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by

Peter Proksch

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GENERALIZATION OF THE MANLEY-ROWE RELATIONS
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Peter Proksch*

General relations for the energy spectral densities of non-periodic signals constrained in bandwidth in nonlinear reactances are derived. It is assumed that the charge-voltage characteristics of the reactances are polynomials. In addition certain conditions for the frequency bands of the signals must be met.

By contrast with the Manley-Rowe relations integrals with respect to frequency appear instead of the ratios power over frequency. The integrands are energy spectral densities divided by frequency.

For parametric devices inequalities can be derived for the ratios of the energy levels in the different circuits and the limiting frequencies of the energy spectral densities. With these inequalities it is possible to determine limits for the energy levels.

1. Introduction

Manley and Rowe [1] gave general power relationships for nonlinear reactances. They assumed that the electrical variables consist of sinusoidal functions with two non-commensurable basic frequencies and all of the combination frequencies. These relationships apply for nonlinear reactances with unique but otherwise arbitrary characteristics and for arbitrary values of the power components corresponding to the individual frequencies. Rowe [2] also showed that the relationships apply for linear reactances which change in time. Several authors derive the Manley-Rowe relationships in other ways and have extended them for the general case of an arbitrary number of incommensurable basic frequencies [3] - [7]. The most important feature of the Manley-Rowe relationships is the fact that they give a simple explanation for the operation of parametric amplifiers and mixers and allow one to obtain important information about their operation. Manley and Rowe [1] also found relationships for the reactive power components which are associated with a nonlinear active re-

*Dr. P. Proksch, Institute for High Frequency Technology, Technical University, Gusshausstrasse 25, A-1040, Vienna, Austria. Communication from the Institute for High Frequency Technology, Technical University, Vienna.

sistants.

In the present paper we will derive relationships for the components of the spectral energy density for nonlinear reactances. We will assume that the time functions which occur are non-periodic and have a limited frequency band and that the characteristics of the nonlinear reactances are polynomials. Using these relationships, we can obtain information about the energy components in the circuits of parametric circuits if non-periodic electrical variables occur in them. This seems to be of interest both for stochastic processes (noise) as well as for communication functions, because for information transmission one requires non-periodic functions.

In addition to the assumption on the nonlinearity of the reactance according to a polynomial, certain assumptions about the position of the components of the spectral functions of the current and the voltage along the frequency axis must be satisfied.

If information about the power and energy components is to be obtained which corresponds to the individual circuits of parametric amplifiers and mixers, then from this paper we cannot derive equations from the relationships derived. Instead, we can derive inequalities which can be used to give limits for the energy components. From the Manley-Rowe relationships, one obtains results for the power levels only when if one also assumes that only power components occur as individual frequencies due to the use of ideal filters. The relationships derived in this paper then allow one to derive information when no ideal filters are used.

2. Sinusoidal Processes

We will briefly describe the relationships when there are sinusoidal functions with q non-commensurable basic frequencies ω_i $(1 \leq i \leq q)$ and the infinite number of combination frequencies, before we discuss non-periodic time functions with limited frequency band. It is assumed that we have a nonlinear capacitor, whose voltage depends on the charge as follows

(1)

and r is a natural number. The constant a is said equal to 1 in the following because this changes nothing to the results.

We will use the following abbreviation for the combination frequencies

$$\omega_{n_i} = \omega_{n_1, n_2, \ldots, n_q} = \sum_{i=1}^{q} n_i \omega_i, \quad -\infty \leq n_i \leq \infty \tag{2}$$

The charge and the current can then be written in the form

$$q(t) = \sum_{n_i} Q'_{n_i} \exp(j \omega_{n_i} t), \tag{3}$$

$$i(t) = \sum_{n_i} I'_{n_i} \exp(j \omega_{n_i} t) \tag{4}$$

We are taking a sum of q terms over the subscripts n_1 to n_q (the sums are taken between $-\infty$ to $+\infty$).

$$\sum_{n_i} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_q=-\infty}^{\infty} \tag{5}$$

and we will use the following for the complex Fourier amplitudes of the charge and the current

$$Q'_{n_i} = Q'_{n_1, n_2, \ldots, n_q}, \quad I'_{n_i} = I'_{n_1, n_2, \ldots, n_q} \tag{6}, (7)$$

We then obtain the following for the voltage from equations (1) and (3)

$$u(t) = \sum_{n_{1i}} \cdots \sum_{n_{ri}} Q'_{n_{1i}} \cdots Q'_{n_{ri}} \exp\left(j \sum_{k=1}^{r} \omega_{n_{ki}} t\right). \tag{8}$$

