Q\sqrt{2(G+H)}}{2G}$$

$$\chi_7 = (\sigma_6\sigma_3 - \sigma_7\sigma_2)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_7 - \frac{Q}{G}G_7 + \frac{Q}{2(G+H)}(G_7 + H_7)\right] - \frac{\sigma_2 Q\sqrt{2(G+H)}}{2G}$$

$$\psi_k = (\sigma_6\sigma_7 + \sigma_2\sigma_3)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_k - \frac{Q}{G}G_k + \frac{Q}{2(G+H)}(G_k + G_k)\right]$$

$$k = 1,4,5,8 \tag{36}$$

$$\psi_2 = (\sigma_6\sigma_7 + \sigma_2\sigma_3)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_2 - \frac{Q}{G}G_2 + \frac{Q}{2(G+H)}(G_2 + H_2)\right] + \frac{\sigma_3 Q\sqrt{2(G+H)}}{2G}$$

$$\psi_3 = (\sigma_6\sigma_7 + \sigma_2\sigma_3)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_3 - \frac{Q}{G}G_3 + \frac{Q}{2(G+H)}(G_3 + H_3)\right] + \frac{\sigma_2 Q\sqrt{2(G+H)}}{2G}$$

$$\psi_6 = (\sigma_6\sigma_7 + \sigma_2\sigma_3)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_6 - \frac{Q}{G}G_6 + \frac{Q}{2(G+H)}(G_6 + H_6)\right] + \frac{\sigma_7 Q\sqrt{2(G+H)}}{2G}$$

$$\psi_7 = (\sigma_6\sigma_7 + \sigma_2\sigma_3)\,\frac{\sqrt{2(G+H)}}{2G}\left[Q_7 - \frac{Q}{G}G_7 + \frac{Q}{2(G+H)}(G_7 + H_7)\right] + \frac{\sigma_6 Q\sqrt{2(G+H)}}{2G}$$

The partial derivatives of $\hat{S}$ with respect to p , b , e^2 , ψ and χ can be found in Reference 3.

3.3 Derivative of F_2'' with Respect to DSϕ Elements

From Reference 1, one can find that

$$F_2'' = \frac{f^2}{288}\,\delta + \hat{H}/\mu \tag{37}$$

$$\delta = \frac{e^2}{q}(-3b^2 + 24b - 8) + 18\,\frac{b^2}{q} - \frac{1}{\mu}\,d\left(\frac{e^2}{p} + \frac{L}{\mu}\right) \tag{38}$$

$$(60b^2 - 96b + 32) - Bb(24e^2 + 36)$$

$\hat{H}$ is the Hamiltonian of higher harmonics, see Reference 7 for detailed description. Because the new Hamiltonian is a function of only action DSϕ elements, from now on the subscript k represents partial derivative with respect to those DSϕ action elements.

$$\delta_k = \frac{1}{q^2}\left[(e_k^2 q - e^2 q_k)(-3b^2 + 24b - 8)\right] + \frac{e^2}{q}\left[(-6b + 24)b_k\right]$$

$$+ \frac{36b}{q^2}\left(b_k q - \frac{bq_k}{2}\right) - \frac{1}{\mu}\left\{\left[\left(d_k\left(\frac{e^2}{p} + \frac{L}{\mu}\right) + d\left(\frac{e_k^2 p - e^2 p_k}{p^2}\right.\right.\right.\right. \tag{39}$$

$$\left.\left.\left. + \frac{L_k}{\mu}\right)\right](60b - 96b + 32) + d\left(\frac{e^2}{p} + \frac{L}{\mu}\right)(120bb_k - 96b_k)\right\}$$

$$- (24e^2 + 36)(B_k b + Bb_k) - 24e_k^2 Bb$$

where

$$B_1 = 0 \tag{40}$$

$$B_2 = -\frac{6H^2}{G^4}$$

$$B_3 = \frac{4H}{G^3}$$

$$B_4 = 0$$

$$
\begin{aligned}
d_1 &= -1. \\
d_2 &= 1. \\
d_3 &= 0. \\
d_4 &= -\mu(2L)^{-3/2}
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
p_1 &= -2\left(\frac{p}{\mu}\right)^{\frac{1}{2}} \\
p_2 &= -p_1 \\
p_3 &= 0. \\
p_4 &= -2(p\mu)^{\frac{1}{2}}(2L)^{-3/2}
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
e_1^2 &= -\frac{2L}{\mu} p_1 \\
e_2^2 &= -e_1^2 \\
e_3^2 &= 0. \\
e_4^2 &= -\frac{2}{\mu}(p + Lp_4)
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
q_1 &= -0.5 \\
q_2 &= 1.0 \\
q_3 &= 0 \\
q_4 &= -0.5\mu(2L)^{-3/2}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= \frac{2}{G}\left(\frac{H}{G}\right)^2 \\
b_3 &= -\frac{2}{G}\left(\frac{H}{G}\right) \\
b_4 &= 0.
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
L_1 &= 0 \\
L_2 &= 0 \\
L_3 &= 0 \\
L_4 &= 1
\end{aligned} \tag{46}
$$

$$F''_{2k} = \frac{f}{288}\,(2f_k\delta + f\delta_k) + \frac{1}{q}\,(\hat{H}_k - \frac{q_k}{q}\,\hat{H}) \tag{47}$$

$$\hat{H}_k = \frac{\partial\hat{H}}{\partial p}\,p_k + \frac{\partial\hat{H}}{\partial e^2}\,e^2_k + \frac{\partial\hat{H}}{\partial b}\,b_k \tag{48}$$

where

$$\begin{aligned} f_1 &= \frac{f^2}{\mu}\,(\tfrac{1}{2}\,\mu p + 2q\,\sqrt{\mu p}) \\ f_2 &= -\frac{f^2}{\mu}(\,\mu p + 2q\,\sqrt{\mu p}) \\ f_3 &= 0 \\ f_4 &= -\frac{f^2}{(2\sigma_8)^{3/2}}\,(\tfrac{1}{2}\,\mu p + 2q\,\sqrt{\mu p}) \end{aligned} \tag{49}$$

