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ION BEAM PROBING OF ELECTROSTATIC FIELDS

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16. Abstract

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ION BEAM PROBING OF ELECTROSTATIC FIELDS

by Hans Persson*

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SUMMARY

The determination of a cylindrically symmetric, time-independent electrostatic potential V in a magnetic field $\underline{B}$ with the same symmetry by measurements of the deflection of a "primary" beam of ions is analyzed and substantiated by examples. Special attention is given to the requirements on canonical angular momentum and total energy set by an arbitrary, nonmonotone V, to scaling laws obtained by normalization, and to the analogy with ionospheric sounding. The inversion procedure with the Abel analysis of an equivalent problem with a one-dimensional fictitious potential is used in a numerical experiment with application to the NASA Lewis Modified Penning Discharge; the assumed potential can be well reconstructed by simple means.

The determination of V from a study of "secondary" beams of ions with increased charge produced by hot plasma electrons is also analyzed, both from a general point of view and with application to the NASA Lewis SUMMA experiment. Like in the primary beam method there are requirements on the beam energy set by the penetration of the ions through $\underline{B}$, the possibility of repulsive potentials, and the special requirements set by the uniqueness in the determination of V and the computational procedure. Simple formulas and geometrical constructions are given for the minimum energy necessary to reach the axis, the whole plasma, and any point in the magnetic field.

The common, simplifying assumption that V is a small perturbation is critically and constructively analyzed; an iteration scheme for successively correcting the orbits and points of ionization for the electrostatic potential is suggested, and elaborated in the cylindrically symmetric case in terms of a nonlinear, weakly singular integral equation coupled with an empirical relation, and a mapping T in V-space. Conditions are given for T to be contractive, which gives a unique determination of V, and for the first iterate - which corresponds to the simplified solution - to be a good approximation.

It is found that the pertinent smallness quantity ε has the physical significance of the ratio of electrostatic to magnetic force in the

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plasma, or equivalently of E/B-velocity to beam velocity, or equivalently of electric field times magnetic scale length to twice the geometric mean of beam energy and minimum necessary beam energy for penetration through $\underline{B}$. The simplified solution is asymptotically correct when $\varepsilon \to 0$, and for $\varepsilon \lesssim 1$ and a suitable 0:th iterate the iteration converges to a unique limit, the true potential.

A numerical example with low beam energy with application to the SUMMA experiment illuminates the importance of V; the simplified solution is found to be unsatisfactory.

1. INTRODUCTION

Beams of charged particles can be used to determine an unknown electric field. In the first part (sec. 2-3) of the present paper, we shall discuss a "primary beam method" based upon measurements of beam deflection, and in the second part (sec. 4-6) we shall analyze a method based upon a study of the "secondary beams" of ions in more highly ionized states produced in the interaction between the primary beam and the hot electrons of a plasma assumed to be the object of study. Our interest will mainly be a time-independent electrostatic field directed perpendicular to a magnetic field $\underline{B}$. The first method is - up to now - essentially limited to cylindrical symmetry, while the second can be used in arbitrary geometries.

Methods and experiments with studies of deflection with constant beam energy and variable angular momentum were reported by Stallings (ref. 1) and by Black and Robinson (ref. 2). A similar approach is found in the tracing of light rays (Rockett and Deboo, ref. 3). In the first part of the present paper we shall mainly be concerned with the opposite case, when the angular momentum is constant and the energy varied. The general cylindrically symmetric problem of this kind was treated by Whipple (ref. 4), the special case with vanishing magnetic field by Dracott (ref. 5), and another special case (with nonzero magnetic field but the electric field treated as a small perturbation) by Konstantinov and Tselnik (ref. 6). In the experimental investigations and theoretical calculations by Kambic (ref. 7), (that also include the secondary beam method, see below) the unknown potential was parametrized and the parameters determined, as in the work by Borodkin (ref. 8). Swanson, et al. (ref. 9), arrived at their conclusions by comparison with solutions of forward problems with deflection in potential wells. A method for determining the electric field perpendicular to $\underline{B}$ with a beam parallel to $\underline{B}$ was used by Dow (ref. 10), and for determination of the electric field parallel to $\underline{B}$ with a beam parallel to $\underline{B}$ we wish to refer to Johansson (ref. 11) and cited references, especially Ehrenberg and Kentrschynskyj (ref. 12).

The essential beam deflection reference to us is the laboratory report by Whipple (ref. 4). We shall present a derivation of some basic relations similar to his, we shall introduce a normalization of the physical variables, which will allow us to obtain scaling laws and solve "several problems in one," and we shall substantiate the theory with discussion and examples, both analytical and numerical. The primary beam method has similarities with the determination of intermolecular potentials from particle scattering data, and especially with electron density determinations with radio waves in the ionosphere, and we shall make some use of this latter analogy.

In the secondary beam method by Jobes and Hickok one utilizes the change in charge to mass ratio that occurs when hot plasma electrons produce a sudden change of the charge of primary ions (or mass of molecular primary ions). Several important plasma quantities can be measured by studying these "secondaries," notably the electrostatic potential (see for instance ref. 13 and cited references). In a time-independent electric field the secondaries will, in an invariant way, carry information about the plasma potential at the point of ionization (or dissociation) to a detector placed outside the plasma, without any other distortion than that caused by systematic "errors" and stochastic components. They also carry information about the location of the point of ionization. However, this information is in general distorted by the magnetic field and the unknown electric field.

We shall investigate a few factors influencing the requirements on momentum and energy of the primary beam particles, notably those set by the need for penetration into and out of the magnetic field, and the special questions associated with very strong electric fields. It is customary to make the simplifying assumption that the unknown electric field is a small perturbation, when experimental data are analyzed; this allows a determination of the particle orbits - a step in the analysis - without knowledge of the potential. Kambic (ref. 7), in his investigations on the NASA Lewis Modified Penning Discharge, encountered a situation when this simplified approach could not be used. We shall analyze the conditions for validity and uniqueness of the simplified approach and a procedure for improving it, and establish basic criteria herefor. The aim will be towards generality, but specialization to cylindrical symmetry and the conditions in another NASA Lewis experiment - the SUMMA experiment - will be essential for the analysis. However, it is believed that methods as well as criteria can readily be extended to general situations.

In section 2 the theory for the primary beam method for probing an axisymmetric field is displayed, and section 3 contains a simple analytical example and a numerical experiment, the reconstruction of an assumed potential, with application to the above mentioned Modified Penning Discharge. In section 4 the secondary beam method is briefly described, the momentum (and energy) requirement set by a cylindrical magnetic field is analyzed, and the possible importance of two different smallness quantities for the electric field is tentatively discussed. Section 5 consists of an analysis of a suggested iteration scheme, applicable to arbitrary potentials, with no need for parametric representation. The pertinent smallness quantity is found as well as the condition for the electric field to be a perturbation and for the iteration to converge. Section 6 is a numerical illustration of the importance of the electric field.

2. THEORY FOR A PRIMARY BEAM METHOD

A. Formulation

Consider a cylindrically symmetric, time-independent magnetic field B, whose strength may vary with the distance r from the axis of symmetry; $B = B(r)$. See figure 1, in which $\overline{B}$ is directed outwards from the paper, at least on and near the axis. An ion orbit has been drawn in the figure. With such a field, the magnetic vector potential $\overline{A}$ has only one component, A_ϕ, which is positive. The gun is located at the point (r_1,ϕ_1) and the detector at (r_2,ϕ_2). The angle ϕ_2 can be varied. The electric field is assumed to be derivable from a cylindrically symmetric, conservative potential $V(r)$, which is supposed to be known between r_1 and r_2. (Usually, we shall consider the case that it is a constant that may be put equal to zero.) We wish to determine the function $V(r)$ from measurements of $\phi_2 - \phi_1$, the change in polar angle, for ions with mass m and charge q injected with various energies in various directions.

The velocity components are denoted by v_r and v_ϕ, and Φ is the net magnetic flux enclosed inside a circle with radius r and center on the axis. Then the following two expressions hold for the total energy W_o and the canonical angular momentum L_o - the two constants of motion:

$$W_o = \frac{1}{2} mv_\phi^2 + \frac{1}{2} mv_r^2 + qV \tag{1}$$

$$L_o = mrv_\phi + qrA_\phi = mrv_\phi + \frac{q\Phi}{2\pi} \tag{2}$$

where we have used the relation $2\pi rA_\phi = \Phi$, which follows from Stokes' theorem. With the magnetic field direction given in figure 1, the flux Φ is certainly positive. From equation (2) v_ϕ is solved and inserted into equation (1), which gives

$$\frac{1}{2} mv_r^2 + \Psi(r;L_o) = W_o \tag{3}$$

where

$$\Psi = \frac{1}{2mr^2}\left(L_o - \frac{q\Phi}{2\pi}\right)^2 + qV$$

is a function of r containing L_o as a parameter. It has been shown in the literature (see for instance ref. 14) that the radial motion described by equation (3) can be interpreted as a fictitious one-dimensional motion of the particle (with total energy W_0) in the fictitious or effective one-dimensional potential Ψ. In addition to the potential energy associated with the electrostatic field, Ψ contains a term accounting for the azimuthal motion,

$$\Psi_m = \frac{1}{2mr^2}\left(L_o - \frac{q\Phi}{2\pi}\right)^2 = \frac{1}{2}\, mv_\phi^2$$

The additional term is given by the kinetic energy associated with the azimuthal motion, and it consists of coupled contributions from the centrifugal and magnetic forces.

The two quantities W_0 and L_0 can be varied independently by arranging the variation of the energy and the direction of the beam. But there is only one unknown function $V(r)$ to be determined. By formalizing the discussions by Dracott and Whipple (refs. 4 and 5), we shall use this redundance in such a way that the effective potential Ψ governing the radial motion will be monotonic, $d\Psi/dr < 0$, even if the electric field is not, and even if the magnetic field is allowed to change its direction in the pertinent region of space. This is achieved by choosing L_0 sufficiently large. The redundance may also be used to obtain systematic procedures and direct local measurements of the electric field.

We shall prove that for arbitrarily given $\Phi(r)$ and $V(r)$ it is always possible to find an L_0 such that $d\Psi/dr < 0$ for all r between 0 (exclusive) and r_1 and r_2. In practice, there is an infinite number of such L_0. Even if, in principle, any of them can be used for determining an unknown $V(r)$ with an upwards bounded derivative, it may be more practical to use potentials that are nonmonotonic in regions where V is already known (compare sec. 3D).

$V(r)$ can be solved from an integral equation of Abel type. The flux $\Phi(r)$ is assumed to be known, as well as the total change in polar angle, as a function of the total energy W_0, for particles with a constant L_0 among those values giving a monotonic fictitious potential Ψ. Part of the Abel inversion procedure can be made analytically, and essentially only quadratures and the solution of an implicit relation are needed.

B. Normalization

In spite of the fact that the immediate intuitive meaning of the various symbols gets lost, it proves to be practical to normalize all quantities. The analytical work becomes more clear, and only the variables and parameters pertinent to the solution enter into the calculations. Every solution of a problem in normalized variables corresponds to a wide variety of solutions in the original variables, and scaling laws can be found.

A convenient basis for the normalization is the magnetic flux enclosed inside some radius r_n. The quantities r', Φ', L', v', W' are introduced as

$$r' = r_n$$

$$\Phi' = \Phi(r')$$

$$L' = \frac{q\Phi'}{2\pi}$$

$$v' = \frac{L'}{mr'}$$

$$W' = \frac{1}{2} mv'^2$$

Putting

$$r/r' = s$$

$$L_o/L' = \ell_o$$

$$v_\phi/v' = u_\phi$$

$$v_r/v' = u_r$$

$$\Phi/\Phi' = \tau$$

$$qV/W' = v$$

$$W_o/W' = w_o$$

$$\Psi/W' = \psi$$

we obtain the following expressions for the constants of motion and the fictitious potential:

$$w_o = u_\phi^2 + u_r^2 + v$$

$$\ell_o = su_\phi + \tau$$

$$\psi = \frac{1}{s^2} (\ell_o - \tau)^2 + v \tag{4}$$

$$\psi_m = \frac{1}{s^2} (\ell_o - \tau)^2 = u_\phi^2$$

Here, ψ_m is the part of ψ owing to the azimuthal motion.

If the electrostatic potential is set equal to zero at r_n, the point of normalization, one gets for $r = r_n$: $s = 1$; $u_\phi = u_{o\phi}$; $u_r = u_{or}$; $\tau = 1$; $v = 0$; $\psi = (\ell_o - 1)^2$. Clearly, $u_{o\phi} = \ell_o - 1$; and $\ell_o = 1$ corresponds to a purely radial injection.

A natural choice of r_n is the gun location r_1 (or the detector location r_2). However, if the gun is located outside the magnetic field, this is not possible since Φ' would then be zero, and if $\Phi(r_1)$ is very small (in a certain sense) normalization to r_1 is impractical. If the magnetic field changes its direction only once, the corresponding point may be chosen for the normalization. At that point, Φ then has its maximum value.

In any case, the location of the gun will correspond to $r_1/r_n = \alpha$, and that of the detector to $r_2/r_n = \beta$; $\max(\alpha,\beta) = \gamma$.

C. Monotonic Fictitious Potential

Differentiating (4), one obtains

$$\frac{d\psi}{ds} = -\frac{2}{s^2}(\ell_o - \tau)\left[\frac{1}{s}(\ell_o - \tau) + \frac{d\tau}{ds}\right] + \frac{dv}{ds}$$

Inspection of this expression reveals that for given τ and v the momentum ℓ_o can be chosen so large that both parentheses in the first term are positive. In fact, $\lim\limits_{\ell_o \to \infty} \frac{d\psi_m}{ds} = -\infty$, uniformly in s. Thus, the first term can always be made sufficiently negative to dominate over the second term, which may be positive, so that $d\psi/ds$ becomes negative for all s considered.

For a formal proof, let us put

$$\left.\begin{aligned} \operatorname{Sup} \tau &= k_1 \\ \operatorname{Sup}\left[0, \left(-\frac{d\tau}{ds}\right)\right] &= k_2 \\ \operatorname{Sup} \frac{dv}{ds} &= k_3 \end{aligned}\right\}$$

when s varies in the interval $(0,\gamma)$ and let us consider ℓ_o-values at least fulfilling the condition

$$\ell_o > k_1 + k_2\gamma \tag{5}$$

Then,

$$-\frac{d\psi}{ds} = \frac{2}{s^2}(\ell_o - \tau)\left[\frac{1}{s}(\ell_o - \tau) + \frac{d\tau}{ds}\right] - \frac{dv}{ds} \geq \frac{2}{\gamma^2}(\ell_o - k_1)\left[\frac{\ell_o - k_1}{\gamma} - k_2\right] - k_3$$

Since the first term in the right-hand member of the second inequality tends to infinity with ℓ_o, the latter quantity can be chosen so large that the same member becomes positive, which implies that

$$\frac{d\psi}{ds} < 0 \qquad \text{for all} \quad s\varepsilon(0,\gamma)$$

For the considered ℓ_o-values (fulfilling (5) above) this occurs when

$$\ell_o > k_1 + \gamma\left[\frac{k_2}{2} + \sqrt{\frac{k_2^2}{4} + \frac{k_3\gamma}{2}}\right] \tag{6}$$

Conditions (5) and (6) contain the maximum magnetic flux, its maximum rate of decrease with s, and the maximum inwards directed electric field. By using the most stringent of these two conditions, a sufficiently large value of ℓ_o can be determined.

D. Determination of Electrostatic Potential

The rate of change of the azimuthal angle with respect to radius is given by $\frac{d\phi}{dr} = \frac{v_\phi}{rv_r}$, in normalized variables by $\frac{d\phi}{ds} = \frac{u_\phi}{su_r}$. The total angular change $\Delta\phi$ is obtained by integrating from α to the turning-point s_t and then from s_t to β. To be specific, we assume $\alpha = \beta = 1$; if α and β have other values, the corresponding contributions to $\Delta\phi$ will be completely known if v is known between 1 and γ.

Using the symmetry between the inwards and outwards directed motion and inserting $u_\phi = (\ell_o - \tau)/s$; $u_r = \sqrt{w_o - \psi}$ (for motion outwards), we obtain

$$\Delta\phi = 2 \cdot \int_{s_t}^{1} \frac{(\ell_o - \tau)ds}{s^2\sqrt{w_o - \psi}} \tag{7}$$

where s_t is defined through $w_o - \psi(s_t) = 0$, which is unique for monotonic ψ. Let us denote $\Delta\phi/2$ by $f(w_o)$, and in the integral we regard s as a function of ψ instead of the reverse. (This is possible due to the monotone relation.) We then get

$$\int_{(\ell_o-1)^2}^{w_o} \frac{(\ell_o - \tau)(-s')d\psi}{s^2\sqrt{w_o - \psi}} = f(w_o) \tag{8}$$

Suppose that $f(w_o)$, half the change in polar angle, is known for all values of the total energy in an interval extending from $(\ell_o - 1)^2$ up to some maximum value w_{max}. The canonical angular momentum is supposed to be constant for all these w_o and chosen to furnish a monotonic ψ. Then, since τ does not contain w_o, equation (8) is an Abel integral equation with the solution

$$\frac{\ell_o - \tau}{s^2}\frac{ds}{d\psi} = -\frac{1}{\pi}\frac{d}{d\psi}\int_{(\ell_o-1)^2}^{\psi} \frac{f(w_o)dw_o}{\sqrt{\psi - w_o}}$$

We then integrate s from s to 1, whereby ψ varies from ψ to $(\ell_o - 1)^2$. One obtains

$$\int_{s}^{1} \frac{\ell_o - \tau(q)}{s^2}\,ds = \frac{1}{\pi}\int_{(\ell_o-1)^2}^{\psi} \frac{f(w_o)dw_o}{\sqrt{\psi - w_o}} \tag{9}$$

which is a remarkably simple and straightforward solution. It defines s as a unique and monotonic function of ψ, the latter quantity varying from $(\ell_o - 1)^2$ to w_{max}. Indeed, denoting the left-hand member of equation (9) by $F(s)$ and the right-hand member by $G(\psi)$, the derivative

$$\frac{\partial}{\partial s}[F(s) - G(\psi)] = -\frac{\ell_o - \tau}{s^2}$$

becomes nonzero (negative) in virtue of the assumption on ℓ_o. Thus, $s = s(\psi)$ is unique. But the relation between s and ψ has already been shown to be one-one. Hence, $\psi = \psi(s)$, and $G(\psi)$ must be a monotonic function of ψ. Considering $\psi(1) = (\ell_o - 1)^2$, $\psi(0) = +\infty$ and $F(0) = \infty$, $(\ell_o \neq 0)$, F and G must have the general appearance shown in figure 2.