We have the following abbreviations

$$\sum_{n_{ki}} = \sum_{n_{k1}=-\infty}^{\infty} \sum_{n_{k2}=-\infty}^{\infty} \cdots \sum_{n_{kq}=-\infty}^{\infty}, \tag{9}$$

$$Q'_{n_{ki}} = Q'_{n_{k1}, n_{k2}, \ldots, n_{kq}}, \quad \omega_{n_{ki}} = \sum_{i=1}^{q} n_{ki} \omega_i. \tag{10}, (11)$$

$$1 \leq k \leq r.$$

For those sum and equations in (8) which have the same frequency ω_{ni}, we must have the following according to equation (11)

$$\sum_{k=1}^{r} n_{ki} = n_i \tag{12}$$

We can summarize these sum terms in the simplest manner by setting the following for the subscripts n_{1i}

$$n_{1i} = n_i - n_{2i} - \cdots - n_{ri} \tag{13}$$

From equation (8) we then obtain

$$u(t) = \\ = \sum_{n_i} \sum_{n_{2i}} \cdots \sum_{n_{ri}} Q'_{n_{2i}} \cdots Q'_{n_{ri}} Q'_{n_i - n_{2i} - \cdots - n_{ri}} \exp(j \omega_{n_i} t). \tag{14}$$

We then obtain the following for the complex amplitude of the part of the voltage with the frequency ω_n

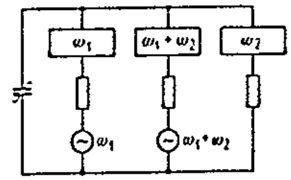

Fig. 1. Greatly simplified replacement circuit of a parametric amplifier with filters which are tuned to the frequencies ω_1, ω_2 and $\omega_1 + \omega_2$.

$$U'_{n_i} = \sum_{n_{2i}} \cdots \sum_{n_{ri}} Q'_{n_{2i}} \cdots Q'_{n_{ri}} Q'_{n_i - n_{2i} - \cdots - n_{ri}} . \quad (15)$$

Here we have

$$\cdots n_{ri} . \quad (16)$$

In the case of parametric amplifiers and mixers, components of the charge and the current only occur at the three frequencies $\omega_{1,0} = \omega_1$, $\omega_{0,1} = \omega_2$ and $\omega_{1,1} = \omega_1 + \omega_2$. Figure 1 shows the greatly simplified replacement circuit for a parametric amplifier. Figure 2,a shows the positions of the spectra of the current, charge and voltage if we assume quadratic nonlinearities of the capacity ($r = 2$) and also that $\omega_{0,1} < 2\omega_{1,0}$. We also have a second case with an arrangement of the frequencies along the frequency axis which is basically different for the one shown in Fig. 2,a. For this case we have $\omega_{0,1} > 2\omega_{1,0}$.

3. Non-periodic Frequency Band Limited Processes with a Finite Energy Content.

We will now consider processes which consist of non-periodic and frequency band limited time processes, in such a manner that the sine function in the time functions are replaced by equations (3) and (4) in each case. The nonlinearity of the capacity is again represented by equation (1). The total current through the nonlinear capacitor and the corresponding charge can then first be written in the following form

$$q(t) = \sum_{n_i} q'_{n_i}(t), \quad i(t) = \sum_{n_i} i'_{n_i}(t) \quad (17), (18)$$

where

$$i'_{n_i}(t) = dq'_{n_i}(t)/dt \quad (19)$$

Instead of each of the exponential functions in equations (3) and (4), we now have a time function with a limited frequency band, which has the same subscripts. These time functions are complex. However, since we are only considering real processes, we have the following for the components of q(t)

$$q'_{-n_i}(t) = q'^{*}_{n_i}(t), \quad (20)$$

and two components with the subscripts n_i and $-n_i$ can be combined into a real time function. The corresponding equations also apply for the components of all of the other physical variables.

From equations (1) and (17), we obtain the following for the voltage at the capacitor

$$u(t) = \sum_{n_{1t}} \cdots \sum_{n_{rt}} q'_{n_{1t}}(t) \cdots q'_{n_{rt}}(t) . \qquad (21)$$

By converting the subscripts according to equation (13), u(t) can be decomposed as follows into components which will be called $u'_{n_t}(t)$

$$u(t) = \sum_{n_t} u'_{n_t}(t) , \qquad (22)$$

$$u'_{n_t}(t) = \sum_{n_{2t}} \cdots \sum_{n_{rt}} q'_{n_{2t}}(t) \cdots q'_{n_{rt}}(t)\, q'_{n_t - n_{2t} - \cdots - n_{rt}}(t) . \qquad (23)$$

Compared with the case of sine functions, the voltage also shows a non-periodic time function with the same subscripts as the exponential function, instead of each of the exponential functions in equation (14).

