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FINAL REPORT FOR

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"Numerical Methods For Problems Involving

The Drazin Inverse"

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Submitted By

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The purpose of the grant was to provide support to the author so that he could collaborate with Professor Gene H. Golub at Stanford University during the summer of 1978 on numerical problems relating to the Drazin inverse and applications thereof. In particular, the objective was to try to develop a useful numerical algorithm for the Drazin inverse and to analyze the numerical aspects of the applications of the Drazin inverse relating to the study of homogeneous Markov chains and systems of linear differential equations with singular coefficient matrices.

It is felt that all objectives were accomplished with a measurable degree of success.

<u>A Stable Algorithm For the Drazin Inverse</u>: During the author's stay at Stanford, the following algorithm was derived and reported on. The algorithm seems to be the best (in the sense of stability) in current existence. The algorithm is based on the work of Golub and Wilkinson (Ill-Conditioned Eigensystems and the Computation of the Jordan Form, SIAM Rev., Vol. 18, Oct. 1976, pp. 578-619.) and is described below.

The first half of the algorithm determines the index of a square matrix and returns the "index decomposition" of the matrix.

I. The Index Decomposition.

(0) Given $A_{n\times n} \neq 0$, apply the singular value decomposition to obtain

$A = U\begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} V$. If $r = n$, then $\mathrm{Ind}(A) = 0$ (i.e., A is nonsingular)

and $A^D = A^{-1} = V^*\Sigma^{-1}U^*$ and process stops. If $r < n$, then continue.

(1) Form $VAV^* = VU\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}A_{11}^{(1)} & 0\\ A_{21}^{(1)} & 0\end{bmatrix}$. If $A_{11}^{(1)} = 0$, then $A^D = 0$.

If $A_{11}^{(1)} \neq 0$, apply the singular value decomposition to $A_{11}^{(1)}$ to obtain

$$A_{11}^{(1)} = U_1\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}V_1$$

where $A_{11}^{(1)}$ is $n_1 \times n_1$ and $\Sigma_1$ is $r_1 \times r_1$. Define $P_1 = V$ so that

$$P_1AP_1^* = \begin{bmatrix}A_{11}^{(1)} & 0\\ A_{21}^{(1)} & 0\end{bmatrix}.$$

If $r_1 = n_1$, then Ind(A) = 1 and Go To II. If $r_1 < n_1$, then continue.

(2) Form $V_1A_{11}^{(1)}V_1^* = V_1U_1\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}A_{11}^{(2)} & 0\\ A_{21}^{(2)} & 0\end{bmatrix}$. If $A_{11}^{(2)} = 0$, then $A^D = 0$. If $A_{11}^{(2)} \neq 0$, apply the singular value decomposition to $A_{11}^{(2)}$ to obtain

$$A_{11}^{(2)} = U_2\begin{bmatrix}\Sigma_2 & 0\\ 0 & 0\end{bmatrix}V_2 \text{ where}$$

$A_{11}^{(2)}$ is $n_2 \times n_2$ and $\Sigma_2$ is $r_2 \times r_2$. Define

$$P_2 = \begin{bmatrix}V_1 & 0\\ 0 & I\end{bmatrix}P_1 = \begin{bmatrix}V_1 & 0\\ 0 & I\end{bmatrix}V$$

so that $P_2AP_2^*$ has the form

$$P_2AP_2^* = \left[\begin{array}{c:cc}A_{11}^{(2)} & 0 & 0\\ \hdashline A_{21}^{(2)} & 0 & 0\\ A_{31}^{(2)} & A_{32}^{(2)} & 0\end{array}\right].$$

If $r_2 = n_2$, then Ind(A) = 2 and Go To II. If $r_2 < n_2$, then continue.

(3) * * * * * *

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(k)

Form $V_{k-1}A_{11}^{(k-1)}V_{k-1}^{*} = \begin{bmatrix} A_{11}^{(k)} & 0 \\ A_{21}^{(k)} & 0 \end{bmatrix}$. If $A_{11}^{(k)} = 0$, then $A^D = 0$. If $A_{11}^{(k)} \neq 0$, apply singular value decomposition to $A_{11}^{(k)}$ to obtain

$$A_{11}^{(k)} = U_k \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} V_k$$

where $A_{11}^{(k)}$ is $n_k \times n_k$ and $\Sigma_k$ is $r_k \times r_k$. Define

$$P_k = \begin{bmatrix} V_{k-1} & 0 \\ 0 & I \end{bmatrix} P_{k-1}$$

so that $P_k A P_k^*$ now has the form

$$P_k A P_k^* = \left[\begin{array}{c|cccc} A_{11}^{(k)} & 0 & 0 & \cdots & 0 \\ \hline A_{21}^{(k)} & 0 & 0 & \cdots & 0 \\ A_{31}^{(k)} & A_{32}^{(k)} & 0 & \cdots & 0 \\ A_{41}^{(k)} & A_{42}^{(k)} & A_{43}^{(k)} & \ddots & \\ \vdots & \vdots & \vdots & & \vdots \\ A_{k+1,1}^{(k)} & A_{k+1,2}^{(k)} & A_{k+1,3}^{(k)} & \cdots & 0 \end{array}\right]$$

If $r_k = n_k$, then Ind(A) = k and GO TO II. If $r_k < n_k$, then continue.

