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NONLINEAR EQUATIONS OF EQUILIBRIUM FOR ELASTIC HELICOPTER OR WIND TURBINE BLADES UNDERGOING MODERATE DEFORMATION

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CONTENTS

NOMENCLATURE

Symbol	Definition
A	cross sectional areas of the blade
$B_1 - B_6$	terms defined by Equation (D-26)
$\tilde{B}_1 - \tilde{B}_6$	terms defined by Equation (D-43)
$\bar{d}$	vectorial distance between a point on the cross section of the blade and the shear center of the cross section
E	Young's modulus
E_1, E_2, E_3	Young's moduli of an orthotropic material
$\bar{E}_x, \bar{E}_y, \bar{E}_z$	the base vectors on the elastic axis of the deformed blade
$\hat{e}_x, \hat{e}_y, \hat{e}_z$	unit vectors in the directions of the coordinates x_0, y_0, z_0, respectively, before the deformation
$\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$	the triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$ after deformation
$\hat{e}''_x, \hat{e}''_y, \hat{e}''_z$	the triad $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$ after the virtual motion
e_1	blade pitch bearing offset defined in Figure 2
$\hat{e}_\eta, \hat{e}_\zeta$	unit vectors in the directions η, ζ, respectively, before deformation
$\hat{e}'_\eta, \hat{e}'_\zeta$	the vectors $\hat{e}_\eta, \hat{e}_\zeta$ after the deformation
$\bar{F}$	the resultant force which acts on a cross section of the blade
G	modulus of shear
G_{12}, G_{13}, G_{23}	moduli of shear of an orthotropic material

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$\bar{G}_x, \bar{G}_y, \bar{G}_z$ — the base vectors of the deformed blade in the directions x_0, y_0, z_0, respectively

$\bar{g}_x, \bar{g}_y, \bar{g}_z$ — the base vectors of the undeformed blade in the directions x_0, y_0, z_0, respectively

H — the height of the tower supporting the wind turbine (Figure 1b)

I_{22}, I_{33}, I_{23} — moments of inertia of the cross section (effective in carrying tensile stresses) around axes parallel to the directions $\hat{e}_y, \hat{e}_z$ which pass through the tensile center. Defined by Equation (5)

I_3, I_2 — the principal moments of inertia of the cross section around the principal axes (for a symmetric profile around the symmetry axes, and an axis perpendicular to it, respectively, Eqs. 6)

$\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ — unit vectors in the directions x, y, and z, respectively

J — torsional stiffness of the blade (Figure 2)

ℓ — length of the elastic part of the blade

$\bar{M}$ — the resultant moment which acts on the cross section of the blade

M_x, M_y, M_z — the components of the resultant moment which acts on a cross section of the blade, $\bar{M}$, in the directions $\hat{e}'_x, \hat{e}'_y$, and $\hat{e}'_z$, respectively

$\tilde{M}_x, \tilde{M}_y, \tilde{M}_z$ — the components of the resultant moment which acts on a cross section of the blade, $\bar{M}$, in the directions $\hat{e}_x, \hat{e}_y$, and $\hat{e}_z$, respectively

n_x, n_y, n_z — the components of the virtual rotation; Equation (D-11)

$\bar{p}$ — distributed external force to unit length of the axis of the blade

p_x, p_y, p_z — the components of the distributed external force in the directions $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$, respectively

$\tilde{p}_x, \tilde{p}_y, \tilde{p}_z$	the components of the distributed external force in the directions $\hat{e}_x, \hat{e}_y, \hat{e}_z$, respectively
$\bar{q}$	distributed external moment per unit length along the axis of the blade
q_x, q_y, q_z	the components of the distributed external moment $\bar{q}$, in the directions $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$, respectively
$\tilde{q}_x, \tilde{q}_y, \tilde{q}_z$	the components of the distributed external moment $\bar{q}$, in the directions $\hat{e}_x, \hat{e}_y, \hat{e}_z$, respectively
$R_1, \ldots, R_4$	terms defined by Equation (D-25)
$\tilde{R}_1, \ldots, \tilde{R}_4$	terms defined by Equation (D-42)
$\bar{R}$	the position vector of a point of the blade after the deformation
${}_o\bar{R}$	the position vector of a point on the deformed elastic axis of the blade
$\bar{r}$	the position vector of any point of the blade before the deformation
${}_o\bar{r}$	the position vector of points on the elastic axis of the blade, before deformation
S_{ij}	the elements in the matrix which describes the transformation between the triads $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ and $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$
[T]	the matrix which gives the transformation between $\hat{e}_x, \hat{e}_\eta, \hat{e}_\zeta$ and $\hat{e}'_x, \hat{e}'_\eta, \hat{e}'_\zeta$
T	the component of the resultant force, $\bar{F}$, which acts in the direction $\hat{e}'_x$ (axial tension)
$\tilde{T}$	the component of the resultant force, $\bar{F}$, which acts in the direction $\hat{e}_x$

$\bar{t}$	resultant force per unit area of the cross section of the blade
U	elastic energy
u,v,w	the components of the displacement, $\bar{W}$, of a point on the elastic axis of the blade in the directions $\hat{e}_x, \hat{e}_y$, and $\hat{e}_z$, respectively
V_y, V_z	the components of the resultant force, $\bar{F}$, which act in the directions $\hat{e}_y'$ and $\hat{e}_z'$, respectively
$\tilde{V}_y, \tilde{V}_z$	the components of the resultant force, $\bar{F}$, which act in the directions $\hat{e}_y$ and $\hat{e}_z$, respectively
$\bar{V}$	the displacement of any point of the blade
$\bar{W}$	the displacement of a point on the elastic axis of the blade
W_E	the work of the external forces which act on the system
W_I	the work of the internal forces of the system
x_0, y_0, z_0	the initial system of coordinates of the blade
x,y,z	a rotating system of coordinates (Figure 1)
x_1, y_1, z_1	a system of coordinates fixed with respect to the ground (Figure 1)
X_A	the offset between the shear center and the aerodynamic center of a cross section of the blade; positive when in the positive direction of η
X_I	the offset between the shear center and the center of gravity of a cross section of the blade; positive when in the positive direction of η

X_{II}	the offset between the shear center and the tension center of a cross section of the blade; positive when in the positive direction of η
y_{oc}, z_{oc}	the position of the tension center of a cross section of the blade with respect to the coordinates y_0 and z_0, respectively
$\beta_{,x}$	rate of change of a pretwist (equal to $\theta_{G,x}$ in the present study)
β_p	preconing angle; inclination of the feathering axis with respect to the hub plane (Figure 2)
ε	typical symbolic quantity used in the ordering scheme
$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, $\varepsilon_{yz}, \varepsilon_{xy}, \varepsilon_{xz}$, $\varepsilon_{x\eta}, \varepsilon_{x\zeta}, \varepsilon_{ij}$	strain components
$\tilde{\varepsilon}_{xx}$	strain of the elastic axis
η, ζ	principal coordinates of a cross section of the blade (η is the axis of symmetry in the present study)
θ_G, β	total geometric pitch angle of the blade cross section (angle between $\hat{e}_y$ and $\hat{e}_\eta$)
$\theta_x, \theta_y, \theta_z$	the rotation component of the triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$, during the deformation, about $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$, respectively
$\delta\bar{\theta}$	the virtual rotation of any point on the elastic axis, during a virtual movement
κ_y, κ_z	curvature of the deformed rod in the directions $\hat{e}'_y, \hat{e}'_z$, respectively. Defined by Equation (B-15)

λ	$= -\tilde{\tilde{\varphi}}/\phi_{,x}$
ν	Poisson's ratio
$\nu_{12}, \nu_{21}, \nu_{13}, \nu_{31}$ ν_{23}, ν_{32}	Poisson's ratios of an orthotropic material
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ $\sigma_{yz}, \tau_{xy}, \tau_{xz}$ $\sigma_{x\eta}, \sigma_{x\zeta}$,	stress components
τ	the twist of the deformed blade, defined by Equation (B-15)
$\tilde{\tau}$	the twist while using coordinates η, ζ instead of y_0, z_0
ϕ	the rotation of a cross section of the blade around the elastic axis (equivalent to θ_x)
φ	warping function
$\tilde{\tilde{\varphi}}$	the warping function as chosen by Hodges and Dowell (Ref. 8)
$\tilde{\varphi}$	φ/τ
ψ	azimuth angle of the blade, measured from straight down position (Figure 1)
Ω	angular speed of rotation
$(\ )_{,x}, (\ )_{,y}, (\ )_{,z}$	differentiation by x_0, y_0, and z_0, respectively
$d(\ )$	differential
$\delta(\ )$	variation
$(\bar{\ })$	vector
$(\hat{\ })$	unit vector
$(\bar{\ }) \times (\bar{\ })$	the cross product of two vectors
$(\ )'$	differentiation with respect to x_0

SUMMARY

A set of nonlinear equations of equilibrium for an elastic wind turbine or helicopter blade are presented. These equations are derived for the case of small strains and moderate rotations (slopes). The derivation includes several assumptions which are carefully stated. For the convenience of potential users the equations are developed with respect to two different systems of coordinates, the undeformed and the deformed coordinates of the blade. Furthermore, the loads acting on the blade are given in a general form so as to make them suitable for a variety of applications. The equations obtained in the present study are compared with those obtained in previous studies. Finally, it should be noted that this report represents the first in a series of three reports documenting the research performed under the grant. The second report (UCLA-ENG-7880) deals with the aeroelastic stability and response of an isolated horizontal axis wind turbine blade. The third report (UCLA-ENG-7881) deals with the aeroelastic stability and response of the complete coupled rotor/tower system simulating essentially the dynamics of the NASA/DOE Mod-0 configuration.

1. INTRODUCTION

Recent investigations on the behavior of elastic slender rotor blades undergoing relatively large deformations during operation, show that non-linear phenomena have considerable influence on this behavior. These non-linear phenomena are due to the inclusion of moderately large deformations in the elastic, inertial and aerodynamic operators associated with this problem. A detailed review of recent research on rotary wing aeroelasticity with an emphasis on the importance of moderately large deformations has been presented in Ref. 1 and is beyond the scope of this report.

Recent emphasis on wind energy conversion, using large horizontal axis wind turbines, employing two-bladed, hingeless rotor configurations having rotor diameters varying between 120 to 300 ft. have added a new stimulus to the study of large flexible, highly pretwisted blades. In a recent study by Friedmann (Ref. 2) it has been noted that efficient construction and operation of wind turbines requires that the vibratory loads and stresses on the rotor itself and the combined rotor tower system be reduced to the lowest possible levels. Thus, structural dynamic and aeroelastic considerations are of primary importance for both the design of wind turbines and the comparison of various potential wind turbine configurations.

In view of this need, it was felt that a careful, fundamental derivation of the equations of motion for slender rotor blades, possibly highly pretwisted, undergoing relatively large deformations during their operation, is required. These equations could be used as a basis for future studies, into which nonisotropic material behavior, such as required for the treatment of composite blades, could be easily incorporated. The main objective of the present study is the derivation of such a set of equations.

A fundamental work in this field was that of Houbolt and Brooks (Ref. 3) where equations of equilibrium for the coupled bending and torsion of twisted nonuniform blades were derived. Although some nonlinear effects were included in their derivation, their final results can be considered as a linear representation of the problem. Following this work, other researchers presented derivations of equations which include additional nonlinear terms. These include, for example, the work of Arcidiacono (Ref. 4), Friedmann and Tong (Ref. 5), Hodges, et al (Ref. 6 - 8) and a recent work by Friedmann (Ref. 9). The most detailed and comprehensive derivation of a set of nonlinear elastic equilibrium equations is presented by Hodges and Dowell (Ref. 8). There the equations are obtained by two complementary methods, Hamilton's principle and the Newtonian method.

In the present work a set of nonlinear elastic equilibrium equations of a blade are presented. These equations are derived for the case of small strain and finite rotations. The derivation includes several assumptions which are presented during the presentation. The equations are developed with respect to two different systems of coordinates. In each case the derivation is done using two complementary methods; the Newtonian method and the principle of virtual work. Finally, the equations obtained in the present study are compared with those obtained in the previous studies.

This report is a modified and abbreviated version of Reference 10, which contains a considerable amount of additional details. Furthermore, it should be noted that this report represents the first in a series of three reports which document the research which has been performed under the grant. The second report (Ref. 20) deals with the aeroelastic stability and response problem of an isolated horizontal axis wind turbine blade. This report also contains typical single-blade aeroelastic stability boundaries together with blade response studies at operating conditions for the MOD-0 wind turbine, currently in operation at NASA Lewis Research Center. The third report (Ref. 21) deals with the aeroelastic response and stability of a coupled rotor-tower configuration corresponding to the NASA/DOE Mod-0 machine.

2. BASIC ASSUMPTIONS

The geometry of the problem is shown in Figures 1 through 3. The following assumptions will be used in deriving the equations of motion.

1) The blade is cantilevered at the hub, the feathering axis of the blade is preconed by an angle β_p.

2) The blade can bend in two mutually perpendicular directions normal to the elastic axis of the blade, and can also twist around the elastic axis. The boundary conditions are those of a cantilevered beam.

3) The blade has an arbitrary amount of pretwist which is assumed to be built in about the elastic axis of the blade.

4) The blade cross section is symmetrical about the major principal axis. It has four distinct points:

    I) Elastic Center (E.C.) - the intersection point between the Elastic Axis (E.A.) and the cross section of the blade

    II) Center of Mass (C.G.)

    III) Tension Center (T.C.) - the intersection point between the Tension Axis (T.A.) and the cross section of the blade

    IV) The Aerodynamic Center (A.C.)

As shown in Figure 3 the C.G - E.C offset is denoted by X_I, the T.C - E.C offset is denoted by X_{II}, and the A.C - E.C offset is denoted by X_A, where it is understood that the offsets shown in Figure 3 are considered to be positive.

5) The strains in the blade are always small, but the rotations can be finite (for additional details see Appendix B).

3. ELASTIC EQUILIBRIUM EQUATIONS OF THE BLADE

In this section, the equilibrium equations of the deformed blade are given. It is assumed that the blade can be considered to be a deformable, slender rod, made of linearly isotropic, homogeneous material. As formerly indicated, the analysis is restricted to the case of small strains and finite rotations. Appendix A gives a brief summary of some well known relations of nonlinear deformations. In Appendix B expressions for rotating and strains of a deformed slender rod are derived and the force and moment resultants are obtained. In Appendix C the equilibrium equations are derived systematically with respect to the deformed as well as the undeformed system of coordinates, using the Newtonian method. In Appendix D the same equations are derived using the principle of virtual work.

In this study the Bernoulli-Euler hypothesis is assumed to apply. This hypothesis is usually stated as: "Plane cross sections which are normal to the elastic axis before deformation remain plane after deformation (except for negligible errors due to warping) and normal to the deformed axis." Furthermore, it is also assumed that strains within the cross section can be neglected, and the warping is very small so that its influence is negligible, besides its effect on the torsional stiffness. (For a more accurate approach other warping effects can be included as shown in Appendix B.)

As shown in Figures 2 and 3, before the deformation of the elastic axis of the blade, which is the line that connects the shear centers of the blade cross sections, coincides with the x_0 axis. The y_0 axis is orthogonal to x_0 and lies in a plane parallel to the hub plane, while z_0

is perpendicular to x_0 and y_0. It is clear that x_0, y_0, z_0 is a rectangular Cartesian system. As shown in Figures 2 and 3, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are unit vectors in the directions x_0, y_0, z_0, respectively. According to the Bernoulli-Euler hypothesis and the other accompanying assumptions, during the deformation the triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$ is carried in a rigid form, composed of translation and rotation, to the new orthogonal triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$ (shown in Figures 2 and 3). The unit vector $\hat{e}'_x$ is tangent to the deformed elastic axis, while $\hat{e}_y$ and $\hat{e}_z$ are rotated around it to the position of $\hat{e}'_y$ and $\hat{e}'_z$.

It is assumed that the blade is acted upon by a distributed load, $\bar{p}$, per unit length of its undeformed axis, given in component form by:

$$\bar{p} = p_x \hat{e}'_x + p_y \hat{e}'_y + p_z \hat{e}'_z \quad . \tag{1}$$

This load, $\bar{p}$, includes body forces, surface tractions and inertial loading. There is also a distributed moment, $\bar{q}$, per unit length of the undeformed axis of the rod, given by:

$$\bar{q} = q_x \hat{e}'_x + q_y \hat{e}'_y + q_z \hat{e}'_z \quad . \tag{2}$$

This also includes body couples, moments of surface tractions and moments of inertial loading. The loads and couples, $\bar{p}$ and $\bar{q}$, and their derivatives are assumed to be continuous.

Then the exact equilibrium equations are obtained in Appendix C as Equations (C-7) and (C-8):

$$
\left.
\begin{aligned}
& T_{,x} + \kappa_y(M_{z,x} + \tau M_y + q_z) \\
& \qquad - \kappa_z(M_{y,x} - \tau M_z + q_y) + p_x = 0 \\
& -(M_{z,x} + \kappa_z M_x + \tau M_y + q_z)_{,x} \\
& \qquad + \kappa_y T - \tau(M_{y,x} + \kappa_y M_x - \tau M_z + q_y) + p_y = 0 \\
& (M_{y,x} + \kappa_y M_x - \tau M_z + q_y)_{,x} \\
& \qquad + \kappa_z T - \tau(M_{z,x} + \kappa_z M_x + \tau M_y + q_z) + p_z = 0 \\
& M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x = 0
\end{aligned}
\right\}
\qquad (3)
$$

The first three equations are basically from force equilibrium relations in the $\hat{e}'_x, \hat{e}'_y$, and $\hat{e}'_z$ directions, respectively. The fourth equation is the moment equilibrium relation in the $\hat{e}'_x$ direction. The equilibrium of moments in the directions $\hat{e}'_y$ and $\hat{e}'_z$ are also satisfied.

T is the axial tension in the blade. M_x, M_y, and M_z are the components of the elastic moments, M_x is the torque, while M_y and M_z are the bending moments. κ_y and κ_z are the curvatures, while τ is the twist of the deformed elastic axis. The expressions for the components of the moments are obtained from Equation (B-43) of Appendix B:

$$
\left.
\begin{aligned}
M_x &= GJ\tau \\
M_y &= -EI_{23}\kappa_y - EI_{33}\kappa_z + Tz_{oc} \\
M_z &= EI_{22}\kappa_y + EI_{23}\kappa_z - Ty_{oc}
\end{aligned}
\right\}
\qquad (4)
$$

where E is young's modulus, G is the shear modulus, J is the torsional stiffness of the blade, and I_{22}, I_{33} and I_{23} are the flexural moments of inertia of the cross section (effective in carrying tensile stresses) around axes parallel to the directions $\hat{e}_y$ and $\hat{e}_z$ which pass through the point $(y_{oc}\ z_{oc})$. This point, whose coordinates are (y_{oc}, z_{oc}), is the tensile center. The moments of inertia are given by Equation (B-42), which is:

$$\left.\begin{aligned} I_{22} &= \iint_A (y_{oc} - y_0)^2 \, dy_0 \, dz_0 \\ I_{23} &= \iint_A (y_{oc} - y_0)(z_{oc} - z_0) dy_0 \, dz_0 \\ I_{33} &= \iint_A (z_{oc} - z_0)^2 \, dy_0 \, dz_0 \quad . \end{aligned}\right\} \tag{5}$$

Furthermore, it is assumed that the blade cross section is symmetric about the η axis (see Figure 3). The moments of inertia about η and an axis perpendicular to η, which pass through the tensile center, are denoted by I_3 and I_2, respectively. Then (see Figure 3):

$$\left.\begin{aligned} I_{22} &= I_2 \cos^2 \theta_G + I_3 \sin^2 \theta_G \\ I_{23} &= (I_2 - I_3) \sin \theta_G \cos \theta_G \\ I_{33} &= I_2 \sin^2 \theta_G + I_3 \cos^2 \theta_G \quad . \end{aligned}\right\} \tag{6}$$

From Figure 3, the following relations follow:

$$y_{oc} = X_{II} \cos \theta_G \quad ; \quad z_{oc} = X_{II} \sin \theta_G \quad . \tag{7}$$

It is assumed that X_{II} is small enough so that the expressions Tz_{oc} and Ty_{oc} in Equation (4) are at most of the magnitude of the other terms in the equation.

Furthermore, it should be emphasized that Equations (4) were obtained after neglecting terms of order ε^2 compared to unity (for more details about the ordering scheme, see Appendices B and C). According to the assumptions of Appendix B, rotations are of order ε.

Within the order of approximation implied by neglecting terms of order ε^2, compared to unity, the equations of equilibrium are simplified and are given in their final form by Equations (C-10):

$$\left.\begin{aligned}
&T_{,x} + \kappa_y M_{z,x} - \kappa_z M_{y,x} + \tau(\kappa_y M_y + \kappa_z M_z) \\
&\qquad + \kappa_y q_z - \kappa_z q_y + p_x = 0 \\
&-M_{z,xx} - (\kappa_{z,x} + \tau\kappa_y)M_x - (\tau_{,x} + \kappa_y\kappa_z)M_y - 2\tau M_{y,x} \\
&\qquad + \kappa_y T + \kappa_z q_x - \tau q_y - q_{z,x} + p_y = 0 \\
&M_{y,xx} + (\kappa_{y,x} - \tau\kappa_z)M_x - (\tau_{,x} - \kappa_y\kappa_z)M_z - 2\tau M_{z,x} \\
&\qquad + \kappa_z T - \kappa_y q_x + q_{y,x} - \tau q_z + P_z = 0 \\
&M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x = 0
\end{aligned}\right\} \tag{8}$$

In order to complete the formulation of the problem, the boundary conditions must be also taken into account. For the present case of a cantilevered blade, the boundary condition at $x_0 = 0$ corresponds to a clamped root and at the tip, $x_0 = \ell$ (where ℓ is the length of the blade), free end conditions apply. Thus

$$\begin{aligned} &\text{for } x_0 = 0: \quad v = w = v_{,x} = w_{,x} = \phi = 0 \quad ; \\ &\text{for } x_0 = \ell: \quad T = V_y = V_z = M_x = M_y = M_z = 0 \quad . \end{aligned} \tag{9}$$

V_y and V_z are the resultant shearing forces at the blade cross section and they are given by (see Equation (C-6) of Appendix C):

$$\begin{aligned} V_y &= -(M_{z,x} + \underline{\kappa_z M_x} + \underline{\tau M_y} + q_z) \quad ; \\ V_z &= M_{y,x} + \underline{\kappa_y M_x} - \underline{\tau M_z} + q_y \quad . \end{aligned} \tag{10}$$

The underlined terms in Equation (10) disappear in the case of a free edge when $M_x = M_y = M_z = 0$, also.