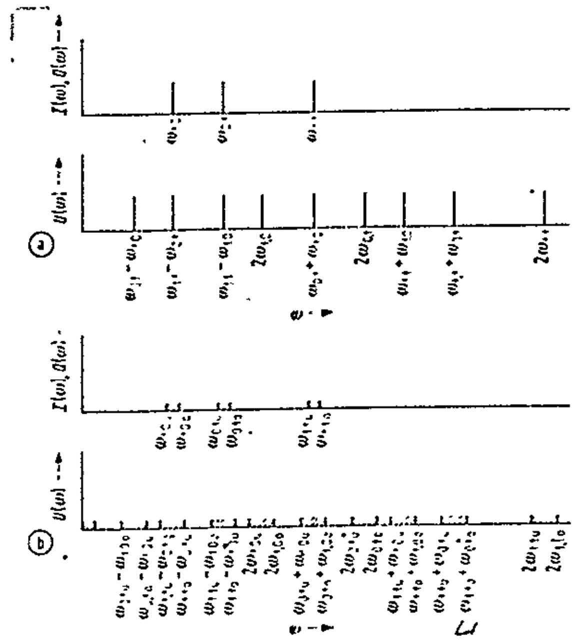

Fig. 2. Position of the spectra of current, charge and voltage for a capacity with quadratic nonlinearity, (a) for a current which consists of three sine functions with the frequencies $\omega_{1,0} = \omega_1, \omega_{0,1} = \omega_2$ and $\omega_{1,1} = \omega_1 + \omega_2$ b) for a current which consists of three non-periodic components which have a limited frequency band.

We will assume that the Fourier transforms (spectral functions) $I(\omega)$, $I'_{n_t}(\omega)$, $Q(\omega)$, $Q'_{n_t}(\omega)$, $U(\omega)$ and $U'_{ni}(\omega)$ corresponding to the time functions $i(t)$, $i'_{n_t}(t)$, $q(t)$, $q'_{n_t}(t)$, $u(t)$ and $U'_{ni}(t)$ exist. From equations (17) and (20), we then obtain the following

$$Q(\omega) = \sum Q'_{n_t}(\omega) , \quad Q'_{-n_t}(\omega) = Q'^{*}_{n_t}(-\omega). \qquad (24), (25)$$

Similar results apply for the current and the voltage. Also, we have

$$I'_{n_t}(\omega) = j\,\omega\, Q'_{n_t}(\omega) . \qquad (26)$$

By using the Fourier transformation and applying it to equation (23), we obtain the following for the components $U'_{ni}(\omega)$ with the help of the convolution theorem

$$U'_{n_t}(\omega) = \frac{1}{(2\pi)^{r-1}} \sum_{n_{2t}} \cdots \sum_{n_{rt}} \int_{-\infty}^{\infty} d\Omega_2 \cdots \int_{-\infty}^{\infty} d\Omega_r\, Q'_{n_{2t}}(\Omega_2) \cdots Q'_{n_{rt}}(\Omega_r) \cdot Q'_{n_t - n_{2t} - \cdots - n_{rt}}(\omega - \Omega_2 - \cdots - \Omega_r). \qquad (27)$$

If the components $q'_{ni}(t)$ are frequency band limited, then from equation (27) it can be seen that the components $u'_{ni}(t)$ are also frequency band limited.

If for the case shown in Fig. 2,a we have non-periodic band limited time functions instead of sine functions, then the spectral functions $I(\omega)$, $Q(\omega)$ and $U(\omega)$ consist of components with the frequency axis position shown in Fig. 2,b. These spectral functions are obtained, for example, if in the replacement circuit of Fig. 1, the AC voltage sources are replaced by generators which produce non-periodic voltages with a limited frequency band. The limited frequencies of the current components are $\omega_{1,0u}$, $\omega_{1,0o}$, $\omega_{0,1u}$, $\omega_{0,1o}$, $\omega_{1,1u}$ and $\omega_{1,1o}$, and in Fig. 2,b we have assumed $\omega_{0,1o} < 2\omega_{1,0u}$. Here again there is an additional case with another configuration along the frequency axis. For this case we have $\omega_{0,1u} > 2\omega_{1,0o}$. The later discussions can be applied to both cases in the same way.