Equation (9) takes on the form

$$F(s) = G(\psi) \tag{10}$$

and corresponding values of s and ψ can be obtained graphically by drawing horizontal lines and note the s and ψ values at the intersections with the curves, as indicated in figure 2. Then, v is obtained as

$$v = \psi - \psi_m$$

In summary, to obtain the electrostatic potential v, the function $F(s)$ is calculated as an integral of a known function involving the magnetic flux and the canonical angular momentum, which must be large enough, as a parameter. The integral is then put equal to $G(\psi)$, that may be called the Abel transform of the function representing half the change in polar angle versus total energy (with constant canonical angular momentum). This gives in implicit form the fictitious potential as a function of radius. After subtracting the part corresponding to the azimuthal motion, only the electrostatic potential remains.

3. Examples

For a given shape $\tau(s)$ of the magnetic field, the part $\psi_m(s;\ell_o)$ of the fictitious potential is uniquely defined as a function of s by the value of the single parameter ℓ_o. The initial value of the azimuthal velocity of the particles, and the pertinent energy interval necessary for an inversion can be obtained in a simple way from the magnetic field strength, the appropriate linear dimension and the mass and charge of the particles by denormalization, using the formulas of the preceding section.

We shall illustrate the theory by analyzing two examples in normalized variables; the first is sufficiently simple to permit analytical methods to be used, while the second, referring to a particular laboratory experiment, requires a numerical treatment.

Among the properties of ψ_m, the following three are generally valid:

(i) $\psi_m \geq 0$, with equality exactly when $\tau = \ell_o$

(ii) $\ell_o \neq 0$ implies that $\psi_m \to \infty$ when $s \to 0$

(iii) for orbits through the axis, $\ell_o = 0$. For most physically interesting magnetic fields (with $\tau(s) = o(s)$ when $s \to 0$; typically, $\tau(s) = O(s^2)$), we then have $\psi_m(0) = 0$ when $\ell_o = 0$, in contrast to the property (ii) above. This is caused by the singularity for $r = 0$ of the transformation between Cartesian and polar coordinates; one either prescribes that a negative value of s corresponds to adding π to the polar angle, or the ψ_m-axis is simply regarded as a part of the ψ_m-curve.

A. Homogeneous Magnetic Field with No Electric Field

For a homogeneous magnetic field we have $\tau = s^2$ and $\psi_m = [(\ell_o/s) - s]^2$. By analyzing the function ψ_m the following additional properties may be derived for injection at $s = 1$ into a homogeneous B:

(iv) $\ell_o > 1$ implies $d\psi_m/ds < 0$; the fictitious potential is monotonically decreasing; at all points of the orbit the magnetic field and centrifugal force accelerate the particle away from the axis.

(v) $\ell_o = 1$ means that the particle is injected exactly radially, and ψ_m has a minimum (=0) at $s = 1$, the point of injection.

(vi) for $-1 < \ell_o < +1$ the particle is initially accelerated by $\overline{B}$ towards the axis.

(vii) for $0 < \ell_o < +1$, the positive half of the interval in (vi), ψ_m has a minimum $\psi_{min} = 0$ at $s = \sqrt{\ell_o}$.

(viii) for $-1 < \ell_o < 0$, the negative half of the interval in (vi), ψ_m has a nonzero minimum $\psi_{min} = 4 \cdot |\ell_o|$ at $s = \sqrt{|\ell_o|}$.

(ix) for $\ell_o = -1$ the centrifugal and magnetic forces exactly cancel initially; ψ_m has a minimum at $s = 1$; if the particles have no radial energy initially, they will simply perform a gyrational motion in a circle with radius 1 and center at the origin.

(x) $\ell_o < -1$ implies that ψ_m is monotonically decreasing.

Representative examples of the various curves possible are shown in figure 3. Suppose, for illustration, that a particle has $\ell_o = -0.4$ and $w_o = 3$. The pertinent ψ_m-curve is then f in figure 3. The initial kinetic energy of the azimuthal motion is then given by the distance np = 1.96 in the figure; (with the present normalization, this is given by $(\ell_o - 1)^2$]. The distance mn, which is 1.04, thus represents the initial kinetic energy of the radial motion. If the particle is initially moving inwards, it will proceed as far as the point t with $s \approx 0.280$, the intersection between the curve f and the straight line $\psi_m = w_o$, and then be reflected back out. The radial velocity is simply $u_r = \pm\sqrt{w_o - \psi_m} = \pm\sqrt{3 - \psi_m}$, and the azimuthal velocity is $u_\phi = \ell_o\tau/s = (-0.4 - \tau)/s$. If w_o is smaller than the distance np, the particle cannot exist at $s = 1$ and have the potential f. Clearly, in the present case the potential is initially attractive, while for instance curve h corresponds to a repulsive potential.

For other shapes of B than homogeneous, other types of fictitious potential curves may appear, especially when $\underline{B}$ changes its direction.

The validity of the inversion equation (9) can easily be checked analytically in the present case.

The left-hand member becomes

$$F(s) = \int_s^1 \frac{\ell_o - s^2}{s^2}\, ds = s - (\ell_o + 1) + \frac{\ell_o}{s}$$

To calculate the right-hand member, the function $f(w_o)$ is first determined from geometrical considerations (see fig. 4). The normalized gyro radius is found to be $u_o/2$, where u_o is the normalized initial velocity. By successively using the cosine and sine theorems, one obtains

$$\sin\frac{\Delta\phi}{2} = \frac{u_o}{2d}\cos\kappa = \frac{\frac{u_o}{2}\cos\kappa_1}{\sqrt{1 + \frac{u_o^2}{4} + u_o\sin\kappa_1}}$$

By utilizing $u_{o\phi} = \ell_o - 1$; $u_{or}^2 = w_o - (\ell_o - 1)^2$; $w_o = u_o^2$, one obtains

$$\cos^2\frac{\Delta\phi}{2} = \frac{(\ell_o + 1)^2}{w_o + 4\ell_o}$$

Thus,

$$f(w_o) = \frac{\Delta\phi}{2} = \arccos\frac{\ell_o + 1}{\sqrt{w_o + 4\ell_o}}$$

This function is plotted for the special choice of $\ell_o = 2$ (curve b in fig. 3), which gives a monotonic potential, in figure 5; the total energy w_o varies between its minimum possible value $(\ell_o - 1)^2 = 1$ and 10. In this case, $f(w_o)$ is monotonic, varying from 0 to 0.78.

Inserting the above expression for $f(w_o)$ and performing a partial integration, one obtains

$$G(\psi) = \frac{\ell_o + 1}{\pi}\int_{(\ell_o-1)^2}^{\psi}\frac{\sqrt{\psi - w_o}\,dw_o}{(w + 4\ell_o)\sqrt{w_o - (\ell_o - 1)^2}}$$

The substitution

$$w_o = (\ell_o - 1)^2\cos^2\xi + \psi\sin^2\xi$$

often used in connection with Abel integrals, transforms the integral to

$$G(\psi) = \frac{2[\psi - (\ell_o - 1)^2]}{\pi(\ell_o + 1)} \int_0^{\pi/2} \frac{d\xi}{1 + \beta^2 \tan^2 \xi}$$

where

$$\beta = \frac{\sqrt{\psi + 4\ell_o}}{\ell_o + 1}$$

The integral is calculated, using integral tables or the calculus of residues, and its value is found to be $\pi/2(\beta + 1)$. After some algebraic manipulations, one obtains

$$F(\psi) = \sqrt{\psi + 4\ell_o} - (\ell_o + 1)$$

The equation $G(s) = F(\psi)$ then gives

$$\psi = \left(\frac{\ell_o}{s} - s\right)^2$$

which is exactly the form of $\psi = \psi_m$ for the assumed, homogeneous magnetic field. Thus, the correct result is obtained in this case.

B. The NASA Lewis Modified Penning Discharge

The theory exposed in section 2 will now be applied to a practical case, the NASA Lewis Modified Penning Discharge, in order to illustrate the procedure and test the accuracy obtainable in the numerical work. To simulate some of the numerical and experimental errors, the procedure has been made deliberately coarse. The actual magnetic field and ion species, and an assumed, physically relevant, nonmonotonic electrostatic potential are used in the calculations.

First, a sufficiently large canonical momentum is chosen, that gives a monotonic fictitious potential. Then the function $f(w_o)$ representing half the change in polar angle is calculated. The inversion procedure giving $\psi(s)$ is performed, and the contribution ψ_m is subtracted. The difference $v = \psi - \psi_m$ is then compared with the assumed function, and it is found that a very good agreement can be obtained by simple means.

The discharge is a Penning discharge in a magnetic mirror field (ref. 15). The anode consists of two closely spaced rings, arranged in a symmetric way parallel to the midplane of the magnetic field. Thus, in the midplane between the rings the magnetic field is locally cylindrically symmetric. The anode radius is 7.6 centimeters, and at $r = 13.5$

centimeters there is a grounded mesh screen defining the potential zero all the way out to the grounded vacuum vessel. A thallium ion gun (ref. 7) is located 86 centimeters from the axis. This point will be the mainpoint of normalization in the following calculations.

Normalized to the gun location, the function $\tau(s)$ is shown in figure 6. There is a change in the direction of $\overline{B}$ with a corresponding maximum of τ at $r = 28$ centimeters ($s = 0.33$), but there is still a significant net flux at the gun location.

We assume Tl-ions with mass number 205, and $B = 0.47\ \mathrm{Vs/m^2}$ on the axis, close to the value used as the typical magnetic field by Kambic (ref. 7) in his measurements mentioned in section 1. The normalization then yields the following values for the units of length, magnetic flux, angular momentum, velocity, and energy:

$$r' = 0.86\ \mathrm{m}; \qquad \varphi' = 2.17\times10^{-2}\ \mathrm{V\ sec}; \qquad L' = 5.53\times10^{-22}\ \mathrm{VA\ sec^2},$$

$$v' = 1.58\times10^{3}\ \mathrm{m/sec}; \qquad W' = 4.97\times10^{-19}\ \mathrm{Nm}$$

For the potential V we shall adopt a variation suggested by Roth (ref. 16); its general character agrees with the potential found experimentally by Kambic (ref. 7). The potential is assumed to increase parabolically from zero at $r = 0$ up to 10 kV at the anode location, and fall linearly to zero at and beyond the mesh. The fully drawn curve in figure 7 shows this variation, in normalized variables.

To determine a sufficient but not too high value of ℓ_o to give an everywhere monotonic potential, the following procedure was followed: All quantities were first temporarily normalized to $r = 13.5$ centimeters, the innermost point at which the electric field vanishes identically. The values of the suprema k_1 through k_3 (sec. 2C) were determined, and a sufficient value of ℓ_o was obtained, using the inequalities (5) and (6). This value was then adjusted downwards by trial and error, and renormalization to $r = 86$ centimeters was carried out. It was found that $\ell_o = 5$, corresponding to $L_o = 2.76\times10^{-21}\ \mathrm{VA\ sec^2}$, was sufficient, whereas $\ell_o = 3$ does not give a monotonic ψ. The two curves are shown in figure 8.

Adopting the value $\ell_o = 5$, the function $f(w_o) = \Delta\varphi/2$ was determined by graphical calculation of the integral

$$\int_{s_t}^{1} \frac{(\ell_o - \tau)\,ds}{s^2\sqrt{w_o - \psi}}$$

for a limited number (17) of values of the total energy w_o. For each value of w_o, the integrable singularity associated with the square root in the denominator was controlled by integrating analytically from s_t to a suitably chosen, slightly higher value of s. On this interval, the rest of the integrand is approximately constant.

Figure 9 shows $f(w_o)$. We see that f increases rapidly from 0 at $(\ell_o - 1)^2 = 16$ up to an essentially constant value of about 1.3. For $w_o > 620$ the energy is sufficient to allow the particles to enter the region of electric field, and f decreases to a minimum of 0.7. When the peak electrostatic potential is passed, the particles are turning in the region of inwards directed electric field. In the outer part of that region ψ has locally a very small modulus of its derivative due to the critical choice of ℓ_o, and f jumps in a discontinuous fashion from 0.9 to a high value, $f \approx 2.25$, which takes place for $w_o = 5740$. This is indeed typical for particles turning at points where $d\psi/ds$ is close to zero, and this is suggested (see sec. 3D) to be actively used as a diagnostic method. For still larger w_o, $f(w_o)$ decreases to a minimum and then tends to its asymptotic value $\pi/2$. The particles have a given, finite azimuthal energy, and in the limit of very high w_o the radial energy becomes very high, and the particles move - essentially uninfluenced by the fields - along curves tending to straight lines through the origin. This gives $(\Delta\phi/2) = \pi/2$.

To perform the inversion procedure (see eq. (9)) the function $(\ell_o - \tau)/s^2$ is calculated and graphically integrated from a variable lower limit to the upper limit ℓ. This gives the function $F(s)$, the left-hand member of equation (9). To calculate the right-hand member $G(\psi)$ from the earlier obtained $f(w_o)$, the integration was adapted to a standard Abel inversion program used in spectroscopy, see Lochte-Holtgreven (ref. 17) and cited references. Putting $w_o = w_{max} - y^2$; $\psi = w_{max} - r^2$; $G(\psi)$ takes on the form

$$- \frac{1}{\pi} \int_r^{\sqrt{w_{max}-(\ell_o-1)^2}} (-2yf) \frac{dy}{\sqrt{y^2 - r^2}}$$

which agrees with the formula in reference 17, if

$$R = \sqrt{w_{max} - (\ell_o - 1)^2}; \qquad I = \int_{(\ell_o-1)^2}^{w_o} f(w_o')dw_o'$$

Thus, an integration of $f(w_o)$ was needed, and it was performed graphically. For frequent use it should be simpler to construct a special program for calculating $G(\psi)$.

F(s) was monotonic, but there turned out to appear some deviations from monotonicity in G. However, monotonic curves could be drawn fitting the points reasonably well. The equation $F(s) = G(\psi)$ was then solved graphically, and ψ_m was subtracted. Four Abel inversions were made altogether, corresponding to different values of the maximum beam energy w_{max}. The points obtained for the potential $v(s)$ are plotted in figure 7; the fully drawn curve is the assumed potential. Clearly, it is possible to reconstruct numerically, with good accuracy, a potential of characteristic shape of relevance to the Modified Penning Discharge. The moderately good agreement around $s = 0.15$ is probably due to the low number of points chosen initially in this region.

Under laboratory conditions it is usually the energy w_o and direction κ of the beam that are varied. It should be noted, that a variation of w_o with constant ℓ_o leads to a simultaneous variation of w_o and κ, according to a definite law. With normalization to the gun position, we have $u_{o\phi} = \ell_o - 1 = \sqrt{w_o} \sin \kappa_1$, where κ_1 is the angle (with sign) between the gun line of sight and the radius, see figure 1. Thus, the angle should be varied as

$$\kappa_1 = \arcsin \frac{\ell_o - 1}{\sqrt{w_o}}$$

With $\ell_o = 5$, the value $w_o = 620$ was shown to allow the particles to reach the outermost part of the electric field region, and for w_o greater than 5740 the region immediately inside the electrostatic potential maximum was accessible. These values of w_o correspond to $\kappa_1 = 9.3$ and 3.0 degrees, respectively.

Even if the data in the numerical experiment above were treated in a rough way, ℓ_o had the exact value 5.0 throughout the calculations. In practice, however, there will always be a spread in ℓ_o if the beam is incompletely or incorrectly focused. Even if all particles move in parallel orbits when leaving the gun, there will be a spread in the angle of injection, due to the finite thickness of the beam. This spread is estimated to δ/r_1, where δ is the beam thickness. With $\delta = 3$ millimeters, $r_1 = 86$ centimeters, this becomes approximately 0.2 degrees, which is about 3 percent of the interesting interval of 6 to 7 degrees. The spread can be reduced if the beam injected is convergent, with its nominal focus close to the axis of the configuration.

To investigate the influence of varying ℓ_o, the function $f(w_o)$ was calculated not only for $\ell_o = 5.0$ (fig. 9) but also for $\ell_o = 5.1$, corresponding to a 2 percent variation. Apart from the very jump at the point where $|d\psi/ds|$ is small and a variation with ℓ_o is expected, the two curves are almost exactly identical.

C. Estimate of Necessary Ion Energy

A characteristic of importance for the design and cost of an ion beam system is the maximum ion energy $\hat{W}_o$ with which the system is to operate. This will be briefly discussed below for the primary beam method, with special reference to the Modified Penning Discharge. A corresponding discussion for the secondary beam method, with application to another laboratory experiment - the SUMMA experiment - is found in sections 4B-C.

(i) First of all, the momentum of the particles must be sufficient so that they can reach the whole region of interest. The function τ, suitable values of ℓ_o and the "region of interest" define a minimum necessary value of the normalized energy w_o as the maximum of $\psi_m = (\ell_o - \tau)^2/s^2$, and for given τ and region of interest this value is the same for all particles, irrespective of mass and charge. Details are found in section 4B.

For given charge q, the unit P'_{mech} for the mechanical part mv' of the linear momentum is given solely by the units for length and magnetic flux as

$$P'_{mech} = \frac{q\Phi'}{2\pi r'}$$

and the unit for energy by $W' = 1/2m\,(P'_{mech})^2$, which also contains the mass m.

A necessary value of W_o is then obtained by multiplying W' by the value of w_o found above. Clearly, as heavy particles as possible should be used if the energy is to be kept low. However, an upper limit of the mass is set by the maximum mass number that can be used. (Since the system is supposed to analyze the electromagnetic field in a plasma we do not expect it to be possible to use charged droplets or other "superparticles"; such entities are expected to interact with the plasma.)