The deflection of a point on the elastic axis is given by $\bar{W}$ where (see Equation (B-6) of Appendix B):

$$\bar{W} = u\hat{e}_x + v\hat{e}_y + w\hat{e}_z \quad . \tag{11}$$

The new triad, $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$, which is tangent to the deformed coordinates of the blade is given by Equation (C-11):

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + S_{12}\hat{e}_y + S_{13}\hat{e}_z \\ \hat{e}'_y &= S_{21}\hat{e}_x + \hat{e}_y + S_{23}\hat{e}_z \\ \hat{e}'_z &= S_{31}\hat{e}_x + S_{32}\hat{e}_y + \hat{e}_z \end{aligned}\right\} \qquad (12)$$

where S_{ij} are functions of $w_{,x}$, $v_{,x}$ and ϕ. When finite rotations are considered, these S_{ij} depend on the sequence of rotations which transform $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ to $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$, as one can see from Equations (B-10), (B-13) and (B-14) of Appendix B. If the sequence chosen is a finite rotation about $\hat{e}_z$, followed by rotation about $\hat{e}_y$, followed by rotation about $\hat{e}_x$, then, neglecting terms of order ε^2 compared to unity, the transformation given in Equation (12) is defined by Equation (B-13):

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + v_{,x}\hat{e}_y + w_{,x}\hat{e}_z \\ \hat{e}'_y &= -(v_{,x} + \phi w_{,x})\hat{e}_x + \hat{e}_y + \phi\hat{e}_z \\ \hat{e}'_z &= -(w_{,x} - \phi v_{,x})\hat{e}_x - (\phi + v_{,x}w_{,x})\hat{e}_y + \hat{e}_z \end{aligned}\right\} . \qquad (13)$$

With relations (13) the curvatures and twist are then given by (B-16):

$$\kappa_y = v_{,xx} + \phi w_{,xx} \qquad (14a)$$

$$\kappa_z = w_{,xx} - \phi v_{,xx} \tag{14b}$$

$$\tau = \phi_{,x} + v_{,xx} w_{,x} \quad . \tag{14c}$$

The term $v_{,x} w_{,x}$ in Eqs. (13) and (14) was introduced by Wempner (Ref. 11) and was later shown to be significant for rotor blades by Kaza and Kvaternik (Refs. 15 and 17). Substitution of Equations (14), (6) and (7) into Equation (4) implies:

$$\begin{aligned}
M_x &= GJ(\phi_{,x} + v_{,xx} w_{,x}) \\
M_y &= -E(I_2 - I_3) \sin \Theta_G \cos \Theta_G (v_{,xx} + \phi w_{,xx}) \\
&\quad -E(I_2 \sin^2 \Theta_G + I_3 \cos^2 \Theta_G)(w_{,xx} - \phi v_{,xx}) + T X_{II} \sin \Theta_G \\
M_z &= E(I_2 \cos^2 \Theta_G + I_3 \sin^2 \Theta_G)(v_{,vv} + \phi\, w_{,xx}) \\
&\quad + E(I_2 - I_3) \sin \Theta_G \cos \Theta_G (w_{,xx} - \phi v_{,xx}) - T X_{II} \cos \Theta_G
\end{aligned} \quad . \tag{15}$$

Substitution of Equations (14) and (15) into Equation (8), using Equations (6) and (7) and neglecting terms of order ε^2 compared to unity, implies:

$$\begin{aligned}
& T_{,x} + v_{,xx}[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - \phi v_{,xx})]_{,x} \\
& + w_{,xx}\{\phi[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}\, w_{,xx}]\}_{,x} \\
& + w_{,xx}[EI_{23}(v_{,xx} + \phi w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx})]_{,x} \\
& - v_{,xx}\{\phi[EI_{23}\, v_{,xx} + EI_{33}(w_{,xx} - \phi v_{,xx})]\}_{,x} \\
& + (v_{,xx} + \phi w_{,xx}) q_z - (w_{,xx} - \phi v_{,xx}) q_y + p_x = 0
\end{aligned} \tag{16a}$$

$$
\begin{aligned}
&- [EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - \phi v_{,xx}) - Ty_{oc}]_{,xx} \\
&- GJ(\phi_{,x} w_{,xxx} + v_{,xx} w_{,x} w_{,xxx} - \phi\phi_{,x} v_{,xxx}) + EI_{23}\ \phi_{,xx} v_{,xx} \\
&+ EI_{33}(\phi_{,xx} w_{,xx} + v_{,xxx} w_{,x} w_{,xx} - \phi\phi_{,xx} v_{,xx} + 2 v_{,xx} w^2_{,xx}) \\
&+ 2(\phi_{,x} + v_{,xx} w_{,x})[EI_{23}\ v_{,xx} + EI_{33}(w_{,xx} - \phi v_{,xx}) - Tz_{oc}]_{,x} \\
&+ (v_{,xx} + \phi w_{,xx} - z_{oc}\ \phi_{,xx})T + (w_{,xx} - \phi v_{,xx})q_x \\
&- (\phi_{,x} + v_{,xx} w_{,x})q_y - q_{z,x} + p_y = 0
\end{aligned}
\tag{16b}
$$

$$
\begin{aligned}
&- [EI_{23}(v_{,xx} + \phi w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx}) - Tz_{oc}]_{,xx} \\
&+ GJ(\phi_{,x} v_{,xxx} + \phi\phi_{,x} w_{,xxx} + v_{,xx} v_{,xxx} w_{,x}) - EI_{23}\ \phi_{,xx} w_{,xx} \\
&- EI_{22}(\phi_{,xx} v_{,xx} + \phi\phi_{,xx} w_{,xx} + v_{,xx} v_{,xxx} w_{,x}) \\
&- 2(\phi_{,x} + v_{,xx} w_{,x})[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}\ w_{,xx} - Ty_{oc}]_{,x} \\
&+ (w_{,xx} - \phi v_{,xx} + y_{oc}\ \phi_{,xx})T - (v_{,xx} + \phi w_{,xx})q_x \\
&- (\phi_{,x} + v_{,xx} w_{,x})q_z + q_{y,x} + p_z = 0
\end{aligned}
\tag{16c}
$$

$$
\begin{aligned}
&[GJ(\phi_{,x} + v_{,xx} w_{,x})]_{,x} + EI_{23}(v^2_{,xx} - w^2_{,xx} + 4\phi\ v_{,xx} w_{,xx}) \\
&\quad + (EI_{33} - EI_{22})[v_{,xx} w_{,xx} + \phi(w^2_{,xx} - v^2_{,xx})] \\
&\quad + T[y_{oc}(w_{,xx} - \phi v_{,xx}) - z_{oc}(v_{,xx} + \phi w_{,xx})] + q_x = 0
\end{aligned}
\tag{16d}
$$

The system of Equations (16) contains four equations with four unknowns being represented by v, w, ϕ and T. The boundary conditions (9) together with Equations (10) and (15) become:

$$\text{for} \quad x_0 = 0: \qquad v = w = v_{,x} = w_{,x} = \phi = 0 \ , \tag{17a}$$

$$\text{for} \quad x_0 = \ell: \quad T = 0 \ , \tag{17b}$$

and

$$-V_{y(M_x=M_y=0)} = [E(I_2 \cos^2 \theta_G + I_3 \sin^2 \theta_G)(v_{,xx} + \phi w_{,xx})$$

$$+ E(I_2 - I_3) \sin \theta_G \cos \theta_G (w_{,xx} - \phi v_{,xx}) - T X_{II} \cos \theta_G]_{,x}$$

$$+ q_z = 0 \qquad (17c)$$

$$V_{z(M_x=M_z=0)} = -[E(I_2 - I_3) \sin \theta_G \cos \theta_G (v_{,xx} + \phi w_{,xx})$$

$$+ E(I_2 \sin^2 \theta_G + I_3 \cos^2 \theta_G)(w_{,xx} - \phi v_{,xx}) - T X_{II} \sin \theta_G]_{,x}$$

$$+ q_y = 0 \qquad (17d)$$

$$M_x = GJ(\phi_{,x} + v_{,xx} w_{,x}) = 0 \qquad (17e)$$

$$-M_y = E(I_2 - I_3) \sin \theta_G \cos \theta_G (v_{,xx} + \phi w_{,xx})$$

$$+ E(I_2 \sin^2 \theta_G + I_3 \cos^2 \theta_G)(w_{,xx} - \phi v_{,xx}) = 0 \qquad (17f)$$

$$M_z = E(I_2 \cos^2 \theta_G + I_3 \sin^2 \theta_G)(v_{,xx} + \phi w_{,xx})$$

$$+ E(I_2 - I_3) \sin \theta_G \cos \theta_G (w_{,xx} - \phi v_{,xx}) = 0 \quad . \qquad (17g)$$

The equations of equilibrium presented above were derived in the directions of the deformed coordinates. As pointed out in Appendix C, the equations can also be derived in the directions of the undeformed coordinates $\hat{e}_x, \hat{e}_y, \hat{e}_z$. In this case the distributed force, given previously by Equation (1), can be taken in the form:

$$\bar{p} = \tilde{p}_x \hat{e}_x + \tilde{p}_y \hat{e}_y + \tilde{p}_z \hat{e}_z \qquad (18)$$

while from similar considerations the distributed moment can be written as:

$$\bar{q} = \tilde{q}_x \hat{e}_x + \tilde{q}_y \hat{e}_y + \tilde{q}_z \hat{e}_z \quad . \tag{19}$$

The equilibrium equations are given by (C-24). In the present case the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$ is given by Equation (13). Then Equations (12) and (13) imply:

$$\begin{aligned} S_{12} &= v_{,x} \quad ; & S_{13} &= w_{,x} & S_{23} &= \phi \quad ; \\ S_{21} &= -(v_{,x} + \phi w_{,x}); & S_{31} &= -(w_{,x} - \phi v_{,x}); & S_{32} &= -(\phi + v_{,x} w_{,x}). \end{aligned} \tag{20}$$

Substitution of Equation (20) into Equation (C-24) and neglecting terms of order ε^2 compared to unity, yields the following equilibrium equations:

$$\begin{aligned} &[T + (v_{,x} + \phi w_{,x}) M_{z,x} - (w_{,x} - \phi v_{,x}) M_{y,x} + w_{,x} \phi_{,x} M_z \\ &+ v_{,x} \phi_{,x} M_y + (v_{,x} w_{,xx} - v_{,xx} w_{,x}) M_x]_{,x} \\ &+ (v_{,x} \tilde{q}_z)_{,x} - (w_{,x} \tilde{q}_y)_{,x} + \tilde{p}_x = 0 \end{aligned} \tag{21a}$$

$$\begin{aligned} &- [M_{z,x} + w_{,xx} M_x + (\phi_{,x} + v_{,xx} w_{,x}) M_y + (\phi + v_{,x} w_{,x}) M_{y,x}]_{,x} \\ &+ (v_{,x} T)_{,x} - \tilde{q}_{z,x} + (w_{,x} \tilde{q}_x)_{,x} + \tilde{p}_y = 0 \end{aligned} \tag{21b}$$

$$\begin{aligned} &[M_{y,x} + v_{,xx} M_x - (\phi_{,x} + v_{,xx} w_{,x}) M_z - \phi M_{z,x}]_{,x} \\ &+ (w_{,x} T)_{,x} - (v_{,x} \tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0 \end{aligned} \tag{21c}$$

$$M_{x,x} - (v_{,xx} + \phi w_{,xx})M_y - (w_{,xx} - \phi v_{,xx})M_z$$
$$+ v_{,x}\tilde{q}_y + w_{,x}\tilde{q}_z + \tilde{q}_x = 0 \tag{21d}$$

Substitution of Equations (15) into Equations (21), using Equations (6) and (7), and neglecting terms of order ε^2 when compared to unity, yields the following equations

$$\langle T + v_{,x}[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - \phi v_{,xx})]_{,x}$$
$$+ w_{,x}[EI_{23}(v_{,xx} + \phi w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx})]_{,x}$$
$$+ w_{,x}\{\phi[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}\, w_{,xx}]\}_{,x}$$
$$- v_{,x}\{\phi[EI_{23}\, v_{,xx} + EI_{33}(w_{,xx} - \phi v_{,xx})]\}_{,x}$$
$$+ GJ(\phi_{,x} + v_{,xx}w_{,x})(v_{,x}w_{,xx} - v_{,xx}w_{,x})\rangle_{,x}$$
$$+ (v_{,x}\tilde{q}_z)_{,x} - (w_{,x}\tilde{q}_y)_{,x} + \tilde{p}_x = 0 \tag{22a}$$

$$- [EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - 2\phi\, v_{,xx})]_{,xx}$$
$$- [GJ(\phi_{,x} + v_{,xx}w_{,x})w_{,xx}]_{,x}$$
$$+ [EI_{33}(\phi w_{,xx} + v_{,x}w_{,x}w_{,xx} - \phi^2\, v_{,xx})]_{,xx}$$
$$- (EI_{33}\, v_{,x}\, w^2_{,xx})_{,x} + [v_{,x}\, T + (Ty_{oc})_{,x} - (\phi Tz_{oc})_{,x}]_{,x}$$
$$- \tilde{q}_{z,x} + (w_{,x}\tilde{q}_x)_{,x} + \tilde{p}_y = 0 \tag{22b}$$

$$- [EI_{23}(v_{,xx} + 2\phi\, w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx})]_{,xx}$$
$$+ [GJ(\phi_{,x} + v_{,xx}w_{,x})v_{,xx}]_{,x} - [EI_{22}\,\phi(v_{,xx} + \phi w_{,xx})]_{,xx}$$
$$- (EI_{22}\, v^2_{,xx}w_{,x})_{,x} + [w_{,x}T + (Tz_{oc})_{,x} + (\phi Ty_{oc})_{,x}]_{,x} -$$

$$- (v_{,x} \tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0 \tag{22c}$$

$$\begin{aligned}
&[GJ(\phi_{,x} + v_{,xx}w_{,x})]_{,x} + EI_{23}(v^2_{,xx} - w^2_{,xx} + 4\phi\, v_{,xx}w_{,xx}) \\
&\quad + (EI_{33} - EI_{22})[v_{,xx}w_{,xx} + \phi(w^2_{,xx} - v^2_{,xx})] \\
&\quad - T[z_{oc}(v_{,xx} + \phi w_{,xx}) - y_{oc}(w_{,xx} - \phi v_{,xx})] \\
&\quad + v_{,x}\tilde{q}_y + w_{,x}\tilde{q}_z + \tilde{q}_x = 0
\end{aligned} \tag{22d}$$

The boundary conditions remain the same as in Equation (17) and it is only required to write q_z and q_y as functions of $\tilde{q}_x, \tilde{q}_y$, and $\tilde{q}_z$. According to Equations (C-16) and (C-11):

$$\begin{aligned}
q_y &= \tilde{q}_y + S_{21}\tilde{q}_x + S_{23}\tilde{q}_z \;; \\
q_z &= \tilde{q}_z + S_{31}\tilde{q}_x + S_{32}\tilde{q}_y \;.
\end{aligned} \tag{23}$$

The equilibrium equations in Appendix C were derived by the Newtonian method. In Appendix D the two sets of equations, with respect to the two different systems of coordinates, are derived using the principle of virtual work. The equilibrium equations which are obtained from this procedure are identical to those obtained in Appendix C, within the approximations inherent in the present theory. One of the advantages of the second method is that it also provides the appropriate set of boundary conditions, which is sometimes difficult to obtain using the Newtonian method. It is shown that the boundary conditions of the blade, as stated in Equations (9) or (17), are in agreement with the boundary conditions obtained by the second method.

The equations of equilibrium can be further simplified by taking into account some common properties pertaining to helicopter and wind turbine blades. These blades are usually stiffer in lag than in the flap-wise direction, thus:

$$EI_2 > EI_3\,,\,GJ \quad . \tag{24a}$$

The geometric pitch angle θ_G has an absolute value less than 45°. Therefore, according to Equation (6)

$$EI_{22} > EI_{33}\,,\,GJ \quad . \tag{24b}$$

Using Equation (24b) together with the ordering scheme (ε^2 neglected compared to unity) enables one to neglect a considerable number of additional small terms; the resulting equations are given below:

$$\begin{aligned}
&(T + v_{,x}[EI_{22}(\underline{\underline{v_{,xx}}} + \underline{\phi w_{,xx}}) + EI_{23}(\underline{\underline{w_{,xx}}} - \underline{\phi v_{,xx}})]_{,x} \\
&+ w_{,x}[EI_{23}(\underline{\underline{v_{,xx}}} + \underline{\phi w_{,xx}}) + EI_{33}(\underline{\underline{w_{,xx}}} - \underline{\phi v_{,xx}})]_{,x} \\
&+ \underline{w_{,x}\{\phi[EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}\, w_{,xx}]\}_{,x}} \\
&- \underline{v_{,x}\{\phi[EI_{23}\, v_{,xx} + EI_{33}\, w_{,xx}]\}_{,x}} + \underline{GJ\phi_{,x}(v_{,x}w_{,xx} - v_{,xx}w_{,x}))_{,x}} \\
&+ (v_{,x}\,\tilde{q}_z)_{,x} - (w_{,x}\,\tilde{q}_y)_{,x} + \tilde{p}_x = 0
\end{aligned} \tag{25a}$$

$$\begin{aligned}
&- [EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - 2\phi v_{,xx}) - EI_{33}\,\phi w_{,xx}]_{,xx} \\
&- (GJ\phi_{,x}w_{,xx})_{,x} + [v_{,x}T + (Ty_{oc})_{,x} - (\phi T z_{oc})_{,x}]_{,x} \\
&- \tilde{q}_{z,x} + (w_{,x}\,\tilde{q}_x)_{,x} + \tilde{p}_y = 0
\end{aligned} \tag{25b}$$

$$- [EI_{23}(v_{,xx} + 2\phi\, w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx})]_{,xx}$$

$$+ [GJ(\phi_{,x} + v_{,xx}w_{,x})v_{,xx}]_{,x} - [EI_{22}\, \phi(v_{,xx} + \phi w_{,xx})]_{,xx}$$

$$- (EI_{22}\, v^2_{,xx}w_{,x})_{,x} + [w_{,x}T + (Tz_{oc})_{,x} + (\phi Ty_{oc})_{,x}]_{,x}$$

$$- (v_{,x}\, \tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0 \qquad (25c)$$

$$[GJ(\phi_{,x} + v_{,xx}w_{,x})]_{,x} + EI_{23}(v^2_{,xx} - w^2_{,xx} + 4\phi\, v_{,xx}w_{,xx})$$

$$+ (EI_{33} - EI_{22})[v_{,xx}w_{,xx} + \phi(w^2_{,xx} - v^2_{,xx})]$$

$$- T[z_{oc}(v_{,xx} + \phi w_{,xx}) - y_{oc}(w_{,xx} - \phi v_{,xx})]$$

$$+ v_{,x}\, \tilde{q}_y + w_{,x}\, \tilde{q}_z + \tilde{q}_x = 0 \qquad (25d)$$

Usually, in the case of rotating blades large tensile forces are caused by the centrifugal forces; therefore, the nonlinear contributions of the bending and torsional moments to the tensile force are very small. In this case it seems to be justified to keep only the principal terms of this contribution (twice underlined in Equation (25a)) and neglect all the other terms associated with it (once underlined in Equation (25a)). In fact, it seems that neglecting the twice underlined terms will also not affect the results in a significant manner.

A further simplification can be obtained if all stiffnesses are approximately of the same order of magnitude, which means:

$$\frac{I_{22}}{I_{33}}\,,\ \frac{EI_{22}}{GJ}\,,\ \frac{EI_{33}}{GJ} \cong (0.5 - 2)\ . \qquad (26)$$

In this case, Equations (22) turn out to be:

$$\{T + v_{,x}[EI_{22}(\underline{\underline{v_{,xx}}} + \underline{\phi w_{,xx}}) + EI_{23}(\underline{\underline{w_{,xx}}} - \underline{\phi v_{,xx}})]_{,x}$$

$$+ w_{,x}[EI_{23}(\underline{\underline{v_{,xx}}} + \underline{\phi w_{,xx}}) + EI_{33}(\underline{\underline{w_{,xx}}} - \underline{\phi v_{,xx}})]_{,x}$$

$$+ \underline{w_{,x}[\phi(EI_{22}\, v_{,xx} + EI_{23}\, w_{,xx})]_{,x} - v_{,x}[\phi(EI_{23}\, v_{,xx} + EI_{33}\, w_{,xx})]_{,x}}$$

$$+ \underline{GJ\phi_{,x}(v_{,x}w_{,xx} - v_{,xx}w_{,x})\}_{,x}} + (v_{,x}\,\tilde{q}_z)_{,x} - (w_{,x}\,\tilde{q}_y)_{,x} + \tilde{p}_x = 0 \qquad (27a)$$

$$- [EI_{22}(v_{,xx} + \phi w_{,xx}) + EI_{23}(w_{,xx} - 2\phi\, v_{,xx}) - EI_{33}\,\phi w_{,xx}]_{,xx}$$

$$- (GJ\phi_{,x}\, w_{,xx})_{,x} + [v_{,x}T + (Ty_{oc})_{,x} - (\phi Tz_{oc})_{,x}]_{,x}$$

$$- \tilde{q}_{z,x} + (w_{,x}\,\tilde{q}_x)_{,x} + \tilde{p}_y = 0 \qquad (27b)$$

$$- [EI_{23}(v_{,xx} + 2\phi\, w_{,xx}) + EI_{33}(w_{,xx} - \phi v_{,xx}) + EI_{22}\,\phi v_{,xx}]_{,xx}$$

$$+ (GJ\phi_{,x}\, v_{,xx})_{,x} + [w_{,x}T + (Tz_{oc})_{,x} + (\phi Ty_{oc})_{,x}]_{,x}$$

$$- (v_{,x}\,\tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0 \qquad (27c)$$

$$[GJ(\phi_{,x} + v_{,xx}w_{,x})]_{,x} + EI_{23}(v^2_{,xx} - w^2_{,xx}) + (EI_{33} - EI_{22})v_{,xx}\,w_{,xx}$$

$$- T[z_{oc}(v_{,xx} + \phi w_{,xx}) - y_{oc}(w_{,xx} - \phi v_{,xx})]$$

$$+ v_{,x}\,\tilde{q}_y + w_{,x}\,\tilde{q}_z + \tilde{q}_x = 0 \qquad (27d)$$

The underlined terms in Equation (27a) have the same meaning as those in Equation (25a).