If the frequencies of the sinusoidal functions of the current (Fig. 2,a) lie in the frequency regions in which the corresponding components of $I(\omega)$ are different from zero (Fig. 2,b), then they also will lie in the regions in which components of $U(\omega)$ are different from zero. This is also true for the general case where to each sine function with a frequency ω_{ni}, we have a corresponding current and voltage component with a limited frequency band.

For the general case we will now investigate the assumptions required about the position of the components along the frequency axis so that the energy relationships can be derived. Such assumptions must become the ones used for the derivation of the Manley-Rowe relationships for the case of the sine functions. This is because the sine functions can be considered in special cases of the frequency limited functions. Manley and Rowe [1] assume non-commensurable basic frequencies and arbitrary nonlinearities. From the derivation of the Manley-Rowe relationships [1], one can also see that there the assumption was first made that two frequencies ω_{ni} and ω_{mi}, to which the complex amplitudes I'_{ni} and U'_{mi} correspond, are only equal for the case when all of the subscripts n_i and m_i agree. Therefore, we must have $\omega_{n_i} \neq \omega_{m_i}$ for $n_i \neq m_i$. For the arbitrary nonlinearity case, this assumption can only be satisfied if, just like in [1], one assumes no commensurable basic frequencies. On the other hand, it is found that the basic frequencies must be non-commensurable if one restricts oneself to nonlinearities described by polynomials and where power components do not occur for all of the combination frequencies. This is, for example, the case for what is shown in Fig. 2.

Starting with the assumptions, we can make the following appropriate assumptions for non-periodic time functions: If all the subscripts n_i and m_i of two components of the spectral functions of current and voltage $I'_{ni}(\omega)$ or $U'_{mi}(\omega)$ agree, then for these a frequency region exists in which both are different from zero. If not all the subscripts coincide, then there are no values of ω for which both $I'_{ni}(\omega)$ and $U'_{mi}(\omega)$ are different from zero. If the frequency intervals within which $I'_{ni}(\omega)$ and $U'_{mi}(\omega)$ are different from zero are called $\bar{\omega}_{I.ni}$ and $\bar{\omega}_{U.mi}$, then these assumptions can be written in the following form:

$$\begin{aligned} \bar{\omega}_{I,n_i} \cap \bar{\omega}_{U,m_i} &\neq 0 \quad \text{for } n_i = m_i, \\ \bar{\omega}_{I,n_i} \cap \bar{\omega}_{U,m_i} &= 0 \quad \text{for } n_i \neq m_m. \end{aligned} \tag{28}$$

(0/zero set)

In the following we will investigate the conditions for which these assumptions are satisfied. If we first consider a current, then we can see that a component $I'_{ni}(\omega)$ does not actually have to correspond to each frequency ω_{ni}. Out of the frequencies ω_{ni}, we can always find two with an arbitrarily small separation along the frequency axis. If a frequency region is introduced at the point of each frequency where the component $I'_{ni}(\omega) \neq 0$, then these regions will cover one another in such a way that there is no value of ω, where only one component $I'_{ni}(\omega)$ is different from zero alone. The assumptions (28) can therefore only be satisfied if only certain components $I'_{ni}(\omega)$ are different from zero and the others are zero for all values of ω.

Also, a voltage component $U'_{ni}(\omega)$ must not occur for each frequency. This is only exactly true and possible when the nonlinearity of the capacitor does not have an arbitrary magnitude but instead is given by a power function or a polynomial. Otherwise, in the case where only individual components $I'_{ni}(\omega)$ were present, a component $U'_{ni}(\omega)$ would occur for each frequency ω_{ni}.

In order to test whether the assumption (28) is satisfied, the limiting frequencies of $I'_{ni}(\omega)$ and $U'_{ni}(\omega)$ must be known. The limiting frequencies of the components $U'_{ni}(\omega)$ can then be determined from those of the components $Q'_{ni}(\omega)$ in the following way: According to (27), each component $U'_{ni}(\omega)$ in general again consists of sum terms which are derived by the multiple convolution of components of $Q(\omega)$. The basic frequencies of these sum terms can be determined based on the process during the multiple convolution. The lower limiting frequency of a sum term is found as the algebraic sum of the lower limiting frequencies of the components of $Q(\omega)$ under the integral sine in (27). For the lower limiting frequency of a component $Q'_{-mi}(\omega)$ along the negative ferquency axis, we use the negative upper limiting frequency of $Q'_{ni}(\omega)$. Similar statements apply for the upper limiting frequency. If in this way we determine the limiting frequencies of all the sum terms of $Q'_{ni}(\omega)$, then the lowest of all the limiting frequencies is also the lower limiting frequency of $U'_{ni}(\omega)$. The highest of all the upper limiting frequencies is also the upper limiting frequency of $U'_{ni}(\omega)$.