In the Modified Penning Discharge we may obtain a rough estimate by choosing $\ell_o = 0$, corresponding to an orbit through the axis. This gives $\psi_m = (\tau/s)^2$, which has a maximum value of about 50 (for $s \approx 0.17$). With $W' = 4.97\times10^{-19}$ Nm , this gives the condition $\hat{W}_o$ 155 eV, which is a very liberal requirement indeed.

(ii) If there is a repulsive potential in a region, the beam energy must be sufficient to allow the particles to reach that region. This defines an additional requirement on W_o, in the present case $W_o > 10$ keV.

(iii) If there is a region of inwards directed electric field (and the ions are positively charged) an especially high value of ℓ_o is required to give a monotonic fictitious potential, which is essential for the method to work. This puts a requirement on the kinetic energy of the beam in that region. Especially if the potential energy is high there, the total energy W_o may need to be quite high, as can be seen from the following dimensional analysis.

The ratio between magnetic and centrifugal force is

$$\frac{qvB}{mv^2/r} = \frac{r}{\rho}$$

where $\rho = mv/qB$ is the formal gyro radius. In applications we expect this ratio to be smaller than unity - but perhaps not very much smaller. This justifies us to temporarily neglect the magnetic force against the centrifugal force, at least on a local scale. A monotonic fictitious potential at a certain radius then means that there are particles such that the centrifugal force is strong enough to prevent the electric field from pulling the particles closer to the axis, i.e., we must at least have

$$\frac{mv^2}{r} = qE$$

Multiplication by $r/2$ gives

$$\frac{mv^2}{2} = \frac{qEr}{2}$$

If the potential at this point is V_o, we must have

$$\hat{W}_o \quad q\left(V_o + \frac{Er}{2}\right)$$

where E is directed inwards. Clearly, the effect becomes especially important if we have, at a large radius, both a high potential and a strong inwards electric field. In the present case, with $V_o = 10^4$; $V = ar^2$; $E = 2ar$; $Er = 2V$, we obtain

$$\hat{W}_o > \left(10 + \frac{2 \cdot 10}{2}\right)\text{keV} = 20 \text{ keV}$$

It was found in section 3B that $w_o = 5700$ was enough to probe the region immediately inside the potential maximum, where V, E, and r are all large. This corresponds to an energy of 17.8 keV, which is a little lower than the figure above, the difference being due to the influence of the magnetic field which was neglected above.

With $\ell_o = 5$, sufficient energies to probe the region well inside the point of maximum electrostatic potential, $s = 0.088$, can be obtained from the corresponding fictitious potential curve in figure 8. However, these energies are unnecessarily high (compare sec. 3D).

It should be observed that even if the magnetic field did not give rise to any special requirements on the beam energy in the present case, this is not true in general. Especially if the magnetic field is strong and its influence extends over a region much larger than that of the electric field, the requirement (i) above becomes essential (compare the example (SUMMA) in sec. 4C).

(iv) Additional requirements may arise from considerations concerning ion optics, notably focusing, and scattering, but this is not discussed here.

D. Comments

It should be pointed out that a lower beam energy is sufficient if one uses several values of ℓ_o in succession. For instance, the fictitious potential curve corresponding to $\ell_o = 3$ is monotonic for sufficiently small s, and particles may have turning-points inside $s = 0.052$, where the value of ψ is equal to the local maximum at 0.088. If the potential has been determined from $s = 1$ inwards as far as $s = 0.052$, (e.g., using $\ell_o = 5$) one can then switch to $\ell_o = 3$, and the integral for $f(w_o) = \Delta\phi/2$ can be written as

$$f(w_o) = \int_{s_t}^{0.052} + \int_{0.052}^{1} \qquad \ell_o = 3$$

The second term on the right can be calculated since v is already known on that interval of integration, and the same term is simply subtracted from $f(w_o)$. Then the same inversion scheme is used for the first term.

By this change of ℓ_o one needs a lower energy for the probing; for small s the function $\psi(s;\,3)$ is half the function $\psi(s;\,5)$ or less. (Often ψ is approximately proportional to ℓ_o^2.) It should be possible to develop this method of varying both w_o and ℓ_o to a systematic procedure. It is perhaps possible to use it in an infinitesimal way; if v has been determined in to a certain value of s, we may then proceed inwards, using

$$df = \frac{\partial f}{\partial w_o}\, dw_o + \frac{\partial f}{\partial \ell_o}\, d\ell_o$$

Here, dw_o and $d\ell_o$ would be controlled at the gun and df measured. One difficulty worth mentioning is that in spite of the occurrence of $s_t = s_t(w_o, \ell_o)$ in the lower limit of integration in equation (7), the derivatives above do not contain any local contributions from s_t, since there will be a factor $\sqrt{\psi - w_o}$, which becomes 0 at s_t, in the numerator and thus the differential expression above does not contain the local electric field.

Clearly, one needs to know when ψ ceases to be monotonic; this can be done by observing sudden jumps or peaks in f for certain (w_o, ℓ_o), as in figure 9 (ψ' was not exactly zero in that case, only nearly). It is easy to show that if $\psi'(s_t) = 0$, f becomes infinite. Indeed, Taylor's theorem gives

$$\psi(s) = \psi(s_t) + (s - s_t)\psi'(s_t) + (s - s_t)^2 \frac{\psi''(\xi)}{2}$$

in a neighborhood of s_t. Using $\psi(s_t) = w_o$ and assuming $\psi'(s_t) = 0$ we obtain

$$w_o - \psi = - \frac{\psi''(\xi)}{2} (s - s_t)^2$$

Insertion into (7) gives

$$\frac{\Delta\phi}{2} = f(w_o) = \int_{s_t}^{a} \frac{\ell_o - \tau}{s^2} \cdot \sqrt{- \frac{2}{\psi''(\xi)}} \cdot \frac{ds}{s - s_t} + \int_a^1$$

Clearly, the first integral is at least logarithmically infinite, due to the factor $(s - s_t)$ in the denominator. This behavior of f is analogous to the singularity of the "equivalent height of reflection" of a radio wave reflected at an electron density maximum in an ionospheric layer (ref. 18). Like in the ionospheric case we expect that the singularity in practice becomes a pronounced peak.

4. SECONDARY BEAM METHOD

A. Description of Method

In the secondary beam method by Hickok and Jobes, see for instance reference 13, a beam of "primary" ions is sent through a plasma. Due to collisions with the hot plasma electrons, ions in higher ionized states - "secondary" ions - are produced along the primary beam path through the plasma.

Due to the small mass of the ionizing electron, there is negligibly small change of the mechanical momentum of the ion in the collision. Therefore, its velocity is continuous at the point of ionization. Thus, the orbit of the secondary particle is tangent to the primary beam path. Furthermore, since the charge is changed by a specific ratio, usually 2:1, the electromagnetic force and the inverse of the radius of curvature at the ionization point will be changed by the same ratio.

The change of charge or, more generally, charge to mass ratio (ref. 19) also produces a change in total energy and canonical momentum of the particle. Due to the continuity of $\bar{v}$, these changes are given by

$$\Delta W_o = \Delta q \cdot V(\bar{r}_i)$$

$$\Delta \bar{p} = \Delta q \cdot \bar{A}(\bar{r}_i)$$

where $\bar{p}$ is the linear canonical momentum, $\bar{r_i}$ is the point of ionization, and Δ denotes the difference between the values after and before the ionization.

By systematically studying the secondary particles thus generated, it is possible to draw certain conclusions about the local conditions at the various points of ionization.

B. Cylindrically Symmetric Case

In a cylindrically symmetric situation the change in canonical angular momentum and total energy produced by the ionization are given by

$$\Delta L_o = \Delta q \cdot \frac{\Phi(r_i)}{2\pi}$$

$$\Delta W_o = \Delta q \cdot V(r_i)$$

We notice that the effect of the ionization is to change the values of the two constants of motion, quantities that characterize the motion essentially completely. This need not be true in the general case, however.

We shall exclusively deal with transition from singly to doubly charged positive particles, $\Delta q = 2e - e = e$, and in the process of normalization we shall refer to the singly charged species, $q = e$. Index I will be used for primaries and II for secondaries. Whenever suitable, these indices will be dropped for simplification.

In addition to the previously derived formulas

$$u_{\phi I} = \frac{\ell_{oI} - \tau(s)}{s}$$

$$\psi_{mI} = u_{\phi I}^2$$

$$\psi_I = \psi_{mI} + v$$

$$u_{rI}^2 = W_{oI} - \psi_I$$

for the primaries, we now also have

$$u_{\phi II} = \frac{\ell_{oII} - 2\tau(s)}{s}$$

$$\psi_{mII} = u_{\phi II}^2 \tag{11}$$

$$\psi_{II} = \psi_{mII} + 2v$$

$$u_{rII}^2 = w_{oII} - \psi_{II}$$

for the secondaries. Here, ℓ_{oII} and w_{oII} are given by

$$\ell_{oII} = \ell_{oI} + \tau(s_i) \tag{12}$$

$$w_{oII} = w_{oI} + v(s_i) \tag{13}$$

The same L' and W' have been used in the normalization. The formula $d\phi/ds = u_\phi/ru_r$ has the same form for primaries and secondaries.

In the cylindrically symmetric case the secondary beam method allows an immediate, unique determination of the potential $v(s)$, at least in principle. The potential is given by the change in total energy according to equation (13) above, and the location by the change in ℓ_o (eq. (12)). The primary momentum ℓ_{oI} is known from the initial value of the azimuthal velocity and the magnetic flux enclosed inside the gun location. If the detector is capable of measuring not only the total energy of the secondaries but also the azimuthal component of the velocity, the flux $\tau(s_i)$ can be obtained by combining equations (11) and (12) above and putting $s = \beta$, corresponding to the location of the detector. One then obtains

$$\tau(s_i) = \beta u_{\phi II} + 2\tau(\beta) - \ell_{oI}$$

or

$$\tau(s_i) = (\beta u_{\phi II} - \alpha u_{\phi I}) + [2\tau(\beta) - \tau(\alpha)]$$

where ℓ_{oI} has been expressed in initial values of flux and azimuthal velocity. If, in particular, the magnetic field has a unique direction inside the plasma, s_i can be determined from knowledge of $\tau(s_i)$. Even if it has not, it seems that it should be possible to use the knowledge of the beam deflection to distinguish between two - or a few - values of s_i having the same τ.

It should be pointed out that in practice $u_{\phi II}$ can only be measured with limited accuracy. Especially if the plasma is very small compared to the extent of the magnetic field, see section 4C below, the method does not appear to be feasible, but if the plasma fills the major part of the magnetic field, it should be possible to use this method of locating the point of ionization.

If the total beam deflection from gun to detector would be very sensitive to changes in ℓ_0, the same deflection could perhaps be used to determine the change in ℓ_0 and hence the quantity s_i. However, this possibility has not yet been investigated in detail.

As with the primary beam method, there are certain requirements on the ion energy.

(i') Not only must the momentum of the primaries be sufficient for penetration into $\underline{B}$, but that of the secondaries must also be sufficient to allow them to move to a place where they can be detected. If the charge is doubled in the ionization, this requires - for given mass and magnetic field to pass through - essentially four times as high energy as with the primaries alone. This additional energy requirement is typical for the secondary beam method. The energy necessary can be significantly reduced if the detector can operate in a strong magnetic field (ref. 20). The above considerations will be substantiated in the example below (see sec. 4C).

(ii') Repulsive electrostatic potentials have the same kind of importance as in the primary beam method; it should be noted that the potential energy is doubled for the secondary beam. On the other hand, an attractive potential will give the particles a momentum increase facilitating their entering and leaving the magnetic field region.

(iii') There does not seem to be any requirement directly corresponding to (iii) in section 3C, since there is no need for a monotonic fictitious potential in the secondary beam method. However, there may be some considerations connected with the actual determination of v that put requirements on the beam energy, notably if v is solved by iteration (compare sec. 5 below).

C. The SUMMA Experiment

The NASA Lewis SUMMA experiment (ref. 21) is a burnout device using a discharge in a magnetic mirror field, with two hollow cathodes and two ring-shaped anodes. In the midplane the configuration is locally cylindrically symmetrical. The magnetic field data (ref. 22) show that the same field changes its direction at a distance of 77.78 centimeters from the axis, and the flux function - normalized to this point - is shown in figure 10. In the normalization procedure we assume that the magnetic field on the axis (in the midplane) is 40 kG - although values up to

49 kG are planned. The particles are thallium ions with mass number 205. The unit energy W' then becomes 0.3949×10^5 eV.

An electrostatic potential curve compatible with experimental data (ref. 23) is also shown in figure 10; it corresponds to a potential well of about 14 kV. It has the functional dependence $v = a \sin b_s/s + c$ which with the chosen values of a, b, c gives a potential and electric field that are continuous both on the axis, $s = 0$, and at the suggested plasma boundary, $s = 0.0243$ ($r = 1.89$ cm).

We shall estimate a minimum energy necessary to probe this plasma.

It is necessary that primary particles with $\ell_o = 0$ can pass the maximum of the corresponding fictitious radial potential and that the secondary particles generated by these primaries can reach the detector. The reason is that $\ell_o = 0$ is necessary for particles passing through $s = 0$, the innermost point of the plasma.

As to the sufficiency for the present energy estimate, one may take either of two attitudes:

(a) $\ell_o = 0$ is sufficient to consider. Indeed, all values of s in the plasma are reached by such a beam, and secondaries are generated for all these values of s. It even turns out that the energy requirement obtained by considering $\ell_o = 0$ allows a probing of essentially half the plasma in the present case. However, to keep the detector position even more at our disposal, we may prefer a more stringent attitude - (b) below.

(b) ℓ_o should be allowed to take a set of values such that the whole plasma is covered by primary orbits. For the present purpose, we shall be satisfied with the increased coverage obtained with a set of beams that have radial turning points for all values between 0 and δ, on both sides of the axis.

In the present case, the requirements set by the attitudes (a) and (b) do not differ considerably

In what now follows, we shall neglect the influence of the electrostatic potential on the effective potential. This is justified by the result; the electrostatic potential only has a marginal influence unnecessary to consider in the present energy estimate.

The fictitious potential for a λ-fold ionized particle with $\ell_o = 0$ then becomes

$$\psi = \lambda^2\left(\frac{\tau}{s}\right)^2 = \lambda^2 x(s)$$

The function $x(s)$ is displayed in figure 11. The maximum value is 1.8123 and occurs at $s = 0.565$ ($r = 43.9$ cm). Thus, the minimum energy w_o

necessary for a primary particle to reach the center is $x_{max} = 1.8123$, which with the value of W' given above corresponds to 71.6 keV. Using unnormalized variables, the necessary energy of particles with zero angular momentum is obtained from formula (3) in section 2A as

$$W_o = \frac{q^2}{8m\pi^2} \cdot \max\left(\frac{\Phi}{r}\right)^2 \tag{14}$$

With singly charged ions this energy, expressed in electronvolts, becomes

$$W_o = \frac{e}{8m\pi^2} \cdot \max\left(\frac{\Phi}{r}\right)^2$$

For primary energies in an interval above this lower limit, the secondaries would be trapped inside the magnetic field. Thus, a higher primary beam energy may be necessary.

Secondaries with $\Delta\ell_o \equiv \tau(s_i) = 0$ are produced at $s_i = 0$. These particles then have $\ell_{oII} = 0$, and the potential $\psi_{II} = 4x(s)$. The energy necessary to bring these particles out will thus be $4x_{max}$, corresponding to about 287 keV, which must be furnished by the primary beam (since the influence of v was neglected). For other secondaries than those with $s_i = 0$, there is a small positive increment $\tau(s_i)$ to ℓ_o. It will be shown below that the magnitude of the fictitious potential maximum will be smaller for these particles. Thus, the minimum necessary primary ion energy is 287 keV if both the gun and the detector are located outside the magnetic field, and attitude (a) above is adopted.

This requirement may be relaxed if the detector is located inside $\underline{B}$. Clearly, it would have to be moved inside $s = 0.565$, the location of x_{max}. If primary particles with $w_{oI} = 1.8123$ are considered, the maximum radius s_t of secondaries with $\ell_{oII} = 0$ is obtained from the equation $4x(s_t) = 1.8123$, which gives $s_t = 0.1853$ ($r = 14.4$ cm).

We conclude that any lowering of the necessary energy requires that the detector be moved inside $s = 0.565$ ($r \approx 44$ cm), where $B \approx 13$ kG. With unchanged position of the gun, the minimum energy is gradually lowered from 287 to 71.6 keV if the detector is moved as far as $s = 0.1853$ ($r \approx 14$ cm) where $B \approx 37$ kG, according to the formula $W_o = W' \cdot 4 \cdot x(\beta)$ (where β is the detector position). Moving the detector inside 14 centimeters does not lead to any decrease in necessary primary energy, unless the gun is also moved inside 44 centimeters, or some entirely different arrangement is used, like an electric field guiding the beam through $\underline{B}$ (ref. 24).

To see how a requirement on increased coverage of the plasma influences the minimum beam energy of primaries, we shall now construct, for an aribtrary fixed value of w_o in an interval of admissible values, an interval of ℓ_o sufficient to give turning points between 0 and δ, on both sides of the axis. Then, a minimum value of w_o among the admissible ones will be determined, that is sufficient for probing with all the corresponding ℓ_o. Let s_1 be a number $>\delta$ and let the admissible values of w_o be given by

$$[\tau'(\delta)]^2 \leq w_o < \left[\frac{\tau(s_1) - \tau(\delta)}{\delta}\right]^2 \tag{15}$$

The reason for this choice will become apparent later.