To facilitate the use of these equations for rotor-dynamics applications and to also simplify comparison of the equations in this revised version of the report with the previous version, the Equations (27a-d) are rewritten below using the principal moments of inertia of the cross section (Equations (6) and (7)). It should be noted that in the previous version of the report, few terms in Equations (22b - d) were missing due to an algebraic error. In most copies of the report these terms were added in handwriting.

$$T_{,x} + (v_{,x}\tilde{q}_z)_{,z} - (w_{,x}\tilde{q}_y)_{,x} + \tilde{p}_x = 0 \tag{27aa}$$

$$\begin{aligned}
&- [E(I_2 \cos^2\theta_G + I_3 \sin^2\theta_G)(v_{,xx} + \phi w_{,xx}) \\
&+ E(I_2 - I_3)\sin\theta_G \cos\theta_G(w_{,xx} - 2\phi v_{,xx}) \\
&- E(I_2 \sin^2\theta_G + I_3\cos^2\theta_G)\phi w_{,xx}]_{,xx} - (GJ\phi_{,x} w_{,xx})_{,x} \\
&+ \{v_{,x}T + [TX_{II}(\cos\theta_G - \phi\sin\theta_G)]_{,x}\}_{,x} - \tilde{q}_{z,x} + (w_{,x}\tilde{q}_x)_{,x} + \tilde{p}_y = 0
\end{aligned} \tag{27bb}$$

$$\begin{aligned}
&- [E(I_2 - I_3)\sin\theta_G\cos\theta_G(v_{,xx} + 2\phi w_{,xx}) \\
&+ E(I_2\sin^2\theta_G + I_3\cos^2\theta_G)(w_{,xx} - \phi v_{,xx}) \\
&+ E(I_2\cos^2\theta_G + I_3\sin^2\theta_G)\phi v_{,xx}]_{,xx} + (GJ\phi_{,x} v_{,xx})_{,x} \\
&+ \{w_{,x}T + [TX_{II}(\sin\theta_G + \phi\cos\theta_G)]_{,x}\}_{,x} \\
&- (v_{,x}\tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0
\end{aligned} \tag{27cc}$$

$$\begin{aligned}
&[GJ(\phi_{,x} + v_{,xx}w_{,x})]_{,x} + (I_2 - I_3)\sin\theta_G\cos\theta_G(v^2_{,xx} - w^2_{,xx}) \\
&\quad + (I_3 - I_2)(\cos^2\theta_G - \sin^2\theta_G)v_{,xx}w_{,xx} \\
&\quad - TX_{II}[\sin\theta_G(v_{,xx} + \phi w_{,xx}) - \cos\theta_G(w_{,xx} - \phi v_{,xx})] \\
&\quad + v_{,x}\tilde{q}_y + w_{,x}\tilde{q}_z + \tilde{q}_x = 0
\end{aligned} \tag{27dd}$$

4. COMPARISON BETWEEN THE PRESENT ELASTIC EQUATIONS AND THE EQUATIONS OF OTHER STUDIES

Elastic equilibrium equations for a rotor blade were derived by different researchers during the past twenty years, as was shown in the introduction. In this chapter a comparison will be made between the present derivation and some of the previous ones. For the sake of brevity these comparisons are concise, much more detailed comparisons can be found in Reference 18.

4.1 Comparison with Houbolt and Brooks (Ref. 3)

A set of equations equivalent to those of Reference 3 can be obtained from Equations (21a - d) by neglecting the nonlinear terms associated with the elastic moments. Performing these operations and replacing the moments by the appropriate expressions, as shown in Equation (15), results in the equations given below. It should be noted that nonlinear terms containing the displacements have been neglected in these equations.

$$T_{,x} + (\tilde{q}_z v_{,x})_{,x} - (\tilde{q}_y w_{,x})_{,x} + \tilde{p}_x = 0 \quad , \quad (28a)$$

$$[E(I_2 \cos^2 \theta_G + I_3 \sin^2 \theta_G) v_{,xx} + E(I_2 - I_3) \sin \theta_G \cos \theta_G w_{,xx} - T X_{II} \cos \theta_G]_{,xx} - (T v_{,x})_{,x} + \tilde{q}_{z,x} - (w_{,x} \tilde{q}_x)_{,x} - \tilde{p}_y = 0 \quad , \quad (28b)$$

$$-[E(I_2 - I_3) \sin \theta_G \cos \theta_G v_{,xx} + E(I_2 \sin^2 \theta_G + I_3 \cos^2 \theta_G) w_{,xx} - T X_{II} \sin \theta_G]_{,xx} + (T w_{,x}) + \tilde{q}_{y,x} - (v_{,x} \tilde{q}_x)_{,x} + \tilde{p}_z = 0 \quad (28c)$$

$$[GJ\, \phi_{,x}]_{,x} + \tilde{q}_z\, w_{,x} + \tilde{q}_y\, v_{,x} + \tilde{q}_x = 0 \quad . \tag{28d}$$

If Equations (28) in this report are compared with Equations (15) and (18 - 20) of Reference 3, the following observations can be made:

1) Equation (28a) is identical to Equation (15) of Houbolt and Brooks, except for the terms $(\tilde{q}_z\, v_{,x})_{,x}$ and $(\tilde{q}_y\, w_{,x})_{,x}$ which are not present in Equation (15).

2) Comparing Equation (28b) of the present study with Equation (20) of Reference 3, it follows that in addition to the terms contained in Equation (28b), Houbolt and Brooks' equation contains an additional term involving $\beta_{,x}$, probably resulting from the assumption $\sigma_{11} \cong \sigma_{xx}$.

   Furthermore, the term $[T X_{II}\, \phi \sin \Theta_G]_{,xx}$ which appears in Reference 3 is a physically nonlinear term. This term, which appears in Equation (22b) of the present study is associated with the term $(\phi\, M_y)_{,xx}$ in Equation (21b) of the present study. If this term is retained, all other terms of the same order should also be retained. However, this was not done in the equations presented by Houbolt and Brooks.

   In the loading terms, the term, $-(w_{,x}\, \tilde{q}_x)_{,x}$, which appears in Equation (28b) of the present study does not appear in Equation (20) of Reference 3.

3) The comparison between Equation (28c) of the present study and Equation (19) of Houbolt and Brooks is analogous to the comparison given in the previous section, and will not be repeated

here.

4) Comparison of Equation (28d) and Equation (18) of Reference 3, shows that except for the terms which appear in Equation (28d), additional terms containing $\beta_{,x}$ appear in those of Houbolt and Brooks' equation. These, again, are related to the assumption $\sigma_{11} \cong \sigma_{xx}$. Houbolt and Brooks also retain the term $T k_a^2 \phi_{,x}$ (k_a being the radius of gyration of the cross sectional area). However, this term should be neglected within the assumption that strain is negligible compared to unity. Similar to what has been pointed out already in Item (2) of this comparison, from the nonlinear terms $-v_{,xx} M_y$ and $-w_{,xx} M_z$ in Equation (21d) of the present study, Houbolt and Brooks retain only the terms $-T X_{II} \sin \Theta_G v_{,xx}$ and $T X_{II} \cos \Theta_G w_{,xx}$, while apparently neglecting other terms of the same order.

5) As pointed out in (2) and (4) above, it appears that Houbolt and Brooks assumed that in the expressions for the moments M_z and M_y (Eq. (4) of the present study), the terms Tz_{oc} and Ty_{oc} are much larger than the other terms. This seems to imply that the offset between the shear center and the tension center is the main contributor to the bending moments in the blade. This is a very special case which may be of limited importance from an engineering point of view. The assumption of the present study, that these terms are, at most, of the same

magnitude as that of the other terms, seems to be more realistic.

4.2 Comparison with Hodges and Dowell (Ref. 8)

1) Comparison of the expressions for ε_{xx} — shows that the strain at the elastic axis, and the contributions due to bending all over the cross section, are exactly the same. The expressions for warping are different. Hodges and Dowell also add the terms $(\eta^2 + \zeta^2)\theta_{G,x}\,\phi_{,x}$ and $[(\eta^2 + \zeta^2)\phi^2_{,x}]/2$. The first expression is also present in the derivations of Houbolt and Brooks. The second term, as was pointed out by Hodges and Dowell thermselves, is neglected within the approximation that terms of order ε^2 are negligible compared to unity. They retained this term for the case of very large torsional deformations, which imply that $y_o\,\phi_{,x}$ and $z_o\,\phi_{,x}$ are of order ε . The occurrence of such large elastic twist is unusual for most wind turbine or helicopter blades and is not treated in the present work. It should also be mentioned that the case when squares of strains are not negligible, in comparison to the strains themselves, is very special, and it appears that for this case a more refined theory than the one presented in this study will be required.

2) Comparing the shearing strains $e_{x\eta}$ and $e_{x\zeta}$ — as derived from Equation (B-23) of the present study with Equations

(25, 26) of Hodges and Dowell, shows some discrepancy which appears to be related to the previously discussed differences in the warping function, and the expression for the twist. Furthermore, terms which represent products of the warping and curvatures in this work, are neglected by Hodges and Dowell in theirs. This neglect seems to be justified for the case of slender blades having a closed cross section.

After obtaining the strain components, Hodges and Dowell derived the equations of equilibrium by two complementary methods, the Hamiltonian principle (similar to what is done in Appendix D of this report) and the Newtonian method (similar to what is done in Appendix C of this report). It is obvious that different strain expressions, and different transformation relations between $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ and $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$, will yield different final equations.

5. CONCLUDING REMARKS

A system of consistent, nonlinear, equations of equilibrium of a pretwisted wind turbine or helicopter blade which undergoes moderate deformation was systematically derived. The derivation contains, in addition to the basic assumptions listed in Section 2, some additional assumptions which are gradually introduced in the course of derivation. For the sake of completeness, these additional assumptions are briefly summarized. The blade is slender and its undeformed elastic axis is straight, the blade is made of elastic isotropic material. The Euler-Bernoulli assumptions are valid (for details see Appendix B) and warping of the cross sections due to torsion is neglected.* Axial forces in the blade contribute to the bending moments, due to the offset between the elastic center of the cross section and the tensile center. It is assumed that this offset is sufficiently small such that the magnitude of this contribution is, at most, of the magnitude of the other contributions to the bending moments (e.g., see Equations (B43)). The strains in the blade are always small (less then 0.01), while the slopes due to elastic rotations are of order of magnitude ε where $\varepsilon \approx 0.2$; furthermore, terms of order of magnitude ε^2 are neglected when compared to unity. Finally, it is assumed that deformations are changing gradually along the span of the blade, which implies that a modal expansion representing blade deformations would be restricted to

*Except for the warping contribution to torsional rigidity.

the lower modes.

Nonlinear structural problems require careful distinction between undeformed and deformed systems of coordinates for representing blade deformations. Therefore, in this study the final equations of equilibrium are presented in both the undeformed and deformed system. The general load components are also defined with respect to each of the systems.

The orthogonal system of coordinates x, y, z, used in deriving the equations of equilibrium in this study was found to be slightly more convenient than the curvilinear nonorthogonal x, η, ζ coordinate system used in References 3, 6, 8, 9 and 17. The main advantage being a somewhat simpler derivation and slightly simpler final equations. Additional information on this topic is provided in Reference 18.

An ordering scheme, such as used in this study, can simplify the equations considerably. The equations can be further simplified for certain blade geometries. It should be noted that the loading terms in the equations (forces and moments) were presented in a general form, without any approximations. Substitution of explicit expressions for the loading terms, and the application of an ordering scheme, enables one to identify and neglect a considerable number of additional small terms.

Since their derivation, these equations have been used extensively in a variety of aeroelastic stability and response problems as indicated below:

1) Calculation of coupled flap-lag-torsional aeroelastic stability

of hingeless rotor blades in forward flight (Ref. 19).

2) Aeroelastic stability and response calculations for an isolated horizontal axis wind turbine blade (Ref. 20). It should be noted that in this study, dynamic blade root bending moments were also calculated and found to be in satisfactory agreement with the loads measured on the NASA/DOE Mod-0 machine.

3) Aeroelastic stability and response calculation of a coupled rotor/tower horizontal axis wind turbine, simulating the behavior of the NASA/DOE Mod-0 machine (Ref. 21).

Finally, it is important to note that the equations derived in this study were used to investigate the large deformations of a cantilevered beam loaded by a concentrated transverse load at the free end (Ref. 22). The numerical results obtained were in very good agreement with experimental results, which indicates that these equations are reliable and can be used with confidence in a variety of applications.

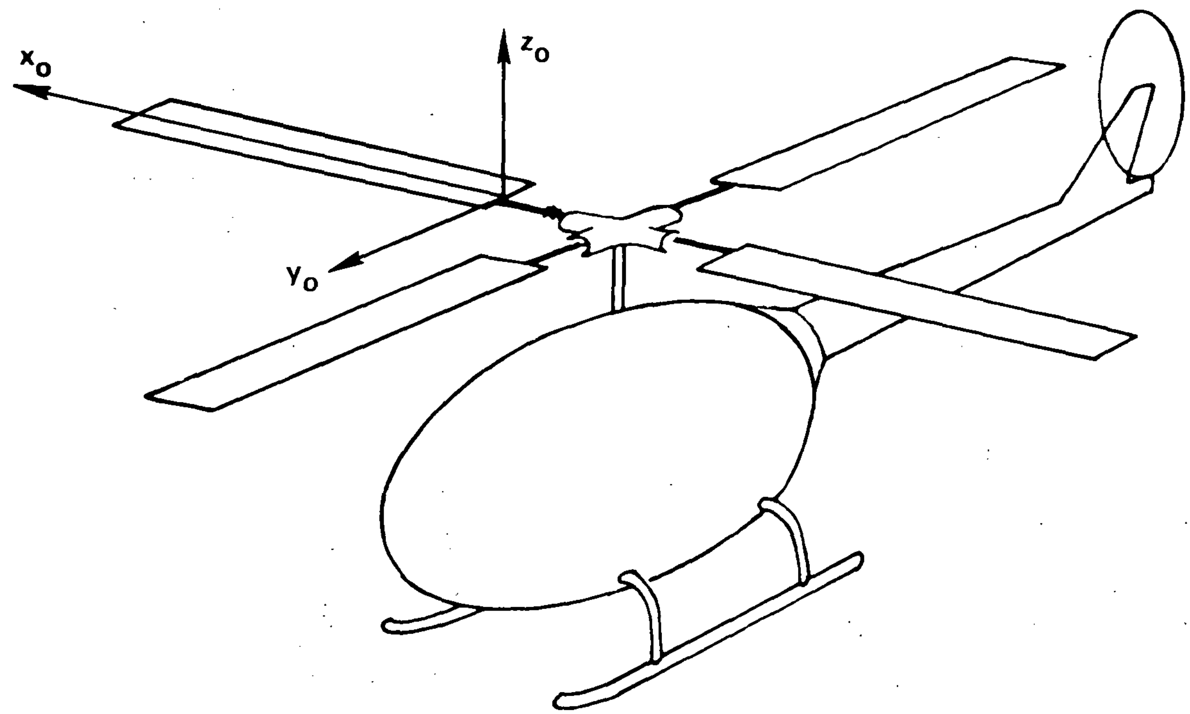

Figure 1a. General Description of Helicopter Rotor Geometry.

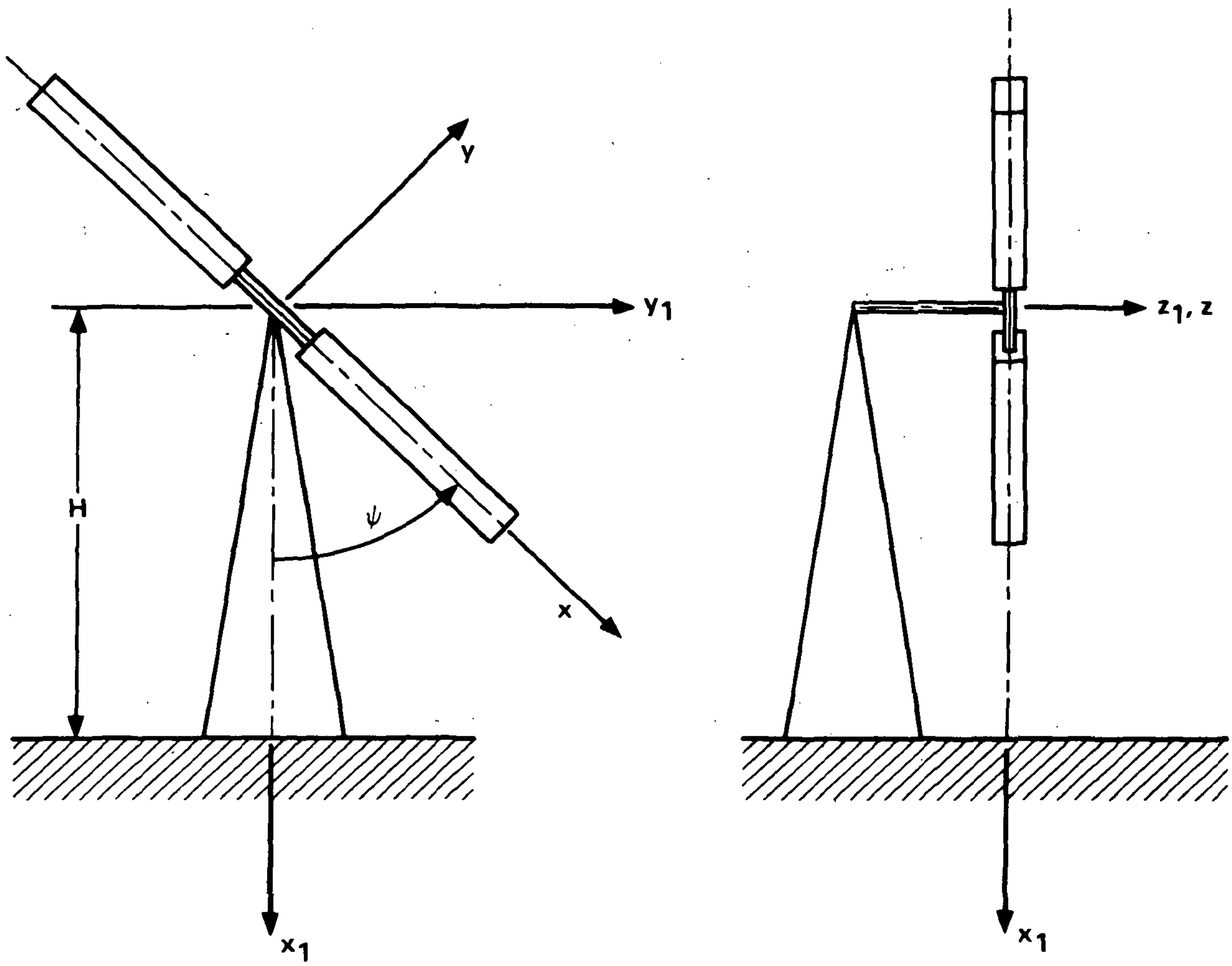

Fig. 1b. General Description of the Wind Turbine Geometry

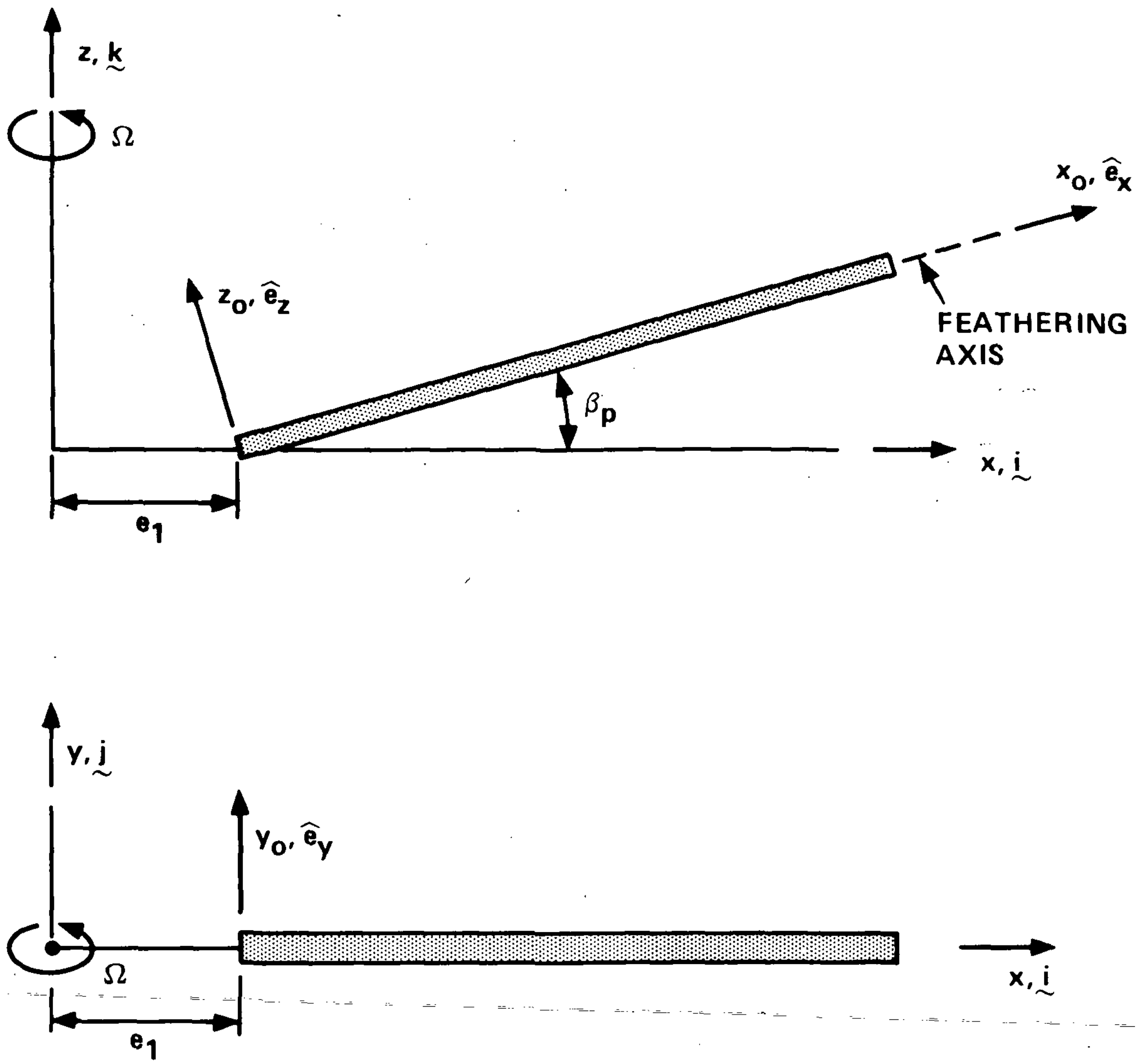

Fig. 2a. Typical Description of the Undeformed Blade in the Rotating System x, y, z (i, j, k)