Figure 2,b shows a case where all the assumptions (28) are satisfied. The limiting frequencies of the voltage components obtained in the above manner are expressed by the limiting frequencies of the three current components. It can be found that the assumptions (28) are only satisfied for this case when the following relationships apply:

$$\begin{aligned} &\omega_{0,1o} < 2\omega_{1,0u}, \quad 2\omega_{1,0o} < \omega_{1,1u}, \\ &\omega_{1,1o} < 2\omega_{0,1u}, \quad \omega_{1,1o} - \omega_{1,1u} < \omega_{1,0u}. \end{aligned} \tag{29}$$

In the other case $(\omega_{0,1u} > 2\omega_{1,0o})$, the following relationships must hold:

$$\begin{aligned} &2\omega_{1,0o} < \omega_{0,1u}, \quad \omega_{1,0o} < 2\omega_{1,0u}, \\ &\omega_{0,1o} - \omega_{0,1u} < \omega_{1,0u}, \\ &\omega_{1,1o} < 2\omega_{0,1u}, \quad \omega_{1,1o} - \omega_{1,1u} < \omega_{1,0u} \end{aligned} \tag{30}$$

4. Relationships for Spectral Energy Density.

In order to derive relationships which are the generalizations of the Manley-Rowe relationships to non-periodic frequency band limited time functions, we will now again consider the general case. According to equation (28), we have the following for the spectral energy density:

$$\frac{1}{2\pi} U(\omega) I^*(\omega) = \frac{1}{2\pi} \sum_{n_i} U'_{n_i}(\omega) I'^*_{n_i}(\omega). \tag{31}$$

The components of the spectral energy density in (31) are divided by ω and are integrated over ω. In addition, the integrals which are formed are then multiplied with a coefficient n_j $(i \leqq j \leqq q)$, and the sums are formed over the subscripts n_i. Using the Parseval equation and using equation (26), we obtain the following system of q equations after exchanging summation and integration functions:

$$\sum_{n_i} n_j \int_{-\infty}^{\infty} \frac{U'_{n_i}(\omega)\, I'^{*}_{n_i}(\omega)}{2\pi\omega}\, d\omega = \\ = (-j) \int_{-\infty}^{\infty} \sum_{n_i} n_j u'_{n_i}(t)\, q'^{*}_{n_i}(t)\, dt\,. \tag{32}$$

If we substitute the expression from equation (23) for $u'_{ni}(t)$, then we find the following for the sum on the right side of equation (32), which will be called S:

$$S = \sum_{n_i} \sum_{n_{2i}} \cdots \sum_{n_{ri}} n_j q'^{*}_{n_i}(t)\, q'_{n_{2i}}(t) \cdots \\ \cdots q'_{n_{ri}}(t)\, q'_{n_i - n_{2i} - \cdots - n_{ri}}(t)\,. \tag{33}$$

We can now show that this sum and therefore the expression on the right side of equation (32) vanish identically. For this purpose, let us assume that we have written down r expressions which are derived from S as follows: In order to obtain the first r -1 expressions, we will only consider one of the r -1 renamed quantities

$$n_i \to -n_{ki}, \quad n_{ki} \to -n_i, \quad 2 \leqq k \leqq r\,.$$

In this renaming, all of the other subscripts remain unchanged. The last expression is obtained by the transitions

$$n_i \to -(n'_i - n_{2i} - \cdots - n_{ri}), \\ n_i - n_{2i} - \cdots - n_{ri} \to -n'_i$$

and the subsequent renaming $n'_i \to n_i$. Since we are only dealing with renaming of the summation subscripts which all run from $-\infty$ to $+\infty$, the sum remains unchanged as a whole. If each subscript has a certain value, then however another sum term appears in each sum. These sum terms, however, agree with the sum terms in the original sum except for the coefficients n_j. This is because in the renaming, two factors of the sum terms go over into one another and the other factors remain unchanged. Therefore, only the coefficients n_j are changed.