Now the abbreviations A_1, A_2, A_3, A_4 in the expressions of analytical integration will be given:

$$\begin{aligned} A_4 = {} & \frac{\varepsilon}{2}\,f_4\,(b - 2/3) + \mu(2L)^{-3/2} + \frac{\varepsilon^2}{288}\,f\,(2f_4\delta + f\delta_4) \\ & + \frac{\varepsilon^2}{q}\,(\hat{H}_4 - \frac{q_4}{q}\,\hat{H}) \end{aligned} \tag{50}$$

$$A_3 = \frac{\varepsilon}{2}\,fb_3 + \frac{\varepsilon^2}{288}\,f^2\delta_3 + \frac{\varepsilon^2}{q}\,(\hat{H}_3 - \frac{q_3}{q}\,\hat{H}) \tag{51}$$

$$\begin{aligned} A_2 = {} & \frac{\varepsilon}{2}\left[f_2(b - 2/3) + fb_2\right] + \frac{\varepsilon^2}{288}\,f\,(2f_2\delta + f\delta_2) \\ & + \frac{\varepsilon^2}{q}\,(\hat{H}_2 - \frac{q_2}{q}\,\hat{H}) + A_3 \end{aligned} \tag{52}$$

$$\begin{aligned} A_1 = {} & 1 + \frac{\varepsilon}{2}\,f_1(b - 2/3) + \frac{\varepsilon^2}{288}f\,(2f_1\delta + f\delta_1) \\ & + \frac{\varepsilon^2}{q}\,(\hat{H}_1 - \frac{q_1}{q}\,\hat{H}) + A_2 \end{aligned} \tag{53}$$

4.0 CONCLUSIONS

The equations described in this report have been implemented into the ASOP program. The program has been checked out and verified with results documented in reference 8. Comparisons with numerical integrations show the long period theory to be accurate to within several meters after 800 revolutions. The extension of ASOP to include the long period terms, allows the solution to maintain a high degree of accuracy even for extremely long prediction intervals.

REFERENCES

1. Scheifele, G. and Graf, O.: "Analytical Satellite Theories Based on a New Set of Canonical Elements". AIAA paper No. 74-838, Aug., 1974.

2. Bond, V.R.: "An Analytical Singularity-free Solution to the J_2 Perturbation Problem", NASA/Johnson Space Center Report JSC-13/28, 1977.

3. Starke, S.E.: "An Analytical Satellite Orbit Predictor (ASOP)", NASA/Johnson Space Center Report JAS-13094, 1977.

4. Scheifele, G., Mueller, A. and Starke, S.: "A Singularity Free Analytical Solution of Artificial Satellite Motion with Drag", ACM Technical Report TR-103, March, 1977.

5. Mueller, A.C.: "An Atmospheric Density Model for Application in Analytical Satellite Theories", ACM Technical Report TR-107, Nov., 1977.

6. Mueller, A.C.: "Perturbations of Non-Resonant Satellite Orbits due to a Rotating Earth", ACM Technical Report TR-112, June, 1978.

7. Mueller, A.C.: "Recursive Analytical Solution Describing Artificial Satellite Motion Perturbed by an Arbitrary Number of Zonal Terms", presented at the 1977 AAS/AIAA Astrodynamics Conference, 1977.

8. Wang, K.C.: "Check Out and Verification of Long Period Perturbations", ACM Memo. 189, Dec., 1978.

APPENDIX

COMPUTATIONAL PROCEDURE

The computational procedure for elimination of long periodic terms and analytical integration of primed variables are described below. First subroutine LONGPP(NN) (long period perturbations) is called with parameter 0 , it will return initialized primed variable. During the procedure subroutine DETERM is called to compute terms related to the higher order harmonics. Then subroutine will be called again with parameter 1 , this time it will return the partial derivatives of primed Hamiltonian with respect to the DSϕ elements. During the procedure subroutine FPRIME is called to compute derivatives of higher order harmonics. The sequence of computation will be given below. The left column gives the quantity to be computed, and the right column references the equation number in the text.

LONGPP(0)

Computating Sequence	From Equation
d	(20)
B	(18)
e	(10)
$\frac{\partial F_1}{\partial G}$	(19)
χ	(12)
ψ	(13)
θ	(14)
$\hat{S}_k$	subroutine DETERM
f_k	(30)
b_k	(31)
d_k	(32)

Computating Sequence (continued)	From Equation (continued)
B_k	(33)
e_k^2	(34)
$\left(\frac{\partial F_1}{\partial G}\right)_k$	(24)
χ_k	(35)
ψ_k	(36)
T_k	(25)
S_{1k}^*	(23)
$\sigma'(0)$, $\rho'(0)$	(3)
LONGPP(1)	
d	(20)
B	(18)
e	(10)
B_k	(40)
d_k	(41)
P_k	(42)
e_k^2	(43)
q_k	(44)
$\hat{H}_k$	subroutine FPRIME
δ_k	(39)
F''_{2k}	(47)
A_1, A_2, A_3, A_4	(50) - (53)