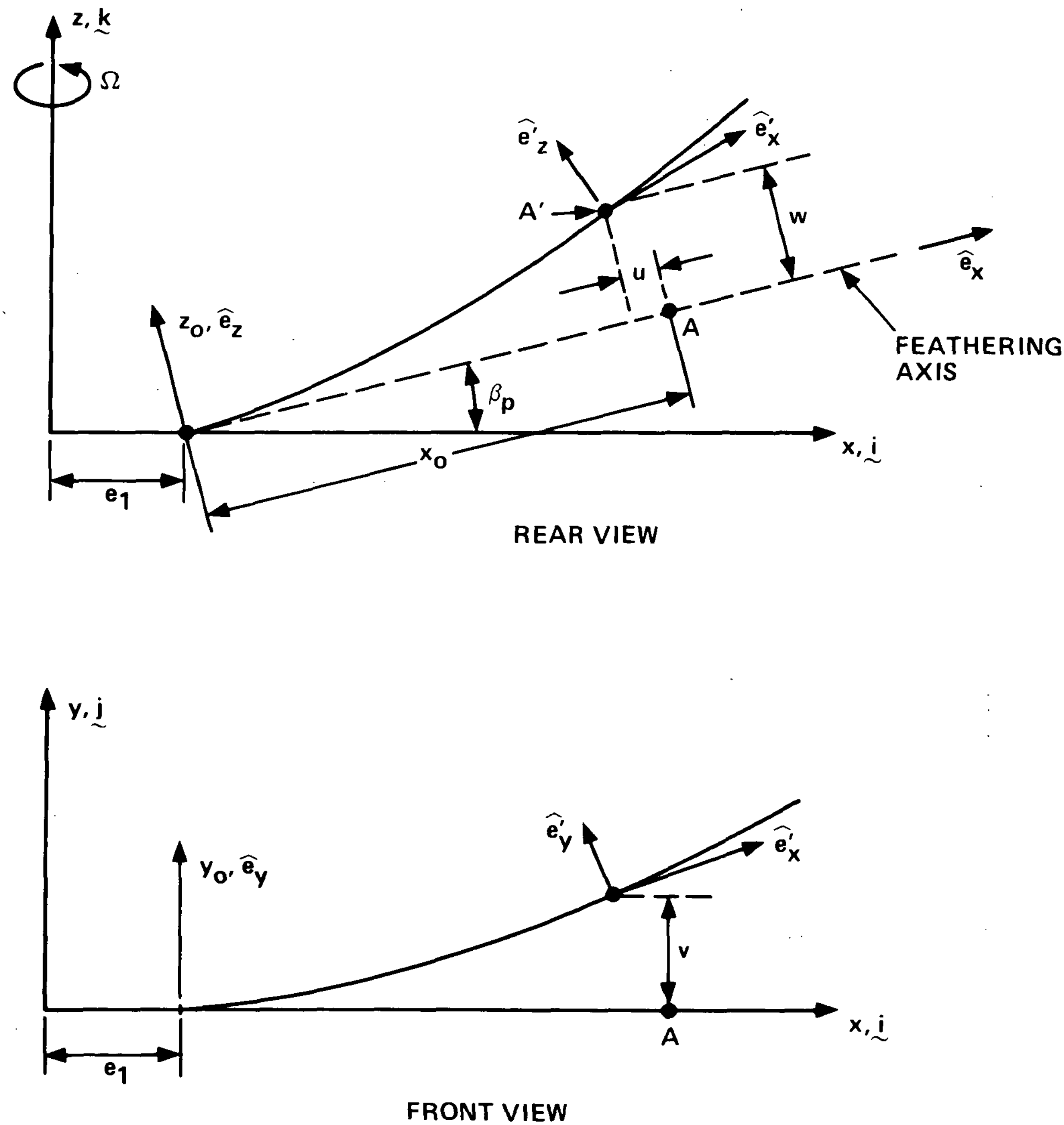

Fig. 2b. Geometry of the Elastic Axis of the Deformed Blade

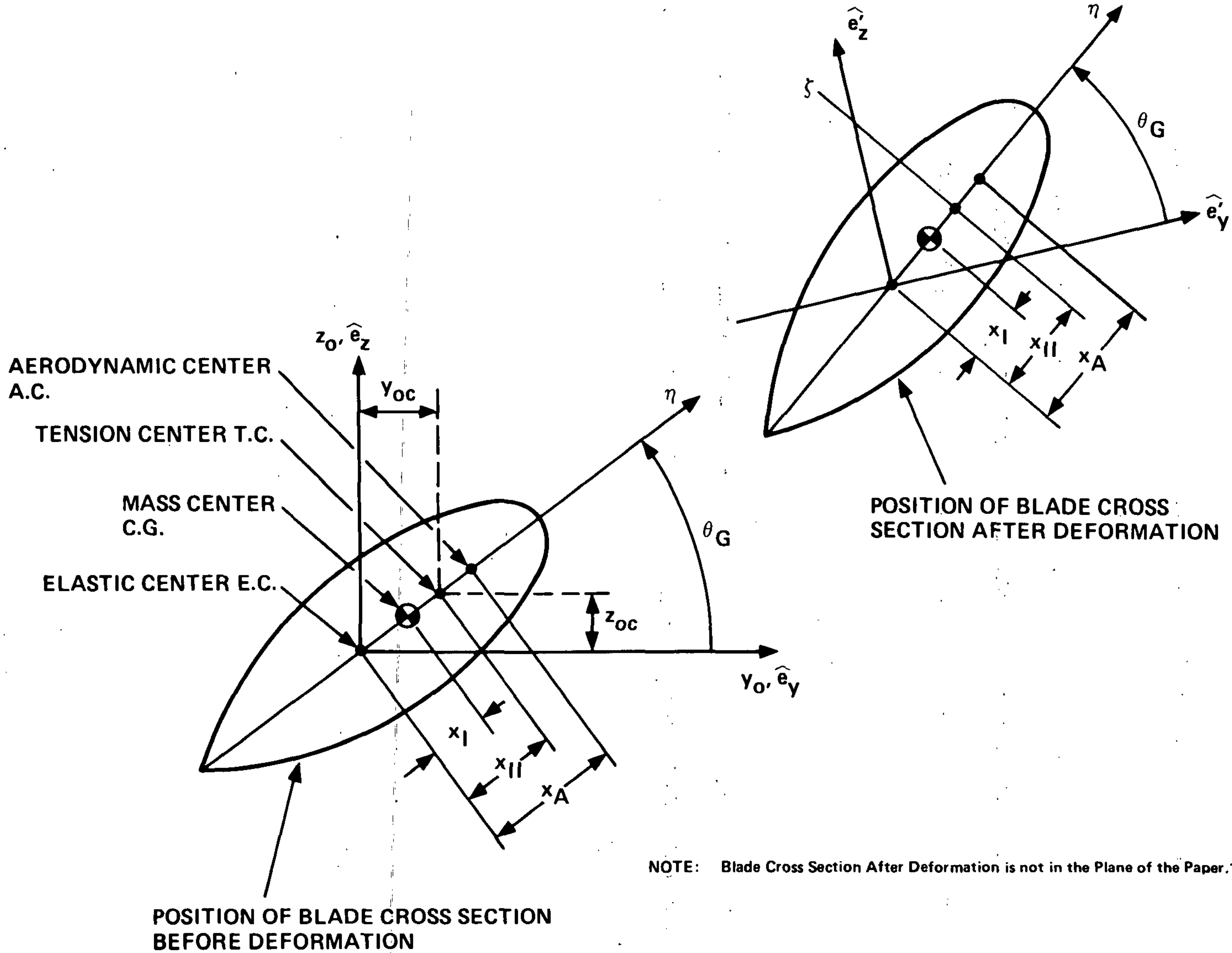

Fig. 3. Blade Cross Section Positions Before and After the Deformation

APPENDIX A

DEFORMATIONS

A detailed development of the expressions in this Appendix can be found in many books on elasticity (for example, Refs. 10, 11); therefore, these are only briefly repeated here for the sake of clarity.

Consider a material point P in an elastic body, where the position before deformation is given by the position vector $\bar{r}$ (Figure A-1). The position vector $\bar{r}$ is a function of three coordinates, such that:

$$\bar{r} = \bar{r}(x_0 , y_0 , z_0) \quad . \tag{A-1}$$

The coordinate system shown in Figure A-1 is an orthogonal Cartesian system with the unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$ in the directions x_0, y_0, z_0, respectively. Thus:

$$\bar{r} = x_0\hat{e}_x + y_0\hat{e}_y + z_0\hat{e}_z \quad . \tag{A-2}$$

After the deformation, the material particle is located at point P' (see Figure A-1) defined by the position vector $\bar{R}$. If the initial coordinates of the particle are used as independent variables, then:

$$\bar{R} = \bar{R}(x_0 , y_0 , z_0 , t) \quad , \tag{A-3}$$

where $\bar{R}$ is also a function of time because the position of the point is a function of time $(x_0, y_0, z_0$ can be considered to be the coordinates of the point at $t = t_0$).

If $\bar{V}$ denotes the displacement of the point, then:

$$\bar{R} = \bar{r} + \bar{V} \quad . \tag{A-4}$$

The base vectors of the point before deformation are defined as:

$$\bar{g}_x = \bar{r}_{,x} \quad ; \quad \bar{g}_y = \bar{r}_{,y} \quad ; \quad \bar{g}_z = \bar{r}_{,z} \quad . \tag{A-5}$$

After the deformation, the base vectors are:

$$\bar{G}_x = \bar{R}_{,x} \quad ; \quad \bar{G}_y = \bar{R}_{,y} \quad ; \quad \bar{G}_z = \bar{R}_{,z} \quad . \tag{A-6}$$

The strain components are given by Equation (2-20) of Wempner in Reference 11 (where x_0, y_0, z_0 is an orthogonal system):

$$\begin{aligned}
\varepsilon_{xx} &\equiv \tfrac{1}{2} (\bar{G}_x \cdot \bar{G}_x - 1) \; ; \quad & \varepsilon_{xy} &= \varepsilon_{yx} = \tfrac{1}{2} (\bar{G}_x \cdot \bar{G}_y) \; ; \\
\varepsilon_{yy} &\equiv \tfrac{1}{2} (\bar{G}_y \cdot \bar{G}_y - 1) \; ; \quad & \varepsilon_{xz} &= \varepsilon_{zx} = \tfrac{1}{2} (\bar{G}_x \cdot \bar{G}_z) \; ; \\
\varepsilon_{zz} &\equiv \tfrac{1}{2} (\bar{G}_z \cdot \bar{G}_z - 1) \; ; \quad & \varepsilon_{yz} &= \varepsilon_{zy} = \tfrac{1}{2} (\bar{G}_y \cdot \bar{G}_z) \; .
\end{aligned} \tag{A-7}$$

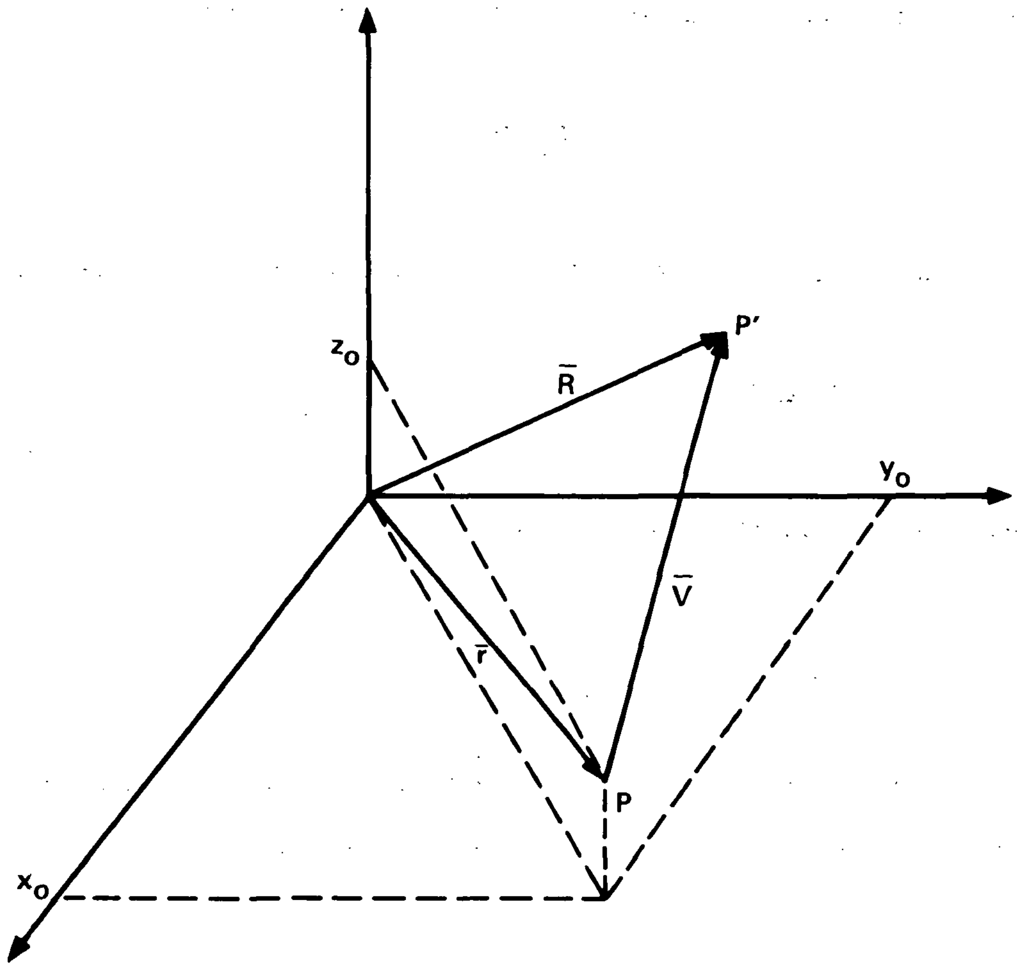

Fig. A1. Position of a Material Particle Before and After the Deformation

APPENDIX B

MECHANICS OF A DEFORMED ROD WHICH IS SLENDER AND STRAIGHT BEFORE DEFORMATION

B.1 General Expressions

A straight slender rod is shown in Figure B-1. Every material point in this rod is described by a rectangular Cartesian system of coordinates, x_0, y_0, z_0. The coordinate x_0 is identical with the elastic axis of the rod, defined as the line which connects the shear centers of the cross sections of the rod. It is assumed that the elastic axis is a straight line. In this case, x_0 denotes length along the elastic axis of the undeformed rod, while y_0 and z_0 denote lengths along lines orthogonal to the undeformed elastic axis.

Before the deformation the position vector of every material point is given by:

$$\bar{r} = x_0\hat{e}_x + y_0\hat{e}_y + z_0\hat{e}_z \quad , \tag{B-1}$$

while after the deformation, at time t, the new position vector is:

$$\bar{R} = \bar{R}(x_0, y_0, z_0, t) \quad . \tag{B-2}$$

The displacement of the particle is:

$$\bar{V} = \bar{R} - \bar{r} \quad . \tag{B-3}$$

Looking at a particle, which before deformation lies on the elastic axis, its initial position vector is:

$$ {}_o\bar{r} = x_0 \hat{e}_x \tag{B-4} $$

and its position after the deformation is given by:

$$ {}_o\bar{R} = \bar{R}(x_0, 0, 0, t) \ . \tag{B-5} $$

The displacement of this point is denoted by:

$$ \bar{W} = \bar{V}(x_0, 0, 0) = u\hat{e}_x + v\hat{e}_y + w\hat{e}_z \ . \tag{B-6} $$

The base vectors of the undeformed rod are simply (according to Eq. (A-5)) the orthonormal triad:

$$ \bar{g}_x = \hat{e}_x \quad ; \quad \bar{g}_y = \hat{e}_y \quad ; \quad \bar{g}_z = \hat{e}_z \ , \tag{B-7} $$

while the base vectors of the deformed rod are (according to (A-6)):

$$ \left. \begin{aligned} \bar{G}_x &\equiv \bar{R}_{,x} = (\bar{r} + \bar{V})_{,x} = \hat{e}_x + \bar{V}_{,x} \\ \bar{G}_y &\equiv \bar{R}_{,y} = (\bar{r} + \bar{V})_{,y} = \hat{e}_y + \bar{V}_{,y} \\ \bar{G}_z &\equiv \bar{R}_{,z} = (\bar{r} + \bar{V})_{,z} = \hat{e}_z + \bar{V}_{,z} \end{aligned} \right\} \ ; \tag{B-8} $$

and at the elastic axis, a set $\bar{E}_x, \bar{E}_y, \bar{E}_z$ is defined as:

$$
\left.
\begin{aligned}
\bar{E}_x &\equiv \bar{G}_x(x_0, 0, 0) = \hat{e}_x + \bar{W}_{,x} \\
&= (1 + u_{,x})\hat{e}_x + v_{,x}\,\hat{e}_y + w_{,x}\,\hat{e}_z \\
\bar{E}_y &\equiv \bar{G}_y(x_0, 0, 0) = \hat{e}_y + \bar{V}(x_0, 0, 0)_{,y} \\
\bar{E}_z &\equiv \bar{G}_z(x_0, 0, 0) = \hat{e}_z + \bar{V}(x_0, 0, 0)_{,z}
\end{aligned}
\right\} . \qquad \text{(B-9)}
$$

The strains at any point are calculated by using Equations (A-7).

The motion which carries the rectangular lines of the undeformed rod into the curved lines of the deformed rod, carries the initial tangent unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$ to the current tangent base vectors, $\bar{G}_x, \bar{G}_y, \bar{G}_z$, respectively. This motion can be looked upon as two successive motions: First, the triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$ is rigidly transformed and rotated to the orientation of an intermediate orthonormal triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$. Next, the intermediate triad is deformed to the triad $\bar{G}_x, \bar{G}_y, \bar{G}_z$ which means changing the angles between the vectors as well as the length of the vectors. The procedure which was described above is illustrated by Figure B-2.

Consider a triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$ which is positioned on the elastic axis of the rod before deformation. In the stage of rigid transformation and rotation this triad is carried to the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$, respectively. Without any loss of generality, let us assume that $\hat{e}'_x$ is carried in this stage to the direction of $\bar{E}_x$, which means that it is tangent to the elastic axis of the rod after deformation.

If the rotation of the triad $\hat{e}_x, \hat{e}_y, \hat{e}_z$ to the position $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$

is relatively large, it cannot be described by a vector and it is treated by means of Euler angles (see, for example, Novozhilov, Ref. 12, Chapter VI, p. 205). If it is described as a finite rotation Θ_z about $\hat{e}_z$, followed by a rotation Θ_y about $\hat{e}_y$, followed by a rotation Θ_x about $\hat{e}_x$, then the triad $\hat{e}'_x \cdot \hat{e}'_y, \hat{e}'_z$ obtained after these three rotations, is given by:

$$\hat{e}'_x = (\cos\Theta_y \cos\Theta_z)\hat{e}_x + (\cos\Theta_y \sin\Theta_z)\hat{e}_y - \sin\Theta_y\, \hat{e}_z \quad , \quad \text{(B-10a)}$$

$$\hat{e}'_y = (\sin\Theta_x \sin\Theta_y \cos\Theta_z - \cos\Theta_x \sin\Theta_z)\hat{e}_x + (\cos\Theta_x \cos\Theta_z + \sin\Theta_x \sin\Theta_y \sin\Theta_z)\hat{e}_y + (\sin\Theta_x \cos\Theta_y)\hat{e}_z \quad , \quad \text{(B-10b)}$$

$$\hat{e}'_z = (\cos\Theta_x \sin\Theta_y \cos\Theta_z + \sin\Theta_x \sin\Theta_z)\hat{e}_x - (\sin\Theta_x \cos\Theta_z - \cos\Theta_x \sin\Theta_y \sin\Theta_z)\hat{e}_y + (\cos\Theta_x \cos\Theta_y)\hat{e}_z \quad . \quad \text{(B-10c)}$$

Equation (B-10) are identical to Equations (A2) of Hodges and Dowell (Ref. 8), after replacing Θ_x, Θ_y and Θ_z by $\bar{\Theta}$, $-\bar{\beta}$, and $\bar{\zeta}$, respectively.

Consider the deformation of an element dx_0 on the elastic axis of the rod, as described in Figure B-3. The procedure is as follows: First, the element is carried in a rigid body translation that does not appear in Figure B-3. Then the element is stretched by an amount

$u_{,x}\, dx_0$ to the position A - (1). Then the element is rotated by θ_z about $\hat{e}_z$ while point (1) moves a distance $v_{,x}\, dx_0$ to location (2), followed by a rotation $-\theta_y$, while the element tip moves from location (2) to (3). Finally, the element in its position A - (3) is rotated by an amount θ_x about itself. From Figure B-3, the following relations are obtained:

$$\sin \theta_y = - \frac{w_{,x}}{\sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2 + w_{,x}^2}} , \qquad \text{(B-11a)}$$

$$\cos \theta_y = \frac{\sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2}}{\sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2 + w_{,x}^2}} , \qquad \text{(B-11b)}$$

$$\sin \theta_z = \frac{v_{,x}}{\sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2}} , \qquad \text{(B-11c)}$$

$$\cos \theta_z = \frac{1 + u_{,x}}{\sqrt{1 + 2u_{,x} + u_{,x}^2 + v_{,x}^2}} . \qquad \text{(B-11d)}$$

The quantity $u_{,x}$ is of the magnitude of strain, as will be shown later. Assuming that the strains are small (say, $\varepsilon_{ij} < 0.01$) which is the case for most engineering materials, then the expression $u_{,x}$ will be neglected in comparison to unity. Next, it is assumed that $v_{,x}, w_{,x}$ and θ_x are quantities of magnitude equal to or less than ε (in our case, $\varepsilon \approx 0.2$), and quantities of the magnitude of ε^2 are negligible compared to unity.