If we obtain the r expressions obtained to obtain the original expression for S, then next to each sum term of the multiple sum we have the coefficient

$$n_j - n_{2j} - n_{3j} - \cdots - n_{rj} - \\ - (n_j - n_{2j} - n_{3j} - \cdots - n_{rj}) = 0\,.$$

The sum of all the expressions is therefore zero. Since this is (r + 1) of S, S vanishes identically. Therefore, from equation (32), we obtain the following relationships (a total of q equations)

$$\sum_{n_i} n_j \int_{-\infty}^{\infty} \frac{U'_{n_i}(\omega)\, I'^{*}_{n_i}(\omega)}{2\pi\omega}\, d\omega = 0, \qquad (34)$$
$$1 \leq j \leq q.$$

Out of the components $I'_{ni}(\omega)$ and $U'_{ni}(\omega)$, we can summarize to each to a component $I_{ni}(\omega)$ or $U_{ni}(\omega)$, in which all of the subscripts ni only differ by the sign. $I_{ni}(\omega)$ and $U_{ni}(\omega)$ are spectral functions of real time functions. Equation (34) can now be converted so that $I_{ni}(\omega)$ and $U_{ni}(\omega)$ occur instead of $I'_{ni}(\omega)$ and $U'_{ni}(\omega)$. In addition, two components of the spectral energy density are summarized into one as follows:

$$U_{n_i}(\omega)\, I^{*}_{n_i}(\omega) = U'_{n_i}(\omega)\, I'^{*}_{n_i}(\omega) + U'_{-n_i}(\omega)\, I'^{*}_{-n_i}(\omega). \qquad (35)$$

In equation (34) the integrand ω must be replaced by $|\omega|$ in the denominator. This would lead to a change in the sign if $\bar{\omega}_{I,ni}$ and $\bar{\omega}_{U,ni}$ lie along the negative frequency axis. Therefore, coefficients must be introduced which take on the values +1 or -1 depending on the position of these frequency regions. Therefore, equation (34) becomes

$$\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_j=0}^{\infty} \cdots \sum_{n_q=-\infty}^{\infty} n_j \int_{-\infty}^{\infty} \frac{e_{n_i}\, U_{n_i}(\omega)\, I^{*}_{n_i}(\omega)}{2\pi|\omega|}\, d\omega = 0, \qquad (36)$$
$$1 \leq j \leq q,$$

with

$$e_{n_i} = \begin{cases} +1, & \text{if } \omega > 0 \text{ for } I'_{n_i}(\omega) \neq 0 \\ -1, & \text{if } \omega < 0 \text{ for } I'_{n_i}(\omega) \neq 0. \end{cases}$$

For a case with three current components (this does not have to be the case with the quadratic nonlinearity) where the components $I'_{1.0}(\omega)$, $I'_{0.1}(\omega)$ and $I'_{1.1}(\omega)$ lie along the positive frequency axis, so that the three coefficients $e_{1.0}$, $e_{0.1}$ and $e_{1.1}$ take on the value + 1, the relationships (36) have the following form:

$$\int_{-\infty}^{\infty} \frac{U_{1,0}(\omega)\, I^{*}_{1,0}(\omega)}{2\pi|\omega|}\, d\omega + \int_{-\infty}^{\infty} \frac{U_{1,1}(\omega)\, I^{*}_{1,1}(\omega)}{2\pi|\omega|}\, d\omega = 0,$$
$$\int_{-\infty}^{\infty} \frac{U_{0,1}(\omega)\, I^{*}_{0,1}(\omega)}{2\pi|\omega|}\, d\omega + \int_{-\infty}^{\infty} \frac{U_{1,1}(\omega)\, I^{*}_{1,1}(\omega)}{2\pi|\omega|}\, d\omega = 0 \qquad (37)$$

Such a case, for example, occurs in a parametric amplifier.

5. Nonlinearity Corresponding to a Polynomial

We will now assume that the characteristic of the nonlinear capacitor is represented by a polynomial of degree r instead of equation (1):

$$u = a_1 q + a_2 q^2 + \cdots + a_r q^r . \tag{38}$$

We can think of this as a dipole having a characteristic corresponding to r switching elements switched in series. Each of these has a characteristic corresponding to a power function which makes up the polynomial. The voltage contributions over the switching element with a characteristic (38) is then equal to the sum of the corresponding components for the individual circuit elements. If the relationships (34) apply for each of the r switching elements, then these apply also for the capacitor with the characteristic (38) because in equation (32) $u'_{ni}(t)$ can be replaced by the sum of the corresponding components from the individual circuit elements and the expressions produced in this way are then zero.

