Geoid Anomalies and the Near-Surface Dipole Distribution of Mass

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Abstract. Although geoid or surface gravity anomalies cannot be uniquely related to an interior distribution of mass, they can be related to a surface mass distribution. However, over horizontal distances greater than about 100 km, the condition of isostatic equilibrium above the asthenosphere is a good approximation and the total mass per unit column is zero. Thus the surface distribution of mass is also zero. For this case we show that the surface gravitational potential anomaly can be uniquely related to a surface dipole distribution of mass. Variations in the thickness of the crust and lithosphere can be expected to produce undulations in the geoid.

Introduction

The gravitational potential and acceleration can in general be obtained by integrating over any specified distribution of mass. In many cases, however, the detailed distribution of mass in the crust and mantle may be unknown. In these cases unique relationships between the gravity and geoid anomalies and surface distreibutions of density may be of considerable use. One example of such a relationship is the Bouguer formula for the gravity anomaly Δg,

$$\Delta g = 2\pi G \sigma(x,y) \tag{1}$$

where G is the gravitational constant and the surface density distribution is

$$\sigma(x,y) = \int_0^h \Delta\rho\ (x,y,z)\,dz \tag{2}$$

where $\Delta\rho$ is the density anomaly. The Bouguer formula is valid if the horizontal scale of the density variation is large compared with the vertical scale h and h<<a where a is the radius of the earth.

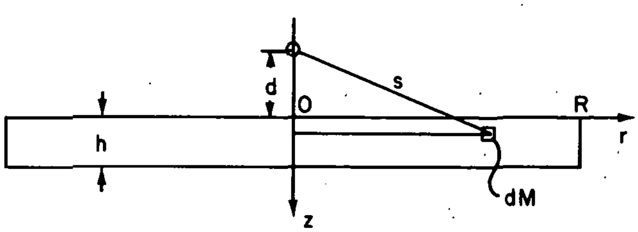

Fig. 1. Illustration of the circular disk formulation.

Using the technique of matched asymptotic expansions, Ockendon and Turcotte (1977) have derived a power series expansion for the gravitational acceleration and potential caused by slowly varying density changes. They find that if the near surface density distribution is in isostatic equilibrium then the gravitational potential anomaly ΔU is given by

$$\Delta U = -2\pi G\delta\ (x,y) \tag{3}$$

where the surface dipole density distribution is

$$\delta(x,y) = \int_0^h z\ \Delta\rho(x,y,z)\,dz \tag{4}$$

The conditions for the validity of this relation are the same as for the Bouguer formula with the additional isostatic requirement that $\sigma = 0$, i.e., that the gravity anomaly given by the Bouger formula (1) is zero.

It is the purpose of this paper to give two elementary planar derivations of (3) and to test its validity for near surface density variations on the earth.

Disk Approximations

We first consider a circular disk of thickness h and radius R as shown in Figure 1. The density of the disk is a function of the vertical coordinate z, $\rho(z)$, but not of r. The contribution to the gravitational acceleration of each element of mass in the disk is

$$d\vec{g} = -\frac{G\vec{s}\,dM}{|s|^3} \tag{5}$$

Integrating over the volume of the disk to obtain the vertical components of the gravitational acceleration on the axis at a distance d above yields

$$g_z = 2\pi G \int_0^h \int_0^R \frac{(d+z)\ r\rho(z)\ dr dz}{[r^2 + (d+z)^2]^{1/2}} \tag{6}$$

First integrating with respect to r and then taking the limit $R \rightarrow \infty$ gives

$$\lim_{R\rightarrow\infty} g_z = 2\pi G \int_0^h \rho(z)\,dz \tag{7}$$

which is the Bouguer formula previously given in (1).

The gravitational potential due to each element of mass is

$$dU = \frac{GdM}{|s|} \tag{8}$$

Proc. of the 9th GEOP Conference, *An International Symposium on the Applications of Geodesy to Geodynamics, October 2-5, 1978.* Dept. of Geodetic Science Rept. No. 280, The Ohio State Univ., Columbus, Ohio 43210.

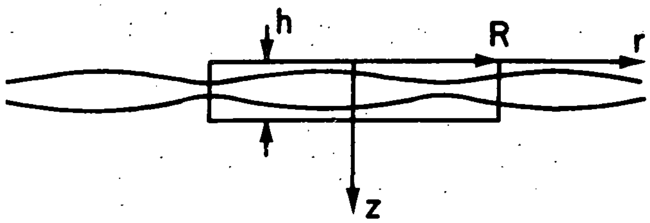

Fig. 2. Illustration of the application of Gauss' theorem to a thin layer of mass anomalies.

Integrating over the volume of the disk to obtain the potential at a height d on the axis of the disk yields

$$U = 2\pi G \int_0^h \int_0^R \frac{r\rho(z)\ dzdr}{[r^2 + (d + z)^2]^{1/2}} \tag{9}$$

Integration with respect to r and expanding for large R gives

$$U = 2\pi G\ [R\int_0^h \rho(z)dz - \int_0^h (d + z)\ \rho(z)\ dz + \frac{1}{2R}\int_0^h (d + z)^2\rho(z)\ dz + 0\ (R^{-3})] \tag{10}$$

First applying the condition of isostasy, i.e.,

$$\int_0^h \rho(z)\ dz = 0$$

and then taking the limit $R \rightarrow \infty$ yields

$$U = -2\pi G\int_0^h z\rho(z)dz \tag{11}$$

which is the formula previously given in (3).