The condition for radial reflection, $\psi(s_t) = w_o$, can be solved for ℓ_o. One obtains $\ell_o = \tau(s_t) \pm s_t\sqrt{w_o}$. For a given w_o, this defines two functions of s_t, namely, $\tau(s_t) + s_t\sqrt{w_o}$ and $\tau(s_t) - s_t\sqrt{w_o}$. The first of these is monotone increasing with s_t, since $\tau' \geq 0$ for $s_t \varepsilon [0,\delta]$; and the second is monotone decreasing, provided that $w_o \geq \max\limits_{0 \leq s_t \leq \delta} [\tau'(s_t)]^2 = [\tau'(\delta)]^2$; i.e., if w_o is admissible. Thus, the interval for ℓ_o bounded by the two functions of s_t above expands on both sides, when s_t is increased, and its maximum extent is reached for $s_t = \delta$. Consequently, the turning points of the orbits with ℓ_o in the interval

$$\tau(\delta) - \delta\sqrt{w_o} \leq \ell_o \leq \tau(\delta) + \delta\sqrt{w_o}$$

form a curve across the plasma, through the axis. This is the pertinent interval for ℓ_o.

Differentiating $\psi = (\ell_o - \tau)^2/s^2$ with respect to s, one gets

$$\psi' = -\frac{2}{s^3}(\ell_o - \tau)(\ell_o - \tau + s\tau')$$

The derivative is zero for $\ell_o - \tau = 0$, but any such solution of $\psi' = 0$ must correspond to a minimum of ψ, since the nonnegative function ψ is then equal to zero. A necessary condition for maximum at a point s_1 inside or outside the plasma is then

$$\ell_o - \tau(s_1) + s_1\tau'(s_1) = 0 \tag{16}$$

The second derivative is

$$\psi'' = -\frac{2}{s^2}(\ell_o - \tau)\tau'' + \Gamma$$

where Γ contains the factor $\ell_o - \tau + s\tau'$. Putting $s = s_1$ and using equation (16) above, we obtain

$$\psi''(s_1) = \frac{2}{s_1}\,\tau'(s_1)\tau''(s_1) \leq 0$$

the inequality being a necessary condition for a maximum. The situation of interest to us is when $\tau'(s_1) \geq 0$ and $\tau''(s_1) \leq 0$. Indeed, from figure 10 it appears that $\tau'(s_1) < 0$ and $\tau''(s_1) > 0$ would certainly require $s_1 > 1$ and $\tau(s_1) > 0.6 > \tau(\delta) \approx 0.0022$. If $\tau'(s_1) < 0$, equation (16) gives $\ell_o > \tau(s_1)$, and

$$\psi(\delta) = \left(\frac{\ell_o - \tau(\delta)}{\delta}\right)^2 > \left(\frac{\tau(s_1) - \tau(\delta)}{\delta}\right)^2$$

and thus, no admissible w_o would allow the particles to reach the point δ.

Considering equation (16), the maximum potential may be written $\psi_{max} = [\tau'(s_1)]^2$. Thus, it appears that an upper limit for the energy necessary to reach all points in the magnetic field is set by $\max(\tau')^2$. The derivative of ψ_{max} with respect to ℓ_o is $2\tau'(s_1) \cdot d/d\ell_o\, \tau'(s_1)$. Differentiating (16) with respect to ℓ_o, one obtains

$$1 + s_1 \frac{d}{d\ell_o}\tau'(s_1) = 0$$

Hence,

$$\frac{d}{d\ell_o}\tau'(s_1) = -\frac{1}{s_1} < 0$$

and

$$\frac{d\psi_{max}}{d\ell_o} = -\frac{2}{s_1}\,\tau'(s_1) < 0$$

since $\tau'(s_1)$ must be positive. We conclude that the maximum maximorum of ψ for various s and various values of ℓ_o in its interval is given with ℓ_o at its minimum value $\tau(\delta) - \delta\sqrt{w_o}$. The minimum possible w_o can then be obtained from

$$\sqrt{w_o} = \frac{\tau(s_1) - \tau(\delta) + \delta\sqrt{w_o}}{s_1}$$

$$\sqrt{w_o} = \sqrt{\hat{\psi}_{max}} = \frac{\tau(s_1) - \tau(\delta)}{s_1 - \delta} = \tau'(s_1) \tag{17}$$

This energy is admissible since it turns out that $s_1 > 2\delta$, implying $s_1 - \delta > \delta$. Equation (17) above has an immediate geometric interpretation: The tangent to the $\tau(s)$-curve at the point s_1 passes through the point δ, and the square of the direction coefficient of this line, shown in figure 10, gives the minimum energy w_o.

The equation (17) was solved for s_1, and $[\tau'(s_1)]^2$ was determined. It was found that w_o must at least be equal to 1.97, W_o = 77.8 keV, for attitude b, which is about 9 percent higher than the attitude a value of 71.6 keV previously found

No detailed analysis of secondaries in attitude b has been performed.

It must be emphasized that the above estimates are only those necessary for the particle to move into and out of the magnetic field; in a practical situation other considerations must also be considered, some of which are discussed in section 5.

It should also be emphasized that even if the electrostatic potential of 14 kV is smaller or even much smaller than the particle energies of 75 to 350 keV necessary to get particles sufficiently rigid in $\underline{B}$, and hence could be neglected in the energy estimate above, this does not immediately mean that electrostatic forces can be neglected compared with magnetic forces in the plasma region. The reason is that the electrostatic forces act over a much smaller scale length, by a factor of the order of 10^{-2}. Indeed, the magnetic force on a 75 keV Tl-ion in a B-field of 40 kG corresponds to a $|\bar{v} \times \bar{B}|$ of 8.4 kV/cm, and that of a 350 keV ion to 18 kV/cm. These figures are of the same order as the electric field in the plasma; the assumed function v(s) corresponds to an average electric field of 7.4 kV/cm and a maximum field of 12 kV/cm. Even if the beam energy is increased to 1 MeV, the value of $|\bar{v} \times \bar{B}|$ is only about 31 kV/cm.

In the secondary beam method it is usually assumed that the electric field is a small perturbation that is neglected when the orbits and points of ionization are determined. In this way, they can be determined without knowledge of the potential. The potential is then determined for each ionization point by the observed change in total energy.

Our conclusion is that the electrostatic force cannot be considered as a small perturbation, without anything further. One should distinguish between two different smallness parameters,

$$\varepsilon_1 = \frac{E}{vB}$$

ratio between electric and magnetic force, and

$$\varepsilon_2 = \frac{eV_{max}}{W}$$

ratio between maximum potential and beam energy. Clearly, in the case of the SUMMA experiment $\varepsilon_1 < 1$ and $\varepsilon_2 << 1$. It will be shown in section 5 that the pertinent smallness quantity is ε_1.

Since it is desirable to have a low ratio $\varepsilon_1 = E/vB$, this can be achieved by varying the mass number of the probing ions. Indeed, apart from any other considerations there exists, for each energy within certain limits, an optimum mass number that corresponds to particles with sufficient rigidity and highest possible velocity. Using a rough estimate one finds $\varepsilon_1 = \frac{E}{B}\sqrt{\frac{m}{2W}}$, so for given fields E and B and beam energy W, a small mass is favorable. However, the gyro radius must at least be equal to half the magnetic field radius, say. We then have

$$\frac{\sqrt{2mW}}{qB} \geq \frac{R_B}{2}$$

which defines a lower limit for m. Clearly, for this limit the mass number $M \propto 1/W$, and $\varepsilon_1 = qER_B/4W$. Thus, we then have $\varepsilon_1 \propto 1/W$, in contrast to the $1/\sqrt{W}$-variation obtained with constant mass. Using the figures above, we assume that we have a possible probing with $M = 205$; $W = 350$ keV; $vB = 18$ kV/cm; $E = 10$ kV/cm; this gives $\varepsilon_1 = 0.56$. Raising the energy to 1.84 MeV and still using thallium ions, ε_1 becomes 0.24. However, if the mass number is reduced by approximately the energy ratio and potassium ($M = 39$) is used instead, we get $\varepsilon_1 = 0.11$, which is an improvement.

The interval for ℓ_{oI} for a 355 keV primary beam ($w_o = 9$) becomes $-0.0707 \leq \ell_{oI} \leq 0.0758$. Using the formulas

$$u_{\phi I}(\alpha) = \frac{\ell_{oI} - \tau(\alpha)}{\alpha} = \sqrt{w_{oI}}\,\sin\kappa_1$$

and the values $w_{oI} = 9$; $\alpha = 4$; $\tau(\alpha) = 0.44$, one obtains for the angle κ_1, the gun direction:

$$-2.44^o \leq \kappa_1 \leq -1.74^o$$

Without any lenses or similar arrangement, the interval for sweeping the entire plasma becomes 0.6^o, which is about the same as the geometrical angle of 0.7^o occupied by the plasma. In the same way, the angle κ_2 at the detector is obtained as

$$\sin\kappa_2 = \frac{\ell_{oI} + \tau(s_1) - 2\tau(\beta)}{\beta\sqrt{w_{oII}}}$$

To see whether measurements of this angle can be used for determining $\tau(s_1)$ and thereby s_i, the equation is varied, whereby $\delta \ell_{oI} = 0$ (no consideration of deflections). One obtains

$$\delta\kappa_2 = \frac{180}{\pi} \cdot \frac{\delta\tau(s_1)}{3\beta}$$

For a reasonable spatial resolution, $\delta\tau(s_i) = 1/10\ \tau(\delta) = 0.00022$. It is found that $\beta = 4$, corresponding to a detector location outside the magnetic field, gives $\delta\kappa_2 = 10^{-3\circ}$, and $\beta = 0.2$ (detection in a strong $\underline{B}$) gives $\delta\kappa_2 = 0.02^\circ$.

Consequently, this method of determining the point of ionization does not seem feasible in the SUMMA. A corresponding estimate for the Modified Penning Discharge tells that the exit angle would have to be measured with an accuracy of about 0.1°; it would have been less stringent if the repulsive electrostatic potential in this latter experiment had not necessitated a much higher beam energy than that needed for penetration through the magnetic field.

5. SECONDARY BEAM METHOD; DETERMINATION OF V

In the present section we shall discuss how the unknown potential can be determined from secondary beam measurements.

To fix the ideas, we shall start by considering the cylindrically symmetric case; the methods appear to be amenable to extension to the general case, and the physical implications are expected to persist under more general conditions.

A. Cylindrically Symmetric Case

The total change in polar angle from the primary beam at the gun ($s = \alpha$) to the secondary beam detector ($s = \beta$) is given by

$$I_a = I_1 + I_2 + I_3$$

for particles ionized before reaching the point s_{t1} of closest approach to the axis, and by

$$I_b = I_1 + I_4 + I_3$$

for particles ionized on their way out. Here,

$$I_1 = \int_{s_i}^{\alpha} \frac{u_{\phi I}}{s\sqrt{w_{oI} - \psi_I}}\, ds$$

$$I_2 = 2 \int_{s_{tII}}^{s_i} \frac{u_{\phi II}}{s\sqrt{w_{oII} - \psi_{II}}} ds$$

$$I_3 = \int_{s_i}^{\beta} \frac{u_{\phi II}}{s\sqrt{w_{oII} - \psi_{II}}} ds$$

$$I_4 = 2 \int_{s_{tI}}^{s_i} \frac{u_{\phi I}}{s\sqrt{w_{oI} - \psi_{I}}} ds$$

The I_a and I_b are functions of the point of ionization s_i and the constants of motion ℓ_{oI} and w_{oI} of the primary beam, and functionals of the potential $v(s)$. Each of them has the form

$$I(s_i, \ell_{oI}, w_{oI}; v) = \int_0^{\gamma} K[s, s_i, \ell_{oI}, w_{oI}; v(s)] ds$$

where $\gamma = \max(\alpha, \beta)$ as before.

With the gun and detector at fixed positions, corresponding to a difference in polar angle of magnitude C, the quantities w_{oI} and ℓ_{oI} are varied in a suitable way (by varying the ion energy and angle of injection), so that all values of s_i between 0 and the plasma boundary δ are covered. For each (ℓ_{oI}, w_{oI}) the potential at the point (or points) of ionization is measured as the change in total energy $w_{oII} - w_{oI}$; clearly, this change becomes a (not necessarily one-valued) function of (ℓ_{oI}, w_{oI}). We thus have the nonlinear equations

$$\left.\begin{aligned} I(s_i, \ell_{oI}, w_{oI}; v) - C &= 0 \qquad (18) \\ v(s_i) - g(\ell_{oI}, w_{oI}) &= 0 \qquad (19) \end{aligned}\right\}$$

This system of equations should be solved for the function v for all values of its argument between 0 and δ.

The method should be determined in each case. One natural suggestion would be to use possible foreknowledge of v to make a model assumption with a certain number of parameters, like Kambic made for the primary beam method (ref. 7), another would be to expand v in a suitable set of functions. In either case, the parameters or expansion coefficients are determined so as to minimize a sum of squares of left-hand members of the type above.

At present, however, we shall only tentatively discuss a technique that has been used by Jobes and Hickok (ref. 13); their technique - like our suggested improvement of it - is by no means limited to cylindrical symmetry but works equally well in arbitrary geometries.

Jobes and Hickok use a high beam energy, which makes it reasonable to attempt to treat the electric field as a small perturbation. This allows them to find the particle orbits and points of ionization $\overline{r_i}$ without knowledge of the potential. The latter is then found from the change in total energy.

In our formalism for the cylindrically symmetric case, this would correspond to solving the equations

$$\left.\begin{aligned} I(s_i,\ell_{oI},w_{oI};0) - C &= 0 \qquad (20) \\ v(s_i) - g(\ell_{oI},w_{oI}) &= 0 \qquad (21) \end{aligned}\right\}$$

Here, s_i is determined from equation (20) and the corresponding value of v from equation (21).

Questions of immediate interest concern the quality and range of applicability of the Jobes-Hickok approximation (JH-approximation), the requirements on beam energy and momentum, in relation to the electric and magnetic fields. It seems especially important to determine in what sense the electric field should be small. In situations when the method needs refinements, it is natural to try to improve it by iteration.

B. Iteration Scheme

Both equations (18) and (19) above contain v. If the dependence of I on v were reasonably strong, we would have a situation similar to the primary beam method; the change I in polar angle would furnish information on the function v. However, we shall be interested in trying to adopt the Jobes-Hickok attitude in the form that equation (18) essentially determines s_i (and eq. (19) determines v). We then wish that I depends strongly on s_i, which would tend to allow a good determination of s_i, and that I depends weakly on v, in order that the determination of s_i is insensitive to our assumption what v is. It seems that the ratio between the variation of I with v and its partial derivative with respect to s_i is pertinent. To get the correct physical dimension for the variation of the potential, it seems that some gradient of v would enter into our performance quantity. This will be more clear in the formal formulation below.

In the iteration both s_i and v are iterated, but ℓ_{oI} and w_{oI} are not; instead, they are controlled by the conditions at the gun. As the scheme for iteration, the following is hereby suggested:

$$\left.\begin{aligned} & I(s_{i,n}, \ell_{oI}, w_{oI}; v_{n-1}) - C = 0 && (22) \\ & v_n(s_{i,n}) - g(\ell_{oI}, w_{oI}) = 0 && (23) \\ & v_n(\cdot) = v_n(s_{i,n});\ s_{i,n} \varepsilon [0,\mu];\ \mu \geq \delta && (24) \\ & (n = 1,2,3, \ldots) \end{aligned}\right\}$$

Thus, starting with a function v_o, the n^{th} iterate v_n is obtained from v_{n-1} by the procedure:

(a) Insert $v = v_{n-1}$ in equation (18) and solve for s_i; this gives the iterate $s_{i,n}$ (according to eq. (22)).

(b) The value of v_n at the pertinent $s_{i,n}$ is obtained from equation 19 (eq. 23).

(c) The steps (a) and (b) are repeated to give the potential at all points of interest. This gives the full function $v_n(s)$, the next iterate (eq. (24)).

C. Condition for Local Convergence

In what now follows, we shall specify a way of varying w_{oI} and ℓ_{oI}; we shall assume that the energy w_{oI} is kept constant, and only ℓ_{oI} is varied (by varying the direction of the injected beam). To be specific, we shall use I_b, and the constant value of w_{oI} will be omitted in the symbols for I_b and g.

Our nonlinear system of equations then becomes

$$\left.\begin{aligned} & I_b(s_i, \ell_{oI}; v) - C = 0 && (25) \\ & v(s_i) - g(\ell_{oI}) = 0 && (26) \end{aligned}\right\}$$

The iteration is understood as a transformation $v_n = Tv_{n-1}$, where $T\colon\ v \to u = Tv$ is defined by the equations

$$\left.\begin{aligned} &I_b(s_i, \ell_{oI}; v) - C = 0 && (27) \\ &u(s_i) - g(\ell_{oI}) = 0 && (28) \end{aligned}\right\}$$

It is of interest to investigate the condition for this mapping to be a contraction mapping (ref. 21), in which case the iteration will converge to a unique limit, the true potential v^* (if a certain closedness condition is fulfilled). Our discussion will be "local" in the sense of some suitable norm $||\cdot||$ on a Banach space of suitably defined functions v on $[0,\mu]$. T is then a contraction mapping if

$$||\delta u|| \leq \alpha \cdot ||\delta v||; \qquad 0 < \alpha < 1 \tag{29}$$

and the error in the iteration can for instance be estimated by

$$||v_n - v^*|| \leq \frac{\alpha^n}{1 - \alpha} ||v_1 - v_o|| \tag{30}$$

where v^* is the true potential (ref. 21). Local convergence means that there is a ball around v^* with an (unknown) radius ρ, such that convergence prevails within that ball. We shall not define any norm sharply, but keep a qualitative element, nor shall we investigate any other property of T than the inequality (eq. (29)) above; it is believed that the essential physical information can be obtained in this way.