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6. Parametric Amplifiers and Mixers

We will now show the information one can obtain using the relationships obtained above for parametric circuits. We will consider the case with three frequency bands limited components of spectral energy density. In the replacement circuit of Fig. 1, for this case the AC voltage sources must be replaced by sources for non-periodic and frequency band limited voltages. If we assume that each component has a real part which has the same sign for all values of ω, we have the following

$$\frac{1}{\omega_{1,0o}} \int_{-\infty}^{\infty} \frac{I_{1,0}(\omega) U^*_{1,0}(\omega)}{2\pi} d\omega < \int_{-\infty}^{\infty} \frac{I_{1,0}(\omega) U^*_{1,0}(\omega)}{2\pi|\omega|} d\omega < \frac{1}{\omega_{1,0u}} \int_{-\infty}^{\infty} \frac{I_{1,0}(\omega) U^*_{1,0}(\omega)}{2\pi} d\omega . \tag{39}$$

For example, for a quadratic nonlinearity we can usually show that this applies for a parametric amplifier. The quantities $\omega_{1,0u}$ and $\omega_{1,0o}$ are the limiting frequencies of $I_{1,0}(\omega)\, U^*_{1,0}(\omega)$. Corresponding relationships apply for $I_{0,1}(\omega)\, U^*_{0,1}(\omega)$ and $I_{1,1}(\omega)\, U^*_{1,1}(\omega)$. If the energy components in the individual circuits are called

$E_{1,0}$, $E_{0,1}$ and $E_{1,1}$, then we obtain

$$\frac{|E_{1,0}|}{\omega_{1,0o}} < \int_{-\infty}^{\infty} \frac{I_{1,0}(\omega)\, U^*_{1,0}(\omega)}{2\pi|\omega|}\, d\omega < \frac{|E_{1,0}|}{\omega_{1,0u}},$$

and the corresponding inequalities for $E_{0,1}$ and $E_{1,1}$. In addition, we have the following inequalities from equation (37)

$$\frac{|E_{1,0}|}{\omega_{1,0o}} < \frac{|E_{1,1}|}{\omega_{1,1u}}, \quad \frac{|E_{1,0}|}{\omega_{1,0u}} > \frac{|E_{1,1}|}{\omega_{1,1o}}, \tag{40}$$

as well as four similar inequalities for $E_{0,1}$ and $E_{1,1}$ or $E_{1,0}$ and $E_{0,1}$. In addition, from equation (37) we find that $E_{1,0}$ and $E_{0,1}$ have the same signs, whereas $E_{1,0}$ and $E_{1,1}$ have different signs. If one of the three energy components is known, then from (40) and for additional inequalities we obtain the limits for the contributions of the two others. If instead of the non-periodoc time function $i_{1,1}(t)$ we have a sine function with the frequencies $\omega_{1,1}$, then in (40), $\omega_{1.1u}$ and $\omega_{1,1o}$ must be replaced by $\omega_{1,1}$.

7. Discussion

First we will show the relationship between relationships (36) and the Manley-Rowe equations. For sinusoidal time functions with complex amplitudes U'_{ni} or I'_{ni} and the frequencies ω_{ni}, we obtain the following spectral functions

$$U_{ni}(\omega) = U'_{ni}\,\delta(\omega - \omega_{ni}) + U'^*_{ni}\,\delta(\omega + \omega_{ni}),$$
$$I_{ni}(\omega) = I'_{ni}\,\delta(\omega - \omega_{ni}) + I'^*_{ni}\,\delta(\omega + \omega_{ni}).$$

Since the spectral energy density does not exist for the sign functions, then in this case in equation (36) the quantities $U_{ni}(\omega)\, I^*_{ni}(\omega)$ must be replaced by the spectral power densities $U'_{ni} I'^*_{ni}\, \delta(\omega - \omega_{ni}) + U'^*_{ni} I'_{ni}\, \delta(\omega + \omega_{ni})$. Then for the integrals in (36), we obtain

$$\int_{-\infty}^{\infty} \frac{e_{ni}[U'_{ni} I'^*_{ni}\,\delta(\omega - \omega_{ni}) + U'^*_{ni} I'_{ni}\,\delta(\omega + \omega_{ni})]}{|\omega|}\, d\omega = \frac{U'_{ni} I'^*_{ni} + U'^*_{ni} I'_{ni}}{\omega_{ni}}. \tag{41}$$

In the denominator on the right side of equation (41), we have the power components P_{ni} and therefore the relationships (36) become the Manley-Rowe relationships:

$$\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_j=0}^{\infty} \cdots \sum_{n_q=-\infty}^{\infty} n_j \frac{P_{ni}}{\omega_{ni}} = 0, \quad 1 \leq j \leq q. \tag{42}$$

In the derivation of equation (36) we assumed a nonlinearity which corresponds to a polynomial, and we show that these relationships do not hold exactly for a strong nonlinearity. In contrast to this, the Manley-Rowe relationships apply for a unique but otherwise arbitrary characteristic if the basic frequencies are non-commensurable [1]. We obtain a number of independent relationships equal to the number of non-commensurable frequencies. Manley and Rowe [1] assume two basic frequencies and therefore obtain two relationships, the relationships (42) for q = 2.