Using these assumptions and expanding $\sin \Theta_x$ and $\cos \Theta_x$ into series, one obtains:

$$\begin{aligned} &\sin \Theta_x \cong \Theta_x \;;\; \sin \Theta_y \cong -w_{,x} \;;\; \sin \Theta_z \cong v_{,x} \;; \\ &\cos \Theta_x \cong 1 \;;\; \cos \Theta_y \cong 1 \;;\; \cos \Theta_z \cong 1 \;. \end{aligned} \tag{B-12}$$

At this stage, in order to be consistent with the usual notation in the literature (for example, Reference 9), Θ_x is replaced by ϕ. Substituting Equations (B-12) into Equations (B-10) implies:

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + v_{,x}\hat{e}_y + w_{,x}\hat{e}_z \\ \hat{e}'_y &= -(v_{,x} + \phi w_{,x})\hat{e}_x + \hat{e}_y + \phi\hat{e}_z \\ \hat{e}'_z &= -(w_{,x} - \phi v_{,x})\hat{e}_x - (\phi + v_{,x}w_{,x})\hat{e}_y + \hat{e}_z \end{aligned}\right\} . \tag{B-13}$$

Because the rotations are treated as finite, it is not surprising that the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$ depends on the sequence of rotations. If the sequence consists of rotation Θ_y about $\hat{e}_y$, followed by a rotation Θ_z about $\hat{e}_z$, and finally, a rotation Θ_x about $\hat{e}_x$, as described in Figure B-4, then the expressions become:

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + v_{,x}\hat{e}_y + w_{,x}\hat{e}_z \\ \hat{e}'_y &= -(v_{,x} + \phi w_{,x})\hat{e}_x + \hat{e}_y + (\phi - v_{,x}w_{,x})\hat{e}_z \\ \hat{e}'_z &= -(w_{,x} - \phi v_{,x})\hat{e}_x - \phi\hat{e}_y + \hat{e}_z \end{aligned}\right\} . \tag{B-14}$$

The triad (B-14) differs from (B-13) by terms of second order. Other different sequences of rotation will yield other triads which will differ from each other by second order terms. Therefore, it is most important to retain one particular triad during a complete derivation, and consistency with this selected triad. In the theory of space curves there are three important quantities, defined as (for example, Wempner, Ref. 11, Eqs. (8-19) - (8-21)):

$$\left.\begin{aligned} \kappa_y &\equiv \hat{e}'_y \cdot \hat{e}'_{x,x} = -\hat{e}'_x \cdot \hat{e}'_{y,x} \\ \kappa_z &\equiv \hat{e}'_z \cdot \hat{e}'_{x,x} = -\hat{e}'_x \cdot \hat{e}'_{z,x} \\ \tau &\equiv \hat{e}'_z \cdot \hat{e}'_{y,x} = -\hat{e}'_y \cdot \hat{e}'_{z,x} \end{aligned}\right\} , \qquad \text{(B-15)}$$

where κ_y and κ_z are curvatures, while τ is the twist. Equation (B-15) represents the exact expressions for the curvature and twist when strains are neglected compared to one. If $\hat{e}'_x$, $\hat{e}'_y$, and $\hat{e}'_z$ are given by Equation (B-13), then:

$$\left.\begin{aligned} \hat{e}'_{x,x} &= v_{,xx}\,\hat{e}_y + w_{,xx}\,\hat{e}_z \\ \hat{e}'_{y,x} &= -(v_{,xx} + \phi_{,x} w_{,x} + \phi\, w_{,xx})\hat{e}_x + \phi_{,x}\,\hat{e}_z \\ \hat{e}'_{z,x} &= -(w_{,xx} - \phi_{,x} v_{,x} - \phi v_{,xx})\hat{e}_x - (\phi_{,x} + v_{,xx} w_{,x} + v_{,x} w_{,xx})\hat{e}_y \end{aligned}\right\} . \qquad \text{(B-16a)}$$

Substituting expressions (B-16a) into Equations (B-15), and assuming that $v_{,xx}$, $w_{,xx}$, and $\phi_{,x}$ are of the same magnitude, and neglecting again terms of the magnitude of ε^2 compared to unity, implies:

$$\left.\begin{aligned} \kappa_y &= v_{,xx} + \phi w_{,xx} \\ \kappa_z &= w_{,xx} - \phi v_{,xx} \\ \tau &= \phi_{,x} + v_{,xx} w_{,x} \end{aligned}\right\} . \tag{B-16b}$$

The term $v_{,xx}w_{,x}$ in the expression for twist is a well known term in the theory of rods (see for example, Reference 11, p. 390, Eq. 8-152d). It has also been used in rotor dynamics by Kaza and Kvaternik (Ref. 15).

From the definitions (B-15) and the orthonormality conditions of the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$, it is clear that:

$$\left.\begin{aligned} \hat{e}'_{x,x} &= \kappa_y \hat{e}'_y + \kappa_z \hat{e}'_z \\ \hat{e}'_{y,x} &= -\kappa_y \hat{e}'_x + \tau \hat{e}'_z \\ \hat{e}'_{z,x} &= -\kappa_z \hat{e}'_x - \tau \hat{e}'_y \end{aligned}\right\} , \tag{B-17}$$

which can be easily verified by substitution of expression (B-17) into the definitions (B-15).

B.2 Bernoulli-Euler Hypothesis

At this stage, it is necessary to find an expression for $\bar{R}$. This always requires certain assumptions. In the present case, the well known Bernoulli-Euler hypothesis will be used. In most cases this hypothesis is stated as follows: During bending, plane cross sections which are normal to the axis before deformation remain plane after deformation, and normal to the deformed axis. Usually, this hypothesis is combined with the assumption, although not always stated, that strains within the cross sections can be neglected. This assumption will be used in the present study also. (This is similar to the case of plate and shells where the analogous Love-Kirchoff hypothesis is used.) This hypothesis leads to the following results:

$$\bar{E}_y = \hat{e}'_y , \quad \bar{E}_z = \hat{e}'_z , \tag{B-18}$$

and

$$\bar{R} = x_0 \hat{e}_x + \bar{W} + y_0 \hat{e}'_y + z_0 \hat{e}'_z + \varphi(x_0, y_0, z_0, t)\hat{e}'_x . \tag{B-19}$$

The last term in Equation (B-19) represents small normal displacement which, as pointed out by Novozhilov (Ref. 12, p. 213), is a generalization of the warping function of St.-Venant torsion. This function contains only quadratic and higher degree terms in y_0 and z_0 and is assumed to be small compared with typical cross-sectional dimensions of the rod.

Substitution of the expression (B-6) for $\bar{W}$ into Equation (B-19), then differentiating (B-17) and using Equation (B-17), implies:

$$\begin{aligned} \bar{G}_x = \bar{R}_{,x} &= (1 + u_{,x})\hat{e}_x + v_{,x} \hat{e}_y + w_{,x} \hat{e}_z \\ &+ y_0(-\kappa_y \hat{e}'_x + \tau \hat{e}'_z) + z_0(-\kappa_z \hat{e}'_x - \tau \hat{e}'_y) \\ &+ \varphi_{,x} \hat{e}'_x + \varphi(\kappa_y \hat{e}'_y + \kappa_z \hat{e}'_z) , \end{aligned} \tag{B-20a}$$

$$\bar{G}_y = \bar{R}_{,y} = \hat{e}'_y + \varphi_{,y}\, \hat{e}'_x \quad , \tag{B-20b}$$

$$\bar{G}_z = \bar{R}_{,z} = \hat{e}'_z + \varphi_{,z}\, \hat{e}'_x \quad . \tag{B-20c}$$

By definition, $\hat{e}'_x$ is a unit vector in the direction of $\bar{E}_x$. Defining $\tilde{\varepsilon}_{xx}$ as the strain of the elastic axis, and then using the first of Equations (B-9), implies:

$$\begin{aligned} \bar{E}_x &= (1 + u_{,x})\hat{e}_x + v_{,x}\, \hat{e}_y + w_{,x}\, \hat{e}_z \\ &\equiv (1 + \tilde{\varepsilon}_{xx})\hat{e}'_x \quad . \end{aligned} \tag{B-21}$$

From Equation (B-21), using the phthagorian rule and neglecting terms of order ε^2 compared to unity, implies:

$$\tilde{\varepsilon}_{xx} = u_{,x} + \frac{1}{2}\,(v^2_{,x} + w^2_{,x}) \quad . \tag{B-22}$$

Substitution of Equations (B-20) into the expressions (A-7) of Appendix A for the strain components, and making use of Equation (B-22), implies:

$$\begin{aligned} \varepsilon_{xx} &= \tilde{\varepsilon}_{xx} - y_0\, \kappa_y - z_0\, \kappa_z + \varphi_{,x} \\ \varepsilon_{xy} &= \frac{1}{2}\,(\varphi_{,y} - z_0\, \tau + \varphi\, \kappa_y) \\ \varepsilon_{xz} &= \frac{1}{2}\,(\varphi_{,z} + y_0\, \tau + \varphi\, \kappa_z) \end{aligned} \quad . \tag{B-23}$$

In deriving the expressions (B-23) use was made of the fact that the quantities $y_0\, \kappa_y$, $z_0\, \kappa_z$, $y_0\, \tau$, $z_0\, \tau$, $\varphi_{,x}$, $\varphi\, \kappa_y$, $\varphi\, \kappa_z$, $\varphi_{,y}$, and $\varphi_{,z}$ are of magnitude of strain and therefore less than ε^2.

B.3 Force and Moment Resultants

In calculating the stresses in the rod, use is made of the constitutive relations of the material from which the rod is made. In the present case it is assumed that the rod is made of isotropic Hookean material which is homogeneous for every cross section. On the other hand, in the present study the assumption that $\sigma_{yy} = \sigma_{zz} = 0$, commonly used for slender rods, is made. However, according to the Bernoulli-Euler hypothesis, ε_{yy} and ε_{zz} should also be zero, and the vanishing of σ_{yy}, σ_{zz}, ε_{yy} and ε_{zz} simultaneously, is inconsistent with Hooke's Law. This inconsistency, which is inherent in the Bernoulli-Euler hypothesis, although not always stated is explained in the literature in different ways. One of the explanations is that one is dealing with a material having a special type of orthotropy.

The constitutive relations for an orthotropic material, as given by Lekhnitskii (Ref. 14, Eq. (3-7)), are:

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E_1}\sigma_{xx} - \frac{\nu_{21}}{E_2}\sigma_{yy} - \frac{\nu_{31}}{E_3}\sigma_{zz} \\
\varepsilon_{yy} &= -\frac{\nu_{12}}{E_1}\sigma_{xx} + \frac{1}{E_2}\sigma_{yy} - \frac{\nu_{32}}{E_3}\sigma_{zz} \\
\varepsilon_{zz} &= -\frac{\nu_{13}}{E_1}\sigma_{xx} - \frac{\nu_{23}}{E_2}\sigma_{yy} + \frac{1}{E_3}\sigma_{zz} \\
\varepsilon_{yz} &= \frac{1}{G_{23}}\tau_{yz} \\
\varepsilon_{xz} &= \frac{1}{G_{13}}\tau_{xz} \\
\varepsilon_{xy} &= \frac{1}{G_{12}}\tau_{xy}
\end{aligned}\right\} \quad . \qquad \text{(B-24a)}$$

It is assumed, that in the present case:

$$\nu_{21} = \nu_{31} = \nu_{12} = \nu_{13} = 0; \quad E_2 = E_3 \to \infty; \quad G_{23} = G_{13} = 2G .^{*} \qquad \text{(B-24b)}$$

By using Equation (B-24b) together with the relations (B-24a), the inconsistency that was mentioned earlier, disappears.

Another inconsistency which is inherent in the Bernoulli-Euler hypothesis, concerns the shearing stresses. If torsion is neglected, the assumption that plane cross sections, before deformation, remain plane after the deformation, means that the shearing strains $\tilde{\varepsilon}_{xz}$ and $\tilde{\varepsilon}_{yz}$ are zero. (These components are due to contributions other than torsion. The contributions due to torsion appear in Equation (B-23).) This means, according to Hooke's Law, that the shearing stresses due to contributions other than torsion disappear. However, these stresses, $\tilde{\tau}_{xy}$ and $\tilde{\tau}_{xz}$, or more accurately their resultants -- the shearing forces -- do not vanish at all. Furthermore, they play an important role in the equilibrium calculations. Sometimes this inconsistency is explained by taking $G_{23} = G_{13} \to \infty$ in Equations (B-24a). In the present case, where shearing strains due to torsion are also present, this assumption will cause some problems. Therefore, a better explanation is that in the case of slender rods the shearing strains are very small, so that they do not violate the hypothesis; however, their integral over the cross section should be taken into account, implying that the shearing forces cannot be neglected.

Using Equations (B-24a), (B-24b) together with the strain relations as given by Equation (B-23), implies:

---

*The two in the expressions for G_{23} and G_{13} is needed because shearing strains in Ref. 14 are defined without the factor 1/2 which appears in Eq. (A-7) of the present study.

$$\left.\begin{aligned}
\sigma_{xx} &= E\,\varepsilon_{xx} \\
&= E(\tilde{\varepsilon}_{xx} - y_0\,\kappa_y - z_0\,\kappa_y + \varphi_{,x}) \\
\\
\tau_{xy} &= \tilde{\tau}_{xy} + 2G\,\varepsilon_{xy} \\
&= \tilde{\tau}_{xy} + G(\varphi_{,y} - z_0\,\tau + \varphi\,\kappa_y) \\
\\
\tau_{xz} &= \tilde{\tau}_{xz} + 2G\,\varepsilon_{xz} \\
&= \tilde{\tau}_{xz} + G(\varphi_{,z} + y_0\,\tau + \varphi\,\kappa_z)
\end{aligned}\right\} \quad . \qquad \text{(B-24)}$$

Detailed expressions for $\tilde{\tau}_{xy}$ and $\tilde{\tau}_{xz}$ are not required, as will be shown later.

The force which acts on the unit area of the cross section of the deformed rod, is:

$$\bar{t} = \sigma_{xx}\,\bar{G}_x + \tau_{xy}\,\bar{G}_y + \tau_{xz}\,\bar{G}_z \quad . \qquad \text{(B-25)}$$

Using Equations (B-20), (B-21), combined with (B-25), one obtains:

$$\begin{aligned}
\bar{t} = {} & [\sigma_{xx}(1 + \tilde{\varepsilon}_{xx} - y_0\,\kappa_y - z_0\,\kappa_z + \varphi_{,x}) + \tau_{xy}\,\varphi_{,y} + \tau_{xz}\,\varphi_{,z}]\hat{e}'_x \\
& + [\sigma_{xx}(-z_0\,\tau + \varphi\,\kappa_y) + \tau_{xy}]\hat{e}'_y \\
& + [\sigma_{xx}(y_0\,\tau + \varphi\,\kappa_z) + \tau_{xz}]\hat{e}'_z \quad . \qquad \text{(B-26)}
\end{aligned}$$

Assuming that the stresses are of the same order and neglecting terms of order ε^2 compared to unity, implies:*

---

*See comment on page 57.

$$\bar{t} = \sigma_{xx}\,\hat{e}'_x + \tau_{xy}\,\hat{e}'_y + \tau_{xz}\,\hat{e}'_z \quad . \tag{B-27}$$

The resultant force, $\bar{F}$, which acts on the cross section, is obtained by integration:

$$\bar{F} = \iint_A \bar{t}\; dy_0\, dz_0 = T\;\hat{e}'_x + V_y\,\hat{e}'_y + V_z\,\hat{e}'_z \quad . \tag{B-28}$$

Before proceeding, the warping function, which until now was treated in a general manner, has to be considered. One possibility it to treat it in an exact fashion as done by Wempner (Ref. 11, Chapter 8). This procedure, however, complicates the derivation considerably. Instead, when dealing with a slender rod, it is possible to introduce an assumption analogous to the one used for the case of the St. Venant torsion, whereby φ can be written as:

$$\varphi = \tau\,\tilde{\varphi}(x_0\,,y_0\,,z_0) \quad , \tag{B-29}$$

where $\tilde{\varphi}$ is still a function of x_0 because it is conceivable that the cross section changes along the span with x_0.

According to Equations (B-28), (B-24), (B-27) and (B-29):

$$T = \iint_A E(\tilde{\varepsilon}_{xx} - y_0\,\kappa_y - z_0\,\kappa_z + \tau_{,x}\,\tilde{\varphi} + \tau\,\tilde{\varphi}_{,x})dy_0\,dz_0 \quad . \tag{B-30}$$

Restricting the derivation to symmetric cross sections (at least about one axis of symmetry) yields:

$$\iint_A \tilde{\varphi}\, dy_0\, dz_0 = \iint_A \tilde{\varphi}_{,x}\, dy_0\, dz_0 = 0 \qquad \text{(B-31)}$$

and Equation (B-30) becomes:

$$T = E\, A(\tilde{\varepsilon}_{xx} - y_{oc}\, \kappa_y - z_{oc}\, \kappa_z) \qquad \text{(B-32)}$$

while:

$$\iint_A y_0\, dy_0\, dz_0 = y_{oc}\, A \; ,$$

$$\iint_A z_0\, dy_0\, dz_0 = z_{oc}\, A \; . \qquad \text{(B-33)}$$

The point (y_{oc}, z_{oc}) is the point of intersection of the tension axis and the cross section. Equation (B-32) together with the first of Equations (B-24) implies:

$$\sigma_{xx} = \frac{T}{A} + (y_{oc} - y_0)E\, \kappa_y + (z_{oc} - z_0)E\, \kappa_z$$

$$+ E\, \tau_{,x}\, \tilde{\varphi} + E\, \tau\, \tilde{\varphi}_{,x} \; . \qquad \text{(B-34)}$$

After examining the force resultants on a cross section of the deformed rod, the moment resultants about the point $(y_0 = z_0 = 0)$, $\bar{M}$, will be considered, where:

$$\bar{M} = \iint_A \bar{d} \times \bar{t}\, dA \; . \qquad \text{(B-35)}$$

According to the Bernoulli-Euler hypothesis:

$$\bar{d} = y_0\,\hat{e}'_y + z_0\,\hat{e}'_z + \tau\,\tilde{\varphi}\,\hat{e}'_x \quad . \tag{B-36}$$

Substitution of Equations (B-27), (B-24) and (B-36) into (B-35) implies:

$$\bar{M} = M_x\,\hat{e}'_x + M_y\,\hat{e}'_y + M_z\,\hat{e}'_z \tag{B-37}$$

where:

$$\left.\begin{aligned} M_x &= \iint_A [y_0\,\tau_{xz} - z_0\,\tau_{xy}]dy_0\,dz_0 \\ M_y &= \iint_A [\sigma_{xx}\,z_0 - \tau\,\tilde{\varphi}\,\tau_{xz}]dy_0\,dz_0 \\ M_z &= \iint_A [-\sigma_{xx}\,y_0 + \tau\,\tilde{\varphi}\,\tau_{xy}]dy_0\,dz_0 \end{aligned}\right\} \quad . \tag{B-38}$$

Substitution of Equation (B-24) into the first of Equations (B-38) implies:

$$\begin{aligned} M_x = {} & \iint_A (y_0\,\tilde{\tau}_{xz} - z_0\,\tilde{\tau}_{xy})dy_0\,dz_0 + G\,\kappa_z\,\tau \iint_A y_0\,\tilde{\varphi}\,dy_0\,dz_0 \\ & - G\,\kappa_y\,\tau \iint_A z_0\,\tilde{\varphi}\,dy_0\,dz_0 \\ & + G\,\tau \iint_A [y_0^2 + z_0^2 + y_0\,\tilde{\varphi}_{,z} - z_0\,\tilde{\varphi}_{,y}]dy_0\,dz_0 \quad . \end{aligned} \tag{B-39}$$

The first integral in Equation (B-39) is the torque which is produced by the shearing forces V_y, V_z around the point $z_{oc} = y_{oc} = 0$. This point is the shear center of the cross section; thus, by definition this integral becomes zero. The last integral is the torsional stiffness, J, of the cross section known from St. Venant torsion. Thus, expression (B-39) simplifies to:

$$M_x = G J \tau + G \tau \kappa_z \iint_A y_0 \tilde{\varphi} \, dy_0 \, dz_0$$

$$- G \tau \kappa_y \iint_A z_0 \tilde{\varphi} \, dy_0 \, dz_0 \quad . \qquad \text{(B-40)}$$

Substitution of Equations (B-34) and (B-24) into the second and third of expressions (B-38) implies:

$$M_y = - E I_{23} \kappa_y - E I_{33} \kappa_z + T z_{oc} + \tau_{,x} E \iint_A z_0 \tilde{\varphi} \, dy_0 \, dz_0$$

$$+ \tau E \iint_A z_0 \tilde{\varphi}_{,x} \, dy_0 \, dz_0 - \tau \iint_A \tilde{\varphi} \tilde{\tau}_{xz} \, dy_0 \, dz_0$$

$$- \tau^2 G \iint_A \tilde{\varphi}\tilde{\varphi}_{,z} \, dy_0 \, dz_0 - \tau^2 G \iint_A y_0 \tilde{\varphi} \, dy_0 \, dz_0$$

$$- \tau^2 \kappa_z G \iint_A \tilde{\varphi}^2 \, dy_0 \, dz_0 \quad ; \qquad \text{(B-41a)}$$

$$M_z = E I_{22} \kappa_y + E I_{23} \kappa_z - T y_{oc} - \tau_{,x} E \iint_A y_0 \tilde{\varphi} \, dy_0 \, dz_0$$

$$- \tau E \iint_A y_0 \tilde{\varphi}_{,z} \, dy_0 \, dz_0 + \tau \iint_A \tilde{\varphi} \tilde{\tau}_{xy} \, dy_0 \, dz_0$$

$$+ \underline{\tau^2 G \iint_A \tilde{\varphi}\tilde{\varphi}_{,y} \, dy_0 \, dz_0} - \tau^2 G \iint_A z_0 \tilde{\varphi} \, dy_0 \, dz_0$$

$$+ \tau^2 \kappa_y G \iint_A \tilde{\varphi}^2 \, dy_0 \, dz_0 \quad ; \qquad \text{(B-41b)}$$

where I_{22}, I_{33}, and I_{23} are flexural moments of inertia given by

$$I_{22} = \iint_A (y_{0c} - y_0)^2 \, dy_0 \, dz_0$$

$$I_{33} = \iint_A (z_{oc} - z_0)^2 \, dy_0 \, dz_0 \qquad \text{(B-42)}$$

$$I_{23} = \iint_A (y_{oc} - y_0)(z_{oc} - z_0) dy_0 \, dz_0$$

The underlined terms in Equations (B-41) become zero in the case of a symmetric cross section, which is the type of cross section being considered.

In the case of slender rods with closed cross sections, the influence of warping is usually neglected and in this case, expressions (B-40) and (B-41) become:

$$\left.\begin{aligned} M_x &= G\,J\,\tau \\ M_y &= -E\,I_{23}\,\kappa_y - E\,I_{33}\,\kappa_z + T\,z_{oc} \\ M_z &= E\,I_{22}\,\kappa_y + E\,I_{23}\,\kappa_z - T\,y_{oc} \end{aligned}\right\} \qquad \text{(B-43)}$$

<u>Comment</u>: In Equation (B-26) a contribution of the axial stress to the shearing forces acting on the cross section of the blade exists. For beams where the torsional stiffness is very small compared to the bending stiffness (like the case of beams with thin open cross sections) this contribution, sometimes called the trapeze effect, can cause considerable influence of axial forces on the torsional rigidity of those beams (see, for example, Ref. 16). However, rotor blades, which are the subject matter of this study are made of either closed or solid cross sections, where the above mentioned effect can be neglected, as pointed out by Goodier (Ref. 16, p. 386, second column, line 18 from the top). Therefore, the assumptions leading from Equation (B-26) to (B-27) seems to be appropriate for the present study.