The Frechet differential (ref. 22), supposed to exist, of the functional I_b in equation (27) must vanish, and so must the variation of the left-hand member in equation (28). This gives

$$\left.\begin{aligned} &\frac{\partial I_b}{\partial s_i}(s_i; v) \cdot \delta s_i + \frac{\partial I_b}{\partial \ell_{oI}}(s_i; v) \cdot \delta \ell_{oI} + \delta I_b(s_i; v) = 0 && (31) \\ &u'(s_i)\delta s_i + \delta u(s_i) - g' \cdot \delta \ell_{oI} = 0 && (32) \end{aligned}\right\}$$

where we have omitted ℓ_{oI} in the arguments, and

$$\delta I_b(s_i; v) = I_b(s_i, \ell_{oI}; v + \delta v) - I_b(s_i, \ell_{oI}; v)$$

The differential $\delta \ell_{oI}$ is then solved from (32) and inserted into equation (31), which gives

$$\frac{\partial I_b}{\partial s_i}(s_i; v)\delta s_i + \frac{\partial I_b}{\partial \ell_{oI}}(s_i; v) \cdot \frac{u'(s_i)\delta s_i + \delta u(s_i)}{g'} + \delta I_b(s_i; v) = 0 \tag{33}$$

In a practical measurement, the function $g'(\ell_{oI})$ is determined empirically. In the present analysis, it can be determined from the system of equations (25) and (26), if we insert the true values v^* and s_i^* of v and s_i, respectively. We then take the variation of this system, remembering that $\delta v^* = 0$, since we are only dealing with the true potential, when the empirical output function g' is to be determined. One obtains after elimination of $\delta\ell_{oI}$ in the same way as above,

$$\frac{1}{g'} = - \frac{\frac{\partial I_b}{\partial s_i}(s_i^*; v^*)}{v^{*\prime}(s_i^*) \cdot \frac{\partial I_b}{\partial \ell_{oI}}(s_i^*; v^*)}$$

Insertion into equation (33) yields

$$\frac{\partial I_b}{\partial s_i}(s_i; v)\delta s_i + \delta I_b(s_i; v)$$

$$- [u'(s_i)\delta s_i + \delta u(s_i)] \cdot \frac{\frac{\partial I_b}{\partial s_i}(s_i^*; v^*) \cdot \frac{\partial I_b}{\partial \ell_{oI}}(s_i; v)}{v^{*\prime}(s_i^*) \cdot \frac{\partial I_b}{\partial \ell_{oI}}(s_i^*; v^*)} = 0$$

This equation gives a general relation among δv (through δI_b), δu, and δs_i. In particular, we want to compare δu and δv with the same s_i; thus we put $\delta s_i = 0$. This does not seem necessary, but practical. Solving for $\delta u(s_i)$, we then obtain

$$\delta u = v^{*\prime}(s_i^*) \cdot \frac{\delta I_b(s_i; v)}{\frac{\partial I_b}{\partial s_i}(s_i^*; v^*)} \cdot \frac{\frac{\partial I_b}{\partial \ell_{oI}}(s_i^*; v^*)}{\frac{\partial I_b}{\partial \ell_{oI}}(s_i; v)}$$

Since we assume that $(s_i; v)$ is in an infinitesimal neighborhood of $(s_i^*; v^*)$, the last factor in the right-hand member is equal to unity, plus a small quantity that only contributes to the higher-order variations, which are neglected. Thus, it can be replaced by unity, and we obtain - finally:

$$\delta u = v^{*\prime}(s_i^*) \cdot \frac{\delta I_b(s_i; v)}{\frac{\partial I_b}{\partial s_i}(s_i^*; v^*)} \tag{34}$$

Clearly, this expression for δu has all the qualitative properties discussed above.

The endpoint contributions to the derivative $\partial I_b/\partial s_i$ cancel, due to the continuity conditions at s_i, and one obtains

$$\frac{\partial I_b}{\partial s_i} = \int_{s_i}^{\beta} \frac{\partial}{\partial s_i}\left(\frac{u_{\phi II}}{s\sqrt{w_{oII} - \psi_{II}}}\right) ds \tag{35}$$

Carrying out the differentiation, one obtains

$$\frac{\partial I_b}{\partial s_i} = \int_{s_i}^{\beta} \frac{\frac{2\tau'(s_i)}{s}(w_{oII} - \psi_{II}) - u_{\phi II}\left[v'(s_i) - \frac{2u_{\phi II}\tau'(s_i)}{s}\right]}{2s(w_{oII} - \psi_{II})^{3/2}} ds$$

$$= \int_{s_i}^{\beta} \frac{(w_{oII} - 2v)\cdot 2\tau'(s_i) - u_{\phi II} s v'(s_i)}{s^2(w_{oII} - \psi_{II})^{3/2}} ds \tag{36}$$

For s_i bounded away from s_{tI} and s_{tII}, the variations of I_1 and I_3 can easily be calculated by changing the order of the operations of variation and integration and forming the partial derivative of the integrands with respect to v. One obtains

$$\delta I_1 = \int_{s_i}^{\alpha} \frac{u\phi_I \delta v}{2s(w_{oI} - \psi_I)^{3/2}} ds \approx \int_{s_i}^{\delta} \frac{u_{\phi I}\delta v}{2s(w_{oI} - \psi_I)^{3/2}} ds \tag{37}$$

if the true potential is equal to zero for $s > \delta$. Furthermore,

$$\delta I_3 = \int_{s_i}^{\beta} \frac{u_{\phi II}[2\delta v - \delta v(s_i)]}{2s(w_{oII} - \psi_{II})^{3/2}} ds$$

$$\approx \int_{s_i}^{\delta} \frac{u_{\phi II}\delta v}{s(w_{oII} - \psi_{II})^{3/2}} ds - \delta v(s_i)\cdot \int_{s_i}^{\beta} \frac{u_{\phi II}}{2s(w_{oII} - \psi_{II})^{3/2}} ds \tag{38}$$

When s_i tends to s_{tI}, the point s_{tII} will also tend to s_{tI}, and δI_1 and δI_3 will become infinite, due to the singularity in the integrand. This will be discussed later on.

Forming δI_4 is more complicated, due to the inevitable singularity at the lower limit of integration. We know that ψ_I is monotonic in a neighborhood of s_{tI}, and we tentatively assume that ψ_I is monotonic on $(0,\delta]$. This is fulfilled for the SUMMA numerical problem in section 6, and no effort has been made to relax this assumption. We may then write I_4 as

$$I_4 = \int_{\psi_I(s_{tI})}^{\psi_I(s_1)} \frac{2u_{\phi I}}{s\,\frac{d\psi_I}{ds}} \frac{d\psi_I}{\sqrt{w_{oI} - \psi_I}}$$

$$= -2\left[\sqrt{w_{oI} - \psi_I}\;\frac{2u_{\phi I}}{s\psi_I'}\right]_{s_i} + \int_{s_{tI}}^{s_1} 2\sqrt{w_{oI} - \psi_I}\,\frac{d}{ds}\left(\frac{2u_{\phi I}}{s\psi_I'}\right)ds \qquad (39)$$

by a partial integration. Denoting the first term in the right-hand member by T and the second by J, one obtains

$$I_4 = T + J$$

$$\delta I_4 = \delta T + \delta J$$

One finds

$$\delta T = \left[\frac{2u_{\phi I}\delta v}{s\psi_I'\sqrt{w_{oI} - \psi_I}}\right]_{s_1} + \left[2\sqrt{w_{oI} - \psi_I}\;\frac{2u_{\phi I}\cdot\delta v'}{s(\psi_I')^2}\right]_{s_i} \qquad (40)$$

To form δJ, one can proceed in a fairly straightforward manner, noticing that if

$$J = \int_{s_0}^{s_1} F(s,\psi,\psi',\psi'')ds$$

we have

$$\delta J = \int_{s_0}^{s_1} \delta\psi \left[F_\psi - \frac{d}{ds} F_{\psi'} + \frac{d^2}{ds^2} F_{\psi''} \right] ds$$

$$+ \int_{s_0}^{s_1} \left[\delta s \cdot F + \delta\psi \left(F_{\psi'} - \frac{d}{ds} F_{\psi''} \right) + \delta_{\psi'} \cdot F_{\psi''} \right] \qquad (41)$$

This formula can for instance be derived, using Taylor's theorem and repeated partial integrations (compare ref. 23).

To control the singularity at s_{tI}, we exclude the limit with an ε-interval, obtaining δJ_ε. Then, δJ is obtained as $\delta J = \lim_{\varepsilon \downarrow 0} \delta J_\varepsilon$.

The Euler expression in the integrand of equation (41) becomes

$$\frac{u_{\phi I} \delta v}{s(w_{oI} - \psi_I)^{3/2}}$$

which is the same as that obtained by straightforward partial differentiation of the integrand in I_4. The endpoint variation becomes

$$\left(\frac{2u_{\phi I} \delta v}{s\psi_I' \sqrt{w_{oI} - \psi_I}} \right)_{s_{tI}+\varepsilon} + (\cdot)$$

where $(\cdot)$ denotes a number of terms such that the expression becomes zero for $\varepsilon = 0$.

Consequently,

$$\delta J = \lim_{\varepsilon \downarrow 0} \left[\int_{s_{tI}+\varepsilon}^{s_i} \frac{u_{\phi I} \delta v}{s(w_{oI} - \psi_I)^{3/2}} ds + \left(\frac{2u_{\phi I} \delta v}{s\psi_I' \sqrt{w_{oI} - \psi_I}} \right)_{s_{tI}+\varepsilon} \right]$$

A change to ψ_I as integration variable, followed by a partial integration shows that all contributions that become singular for $\varepsilon \to 0$ cancel and one obtains, adding δT

$$\delta I_4 = \left[\frac{4u_{\phi I}\delta v}{s\psi_I'\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \left[2\sqrt{w_{oI} - \psi_I}\,\frac{2u_{\phi I}\cdot\delta v'}{s(\psi_I')^2}\right]_{s_i}$$

$$- \int_{s_{tI}}^{s_i} \frac{1}{\sqrt{w_{oI} - \psi_I}}\,\frac{d}{ds}\left(\frac{2u_{\phi I}\delta v}{s\psi_I'}\right)ds \qquad (42)$$

The derivation is straightforward but not elegant. Maybe a simpler derivation can be obtained by using a representation of the integrals in I_b as derivatives with respect to the momenta,* for example,

$$I_4 = \frac{\partial}{\partial \ell_{oI}} \int_{s_{tI}}^{s_i} - 2\sqrt{w_{oI} - \psi_I}\,ds$$

By insertion of the above expressions for $\partial I_b/\partial s_i$ (eq. (36)); δI_1 (eq. (37)); δI_3 (eq. (38)); and δI_4 (eq. (42)) into the basic expression for δu (eq. (34)), and introducing a norm (perhaps $||\delta v|| = \max|\delta v|$, or $||\delta v|| = \max|\delta v| + a\cdot\max|\delta v'|$), it is possible to determine under what conditions the iteration will converge to the true potential and particularly also when already the first iterate - the JH solution - is a good approximation. This will be carried out below for the SUMMA experiment.

D. Application to SUMMA Experiment

Using values corresponding to the SUMMA experiment, we shall roughly estimate the magnitude of the various contributions to δI_b and $\partial I_b/\partial s_i$ and obtain an approximate criterion for convergence. First we shall assume s_i to be bounded away from s_{tI} (and s_{tII}), then we shall let s_i tend to s_{tI}. The same kind of criterion is obtained in both cases.

Taking $(w_{oI} - \psi_I) \sim w_{oI}/2$ we get $\delta I_1 \sim \dfrac{\delta v}{w_{oI}} \displaystyle\int_{s_i}^{\delta} \frac{u_{\phi I}\,ds}{s\sqrt{w_{oI} - \psi_I}}$

If the change in polar angle between s_i and δ is typically of the order of unity, we get

$$\delta I_1 = \kappa_1 \frac{\delta v}{w_{oI}}$$

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*The possibility of such a representation was pointed out to me by W. F. Ford.

where κ_1 is of the order of unity. Similarly, one obtains

$$\delta I_3 = \kappa_2 \frac{\delta v}{w_{oI}}$$

In the first term in δI_4 (eq. (42)), the ratio

$$\frac{u_{\phi I}}{\sqrt{w_{oI} - \psi_I}} = \frac{u_{\phi I}}{u_{rI}}$$

is estimated by unity, and $u_{\phi I}$ by $\sqrt{w_{oI}/2}$. The derivative ψ_I' can be written

$$\psi_I' = -2u_{\phi I}\left(\frac{\tau'}{s} + \frac{u_{\phi I}}{s}\right) + v'$$

For the SUMMA, the dominating term is

$$-\frac{2u_{\phi I}^2}{s} \sim \frac{w_{oI}}{s}$$

whence the first term gives a contribution $\kappa_3\ \delta v/w_{oI}$. By the same token, the second term gives $\kappa_4\ \delta v/w_{oI}$, provided that $s\ \delta v' \sim \delta v$. The integral becomes, roughly,

$$\sqrt{\frac{2}{w_{oI}}}\ \frac{2u_{\phi I}\ \delta v}{s\psi_I'} \sim \kappa_5 \frac{\delta v}{w_{oI}}$$

Summing up, noticing differences in sign, we obtain

$$\delta I_b \sim \kappa_6 \frac{\delta v}{w_{oI}}$$

where κ_6 is essentially of the order of unity.

The first term in $\partial I_b/\partial s_i$ becomes

$$\int_{s_i}^{\beta} \frac{(w_{oII} - 2v)2\tau'(s_i)}{s^2(w_{oII} - \psi_{II})^{3/2}}\, ds \sim \frac{w_{oI}2\tau'(s_i)}{\left(\frac{w_{oI}}{2}\right)^{3/2}} \int_{s_i}^{\beta} \frac{ds}{s^2} \sim \frac{\kappa_7 4\tau'(s_i)}{\sqrt{w_{oI}}\ s_i}$$

Near the axis, the magnetic field is approximately homogeneous, so

$\tau'(s_i) = 2\tau(s_i)/s_i$. Furthermore, $\tau(s)/s^2$ is proportional to the magnetic field. Thus, if we disregard the variation in field strength between $s = 0$ and $s = s_1$, the point of maximum fictitious potential (sec. 4), we have

$$\frac{\tau(s_i)}{s_i^2} \sim \frac{\tau(s_1)}{s_1^2}$$

But

$$\frac{2\tau(s_1)}{s_1} = \sqrt{w_{oo}}$$

where w_{oo} is the minimum energy necessary for the ions to get into and out of the magnetic field. Thus,

$$\frac{\kappa_7 4\tau'(s_i)}{\sqrt{w_{oI}}\, s_i} = \frac{\kappa_7 4 \cdot 2\tau(s_i)}{\sqrt{w_{oI}}\, s_i^2} \sim \frac{4\kappa_7}{\sqrt{w_{oI}}} \frac{2\tau(s_1)}{s_1} \frac{1}{s_1}$$

$$= \frac{4\kappa_7\sqrt{w_{oo}}}{s_1\sqrt{w_{oI}}}$$

The second term in $\partial I_b/\partial s_1$ becomes

$$\int_{s_i}^{\beta} \frac{u_{\phi II} v'(s_i)}{s(w_{oII} - \psi_{II})^{3/2}}\, ds \sim \frac{2v'(s_i)}{w_{oI}} \int_{s_i}^{\beta} \frac{u_{\phi II}}{s\sqrt{w_{oII} - \psi_{II}}}\, ds \sim \frac{2\kappa_8 v'(s_i)}{w_{oI}}$$

The estimates above are inserted in equation (34), which gives

$$\delta u \approx v^{*\prime}(s_i^*) \cdot \frac{\kappa_6 \dfrac{\delta v}{w_{oI}}}{\dfrac{4\kappa_7\sqrt{w_{oo}}}{s_1\sqrt{w_{oI}}} + \dfrac{2\kappa_8 v'(s_i)}{w_{oI}}}$$

$$\approx \frac{\kappa_9}{1 + \kappa_{10} \dfrac{2\sqrt{w_{oo}} \cdot w_{oI}}{s_1 \cdot v'(s_i)}}\, \delta v \tag{43}$$

where we have used the fact that $v^{*\prime}(s_i^*) \approx v'(s_i)$.

The condition for convergence is, essentially, that the absolute value of the factor in front of δv be smaller than unity. Since κ_9 may well be greater than unity (one gets that impression when dealing with δI_b), and/or $\kappa_{10}/v'(s_i)$ may be negative, it appears that the factor after κ_{10} should be greater than a quantity of the order of unity. In other words, its inverse value should be smaller than a quantity of the order of unity. At least, this is necessary to insure rapid convergence. Thus, with

$$\varepsilon = \frac{1}{\omega} = \frac{s_1 v'(s_i)}{2\sqrt{w_{oo} \cdot w_{oI}}} \tag{44}$$

we have

$$\delta u \approx \frac{\kappa_9}{1 + \omega\kappa_{10}} \delta v = \frac{\kappa_9 \varepsilon}{\varepsilon + \kappa_{10}} \cdot \delta v$$

Clearly, we have convergence for

$$\varepsilon < \kappa \qquad \text{where} \quad \kappa \sim 1$$

Furthermore, since $\lim_{w_{oI}\to\infty} \omega = \infty$, δu becomes asymptotically $0 \cdot \delta v$, and the error estimate inequality (30) then yields $\|v_1 - v^*\| \sim 0$ asymptotically when $w_{oI} \to \infty$. Thus, the Jobes-Hickok approximation v_1 (with $v_o \equiv 0$) gives asymptotically the true potential v^* in the limit of very high beam energies (provided that the distance between v^* and 0 is not greater than the unknown radius of the ball of convergence).

It is interesting to study the smallness parameter ε for the electric field. Clearly, it is different from ε_2 mentioned in section 4, the ratio between plasma potential and beam energy, in two respects. First of all, the electric field should not be multiplied by its own scale length s_i, but by that of the magnetic field, s_1. This gives much more stringent requirements for the SUMMA, since the ratio between the two scale lengths is about 50. Furthermore, this energy should not be compared with the beam energy, but with the geometric mean of that energy and the minimum energy necessary for penetration through the magnetic field. It also appears that the $w_{oI}^{-1/2}$-dependence of ε can be improved to a w_{oI}^{-1}-dependence, if w_{oo} and w_{oI} are kept approximately alike, when w_{oI} is varied, i.e., if the mass number is varied with w_{oI}, as was discussed in section 4 in connection with $\varepsilon_1 = E/vB$, the ratio between electric and magnetic forces. Indeed, by normalizing the variables in ε_1 one finds

$$\varepsilon_1 = \frac{E}{vB} = \frac{s_1 v'}{2\sqrt{w_{oo} w_{oI}}} = \varepsilon$$

One can also express the condition for convergence in terms of the formal E/B velocity, since

$$\varepsilon = \frac{v(E/B)}{v(\text{beam})}$$

Summing up, we have found the following results:

If the 0^{th} iterate v_o is (in an unknown sense) sufficiently (not necessarily infinitesimally) close to the true potential v^*, and

(a) The electric force in the plasma is very much smaller than the magnetic force (or, alternatively, if the beam velocity is very much higher than the E/B-velocity) then the Jobes-Hickok solution v_1 is a good approximation to v^*, asymptotically correct in the limit of infinite beam energy.