Relationships such as (36) and (42) have practical applications in the case where one wishes to have information about the energy and power parts corresponding to the individual subcircuits of a circuit. Such information cannot be obtained using the Manley-Rowe relationships (42) without further restrictions. Manley and Rowe therefore assume that the power components only occur for individual frequencies, whereas the other components are suppressed by ideal filters. For example if in each of these circuits of a parametric amplifier there is only one power component, then we have the following simple relationship for these components:

$$\frac{P_{1,0}}{\omega_1}+\frac{P_{1,1}}{\omega_1+\omega_2}=0, \quad \frac{P_{0,1}}{\omega_2}+\frac{P_{1,1}}{\omega_1+\omega_2}=0. \tag{43}$$

Using equation (43) one can then obtain information about the sign and the magnitude of the power levels in the three circuits. This is possible because in each of the three equations, there are only two power levels, and each of them is equal to the total power corresponding to a single circuit.

If on the other hand we assume that we have filters with throughput regions instead of ideal filters, which have a width which is not equal to zero, then for strong nonlinearities (for example, corresponding to an exponential function) we have not only the three frequencies ω_1, ω_2 and $\omega_1 + \omega_2$, but also all of the other combination frequencies which fall within the pass range of the filters. If we write down the Manley-Rowe relationships for such a case, then in each of the relationships we have power components from all of the three circuits under the sum in each relationship. We cannot obtain information about the power levels in the individual circuits because the power components in the circuit cannot be summed up and cannot be separated from those in the other circuits, because we do not have the same coefficients n_i in the other components.

The relationships derived in the present paper, however, allow one to obtain information about the energy components in the individual circuits even for the case where one does not assume ideal filters. However, certain assumptions must be satisfied for the nonlinearity of the reactance and the width of the pass range of the filters.

We would also like to point out that energy relationships such as the ones derived here can be derived in a similar manner for nonlinear capacitances where q = q(u) is a polynomial, as well as for nonlinear inductances and nonlinear mechanical systems without losses.

In addition, we can generalize the relationships of Manley and Rowe for the reactance components in a nonlinear active resistance to frequency band limited time functions. One then obtains relationships different from (36) due to the fact that integrands are expressions like

$$U'_{n_i}(\omega)\, I'^{*}_{n_i}(\omega) - U'^{*}_{n_i}(\omega)\, I'_{n_i}(\omega)\,.$$

which are the imaginary parts of the spectral energy density components.

In conclusion, we would like to mention that similar relationships to those used here for processes with finite energy can be found for processes with nonfinite energy content such as those used for a treatment of noise. Relationships for both cases apply for linear capacitances which vary in time [8]

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List of symbols used:

$i(t),\ i'_{n_i}(t)$	Time function.	Current
$I'_{n_i} = I'_{n_1, n_2, \ldots, n_q}$	complex in the subscripts	Current
$I(\omega),\ I_{n_i}(\omega),\ I'_{n_i}(\omega)$	Spectral functions	Current
$q(t), q'_{n_i}(t), Q(\omega), Q'_{n_i}(\omega), Q'_{n_i}$		Charge
$u(t), u'_{n_i}(t), U(\omega), U_{n_i}(\omega), U'_{n_i}(\omega), U'_{n_i}$		Voltage
$\delta(\omega)$		Delta function
ω		Circular frequency
$\omega_{n_i} = \omega_{n_1, n_2, \ldots, n_q} = \sum_{i=1}^{q} n_i \omega_i$	Frequency intervals	
$\overline{\omega}_{l,n_i}; \overline{\omega}_{l',n_i};$ $\omega_{n_{iu}}, \omega_{n_{io}}$	Limiting frequencies	
$\sum_{n_i} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_q=-\infty}^{\infty}$		
0	Zero set (empty set)	
$\cap$	Average of two sets	
$*$	Complex conjugate	

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