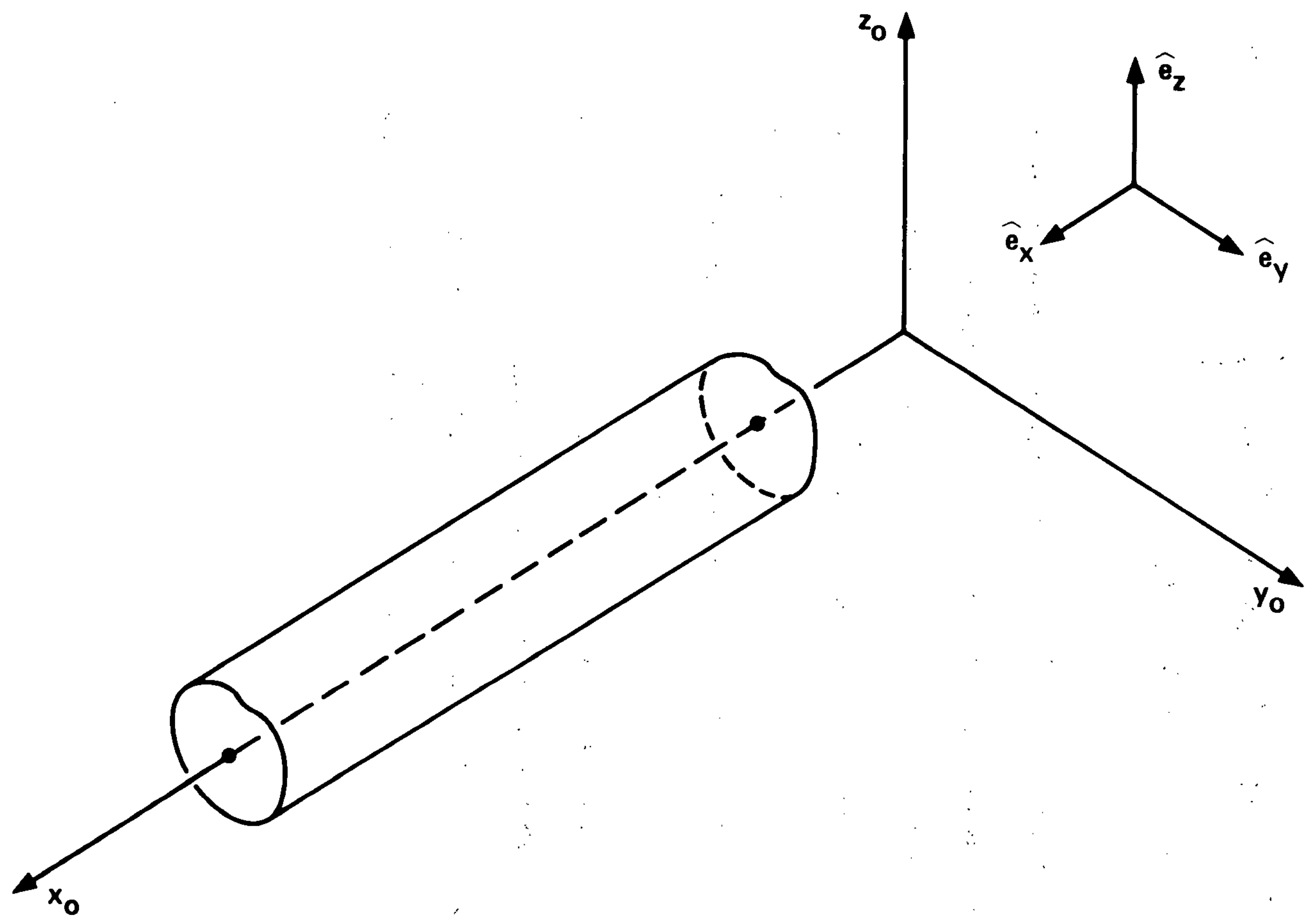

Fig. B1. Geometry of the Rod Before the Deformation

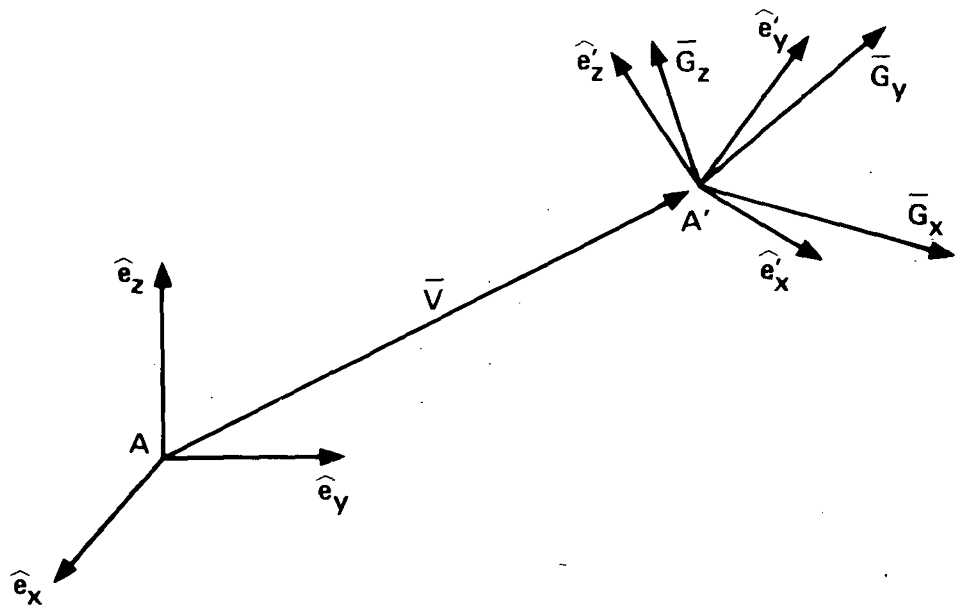

Fig. B2. Procedure of Deformation

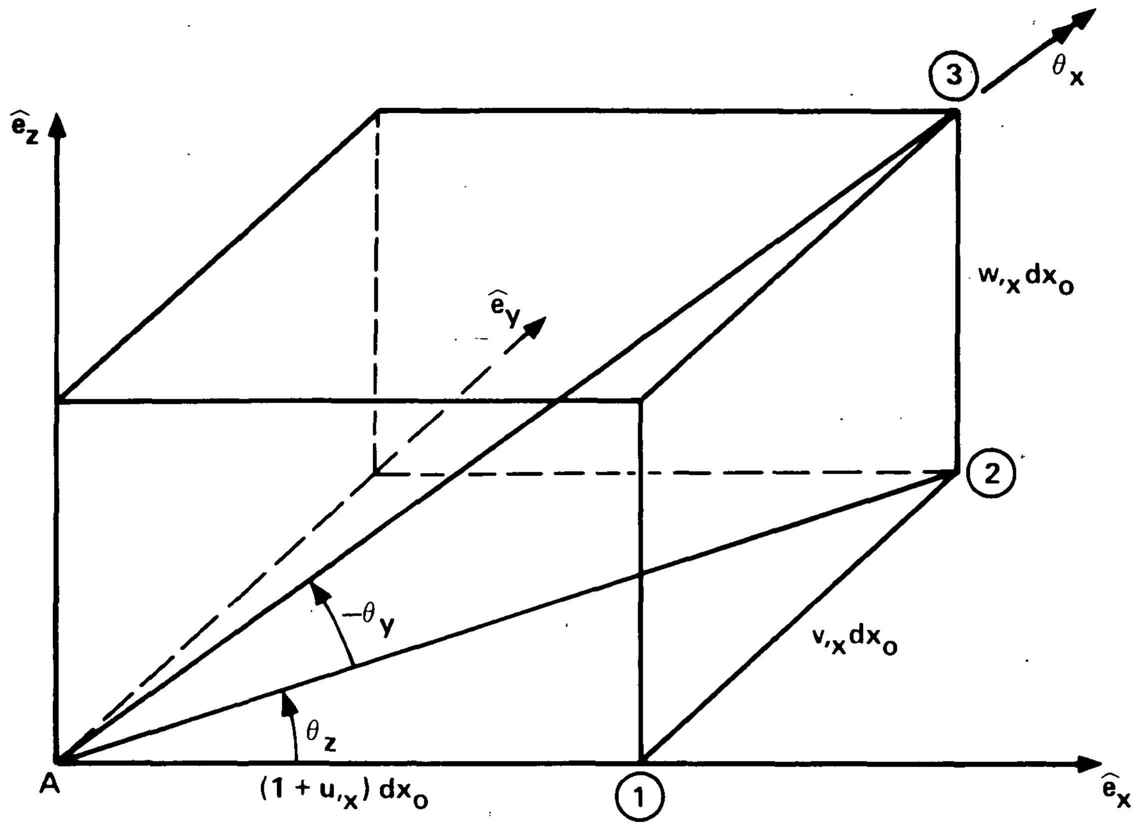

Fig. B3. Euler Angles When the Order of Rotation is $\theta_z, \theta_y, \theta_x$

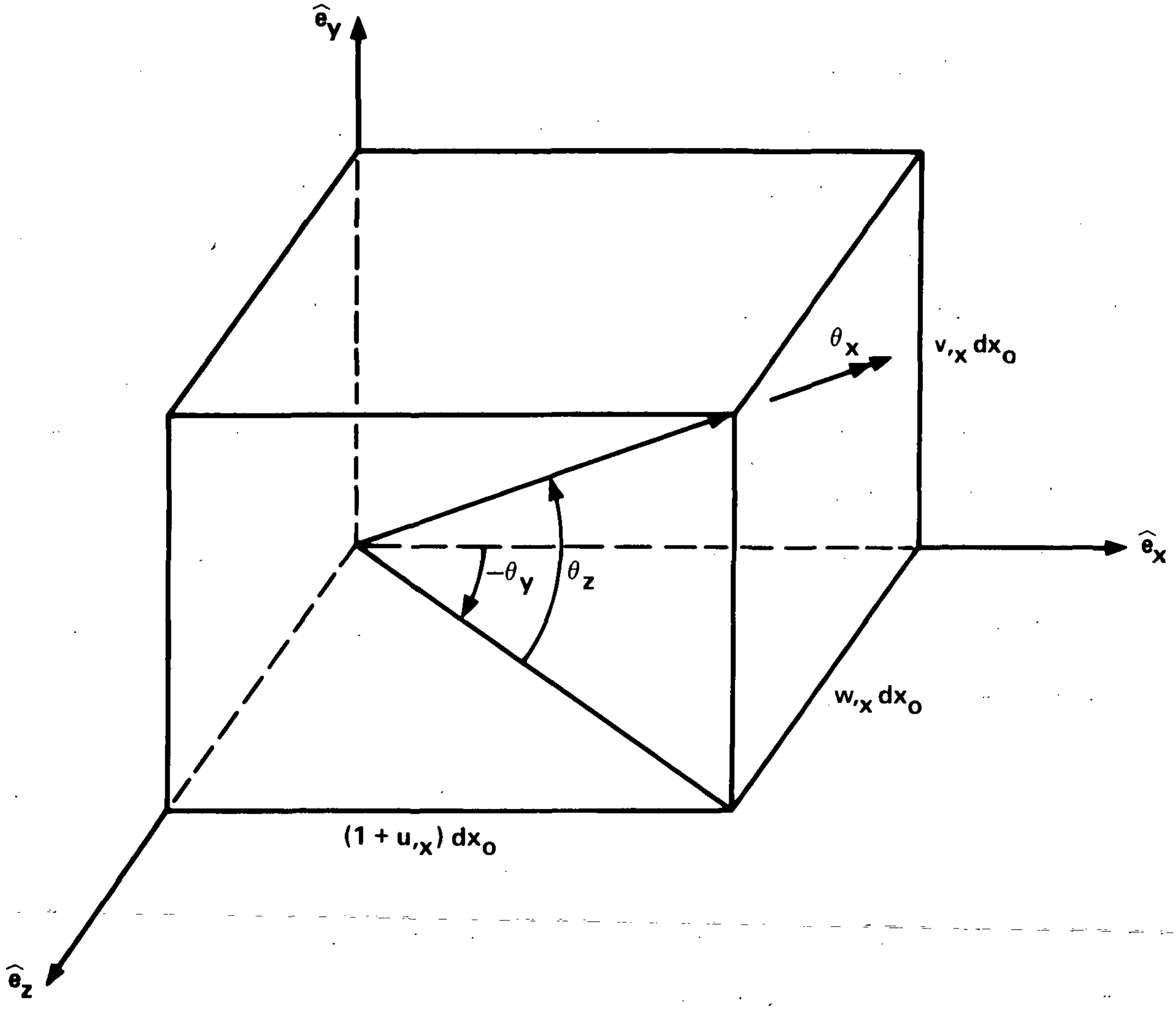

Fig. B4. Euler Angles When the Order of Rotation is $\theta_y, \theta_z, \theta_x$

APPENDIX C

EQUATIONS OF EQUILIBRIUM OF A DEFORMED ROD

In Appendix B the expressions for the resultant forces and moments which act on a cross section of a deformed slender rod, which was initially straight, were calculated. Furthermore, it was assumed that the rod is subjected to a distributed force, $\bar{p}$, per unit length of its undeformed axis. This load $\bar{p}$ includes body forces, surface tractions and inertial loading. There is also a moment $\bar{q}$ per unit length of the undeformed axis of the rod. This includes body couples, moments of surface tractions and moments of inertial loading. The loads $\bar{p}$ and $\bar{q}$ are assumed to be continuous and also having continuous derivatives.

Figure C-1 shows a segment of the deformed rod. From equilibrium of forces the following equation is obtained:

$$\bar{F}_{,x} + \bar{p} = 0 \ . \tag{C-1}$$

From the equilibrium of moments about the point P, letting $dx_0 \to 0$, the following equation is obtained:

$$\bar{M}_{,x} + \bar{q} + \hat{e}'_x \times \bar{F} = 0 \ . \tag{C-2}$$

The load $\bar{p}$ and the couple $\bar{q}$ are described by their components:

$$\bar{p} = p_x e'_x + p_y \hat{e}'_y + p_z \hat{e}'_z \quad ,$$
$$\bar{q} = q_x \hat{e}'_x + q_y \hat{e}'_y + q_z \hat{e}'_z \quad . \qquad \text{(C-3)}$$

Substitution of the first of Equations (C-3) into Equation (C-1), together with Equations (B-17) and (B-28), implies:

$$\left.\begin{aligned}
T_{,x} - \kappa_y V_y - \kappa_z V_z + p_x &= 0 \quad (\hat{e}'_x \text{ direction}) \\
V_{y,x} + \kappa_y T - \tau V_z + p_y &= 0 \quad (\hat{e}'_y \text{ direction}) \\
V_{z,x} + \kappa_z T + \tau V_y + p_z &= 0 \quad (\hat{e}'_z \text{ direction})
\end{aligned}\right\} \quad . \qquad \text{(C-4)}$$

Substitution of the second of equations (C-3) into (C-2), together with Equations (B-17), (B-28) and (B-37) implies:

$$\left.\begin{aligned}
M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x &= 0 \quad (\hat{e}'_x \text{ direction}) \\
M_{y,x} + \kappa_y M_x - \tau M_z + q_y - V_z &= 0 \quad (\hat{e}'_y \text{ direction}) \\
M_{z,x} + \kappa_z M_x + \tau M_y + q_z + V_y &= 0 \quad (\hat{e}'_y \text{ direction})
\end{aligned}\right\} \quad . \qquad \text{(C-5)}$$

Equations (C-4) and (C-5) are exact, and contain no approximations. The procedure of solution is as follows: Expressions for V_y and V_z are obtained from the second and third of Equations (C-5). These are subsequently substituted in Equations (C-4). Following this procedure, the expressions for the shearing forces become:

$$
\begin{aligned}
V_y &= -(M_{z,x} + \kappa_z M_x + \tau M_y + q_z) \quad , \\
V_z &= M_{y,x} + \kappa_y M_x - \tau M_z + q_y \quad .
\end{aligned}
\tag{C-6}
$$

Substitution of expressions (C-6) into Equations (C-4) implies:

$$
\begin{aligned}
&T_{,x} + \kappa_y(M_{z,x} + \tau M_y + q_z) \\
&\quad - \kappa_z(M_{y,x} - \tau M_z + q_y) + p_x = 0 \quad ,
\end{aligned}
\tag{C-7a}
$$

$$
\begin{aligned}
&-(M_{z,x} + \kappa_z M_x + \tau M_y + q_z)_{,x} + \kappa_y T \\
&\quad - \tau(M_{y,x} + \kappa_y M_x - \tau M_z + q_y) + p_y = 0 \quad ,
\end{aligned}
\tag{C-7b}
$$

$$
\begin{aligned}
&(M_{y,x} + \kappa_y M_x - \tau M_z + q_y)_{,x} + \kappa_z T \\
&\quad - \tau(M_{z,x} + \kappa_z M_x + \tau M_y + q_z) + p_z = 0 \quad .
\end{aligned}
\tag{C-7c}
$$

Equations (C-7) represent the equilibrium of forces in the directions $\hat{e}'_x$, $\hat{e}'_y$, and $\hat{e}'_z$, respectively. There is still the moment equation, in the $\hat{e}'_x$ direction that must be satisfied, which is:

$$
M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x = 0 \quad . \tag{C-8}
$$

Equations (C-7) and (C-8) are accurate and contain no approximations. These four equations (three of (C-7) and one of (C-8) must be solved in order to investigate the problem of the deformed rod. In order to obtain a solution, it is necessary to express the moments in terms of the derivatives of the displacements and the rotation of the cross section about the elastic axis. This reduces the problem to one containing four equations and four unknowns.

In trying to simplify the equations, use can be made of the ordering scheme. Performing the differentiation in Equations (C-7a - c) implies:

$$T_{,x} + \kappa_y M_{z,x} - \kappa_z M_{y,x} + \tau(\kappa_y M_y + \kappa_z M_z)$$
$$+ \kappa_y q_z - \kappa_z q_y + p_x = 0 \qquad \text{(C-9a)}$$

$$-M_{z,xx} - (\kappa_{z,x} + \tau\kappa_y)M_x - \kappa_z M_{x,x} - \tau_{,x} M_y$$
$$- 2\tau M_{y,x} + \underline{\tau^2 M_z} + \kappa_y T - q_{z,x} - \tau q_y + p_y = 0 \qquad \text{(C-9b)}$$

$$M_{y,xx} + (\kappa_{y,x} - \tau \kappa_z)M_x + \kappa_y M_{x,x} - \tau_{,x} M_z - 2\tau M_{z,x}$$
$$- \underline{\tau^2 M_y} + \kappa_z T + q_{y,x} - \tau q_z + p_z = 0 \qquad \text{(C-9c)}$$

$$M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x = 0 \quad . \qquad \text{(C-9d)}$$

The underlined terms in Equations (C-9b) and (C-9c) can be neglected according to the ordering scheme. As an example, consider Equation (C-9b). The underlined terms $\tau^2 M_z$ can be neglected, compared to the term $-M_{z,xx}$ which also appears in the equation. As a clarification, recall that according to the ordering scheme one can write:

$$M_z \cong M_z(1 - \varphi^2) \quad .$$

If both sides of the last equality are differentiated twice with respect to x, it implies:

$$M_{z,xx} \cong M_{z,xx} - \underline{M_{z,xx}\varphi^2 - 4M_{z,x}\varphi\varphi_{,x} - 2M_z(\varphi^2_{,x} + \varphi\varphi_{,xx})}$$

The underlined terms in the last equality are negligible compared to $M_{z,xx}$. The term $-2M_z \varphi^2_{,x}$ is of the order of magnitude of $\tau^2 M_z$ and so neglection of the underlined terms in Equations (C-9) is justified. It is clear that the last argument is correct only if deflections and force and moment resultants are changing gradually along the span and do not have very high gradients.

Substitution of the expression for $M_{x,x}$ from Equation (C-9d) into Equations (C-9b,c), and using the ordering scheme, implies the following set of equations:

$$T_{,x} + \kappa_y M_{z,x} - \kappa_z M_{y,x} + \tau(\kappa_y M_y + \kappa_z M_z) + \kappa_y q_z - \kappa_z q_y + p_x = 0 \qquad \text{(C-10a)}$$

$$-M_{z,xx} - (\kappa_{z,x} + \tau\kappa_y)M_x - (\tau_{,x} + \kappa_y\kappa_z)M_y - 2\tau M_{y,x} + \kappa_y T + \kappa_z q_x - \tau q_y - q_{z,x} + p_y = 0 \qquad \text{(C-10b)}$$

$$M_{y,xx} + (\kappa_{y,x} - \tau\kappa_z)M_x - (\tau_{,x} - \kappa_y\kappa_z)M_z - 2\tau M_{z,x} + \kappa_z T - \kappa_y q_x - \tau q_z + q_{y,x} + p_z = 0 \qquad \text{(C-10c)}$$

$$M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x = 0 \quad . \qquad \text{(C-10d)}$$

Sometimes it is more convenient to write the equations of equilibrium in the undeformed directions $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$, instead of the directions after the deformation $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$ as was done in the previous part of this Appendix. The general relation between the two systems is given in the form:

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + S_{12}\, \hat{e}_y + S_{13}\, \hat{e}_z \\ \hat{e}'_y &= S_{21}\, \hat{e}_x + \hat{e}_y + S_{23}\, \hat{e}_z \\ \hat{e}'_z &= S_{31}\, \hat{e}_x + S_{32}\, \hat{e}_y + \hat{e}_z \end{aligned}\right\} . \tag{C-11}$$

Two such transformations, belonging to the class of transformations containing the inherent assumption that quantities of order of ε^2 are negligible compared to unity, are presented in Appendix B as Equations (B-13) and (B-14). The components S_{ij} are of order, ε, or less.