(b) The electric force is not very much smaller than the magnetic, only smaller (beam velocity smaller than, but comparable with E/B-velocity), the Jobes-Hickok solution v_1 is not a good approximation to v^*, but an iteration, meaning that the orbits and points of ionization are successively corrected for the electric field, converges to v^*.

More precise figures than those in the estimate above can easily be obtained when $s_i \to s_{tI}$, since both δI_b and $\partial I_b/\partial s_i$ become singular, and we only need to determine the respective coefficients in front of the singular parts. This limit in general does not correspond to a fixed detector position, but it is believed to be of interest to substantiate the discussion above and emphasize the role played by the smallness parameter ε.

By a partial integration in δI_1, we find

$$\delta I_1 = -\left[\frac{u_{\phi I}\ \delta v}{s\psi_I'\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \ldots$$

Similarly, by using the continuity of u_ϕ and u_r at s_i, one finds

$$\delta I_3 = -\left[\frac{u_{\phi I}\ \delta v}{s\psi_{II}'\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \ldots$$

Equation (24) immediately yields

$$\delta I_4 = \left[\frac{4u_{\phi I}\ \delta v}{s\psi_I'\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \ldots$$

where the dots denote terms that remain finite when $s_i \to s_{tI}$. Addition of the contributions above gives

$$\delta I_b = \left[\frac{u_{\phi I}\ \delta v}{s\psi_I'\sqrt{w_{oI} - \psi_I}}\left(4 - 1 - \frac{\psi_I'}{\psi_{II}'}\right)\right]_{s_i} + \ldots$$

Since the dominating terms in ψ_I' and ψ_{II}' are $-2u_{\phi I}^2/s$ and $-2u_{\phi II}^2/s$, respectively, their ratio will be approximately unity. Thus,

$$\delta I_b = \left[\frac{2u_{\phi I}\ \delta v}{s\psi_I'} \cdot \frac{1}{\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \ldots$$

The pertinent term in $\partial I_b/\partial s_i$ is

$$-\int_{s_i}^{\beta} \frac{u_{\phi II}\left[v'(s_i) - \dfrac{2u_{\phi II}\tau'(s_i)}{s}\right]}{2s(w_{oII} - \psi_{II})^{3/2}}\, ds$$

Using ψ_{II} as integration variable, partial integration, continuity of velocity, and $\psi_I' \approx \psi_{II}'$, one obtains

$$\left[\frac{u_{\phi I}\left(v' - \dfrac{2u_{\phi I}\tau'}{s}\right)}{s\psi_I'}\ \frac{1}{\sqrt{w_{oI} - \psi_I}}\right]_{s_i} + \ldots$$

Insertion in δu and letting $s_i \to s_{tI}$ gives

$$\delta u = \frac{2}{1 - \dfrac{2\sqrt{w_{oI}}\ \tau'}{sv'}}\ \delta v$$

$$\delta u = \frac{2}{1 - \frac{1}{\varepsilon}} \delta v$$

Analysis of the inequality

$$\left| \frac{2}{1 - \frac{1}{\varepsilon}} \right| < 1$$

gives $-1 < \varepsilon < 1/3$, or essentially $|\varepsilon| < 1$. Either of these is a condition on the smallness parameter. And if $|\varepsilon| << 1$, the JH-solution is a good approximation.

E. Comments

There are a few precautions that one should observe, especially if the iteration is applied when there is a finite distance between v_o and v^*.

Multiple secondary orbits to the detector for one single primary orbit may occur. This was found experimentally and explained theoretically by Kambic (ref. 7). However, this does not seem to be a problem; one only gets a little more information from one primary orbit. In any case it will show up in the solution of the nonlinear boundary-value problem with a free boundary (step (a) above) for determining s_i. The only difficulty would arise if different iterates for one and the same potential v^* were associated with different number of solutions s_i for a given primary beam (w_{oI}, ℓ_{oI}).

The proper way of varying ℓ_{oI} and w_{oI} also needs attention. It should be noted, that if a set $\{(w_{oI}, \ell_{oI})\}$ is given, equations (18) and (19) will in general provide consistent information only if v is the true potential v^* (and the measurements g are correctly performed). But if v is an iterate v_{n-1}, two or more pairs (ℓ_{oI}, w_{oI}) corresponding to the same $s_i = s_{i,n}$ may give different values of g (since in reality they correspond to different $s_i = s_i^*$ under v^*). Thus, $v_n(s_{i,n})$ would not be uniquely defined. It seems natural in such a case to choose the arithmetic mean, but it may be better to make $v_n(s)$ a one-valued function by using an empirical procedure. In any case, the set $\{(w_{oI}, \ell_{oI})\}$ should be large enough to provide a coverage of the plasma, not only for v^*, but also for the iterates.

It should over again be emphasized, that our convergence analysis was local around v^*; the finite differences entering in Banach's fixed-point theorem and similar theorems (ref. 21) were replaced by infinitesimal variations. This does not mean that the method only works in these cases; there will be a nonzero, finite (or infinitely large) radius of convergence and uniqueness, although its magnitude is unknown.

To make sure that a solution obtained in a practical case does represent the true solution, it is suggested that the Lipschitz-constant α for T (eq. (29)) is estimated, using a local analysis. With this value of α or a somewhat larger value, and with any v_o chosen (e.g., the JH-choice $v_o \equiv 0$), the quantity $||Tv_o - v_o||$ is calculated. From this, the requirement on the radius ρ is obtained (from eq. (3.37) in ref. 21), and it is then checked that T is contractive all over the ball $\{v| \; ||v - v_o|| \leq \rho\}$, with a constant α not larger than the one obtained from the local analysis. If a larger α would be needed, it may be tried; one would then require a larger ρ within which T is contractive.

These questions, including the definition of a proper norm, seem worthwhile investigating, but numerical and/or experimental experience should be gained first.

Our discussion has been purely deterministic, and the variables continuous. Needless to say, stochastic elements and discretization will introduce new problems and possible modifications.

6. PRELIMINARY NUMERICAL EXPERIMENT

A first, tentative numerical experiment was performed with application to the SUMMA experiment. To illustrate the influence of the electric field, to numerically produce output data g, to choose a suitable value of the constant C giving the detector azimuthal location, and to compare the Jobes-Hickok solution v_1 for this value of C with the true solution v^*, the integrals $I_a(s_1;\ell_{01})$ and $I_b(s_1;\ell_{01})$ were calculated with $\alpha = \beta = 4$ and $w_{0I} = 9$, which is just a little more than $w_{00} \equiv 4 < x_{max} = 7.24$. Two different potentials were used. One was identically equal to zero. The other potential - the true potential (sec. 4C) - and the functional form

$$v = \frac{a \sin bs}{s} + c; \qquad 0 \leq s \leq \delta = 0.0243$$

With $a = -0.0015771918$; $b = 184.91399$; $c = -0.63355173$, v and v' are continuous, and the well depth is -0.355 (corresponding to -14 kV). See the solid curve in figure 10. The value of w_{0I} chosen is motivated by the fact that the quantity $\varepsilon_2 = eV_{max}/w_o$ discussed in section 4C is much smaller than unity, while $\varepsilon_1 = E/vB$ is only marginally smaller than unity. Indeed, v' (mean) $= 0.355/0.0243 \approx 15$ and v' (max) ≈ 24; $s_1 = 0.56$ (fig. 11); and $\sqrt{w_{00}w_{0I}} = 8.1$, which gives $\varepsilon_1 = v's_1/2\sqrt{w_{00}w_{0I}}$ equal to 0.52 (mean) or 0.83 (max). Based on the analysis in section 5 we expect that the electric field should have a noticeable influence on the I-curves, that the JH-solution v_1 should be different from the true solution v^* but perhaps possible to improve by iteration.

Twenty-one positive and twenty-one negative, equidistant values of ℓ_{0I} were chosen (distance 0.0032), with the corresponding primary beams covering the plasma. The value $\ell_{0I} = 0$ was excluded due to the singular behavior of the corresponding potential ψ. The integrals I_a and I_b were calculated for 27 equidistant values of s_i (distance 0.009). Only values of s_i greater than s_{tI} (or s_{tII}) were considered.

The singularities at s_{tII} in I_2 and s_{tI} in I_4 were excluded by intervals of length ε, and the endpoint contributions were taken to be

$$\Delta I_2 = \left(\frac{4u_{\phi II}}{s\sqrt{-\psi'_{II}}}\right)_{s_{tII}} \cdot \sqrt{\varepsilon}$$

and

$$\Delta I_4 = \left(\frac{4u_{\phi I}}{s\sqrt{-\psi'_{I}}}\right)_{s_{tI}} \cdot \sqrt{\varepsilon}$$

respectively.

If ε is too small the integrands in I_2 and I_4 become too large, if ε is too large the Taylor expansion underlying the expressions for ΔI_2 and ΔI_4 becomes inapplicable. Several values of ε were tried, and ε could typically be varied within one order of magnitude without significant change of the curves. The value $\varepsilon = 10^{-5}$ was used for $V \not\equiv 0$ and $\varepsilon = 3\times10^{-5}$ for $V \equiv 0$, but this difference is not important.

The result is represented in figure 12(a) to (h). $V \not\equiv 0$ corresponds to (a) to (d), and $V \equiv 0$ to (e) to (h). Positive angular momenta ℓ_{0I} are found in (a), (b), (e), (f), and negative in (c), (d), (g), (h). I_a is shown in (a), (c), (e), (g) and I_b in (b), (d), (f), (h). From the curves the following conclusions can be drawn:

(i) The electric field has a profound influence on all sets of curves; the curves are changed completely in spite of the low plasma potential compared to the beam energy.

(ii) The values of I_a and I_b jump by about 2π between $\ell_0 = -0.0032$ and $+0.0032$. This is due to the coordinate system; if the $\ell_{0I} < 0$ curves are shifted vertically by 2π, the two sets fit well together without intersecting.

(iii) Several I_a-curves with $V \neq 0$ are essentially horizontal; i.e., I_a is then essentially independent of s_i. Such curves are not suitable for determination of the potential, since no scanning and no spatial resolution is obtained.

(iv) Apart from the horizontal I_a-curves in (a) and (c), all curves corresponding to a given ℓ_{0I} have only one intersection with $I_a = C$ or $I_b = C$. Furthermore, inspection shows that $I_a \neq I_b$ for the same ℓ_{0I}. This means that there is never more than one secondary beam reaching an arbitrarily located detector, for a given primary beam.

(v) In curves (b), (c), (f), (g) there is what appears to be an envelope or focal line $\mathcal{E}$ (or possibly a locus of singular points); the phenomenon looks like a wave motion with wave fronts reflected against the envelope, giving effectively two sets of curves. Near $\mathcal{E}$ the curves depend only slightly on ℓ_0; $\partial I/\partial \ell_{0I} \approx 0$. We thus have a kind of focus; particles sent in different directions appear (for given C in an interval) to be ionized at one and the same radius.

(vi) When there is an envelope, one and the same value of s_i is obtained for more than one ℓ_{0I} near the envelope.

(vii) Apart from (vi), a given s_i is associated with exactly one value of ℓ_{0I}.

(viii) A "hook" at the beginning of the curves is often found. This is believed to correspond to the infinite derivative at s_{tI} or s_{tII} (sec. 5). Since the numerical problem is discretized, the hook does not always show up.

(ix) It is unclear whether or not the two strange curves in (e) are due to numerical errors.

In the present numerical experiment we can choose the detector location (the value of C) in the most favorable way. This is not the case in practice. Unlike the primary beam method, it is not possible to perform the experiment with only the magnetic field, since no secondaries are generated without the plasma. (Maybe it would be possible to perform a supplementary, primary beam experiment to see if the electric field has an important influence on the orbits.)

To determine the JH-solution v_1 we choose

$$C = -5.8500 \doteq 0.43319 \text{ (modulo } 2\pi)$$

Curves (d) (and some of (b)) provide the output function g, and the JH-solution is obtained from (h) (and (e)). Table I summarizes the evaluations. For the given values of ℓ_{0I}, the true values s_i^* are obtained from the diagrams (with $V \neq 0$) as the intersections between the respective I-curves and the straight line $I = C$. The output function $g(\ell_{0I})$, which is also the value of the true potential v^* at s_i^* is then obtained either from the analytic representation of v or from figure 10. The first iterates $s_{i,1}$ are then obtained as the intersections between $I = C$ and the I-curves corresponding to $V \equiv 0$, and the values of v_1 at $s_{i,1}$ are also given by $g(\ell_{0I})$. The points obtained for v_1 are marked in figure 10. Clearly, the agreement is poor, but there is at least some qualitative similarity.

The calculations illustrate the necessity to consider the influence of the electric field on the orbits. It would be interesting to interpolate a potential v_1 and proceed with the iterations, to try a higher value of w_{0I}, with or without a simultaneous change of mass, and to investigate the role played by the focal line and the equation $\partial I/\partial \ell_{0I} = 0$.

7. CONCLUDING REMARKS

The primary beam method, based upon a study of the total angular deviation of charged particles and most readily used with cylindrical symmetry, was found to work satisfactorily in the numerical experiment with conditions taken from the Modified Penning Discharge laboratory experiment. The necessary beam energy in this case was not set by the magnetic field, but by the repulsive electrostatic potential plus the necessity for a sufficiently strong centrifugal force for the unfolding procedure to be unique and effective.

Among important things not treated here we wish to mention the beam optics, especially the possibility of defocusing when the change in polar angle is large - which occurs when the gradient of the fictitious potential is numerically small. However, the calculations with two almost equal ℓ_0-values strongly indicate that away from such points defocusing need not be a problem. Needless to say, influence of random perturbations is of great interest for the practical use of the method, and consideration of more general geometries is also worthwhile. The latter problem can probably be handled by ray-tracing techniques. The nonlocal character of the process of deviation of the particles was mentioned, leading to difficulties to obtain local information about the electric field by varying the constants of motion by small amounts. However, this is believed to be a tractable problem.

Advantages of the primary beam method are the low energy necessary (the secondary beam method requires typically four times as high energy), the high intensity of the detected particles, and the simplicity of detecting them - one only needs to determine where the beam comes out.

On the other hand, the secondary beam method works readily in different geometries, and may furnish information on other interesting plasma quantities as well, like density and temperature. Moreover, with the compact building style of big, modern plasma experiments, the limited access may prohibit a determination of the deviations corresponding to a great number of energies. On the other hand, we saw from section 6 and figures 12 (a) to (h) that care may be necessary in locating the detector if an unknown electric field is to be determined, whose influence on the orbits cannot be neglected, it may be necessary to use a number of detectors at different fixed locations.

Throughout the paper we have tried to combine physical interpretations with the mathematical development, notably in the use of normalized quantities, a technique that solves many problems in one by revealing the pertinent quantities determining the nature of the solution. For the secondary beam method, requirements on beam momentum are thus easily obtained by using the constants of motion. It is a general experience that this technique will fail to give conditions that are both necessary and sufficient; the success in the present case is probably due to the fact that the problem was two dimensional rather than three dimensional.

The nature of the Jobes-Hickok approximation (meaning that the electric field is neglected when the orbits are determined) was illuminated, and it was found to be valid when the electric force in the plasma is much smaller than the magnetic (E/B-velocity much smaller than beam velocity). In terms of energies, the electric field should be multiplied by the scale length of the magnetic field rather than the electric, and the corresponding energy should not be compared with the beam energy but with the geometric mean of the same energy and the minimum possible beam energy. The latter fact opens the possibility that if one wants to minimize the influence of the electric field, one should use lower mass numbers when the energy is increased, and in any design this should be considered in relation to other, more technical requirements.

If the pertinent smallness parameter is not negligible small, but smaller than a quantity of order unity, a true solution may still be obtained by iteration, by means of the procedure described in the paper. The conclusions were based upon a local analysis with variations and contraction mappings. Even if the arguments presented should be strong and convincing enough, there is room for increased mathematical rigor, including a precise definition of the norm, consideration of nonlocal problems, and proper handling of the generally multivalued potential functions appearing in the iterations.

A numerical experiment was performed with application to the SUMMA experiment; the conditions were marginal in that the smallness parameter was of the order of unity. The Jobes-Hickok approximation was shown to be unsatisfactory in this case, and the potential became multivalued. However, with a suitable empirical definition of a one-valued potential, the next iterate may well be closer to the assumed function.

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TABLE I. - SECONDARY BEAM METHOD; NUMERICAL EXPERIMENT

[True points of ionization s_i^*, output data g, and first iterate (Jobes-Hickok solution) $s_{i,1}$, for various values of primary beam canonical angular momentum ℓ_{oI}. Both the true potential $v^*(s_i^*)$ and the first iterate $v_1(s_{i,1})$ are given by $g(\ell_{oI})$.]