This process can continue for only a finite number of steps. When the process terminates, the final value of $P_k A P_k^*$ is called the <u>index decomposition</u> of A.

If $PAP^*$ is the index decomposition, then P is unitary and $PAP^*$ has the form

$$PAP^* = \left[\begin{array}{c|c} B & 0 \\ \hline C & N \end{array}\right]$$

where N is lower triangular with 0's on the diagonal (and hence nilpotent) and B is either nonsingular or a zero matrix. If $B = 0$, then $A^D = 0$. If B is nonsingular, then

$$A^D = P^* \begin{bmatrix} B^{-1} & 0 \\ X & 0 \end{bmatrix} P$$

where X is computed by part II of the algorithm.

II. Computation of the X Factor.

Partition X and $-CB^{-1}$ by rows so that

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ and } -CB^{-1} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} .$$

and let $N = [n_{ij}]$.

Successively determine the rows of X as follows. First, solve the system $x_1 B = r_1$ for $x_1$. Then for $k = 2,3, \ldots, m$, successively solve the systems

$$x_k B = r_k - \sum_{j=1}^{k-1} n_{kj} \quad \text{for } x_k.$$

This concludes the algorithm.

The techniques utilized in the above algorithm have subsequently been applied by J. H. Wilkinson ("Note On The Practical Significance of The Drazin Inverse" Preprint) to produce a direct method for obtaining numerical solutions of the system of differential equations $A\dot{x}(t) + Bx(t) = f(t)$ where A and B may both be singular matrices.

<u>A Perturbation Analysis Of The Limiting Probabilities For A Finite Ergodic Markov Chain In Terms Of The Drazin Inverse</u>. Given any finite homogeneous Markov chain with transition matrix T, the author had previously established the fundamental fact that the limiting probabilities were functions of the elements of a Drazin inverse in the following sense. If $A = I - T$, then

$$\lim_{n\to\infty} \frac{I + T + T^2 + \ldots + T^{n-1}}{n} = I - AA^{\#}.$$

where $A^{\#}$ is the group inverse of A ($A^{\#}$ is alternate notation for the Drazin inverse in the case $\mathrm{Ind}(A) \leq 1$.) The second aspect of the research conducted under this grant was to utilize the above relation between the group inverse and the limiting probabilities so as to ascertain the relative sensitivity the limiting probabilities might exhibit when the original chain is subject to a perturbation. It was discovered that a simple looking perturbation bound can be given in terms of $A^{\#}$. The major result which was established is given below.

Let T denote the transition matrix of an ergodic chain $L$ with the limiting probability (row) vector w. Let $\tilde{L}$ be an ergodic chain obtained from $L$ by perturbing T so that $\tilde{T} = T - E$ is the transition matrix of $\tilde{L}$. Let $A = I - T$, $\tilde{A} = I - \tilde{T}$, and $\tilde{w}$ denote the limiting probability vector for $\tilde{L}$. The object is to bound the relative error

$$\frac{\|w - \tilde{w}\|}{\|w\|} .$$

Although it is not a consequence of any of the traditional theory of linear systems, the bound obtainable is exactly analogous to that which appears in the analysis of a perturbed linear system.

If E is "small" in the sense that $\| EA^{\#} \| < 1$ (for an operator norm induced by a vector norm $\| \cdot \|$) then

$$\frac{\|w - \tilde{w}\|}{\|w\|} \leq \frac{\| EA^{\#} \|}{1 - \| EA^{\#} \|} .$$

The relative error in w is bounded by the relative error in A as follows. If $\| E \| \; \| A^{\#} \| < 1$, then

$$\frac{\|w - \tilde{w}\|}{\|w\|} \leq \frac{\frac{\|A - \tilde{A}\|}{\|A\|} G(A)}{1 - \frac{\|A - \tilde{A}\|}{\|A\|} G(A)}$$

where $G(A) = \| A \| \; \| A^{\#} \|$. Moreover, the above inequalities are sharp in the sense that there exist nontrivial examples where equality is actually attained.

The obvious conclusion is that if G(A) is small, then the limiting probabilities are relatively insensitive to perturbations. If G(A) is large, the limiting probabilities may be quite sensitive to small perturbations. The number G(A) was defined by the author to be the <u>condition of the chain</u>. It can be demonstrated that G(A) also arises in dealing with the mean first passage times and other related aspects of a perturbed chain.

In addition to the perturbation ~~bands~~ bounds given above, the following perturbation formula was derived. If $L$ and $\tilde{L}$ are both ergodic, then

$$\tilde{w} = w(I + EA^{\#})^{-1} = w - wEA^{\#}(I + EA^{\#})^{-1},$$

independent of the size of $\| EA^{\#} \|$.

Finally, examples were constructed to show that the bounds involving G(A) give better estimates than the bounds obtainable using the t.aditional theory based on the nonsingular "fundamental matrix" of the chain. This tends to reinforce the belief that the introduction of the Drazin inverse into the theory of finite Markov chains provides a significant advantage over classical methods.

The results obtained under the support of this grant will be published in detail in four journal articles and in a portion of the author's soon to be released text "Generalized Inverses of Linear Transformations" (Pitman Pub. Co. - coauthor with S. Campbell).