The resultant elastic force on a cross section of the rod is given in an analogous manner to Equation (B-28), by the expression:

$$\bar{F} = \tilde{T}\, \hat{e}_x + \tilde{V}_y\, \hat{e}_y + \tilde{V}_z\, \hat{e}_z \tag{C-12}$$

and the resultant elastic moment, which acts on a cross section of the rod is given in an analogous manner to Equation (B-37), by the expression:

$$\bar{M} = \tilde{M}_x\, \hat{e}_x + \tilde{M}_y\, \hat{e}_y + \tilde{M}_z\, \hat{e}_z \ . \tag{C-13}$$

Equations (C-12), (C-13), (B-28) and (B-37) together with Equation (C-11) imply:

$$\left.\begin{aligned} \tilde{T} &= T + S_{21} V_y + S_{31} V_z \\ \tilde{V}_y &= S_{12} T + V_y + S_{32} V_z \\ \tilde{V}_z &= S_{13} T + S_{23} V_y + V_z \end{aligned}\right\} \tag{C-14}$$

and

$$\left.\begin{aligned} \tilde{M}_x &= M_x + S_{21} M_y + S_{31} M_z \\ \tilde{M}_y &= S_{12} M_x + M_y + S_{32} M_z \\ \tilde{M}_z &= S_{13} M_x + S_{23} M_y + M_z \end{aligned}\right\} . \tag{C-15}$$

The distributed force $\bar{p}$ and distributed moment $\bar{q}$ per unit length are given by (compare to Equation (C-3)):

$$\begin{aligned} \bar{p} &= \tilde{p}_x \hat{e}_x + \tilde{p}_y \hat{e}_y + \tilde{p}_z \hat{e}_z \quad , \\ \bar{q} &= \tilde{q}_x \hat{e}_x + \tilde{q}_y \hat{e}_y + \tilde{q}_z \hat{e}_z \quad . \end{aligned} \tag{C-16}$$

From the equilibrium of forces (Equation (C-1)) the following equations are obtained:

$$\left.\begin{aligned} \tilde{T}_{,x} + \tilde{p}_x &= 0 \\ \tilde{V}_{y,x} + \tilde{p}_y &= 0 \\ \tilde{V}_{z,x} + \tilde{p}_z &= 0 \end{aligned}\right\} , \tag{C-17}$$

and the equilibrium of moments (Equation (C-2)) implies:

$$\left.\begin{aligned}
\tilde{M}_{x,x} - S_{13}\tilde{V}_y + S_{12}\tilde{V}_z + \tilde{q}_x &= 0 \\
\tilde{M}_{y,x} + S_{13}\tilde{T} - \tilde{V}_z + \tilde{q}_y &= 0 \\
\tilde{M}_{z,x} - S_{12}\tilde{T} + \tilde{V}_y + \tilde{q}_z &= 0
\end{aligned}\right\} \quad . \qquad \text{(C-18)}$$

From Equations (C-14), neglecting terms of order ε^2 compared to unity, one obtains:

$$\tilde{T} = T + (S_{21} - S_{31}S_{23})\tilde{V}_y + (S_{31} - S_{21}S_{32})\tilde{V}_z \quad . \qquad \text{(C-19)}$$

The second and third expressions of Equations (C-18), together with (C-19), yields, after neglecting terms of the order ε^2 compared to unity:

$$\begin{aligned}
\tilde{V}_y = & -\tilde{M}_{z,x} + S_{12}(S_{31} - S_{21}S_{32})\tilde{M}_{y,x} \\
& + S_{12}\,T - \tilde{q}_z + S_{12}(S_{31} - S_{21}S_{32})\tilde{q}_y \ , \qquad \text{(C-20a)}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{V}_z = & \ \tilde{M}_{y,x} - S_{13}(S_{21} - S_{31}S_{23})\tilde{M}_{z,x} + S_{13}\,T \\
& - S_{13}(S_{21} - S_{31}S_{23})\tilde{q}_z + \tilde{q}_y \quad . \qquad \text{(C-20b)}
\end{aligned}$$

Substitution of Equations (C-19) and (C-20) into (C-17) and the first expression of Equations (C-18), yields:

$$\begin{aligned}
& [T - (S_{21} - S_{31}S_{23})\tilde{M}_{z,x} + (S_{31} - S_{21}S_{32})\tilde{M}_{y,x}]_{,x} \\
& \quad - [(S_{21} - S_{31}S_{23})\tilde{q}_z]_{,x} + [(S_{31} - S_{21}S_{32})\tilde{q}_y]_{,x} + \tilde{p}_x = 0 \ , \qquad \text{(C-21a)}
\end{aligned}$$

$$[-\tilde{M}_{z,x} + S_{12}(S_{31} - S_{21}S_{32})\tilde{M}_{y,x}]_{,x} + (S_{12}\ T)_{,x}$$

$$- \tilde{q}_{z,x} + [S_{12}(S_{31} - S_{21}S_{32})\tilde{q}_y]_{,x} + \tilde{p}_y = 0 \quad , \qquad \text{(C-21b)}$$

$$[\tilde{M}_{y,x} - S_{13}(S_{21} - S_{31}S_{23})\tilde{M}_{z,x}]_{,x} + (S_{13}\ T)_{,x}$$

$$+ \tilde{q}_{y,x} - [S_{13}(S_{21} - S_{31}S_{23})\tilde{q}_z]_{,x} + \tilde{p}_z = 0 \quad , \qquad \text{(C-21c)}$$

$$\tilde{M}_{x,x} + S_{13}\ \tilde{M}_{z,x} + S_{12}\ \tilde{M}_{y,x} + S_{12}\ \tilde{q}_y + S_{13}\ \tilde{q}_z + \tilde{q}_x = 0\ . \qquad \text{(C-21d)}$$

According to the assumption that quantities of the order of ε^2 are negligible compared to unity, and the orthonormality of the transformation represented by Equation (C-11), one has:

$$\left.\begin{aligned}
S_{21} - S_{31}S_{23} &\cong -S_{12} \\
S_{31} - S_{21}S_{32} &\cong -S_{13} \\
S_{12} + S_{31}S_{32} &\cong -S_{21} \\
S_{13} + S_{21}S_{23} &\cong -S_{31} \\
S_{23} - S_{12}S_{31} &\cong -S_{32} \\
S_{32} - S_{13}S_{21} &\cong -S_{23} \\
S_{21} + S_{13}S_{23} &\cong -S_{12} \\
S_{31} + S_{12}S_{32} &\cong -S_{13}
\end{aligned}\right\} \ . \qquad \text{(C-22)}$$

Making use of Equations (C-23), Equations (C-21a - d) turn out to be:

$$(T + S_{12}\tilde{M}_{z,x} - S_{13}\tilde{M}_{y,x})_{,x} + (S_{12}\tilde{q}_z)_{,x} - (S_{13}\tilde{q}_y)_{,x} + \tilde{p}_x = 0 \quad \text{(C23a)}$$

$$(-\tilde{M}_{z,x} - S_{12}S_{13}\tilde{M}_{y,x})_{,x} + (S_{12}T)_{,x} - \tilde{q}_{z,x} - (S_{12}S_{13}\tilde{q}_y)_{,x} + \tilde{p}_y = 0 \quad \text{(C23b)}$$

$$(\tilde{M}_{y,x} + S_{12}S_{13}\tilde{M}_{z,x})_{,x} + (S_{13}T)_{,x} + \tilde{q}_{y,x} + (S_{12}S_{13}\tilde{q}_z)_{,x} + \tilde{p}_z = 0 \quad \text{(C23c)}$$

$$\tilde{M}_{x,x} + S_{13}\tilde{M}_{z,x} + S_{12}\tilde{M}_{y,x} + S_{12}\tilde{q}_y + S_{13}\tilde{q}_z + \tilde{q}_x = 0 . \quad \text{(C-23d)}$$

Substitution of Equations (C-15) into Equations (C-23), using Equation (C-24d) also, to substitute for $M_{x,x}$, and neglecting terms of order ε^2 compared to unity, yields:

$$\{T - S_{21}M_{z,x} + S_{31}M_{y,x} - S_{13}(S_{32})_{,x}M_z + S_{12}(S_{23})_{,x}M_y + [S_{12}(S_{13})_{,x} - S_{13}(S_{12})_{,x}]M_x\}_{,x} + (S_{12}\tilde{q}_z)_{,x} - (S_{13}\tilde{q}_y)_{,x} + \tilde{p}_x = 0 \quad \text{(C-24a)}$$

$$-\{M_{z,x} + (S_{13})_{,x}M_x + [(S_{23})_{,x} - S_{13}(S_{21})_{,x}]M_y - S_{32}M_{y,x}\}_{,x} + (S_{12}T)_{,x} - \tilde{q}_{z,x} + (S_{13}\tilde{q}_x)_{,x} + \tilde{p}_y = 0 \quad \text{(C-24b)}$$

$$\{M_{y,x} + (S_{12})_{,x} M_x + [(S_{32})_{,x} - S_{12}(S_{31})_{,x}]M_z - S_{23} M_{z,x}\}_{,x}$$

$$+ (S_{13} T)_{,x} - (S_{12} \tilde{q}_x)_{,x} + \tilde{q}_{y,x} + \tilde{p}_z = 0 \quad , \qquad \text{(C-24c)}$$

$$M_{x,x} + [(S_{21})_{,x} + S_{13}(S_{23})_{,x}]M_y + [(S_{31})_{,x} + S_{12}(S_{32})_{,x}]M_z$$

$$+ \tilde{q}_x + S_{12} \tilde{q}_y + S_{13} \tilde{q}_z = 0 \quad . \qquad \text{(C-24d)}$$

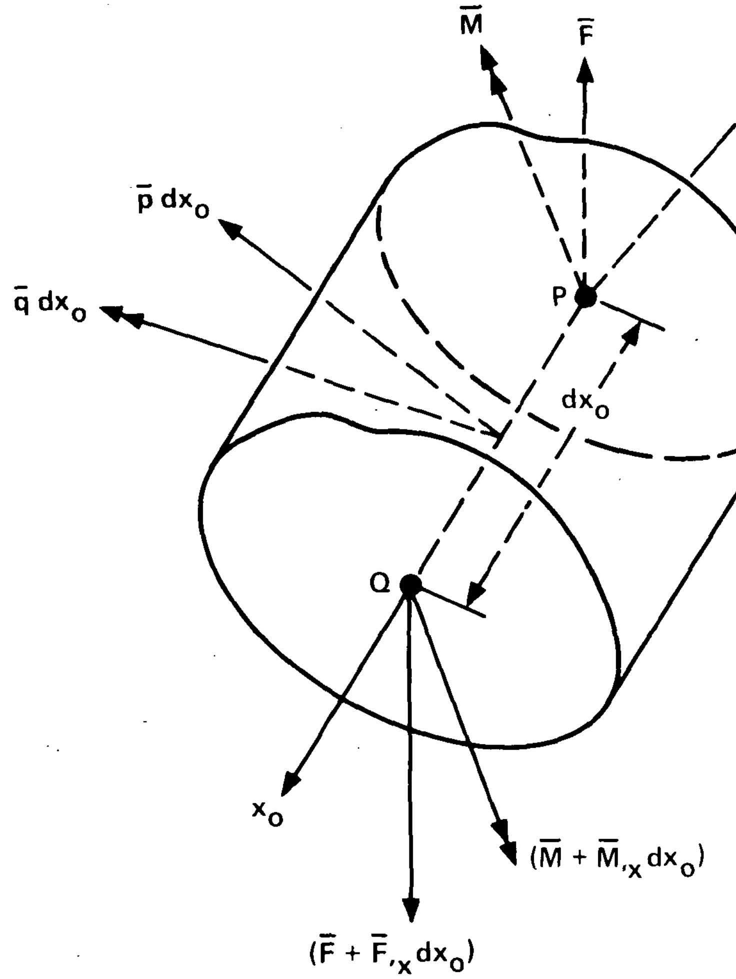

Fig. C-1. Forces and Moments on Segment of the Deformed Rod

APPENDIX D

DERIVATION OF THE EQUATIONS OF EQUILIBRIUM BY THE USE OF THE PRINCIPLE OF VIRTUAL WORK

D.1 Principle of Virtual Work Applied to a Rod

Usually, in a structural system subject to loads, internal forces develop between the various components of the system as a result of the external loads. If the work of the internal forces is denoted by W_I, and that of the external forces W_E, then the principle of virtual work can be simply expressed as:

$$\delta W_I = -\delta W_E \tag{D-1}$$

where δW_I and δW_E are the work done during a virtual displacement by the internal and external forces, respectively.

In the case of an elastic system,

$$U = -W_I \tag{D-2}$$

where U is the elastic energy in the system. Therefore, Equation (D-1) becomes:

$$\delta U = \delta W_E \ . \tag{D-3}$$

Using Equations (D-2) and (D-3) implies (for example, Eq. (9-125) of Wempner, Ref. 11), the following expression for a continuous body:

$$\delta W_I = - \iiint_{\substack{\text{Volume of the} \\ \text{body}}} [\sigma_{xx}\,\delta\varepsilon_{xx} + \sigma_{yy}\,\delta\varepsilon_{yy} + \sigma_{zz}\,\delta\varepsilon_{zz} + 2\tau_{xy}\,\delta\varepsilon_{xy} + 2\tau_{xz}\,\delta\varepsilon_{xz} + 2\tau_{yz}\,\delta\varepsilon_{yz}]dx_0\,dy_0\,dz_0 \ . \qquad \text{(D-4)}$$

For the case of a slender rod of length ℓ and cross section A, within the framework of the Bernoulli-Euler assumptions*

$$\sigma_{yy} = \sigma_{zz} = \varepsilon_{yz} = 0 \ ,$$

Equation (D-4) becomes:

$$\delta W_I = - \int_{x_0=0}^{x_0=\ell} \iint_A [\sigma_{xx}\,\delta\varepsilon_{xx} + 2\tau_{xy}\,\delta\varepsilon_{xy} + 2\tau_{xz}\,\delta\varepsilon_{xz}]dx_0\,dy_0\,dz_0 \ . \qquad \text{(D-5)}$$

The virtual displacement of the rod is given by a displacement $\delta\bar{W}(x_0)$ of every point on the elastic axis, accompanied by rotation $\delta\bar{\theta}(x_0)$ of the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$ at every point of the deformed rod. The virtual rotation $\delta\bar{\theta}$ can be described in this case as a vector because it is infinitesimal. The rod is acted upon by a distributed force, $\bar{p}$, per unit length of its undeformed axis, which includes body forces, surface tractions and inertia loading, and it acts at the elastic axis

*The fact that $\varepsilon_{yz} = 0$ emerges from the assumption that strains within the cross sections of the rod are neglected.

as described in Appendix C. There is also a distributed moment, $\bar{q}$, per unit length of the undeformed axis of the rod. It includes body couples, moments of surface tractions and moments of intertial loading. This moment also acts at the elastic axis of the rod. It is clear, therefore, that:

$$\delta W_E = \int_{x_0=0}^{x_0=\ell} (\bar{p} \cdot \delta \bar{W} + \bar{q} \cdot \delta \bar{\theta}) dx_0 \quad . \tag{D-6}$$

Equations (D-3), (D-5) and (D-6) imply:

$$\int_{x_0=0}^{x_0=\ell} \left\{ \iint_A [\sigma_{xx} \delta \varepsilon_{xx} + 2\tau_{xy} \delta \varepsilon_{xy} + 2\tau_{xz} \delta \varepsilon_{xz}] dy_0 \, dz_0 - \bar{p} \cdot \delta \bar{W} - \bar{q} \cdot \delta \bar{\theta} \right\} dx_0 = 0 \quad . \tag{D-7}$$

According to Equations (B-23) of Appendix B:

$$\left. \begin{aligned} \delta \varepsilon_{xx} &= \delta \tilde{\varepsilon}_{xx} - y_0 \, \delta \kappa_y - z_0 \, \delta \kappa_z \\ \delta \varepsilon_{xy} &= - \frac{1}{2} z_0 \, \delta \tau \\ \delta \varepsilon_{xz} &= \frac{1}{2} y_0 \, \delta \tau \end{aligned} \right\} \quad . \tag{D-8}$$

In deriving Equation (D-8), use was made of the assumption that in the case of a slender rod with closed cross section, the influence of warping is negligible and, therefore, the virtual work is taken as that performed during a rigid body motion of a slice of the rod, and thus, $\delta\varphi$ is assumed to be zero. The terms $\varphi\delta\kappa_y$ and $\varphi\delta\kappa_z$ are also neglected as a consequence of assuming the warping to be negligible.

Substitution of Equation (D-8) into (D-7), use of Equations (B-38) from Appendix B, and neglecting the influence of warping, implies:

$$\int_{x_0=0}^{x_0=\ell} (T\,\delta\tilde{\varepsilon}_{xx} + M_z\,\delta\kappa_y - M_y\,\delta\kappa_z + M_x\,\delta\tau - \bar{p}\cdot\delta\bar{W} - \bar{q}\cdot\delta\bar{\Theta})dx_0 = 0\ , \qquad \text{(D-9)}$$

where $\delta\tilde{\varepsilon}_{xx}$, $\delta\kappa_y$, $\delta\kappa_z$ and $\delta\tau$ are functions of $\delta\bar{W}$ and $\delta\bar{\Theta}$. After substitution of the appropriate values for $\delta\tilde{\varepsilon}_{xx}$, $\delta\kappa_y$, $\delta\kappa_z$ and $\delta\tau$, the equilibrium equations and boundary conditions are obtained by performing an integration by parts of Equation (D-9).

In Chapter 2 of this study it was shown that the equilibrium equations can be obtained with respect to different directions. In the present case, the definition of $\delta\bar{W}$ and $\delta\bar{\Theta}$ determines the directions in which the equations will be valid.

D.2 Equilibrium Equations in the Directions of the Deformed Rod Coordinates $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$

In this case, the virtual displacement is chosen as:

$$\delta\bar{W} = \delta u'\,\hat{e}'_x + \delta v'\,\hat{e}'_y + \delta w'\,\hat{e}'_z\ . \qquad \text{(D-10)}$$

The virtual rotation is given in the form:

$$\delta\bar{\Theta} = n_x\,\hat{e}'_x + n_y\,\hat{e}'_y + n_z\,\hat{e}'_z \quad , \tag{D-11}$$

where n_x, n_y and n_z are the components of the virtual rotation.

Due to $\delta\bar{W}$ and $\delta\bar{\Theta}$ the triad $\hat{e}'_x, \hat{e}'_y, \hat{e}'_z$ is rotated to a new triad $\hat{e}''_x, \hat{e}''_y, \hat{e}''_z$, given by:

$$\left.\begin{aligned} \hat{e}''_x &= \hat{e}'_x + \delta\bar{\Theta} \times \hat{e}'_x = \hat{e}'_x + n_z\,\hat{e}'_y - n_y\,\hat{e}'_z \\ \hat{e}''_y &= \hat{e}'_y + \delta\bar{\Theta} \times \hat{e}'_y = -n_z\,\hat{e}'_x + \hat{e}'_y + n_x\,\hat{e}'_z \\ \hat{e}''_z &= \hat{e}'_z + \delta\bar{\Theta} \times \hat{e}'_z = n_y\,\hat{e}'_x - n_x\,\hat{e}'_y + \hat{e}'_z \end{aligned}\right\} . \tag{D-12}$$

It is clear that $n_x = \delta\phi$, however n_y and n_z are determined by $\delta\bar{W}$. In order to find n_y and n_z let us consider an element dx_0 of the deformed elastic axis, which is shown in Figure D-1. Before the virtual displacement, the element is in position AB, described by $dx_0\,\hat{e}'_x$. After the virtual displacement, the element is in position A'B', given by:

$$\begin{aligned} \overline{A'B'} &= (\bar{R} + dx_0\,\hat{e}'_x + \delta\bar{W} + \delta\bar{W}_{,x}\,dx_0) - (\bar{R} + \delta\bar{W}) \\ &= (\hat{e}'_x + \delta\bar{W}_{,x})dx_0 \quad . \end{aligned} \tag{D-13}$$

Substitution of expression (D-10) into (D-13), and performing the differentiation, while using Equation (B-17) of Appendix B for the derivatives of the unit vectors, implies:

$$\overline{A'B'} = [(1 + \delta u'_{,x} - \kappa_y \delta v' - \kappa_z \delta w')\hat{e}'_x + (\delta v'_{,x} + \kappa_y \delta u' - \tau \delta w')\hat{e}'_y + (\delta w'_{,x} + \kappa_z \delta u' + \tau \delta v')\hat{e}'_z]dx_0 \quad . \qquad \text{(D-14)}$$

Recalling that the virtual displacement $\delta \bar{W}$ is as small as desired, then from Equation (D-14), one obtains after neglecting products of virtual terms:

$$\hat{e}''_x = \hat{e}'_x + (\delta v'_{,x} + \kappa_y \delta u' - \tau \delta w')\hat{e}'_y + (\delta w'_{,x} + \kappa_z \delta u' + \tau \delta v')\hat{e}'_z \quad . \qquad \text{(D-15)}$$

Comparing Equation (D-15) with the first of Equations (D-12), the quantities n_y and n_z are determined. Thus, the rotation components are:

$$\left.\begin{aligned} n_x &= \delta \Phi \\ n_y &= -(\delta w'_{,x} + \kappa_z \delta u' + \tau \delta v') \\ n_z &= (\delta v'_{,x} + \kappa_y \delta u' - \tau \delta w') \end{aligned}\right\} \quad . \qquad \text{(D-16)}$$

Using definitions (B-15) from Appendix B, then:

$$\begin{aligned}
\kappa_y + \delta\kappa_y &= \hat{e}''_{x,x} \cdot \hat{e}''_y \\
\kappa_z + \delta\kappa_z &= \hat{e}''_{x,x} \cdot \hat{e}''_z \qquad . \\
\tau + \delta\tau &= \hat{e}''_{y,x} \cdot \hat{e}''_z
\end{aligned} \tag{D-17}$$

Differentiation of expressions (D-12), using Equations (B-17) of Appendix B, yields:

$$\begin{aligned}
\hat{e}''_{x,x} = {} & \kappa_y \hat{e}'_y + \kappa_z \hat{e}'_z + (n_y \kappa_z - n_z \kappa_y)\hat{e}'_x \\
& + (n_{z,x} + \tau n_y)\hat{e}'_y + (-n_{y,x} + \tau n_z)\hat{e}'_z \quad ,
\end{aligned} \tag{D-18a}$$

$$\begin{aligned}
\hat{e}''_{y,x} = {} & -\kappa_y \hat{e}'_x + \tau \hat{e}'_z - (n_{z,x} + n_x \kappa_z)\hat{e}'_x \\
& - (n_x \tau + n_z \kappa_y)\hat{e}'_y + (n_{x,x} - n_z \kappa_z)\hat{e}'_z \quad .
\end{aligned} \tag{D-18b}$$

Substitution of Equations (D-18) and (D-12) into Equation (D-17), and neglecting nonlinear terms in n_x, n_y and n_z, implies:

$$\begin{aligned}
\delta\kappa_y &= n_{z,x} + \kappa_z n_x + \tau n_y \\
\delta\kappa_z &= -n_{y,x} - \kappa_y n_x + \tau n_z \qquad . \\
\delta\tau &= n_{x,x} - \kappa_y n_y - \kappa_z n_z
\end{aligned} \tag{D-19}$$

Substitution of expressions (D-16) into (D-19) yields:

$$\begin{aligned}
\delta\kappa_y = {} & \delta v'_{,xx} + (\kappa_y \delta u')_{,x} - (\tau \delta w')_{,x} \\
& + \kappa_z \delta\phi - \tau \delta w'_{,x} - \tau \kappa_z \delta u' - \tau^2 \delta v' \quad ,
\end{aligned} \tag{D-20a}$$

$$\delta\kappa_z = \delta w'_{,xx} + (\kappa_z \, \delta u')_{,x} + (\tau \, \delta v')_{,x}$$

$$- \kappa_y \, \delta\phi + \tau \, \delta v'_{,x} + \tau \, \kappa_y \, \delta u' - \tau^2 \, \delta w' \quad , \qquad \text{(D-20b)}$$

$$\delta\tau = \delta\phi_{,x} + \kappa_y \, \delta w'_{,x} + \kappa_y \kappa_z \, \delta u' + \kappa_y \tau \, \delta v'$$

$$- \kappa_z \, \delta v'_{,x} - \kappa_y \kappa_z \, \delta u' + \kappa_z \tau \, \delta w' \quad . \qquad \text{(D-20c)}$$

From Equation (D-14) it is clear that (neglecting products of components of the virtual displacement):

$$\delta\tilde{\varepsilon}_{xx} = \delta u'_{,x} - \kappa_y \, \delta v' - \kappa_z \, \delta w' \quad . \qquad \text{(D-21)}$$

The load $\bar{p}$ and couple $\bar{q}$ are given with respect to the deformed system, in the form of Equations (1) and (2), as:

$$\bar{p} = p_x \, \hat{e}'_x + p_y \, \hat{e}'_y + p_z \, \hat{e}'_z \quad ,$$

$$\bar{q} = q_x \, \hat{e}'_x + q_y \, \hat{e}'_y + q_z \, \hat{e}'_z \quad . \qquad \text{(D-22)}$$