ℓ_{oI} (multiples of 0.0032)	s_i^*	$g(\ell_{oI})$	$s_{i,1}$
-1	0.0111	-0.188	0.0024
-2	.0128	-0.148	.0029
-3	.0143	-0.115	.0076
-4	.0160	-0.082	.0076
-5	.0175	-0.053	.0085
-6	.0192	-0.029	.0098
-7	.0208	-0.014	.0109
-8	.0231	-0.001	.0127
-9	.0238	-0.000	.0141
+1	.0069	-0.281	.0053
+2	.0051	-0.314	.0059

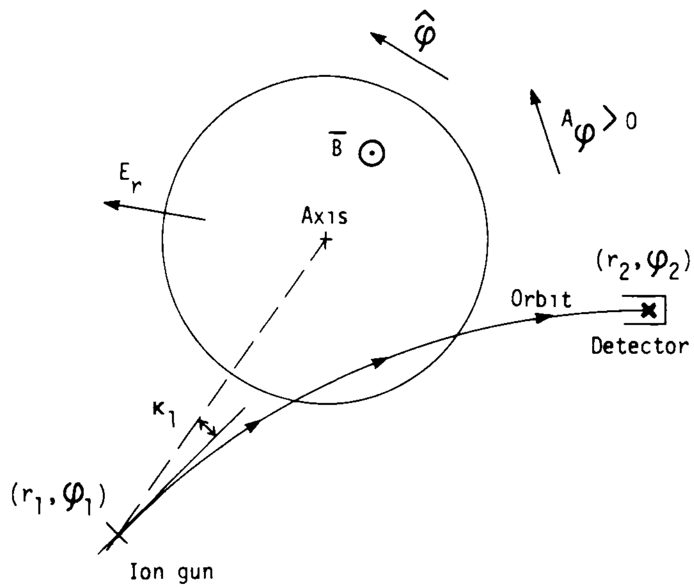


Fig 1 - Cylindrically symmetric magnetic field and electrostatic potential with an ion orbit The ion gun is located at the point (r_1,φ_1) and the detector at (r_2,φ_2) The angle (with sign) between the initial velocity and radius vector is called K_1.

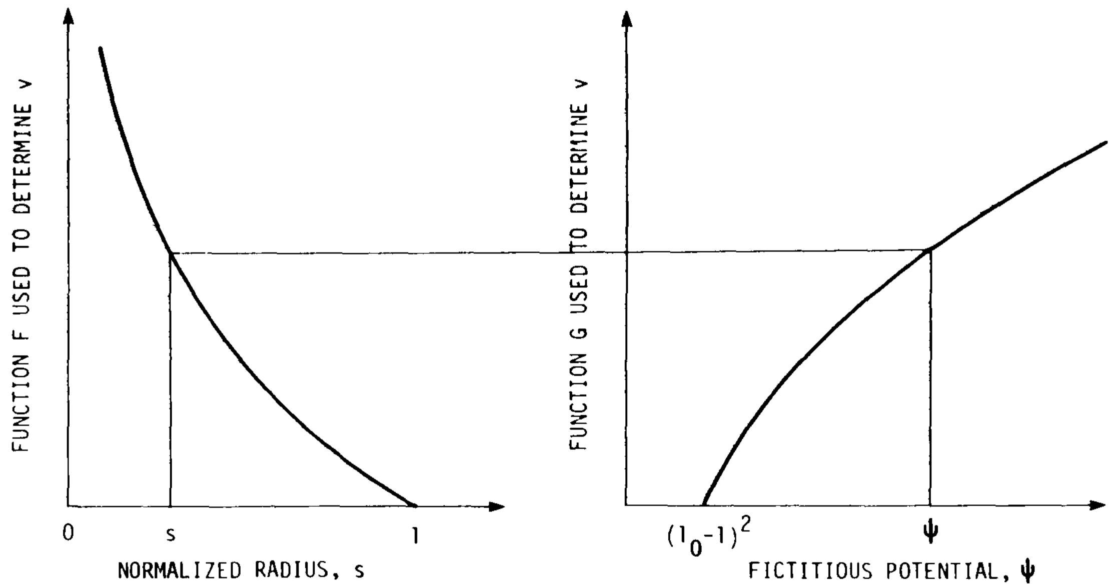


Fig 2 - General appearance of functions F(s) and G(ψ) used in the inversion. Corresponding values of s and ψ can be obtained from F(s) = G(ψ) by the graphical construction shown.

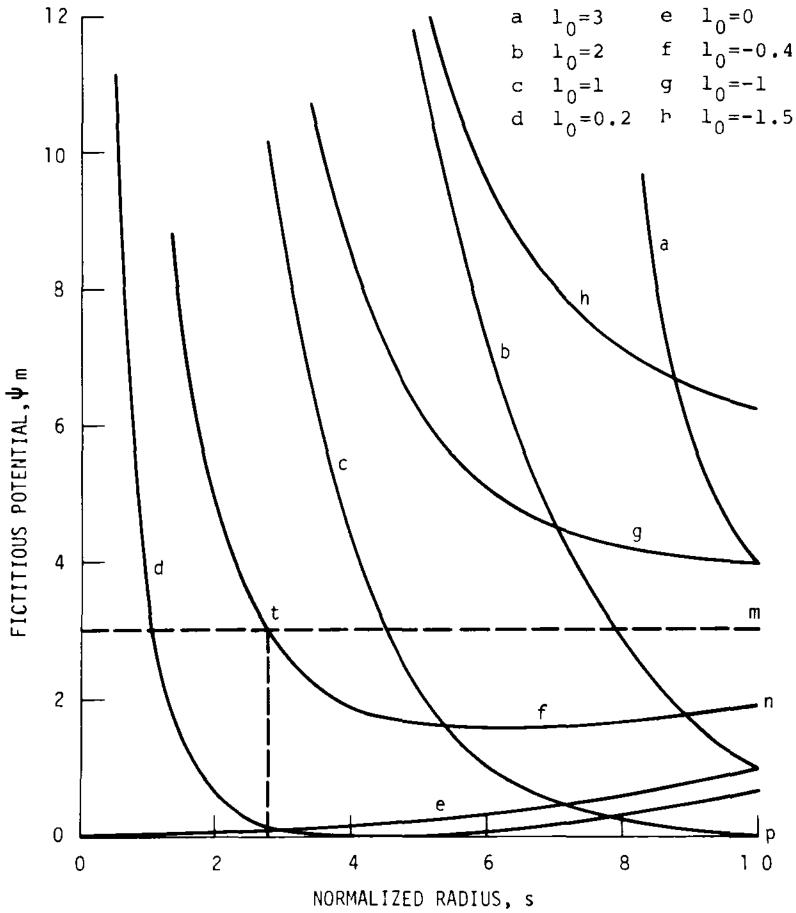


Fig 3 - Fictitious radial potentials Ψm of a homogeneous magnetic field for eight different values of the normalized canonical angular momentum ℓ_0, the coordinate s is the normalized radius

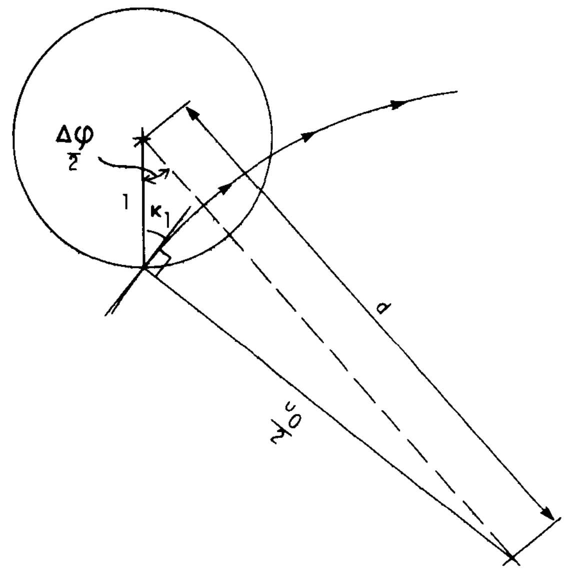


Fig 4 - Geometric determination of $f(w_0)$, half the change in polar angle, for an ion in a homogeneous magnetic field

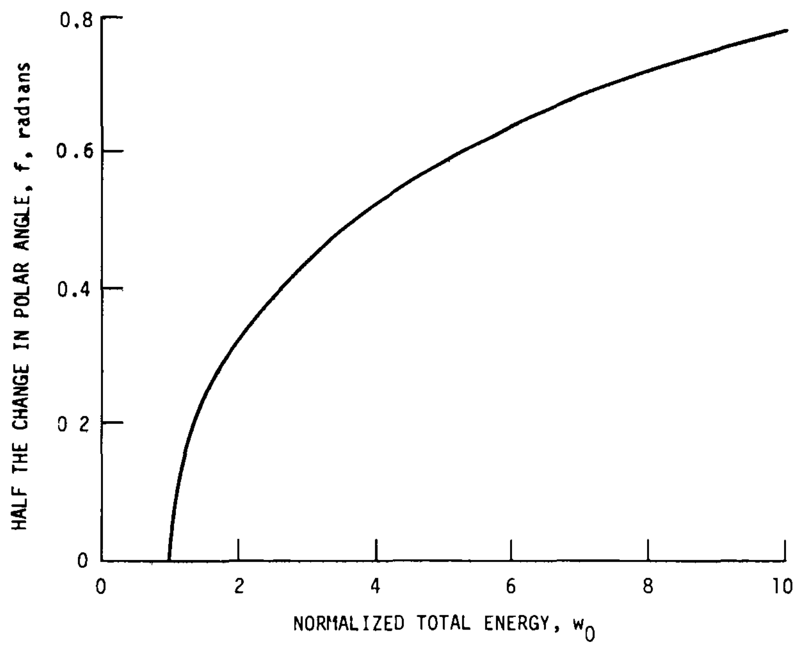


Fig 5 - Half the change in polar angle (in radians) as a function of normalized energy w_0 for homogeneous B and $\ell_0 = 2$

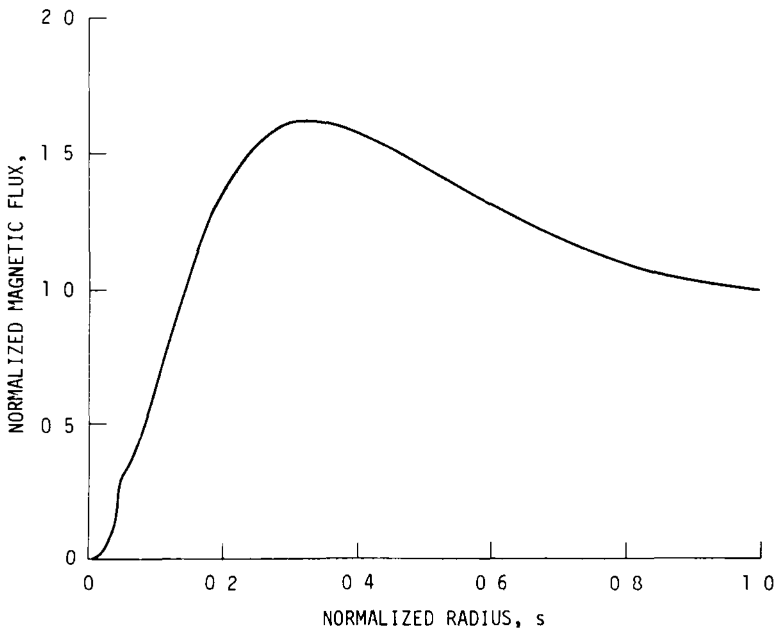


Fig 6 - Flux function (s) = Φ/Φ' for the NASA Lewis Modified Penning Discharge, normalized to gun position

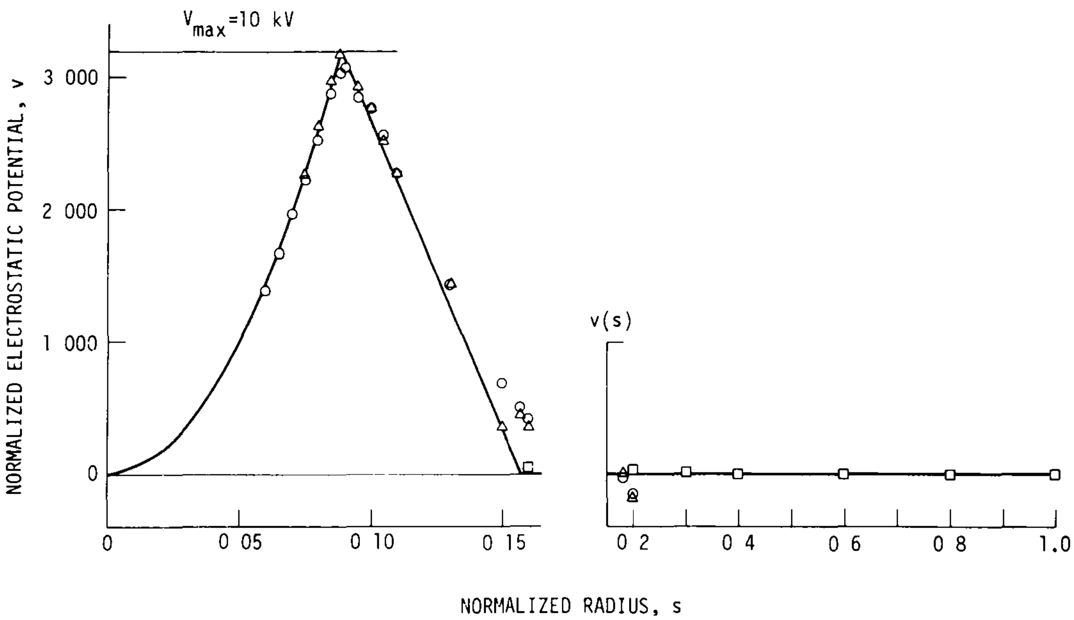


Fig 7 - Normalized potential v(s) in the NASA Lewis Modified Penning Discharge The solid curve represents the assumed potential and the points reconstructed values corresponding to three different values of the maximum ion energy w_{max} used in the Abel inversion

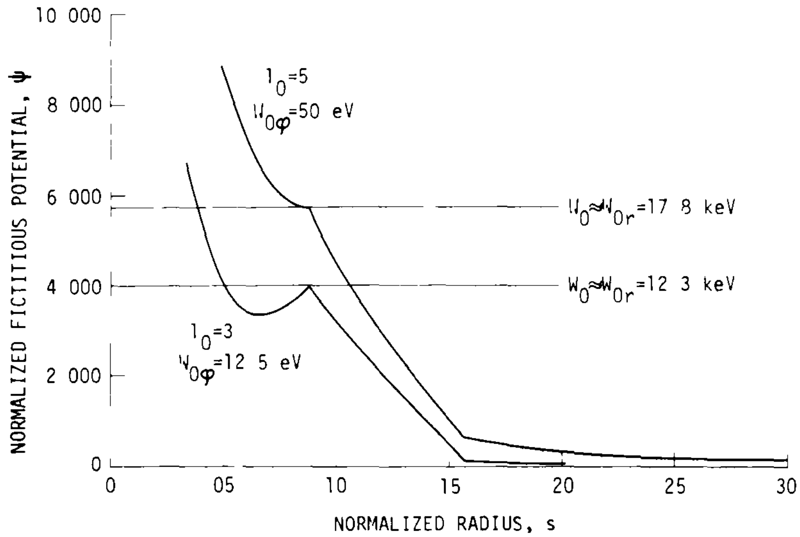


Fig 8 - Fictitious potential for singly charged Tℓ-ions in the NASA Lewis Modified Penning Discharge, v≠0, ℓ_0 = 3 and 5, only the latter value gives an everywhere monotonic potential

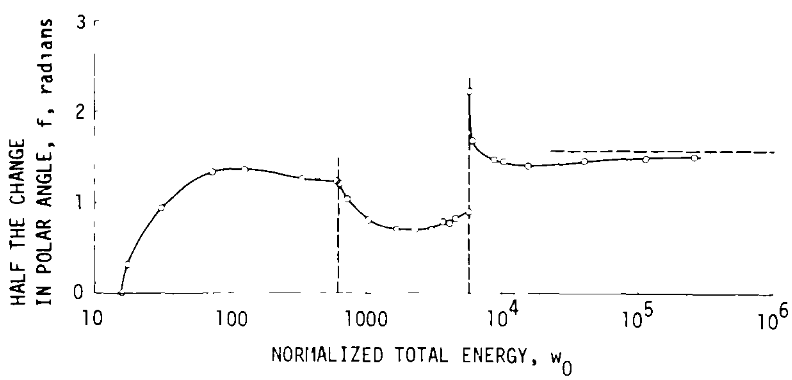


Fig 9 - Semilogarithmic plot of half the change in polar angle (in radians) as a function of normalized energy w_0 for the Modified Penning Discharge

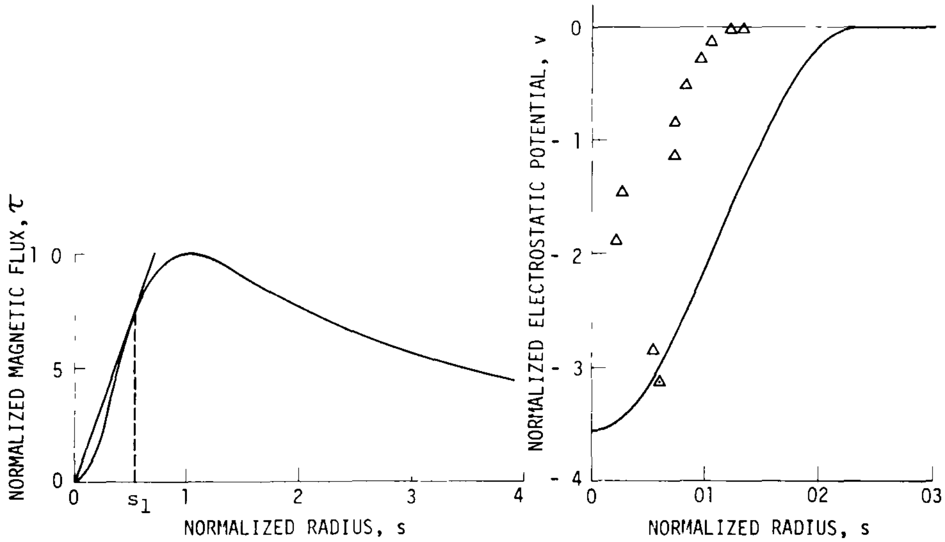


Fig 10 - Normalized flux function $\tau(s)$ and assumed potential $v(s)$ in the SUMMA The straight line (from δ) is tangent to the τ-curve at s_1, the square of the slope is the minimum normalized primary ion energy necessary to reach the whole plasma The points in the v-s diagram represent values reconstructed with the secondary beam method under the assumption of negligible electric field