Substitution of expressions (D-10), (D-11), (D-16), (D-20), (D-21), and (D-22) into Equation (D-9) implies:

$$\int_{x_0=0}^{x_0=\ell} (T \, \delta u'_{,x} - T \, \kappa_y \, \delta v' - T \, \kappa_z \, \delta w' + M_z \, \delta v'_{,xx} + M_z (\kappa_y \, \delta u')_{,x}$$

$$- M_z (\tau \, \delta w')_{,x} + M_z \kappa_z \, \delta\phi - M_z \tau \, \delta w'_{,x} - M_z \tau \, \kappa_z \, \delta u' - M_z \tau^2 \, \delta v'$$

$$- M_y \, \delta w'_{,xx} - M_y (\kappa_z \, \delta u')_{,x} - M_y (\tau \, \delta v')_{,x} + M_y \kappa_y \delta\phi - M_y \tau \, \delta v'_{,x} -$$

$$- M_y \tau \kappa_y \delta u' + M_y \tau^2 \delta w' + M_x \delta \phi_{,x} + M_x \kappa_y \delta w'_{,x} + M_x \kappa_y \kappa_z \delta u'$$

$$+ M_x \kappa_y \tau \delta v' - M_x \kappa_z \delta v'_{,x} - M_x \kappa_y \kappa_z \delta u' + M_x \kappa_z \tau \delta w'$$

$$- p_x \delta u' - p_y \delta v' - p_z \delta v' - q_x \delta \phi + q_y \delta w'_{,x} + q_y \kappa_z \delta u' + q_y \tau \delta v' - q_z \delta v'_{,x}$$

$$- q_z \kappa_y \delta u' + q_z \tau \delta w') dx_0 = 0 \quad . \qquad \text{(D-23)}$$

Integration by parts of Equation (D-23) gives the following variational expression:

$$\int_{x_0=0}^{\ell} (-R_1 \delta u' - R_2 \delta v' - R_3 \delta w' - R_4 \delta \phi) dx_0$$
$$+ \Big[B_1 \delta u' \Big]_{x_0=0}^{x_0=\ell} + \Big[B_2 \delta v' \Big]_{x_0=0}^{x_0=\ell} + \Big[B_3 \delta w' \Big]_{x_0=0}^{x_0=\ell} + \Big[B_4 \delta \phi' \Big]_{x_0=0}^{x_0=\ell}$$
$$+ \Big[B_5 \delta v'_{,x} \Big]_{x_0=0}^{x_0=\ell} + \Big[B_6 \delta w'_{,x} \Big]_{x_0=0}^{x_0=\ell} = 0 \qquad \text{(D-24)}$$

where the various expressions in (D-24) are:

$$R_1 = T_{,x} + \kappa_y (M_{z,x} + \tau M_y + q_z) - \kappa_z (M_{y,x} - \tau M_z + q_y) + p_x \quad , \qquad \text{(D-25a)}$$

$$R_2 = -(M_{z,x} + \kappa_z M_x + \tau M_y + q_z)_{,x} + \kappa_y T$$
$$- \tau (M_{y,x} + \kappa_y M_x - \tau M_z + q_y) + p_y \quad , \qquad \text{(D-25b)}$$

$$R_3 = (M_{y,x} + \kappa_y M_x - \tau M_z + q_y)_{,x} + \kappa_z T$$
$$- \tau (M_{z,x} + \kappa_z M_x + \tau M_y + q_z) + p_z \quad , \qquad \text{(D-25c)}$$

$$R_4 = M_{x,x} - \kappa_y M_y - \kappa_z M_z + q_x \qquad \text{(D-25d)}$$

and

$$\left.\begin{aligned} B_1 &= T + \kappa_y M_z - \kappa_z M_y \\ B_2 &= -(M_{z,x} + 2\tau M_y + \kappa_z M_x + q_z) \\ B_3 &= M_{y,x} + \kappa_y M_x - 2\tau M_z + q_y \\ B_4 &= M_x \\ B_5 &= M_z \\ B_6 &= -M_y \end{aligned}\right\} . \qquad \text{(D-26)}$$

With the virtual displacement being arbitrary, the equilibrium equations turn out to be:

$$R_1 = 0, \quad R_2 = 0, \quad R_3 = 0, \quad R_4 = 0 . \qquad \text{(D-27)}$$

Substitution of Equations (D-25) into (D-27) yields exactly the same equations of equilibrium (C-7,8), which were obtained in Appendix C.

The boundary conditions for this problem can be obtained directly from Equation (D-24), subject to the assumption that the boundary conditions are not varied during the loading.

$$\left.\begin{array}{lll} B_1 = 0 & \text{or} & u = 0 \\ B_2 = 0 & \text{or} & v = 0 \\ B_3 = 0 & \text{or} & w = 0 \\ B_4 = 0 & \text{or} & \phi = 0 \\ B_5 = 0 & \text{or} & v_{,x} = 0 \\ B_6 = 0 & \text{or} & w_{,x} = 0 \end{array}\right\} \quad . \qquad \text{(D-28)}$$

The boundary conditions of free edge and rigid clamping, as stated by Equations (9) and (10) of Chapter 2, are in agreement with Equations (D-28). There is only one item which should be noted. In Equation (9) the boundary condition of $u = 0$ at $x_0 = 0$ is neglected. This is due to the fact that T is used in the equilibrium equations, instead of u. As a result of this difference the first equation contains the first derivative of T instead of a second derivative of u. If expression (B-32) of Appendix B is inserted in the equilibrium equations instead of T, using Equation (B-22) for $\tilde{\varepsilon}_{xx}$, the boundary condition, $u = 0$ at $x_0 = 0$, is needed when the unknown T is replaced by u.

D.3 Equilibrium Equations in the Directions of the Undeformed Coordinates of the Rod $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$

As pointed out in Appendix C (Equation (C-11)), in general:

$$\left.\begin{aligned} \hat{e}'_x &= \hat{e}_x + S_{12}\,\hat{e}_y + S_{13}\,\hat{e}_z \\ \hat{e}'_y &= S_{21}\,\hat{e}_x + \hat{e}_y + S_{23}\,\hat{e}_z \\ \hat{e}'_z &= S_{31}\,\hat{e}_x + S_{32}\,\hat{e}_y + \hat{e}_z \end{aligned}\right\} . \qquad \text{(D-29)}$$

Because of the orthonormality conditions, Equations (D-29) imply:

$$\left.\begin{aligned} \hat{e}_x &= \hat{e}'_x + S_{21}\,\hat{e}'_y + S_{31}\,\hat{e}'_z \\ \hat{e}_y &= S_{12}\,\hat{e}'_x + \hat{e}'_y + S_{32}\,\hat{e}'_z \\ \hat{e}_z &= S_{13}\,\hat{e}'_x + S_{23}\,\hat{e}'_y + \hat{e}'_z \end{aligned}\right\} . \qquad \text{(D-30)}$$

The curvatures and twist, using definitions (B-15) of Appendix B are:

$$\left.\begin{aligned} \kappa_y &= \hat{e}'_{x,x} \cdot \hat{e}'_y = (S_{12})_{,x} + (S_{13})_{,x}\,S_{23} \\ \kappa_z &= \hat{e}'_{x,x} \cdot \hat{e}'_z = (S_{13})_{,x} + (S_{12})_{,x}\,S_{32} \\ \tau &= \hat{e}'_{y,x} \cdot \hat{e}'_z = (S_{23})_{,x} + (S_{21})_{,x}\,S_{31} \end{aligned}\right\} . \qquad \text{(D-31)}$$

In this particular case, the virtual displacement is chosen as (compared to (D-10)):

$$\delta\bar{W} = \delta u\,\hat{e}_x + \delta v\,\hat{e}_y + \delta w\,\hat{e}_z \quad . \qquad \text{(D-32)}$$

Using Equations (D-13), (D-30) and (D-32), implies (compared with (D-14)):

$$\overline{A'B'} = \hat{e}'_x + \delta u_{,x}\,\hat{e}_x + \delta v_{,x}\,\hat{e}_y + \delta w_{,x}\,\hat{e}_z$$

$$= (1 + \delta u_{,x} + S_{12}\,\delta v_{,x} + S_{13}\,\delta w_{,x})\hat{e}'_x$$

$$+ (S_{21}\,\delta u_{,x} + \delta v_{,x} + S_{23}\,\delta w_{,x})\hat{e}'_y$$

$$+ (S_{31}\,\delta u_{,x} + S_{32}\,\delta v_{,x} + \delta w_{,x})\hat{e}'_z \quad . \qquad \text{(D-33)}$$

Neglecting virtual terms compared to unity in Equation (D-33), implies (similar to (D-15)):

$$\hat{e}''_x = \hat{e}'_x + (S_{21}\,\delta u_{,x} + \delta v_{,x} + S_{23}\,\delta w_{,x})\hat{e}'_y$$

$$+ (S_{31}\,\delta u_{,x} + S_{32}\,\delta v_{,x} + \delta w_{,x})\hat{e}'_z \quad . \qquad \text{(D-34)}$$

The virtual rotation around the elastic axis is $\delta\phi\,\hat{e}'_x$, thus, according to Equation (D-12), one obtains:

$$\left.\begin{aligned} n_x &= \delta\phi \\ n_y &= -(S_{31}\,\delta u_{,x} + S_{32}\,\delta v_{,x} + \delta w_{,x}) \\ m_z &= S_{21}\,\delta u_{,x} + \delta v_{,x} + S_{23}\,\delta w_{,x} \end{aligned}\right\} \quad . \qquad \text{(D-35)}$$

Using (D-19), together with (D-31) and (D-35), implies:

$$\delta\kappa_y = \delta v_{,xx} + (S_{21}\,\delta u_{,x})_{,x} - (S_{23})_{,x}\,S_{31}\,\delta u_{,x} + (S_{23}\,\delta w_{,x})_{,x}$$

$$- [(S_{23})_{,x} + (S_{21})_{,x}\,S_{31}]\,\delta w_{,x} + [(S_{13})_{,x} + (S_{12})_{,x}\,S_{32}]\,\delta\phi \quad \text{(D-36a)}$$

$$\delta\kappa_z = \delta w_{,xx} + (S_{31}\,\delta u_{,x})_{,x} + (S_{23})_{,x}\, S_{21}\,\delta u_{,x} + (S_{32}\,\delta v_{,x})_{,x}$$

$$+ [(S_{23})_{,x} + (S_{21})_{,x}\, S_{31}]\delta v_{,x} - [(S_{12})_{,x} + (S_{13})_{,x}\, S_{23}]\delta\phi \qquad \text{(D-36b)}$$

$$\delta\tau = \delta\phi_{,x} + [(S_{12})_{,x}\,(S_{31} - S_{21}S_{32}) - (S_{13})_{,x}\,(S_{21} - S_{23}S_{31})]\,\delta u_{,x}$$

$$- (S_{13})_{,x}\,\delta v_{,x} + (S_{12})_{,x}\,\delta w_{,x} \quad . \qquad \text{(D-36c)}$$

From Equation (D-33), one obtains:

$$\delta\tilde{\varepsilon}_{xx} = \delta u_{,x} + S_{12}\,\delta v_{,x} + S_{13}\,\delta w_{,x} \quad . \qquad \text{(D-37)}$$

The virtual rotation components are given by Equations (D-35) with respect to the rotated triad $(\hat{e}'_x, \hat{e}'_y, \hat{e}'_z)$. Using Equation (D-29), implies:

$$\delta\vec{\theta} = [\,\delta\phi - (S_{21} - S_{23}S_{31})\,\delta w_{,x} + (S_{31} - S_{21}S_{32})\,\delta v_{,x}]\hat{e}_x$$

$$+ [-\,\delta w_{,x} - (S_{31} - S_{32}S_{21})\,\delta u_{,x} + S_{12}\,\delta\phi]\hat{e}_y$$

$$+ [\,\delta v_{,x} + (S_{21} - S_{23}S_{31})\,\delta u_{,x} + S_{13}\,\delta\phi]\hat{e}_z \quad . \qquad \text{(D-38)}$$

With Equation (D-29) and the assumption that terms of order ε^2 are neglected compared to unity:

$$S_{21} - S_{23}S_{31} \cong -S_{12} \quad ,$$

$$S_{31} - S_{32}S_{21} \cong -S_{13} \quad . \qquad \text{(D-39)}$$

The load $\bar{p}$ and moment $\bar{q}$ are given with respect to the undeformed triad $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$, as in Equation (C-16) of Appendix C:

$$\begin{aligned} \bar{p} &= \tilde{p}_x \hat{e}_x + \tilde{p}_y \hat{e}_y + \tilde{p}_z \hat{e}_z \quad , \\ \bar{q} &= \tilde{q}_x \hat{e}_x + \tilde{q}_y \hat{e}_y + \tilde{q}_z \hat{e}_z \quad . \end{aligned} \tag{D-40}$$

Substitution of Equations (D-32), (D-36), (D-37), (D-38) and (D-40) into (D-9), and using (D-39), implies, after integration by parts:

$$\begin{aligned} &\int_{x_0=0}^{x_0=\ell} \left[-\tilde{R}_1 \delta u - \tilde{R}_2 \delta v - \tilde{R}_3 \delta w - \tilde{R}_4 \delta \phi\right] dx_0 - \left[\tilde{B}_1 \delta u\right]_{x_0=0}^{x_0=\ell} \\ &\qquad - \left[\tilde{B}_2 \delta v\right]_{x_0=0}^{x_0=\ell} - \left[\tilde{B}_3 \delta w\right]_{x_0=0}^{x_0=\ell} - \left[\tilde{B}_4 \ \delta\phi\right]_{x_0=0}^{x_0=\ell} \\ &\qquad - \left[\tilde{B}_5 \delta u_{,x}\right]_{x_0=0}^{x_0=\ell} - \left[\tilde{B}_6 \delta v_{,x}\right]_{x_0=0}^{x_0=\ell} - \left[\tilde{B}_7 \delta w_{,x}\right]_{x_0=0}^{x_0=\ell} \\ &= 0 \end{aligned} \tag{D-41}$$

where:

$$\begin{aligned} \tilde{R}_1 = \{&T - S_{21} M_{z,x} + S_{31} M_{y,x} - (S_{23})_{,x} S_{21} M_y - (S_{23})_{,x} S_{31} M_z \\ &+ M_x[(S_{13})_{,x} S_{12} - (S_{12})_{,x} S_{13}]\}_{,x} \\ &+ \tilde{p}_x - (S_{13} \tilde{q}_y)_{,x} + (S_{12} \tilde{q}_z)_{,x} \quad , \end{aligned} \tag{D-42a}$$

$$\tilde{R}_2 = -\{M_{z,x} + (S_{13})_{,x} M_x + [(S_{23})_{,x} + (S_{21})_{,x} S_{31}]M_y - S_{32} M_{y,x}\}_{,x}$$

$$+ (S_{12} T)_{,x} + \tilde{p}_y + (S_{13} \tilde{q}_x)_{,x} - \tilde{q}_{z,x} \qquad , (D\text{-}42b)$$

$$\tilde{R}_3 = \{M_{y,x} + (S_{12})_{,x} M_x - [(S_{23})_{,x} + (S_{21})_{,x} S_{31}]M_z - S_{23} M_{z,x}\}_{,x}$$

$$+ (S_{13} T)_{,x} + \tilde{p}_z + \tilde{q}_{y,x} - (S_{12} \tilde{q}_x)_{,x} \qquad , (D\text{-}42c)$$

$$\tilde{R}_4 = M_{x,x} - [(S_{13})_{,x} + (S_{12})_{,x} S_{32}]M_z - [(S_{12})_{,x} + (S_{13})_{,x} S_{23}]M_y$$

$$+ \tilde{q}_x + S_{12} \tilde{q}_y + S_{13} \tilde{q}_z \qquad , (D\text{-}42d)$$

and:

$$\tilde{B}_1 = T - S_{21} M_{z,x} + S_{31} M_{y,x} - (S_{23})_{,x} S_{21} M_y - (S_{23})_{,x} S_{31} M_z$$

$$+ M_x[(S_{13})_{,x} S_{12} - (S_{12})_{,x} S_{13}] - S_{13} \tilde{q}_y + S_{12} \tilde{q}_z \qquad , (D\text{-}43a)$$

$$\tilde{B}_2 = -M_{z,x} - (S_{13})_{,x} M_x - [(S_{23})_{,x} + (S_{21})_{,x} S_{31}]M_y$$

$$+ S_{32} M_{y,x} + (S_{12} T)_{,x} - \tilde{q}_z + S_{13} \tilde{q}_x \qquad , (D\text{-}43b)$$

$$\tilde{B}_3 = M_{y,x} + (S_{12})_{,x} M_x - [(S_{23})_{,x} + (S_{21})_{,x} S_{31}]M_z$$

$$- S_{23} M_{z,x} + S_{13} T + \tilde{q}_y - S_{12} \tilde{q}_x \qquad , (D\text{-}43c)$$

$$\tilde{B}_4 = M_x \quad , \qquad \text{(D-43d)}$$

$$\tilde{B}_5 = -S_{31} M_y + S_{21} M_z \quad , \qquad \text{(D-43e)}$$

$$\tilde{B}_6 = -S_{32} M_y + M_z \quad , \qquad \text{(D-43f)}$$

$$\tilde{B}_7 = -M_y + S_{23} M_z \quad . \qquad \text{(D-43g)}$$

The virtual displacements are arbitrary, thus, the equations of equilibrium become:

$$\tilde{R}_1 = 0; \quad \tilde{R}_2 = 0; \quad \tilde{R}_3 = 0; \quad \tilde{R}_4 = 0 \; ; \qquad \text{(D-44)}$$

while the boundary conditions (assuming that the boundary conditions are not changed during loading) are:

$$\left.\begin{array}{lll} \tilde{B}_1 = 0 & \text{or} & u = 0 \\ \tilde{B}_2 = 0 & \text{or} & v = 0 \\ \tilde{B}_3 = 0 & \text{or} & w = 0 \\ \tilde{B}_4 = 0 & \text{or} & \phi = 0 \\ \tilde{B}_5 = 0 & \text{or} & u_{,x} = 0 \\ \tilde{B}_6 = 0 & \text{or} & v_{,x} = 0 \\ \tilde{B}_7 = 0 & \text{or} & w_{,x} = 0 \end{array}\right\} \quad . \qquad \text{(D-45)}$$

The equilibirum equations (D-44) are identical to (C-24) of Appendix

C, within the framework of the assumption that terms of order ε^2 are neglected compared to unity, which implies:

$$-S_{21} M_{z,x} - S_{13}(S_{32})_{,x} M_z \cong -S_{21} M_{z,x} - S_{31}(S_{23})_{,x} M_z \quad , \qquad \text{(D-46a)}$$

$$S_{31} M_{y,x} + S_{12}(S_{23})_{,x} M_y \cong S_{31} M_{y,x} - S_{21}(S_{23})_{,x} M_y \quad , \qquad \text{(D-46b)}$$

$$[(S_{23})_{,x} + (S_{21})_{,x} S_{31}] \cong [(S_{23})_{,x} - S_{13}(S_{21})_{,x}] \quad , \qquad \text{(D-46c)}$$

$$-[(S_{13})_{,x} + (S_{12})_{,x} S_{32}] \cong [(S_{31})_{,x} + S_{12}(S_{32})_{,x}] \quad , \qquad \text{(D-46d)}$$

$$-[(S_{12})_{,x} + (S_{13})_{,x} S_{23}] \cong [(S_{21})_{,x} + S_{13}(S_{23})_{,x}] \quad . \qquad \text{(D-46e)}$$

From the boundary conditions (D-45) it seems that due to the change of the coordinate system a new boundary condition appears. $\tilde{B}_5 = 0$ or $u_{,x} = 0$. However, upon checking the sixth and seventh condition, it is clear that in the case of a clamped edge, $v_{,x} = w_{,x} = 0$, or a free edge, $M_y = M_z = 0$, the condition $\tilde{B}_5 = 0$ is satisfied automatically. Thus, the fifth condition is satisfied, by satisfying the other boundary conditions.

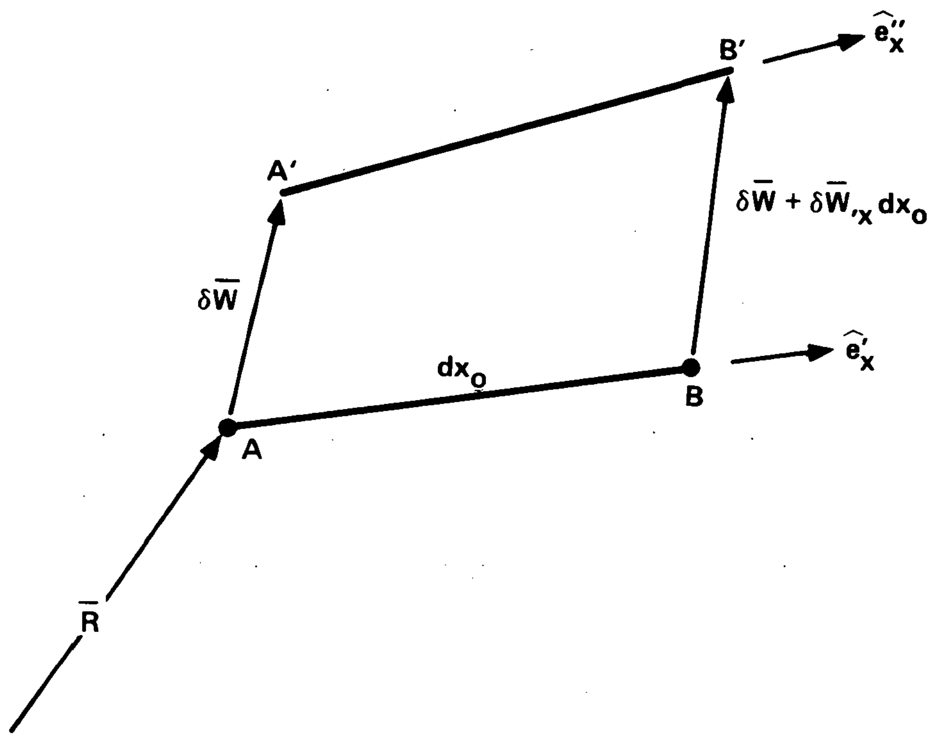

Fig. D1. Displacement of Element on the Elastic Axis During the Virtual Displacement

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16. Abstract

A set of nonlinear equations of equilibrium for an elastic wind turbine or helicopter blade are presented. These equations are derived for the case of small strains and moderate rotations (slopes). The derivation includes several assumptions which are carefully stated. For the convenience of potential users the equations are developed with respect to two different systems of coordinates, the undeformed and the deformed coordinates of the blade. Furthermore, the loads acting on the blade are given in a general form so as to make them suitable for a variety of applications. The equations obtained in the present study are compared with those obtained in previous studies. Finally, it should be noted that this report represents the first in a series of three reports documenting the research performed under the grant. The second report (UCLA-ENG-7880) deals with the aeroelastic stability and response of an isolated horizontal axis wind turbine blade. The third report (UCLA-ENG-7881) deals with the aeroelastic stability and response of the complete coupled rotor/tower system simulating essentially the dynamics of the NASA/DOE Mod-0 configuration.

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