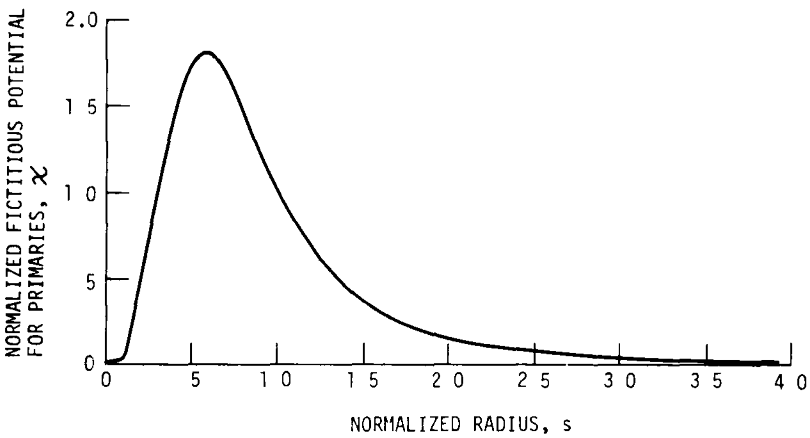
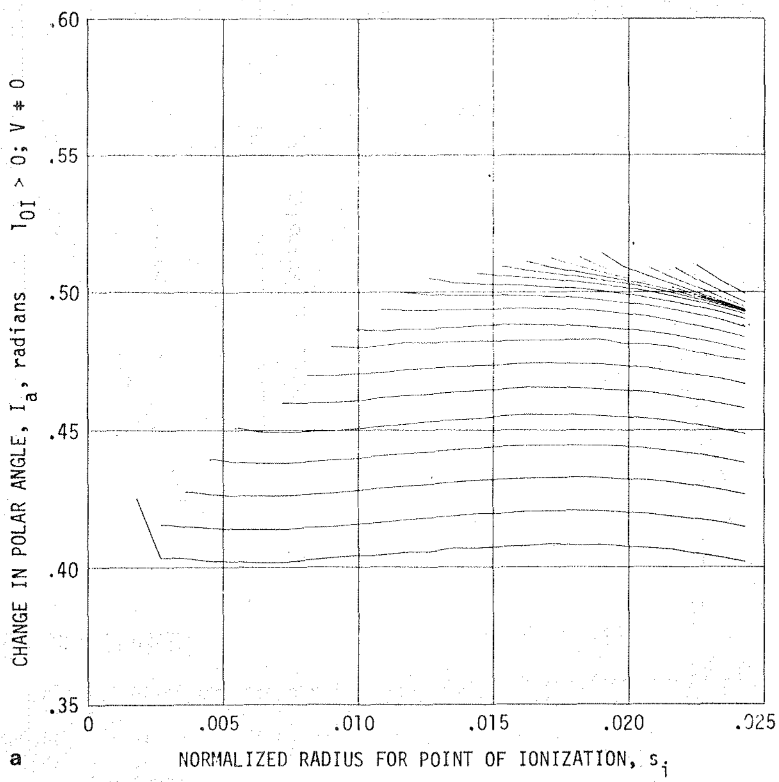


Fig 11 - Function $\chi(s) = (\frac{\tau}{s})^2$ for the SUMMA, with $v \gtrsim 0$ it represents the fictitious potential for singly charged ions with $\ell_0 = 0$, necessary to pass through the axis



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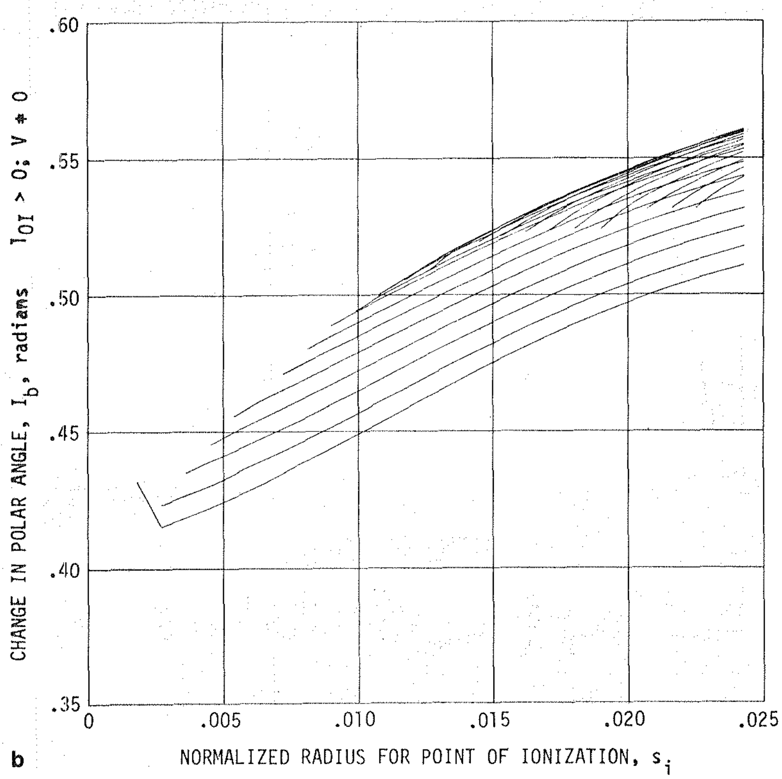
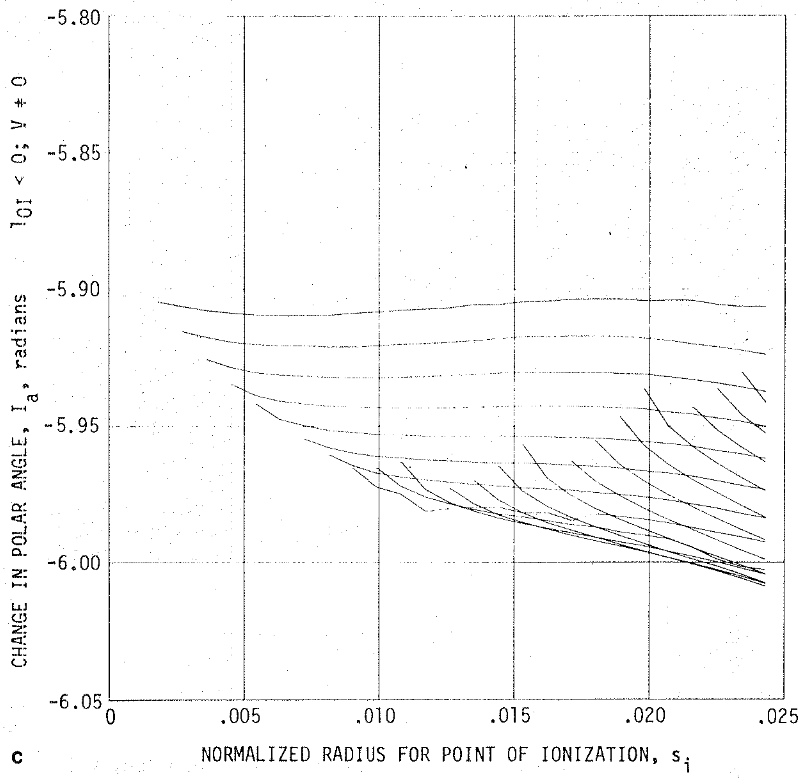


b

Fig 12 - Total change in polar angle from ion gun to secondary ion detector; I_a corresponds to particles ionized before the point of closest approach, and I_b to particles ionized on their way out.



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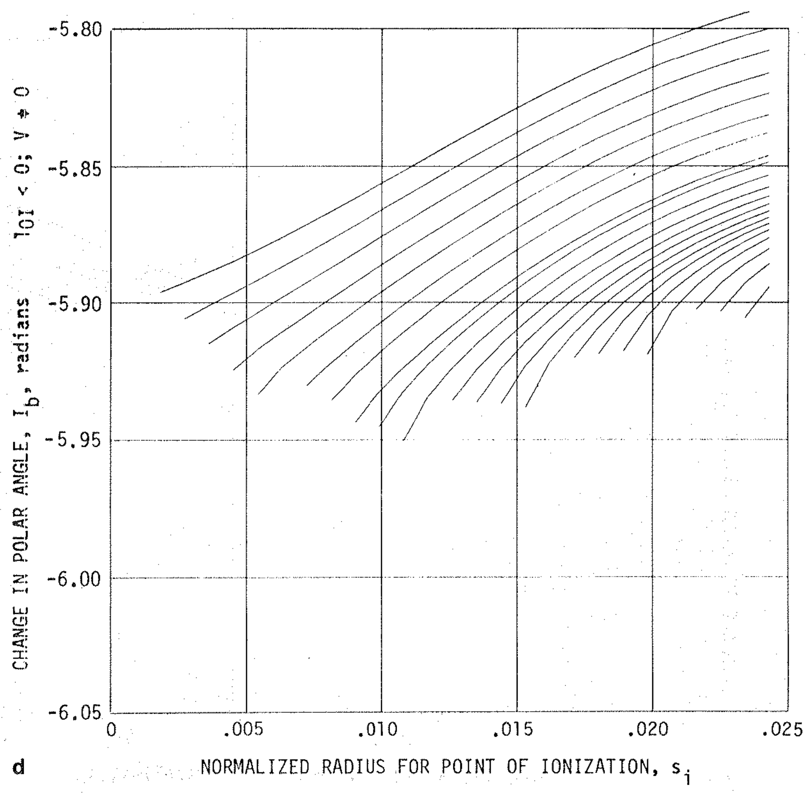
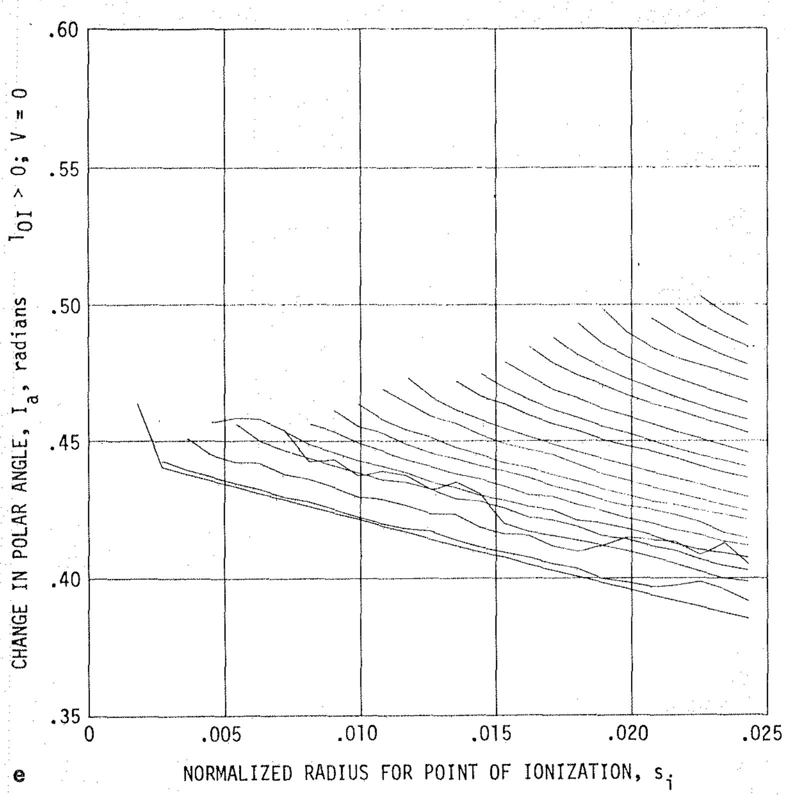
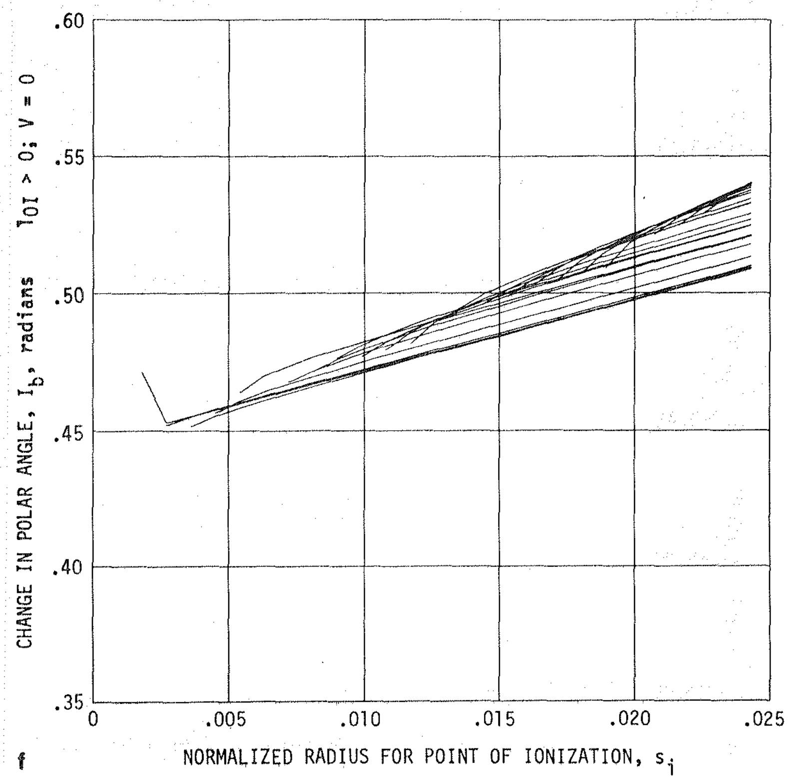


d

Fig 12 - Total change in polar angle from ion gun to secondary ion detector; I_a corresponds to particles ionized before the point of closest approach, and I_b to particles ionized on their way out.

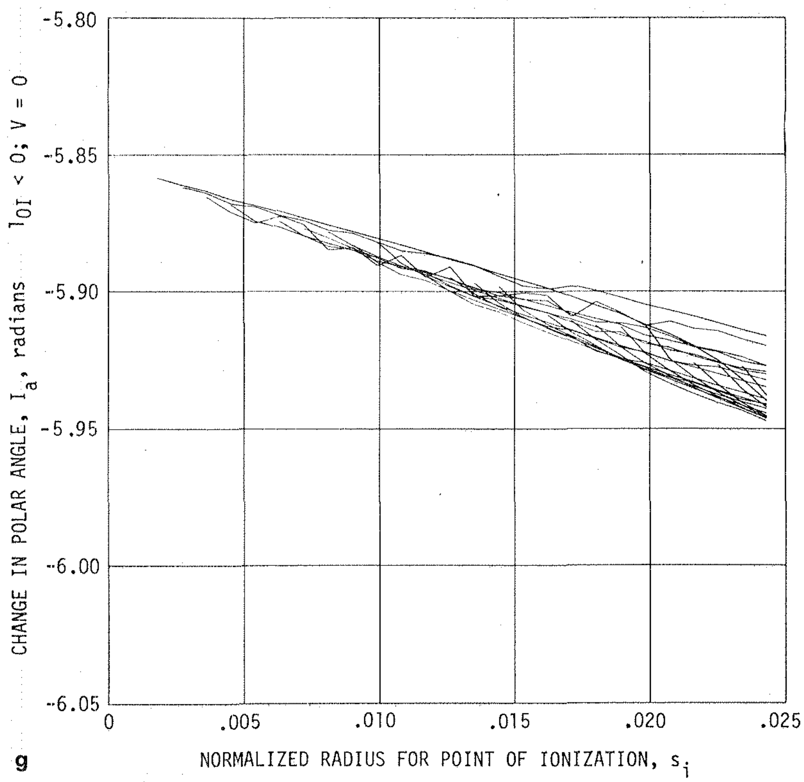


e

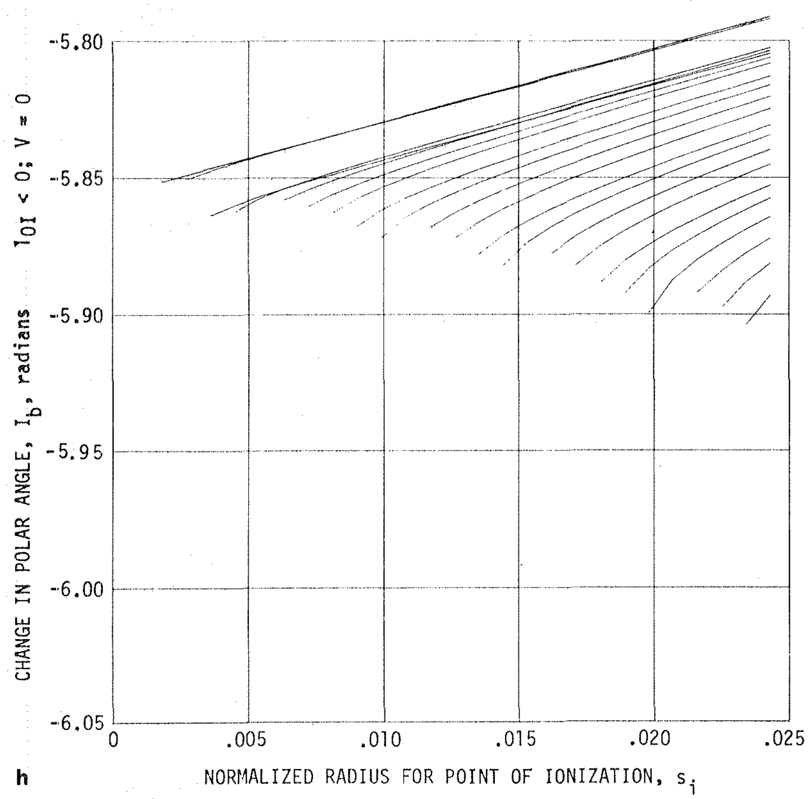


f

Fig 12 - Total change in polar angle from ion gun to secondary ion detector; I_a corresponds to particles ionized before the point of closest approach, and I_b to particles ionized on their way out.



g



h

Fig 12 - Total change in polar angle from ion gun to secondary ion detector; I_a corresponds to particles ionized before the point of closest approach, and I_b to particles ionized on their way out.

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