Mass-Layer Approximations

For mass anomalies confined to thin layers it is useful to integrate (5) over the cylindrical volume illustrated in Figure 2. Gauss' theorem may then be used to convert one of the volume integrals to a surface integral with the result

$$\iint \vec{g} \cdot d\vec{s} = -4\pi G\iiint \rho\ dv \tag{12}$$

In the limit $h \rightarrow 0$ these integrals can be evaluated to yield (Officer, 1974, pp. 262-269)

$$g_z^+ - g_z^- = 4\pi\ G\sigma \tag{13}$$

However by symmetry

$$g_z^+ = -\ g_z^- \tag{14}$$

so that

$$g_z^- = 2\pi\ G\int_0^h \rho\ dz \tag{15}$$

which is the Bouguer formula (1).

A surface dipole mass distribution is obtained by taking two equal surface mass distributions of opposite sign, $\sigma_1 < 0$ on $z = 0$ and $\sigma_2 = -\sigma_1$ on $z = h$ (Fig. 3a); the limits $\sigma_2 \rightarrow +\infty$, $\sigma_1 \rightarrow -\infty$ and $h \rightarrow 0$ are taken such that

$$\int_0^h z\rho dz = \sigma_2 h = \delta$$

is finite. It follows from (13) and (14) that

$$\begin{aligned} g_z &= 0 \quad z > h,\ z < 0 \\ g_z &= 4\pi G\sigma_2, \quad 0 < z < h \end{aligned} \tag{16}$$

Using the relationship between the gravitational field and potential, $g_z = \partial U/\partial z$ the difference in the potential across the dipole layer is

$$U^+ - U^- = 4\pi\ G\sigma_2\ \ h = 4\pi G\delta \tag{17}$$

We choose our origin for U such that

$$U^+ = -U^- \tag{18}$$

so that the distribution of U illustrated in Figure 3c is obtained and

$$U^- = -2\pi\ G\delta = -\ 2\pi G\int_0^h z\rho dz \tag{19}$$

which is the same as (3).

In order to establish the quantitative validity of (3) we consider the gravity and potential fields due to spherical harmonic distributions of mass on spherical surfaces. The gravitational field just outside a spherical surface due to a surface mass distribution $\sigma_n S_n$ (where S_n is the spherical surface harmonic of order n) on the surface is given by (Jeffreys, 1976, p. 234)

$$g = 4\pi\ G\left(\frac{n + 1}{2n + 1}\right)\sigma_n S_n \tag{20}$$

The wavelengths of the surface mass distribution can be related to the order of the harmonic n by

$$\lambda = \frac{2\pi a}{n} \tag{21}$$

where a is the radius of the sphere (of the earth). For short wavelength distributions we take the limit $n \rightarrow \infty$ in (20) with the result

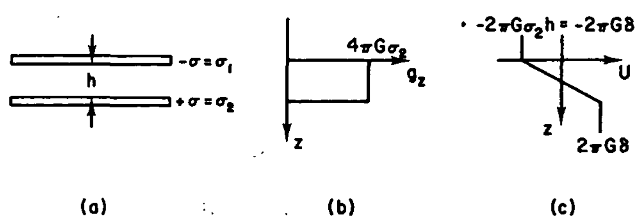

Fig. 3. Gravitational acceleration (b) and potential (c) associated with a dipole mass distribution (a).

$$\lim_{n\to\infty} g = 2\pi G \sigma_n S_n \quad (22)$$

which is the Bouguer formula.

In order to represent a dipole distribution of mass we consider a spherical harmonic distribution of surface mass on a sphere of radius r = a-h with amplitude $K_n\sigma_n S_n$ in addition to the distribution of surface mass $\sigma_n S_n$ on the sphere r = a. The resulting gravitational potential just outside the outer sphere is given by (Jeffreys, 1976, p. 237)

$$U = \frac{-4\pi G}{(2n+1)} \sigma_n S_n \, a \left[1 + K_n \left(1 - \frac{h}{a}\right)^{n+2}\right] \quad (23)$$

The condition of isostasy requires an equal mass defect on the inner sphere to that on the outer sphere. Allowing for the difference in area we require (Jeffreys, 1976, p. 237)

$$K_n = -\frac{a^2}{(a-h)^2} \quad (24)$$

Substitution of (24) into (23) yields

$$U = \frac{-4\pi G}{(2n+1)} \sigma_n S_n \, a \left[1 - \left(1 - \frac{h}{a}\right)^{n}\right] \quad (25)$$

Taking the limit $h/a \to 0$ in (25) gives

$$\lim_{h/a\to 0} U = \frac{-4\pi n G\sigma_n S_n h}{2n+1} \quad (26)$$

Next taking the limit $n \to \infty$ we find

$$\lim_{n\to\infty} U = -2\pi G\sigma_n S_n h \quad (27)$$

and noting that $\sigma_n S_n h$ is the surface dipole distribution of mass this is the same as (3).

By using (25) we can compare the results for finite depths of compensation h with the limiting solution given in (27). This is done in Figure 4. The ratio of the surface potential U from (25) to the value $2\pi G\sigma_n S_n h$ is given as a function of 1/n for h = 25, 50. 100, 200, and 400 km. The corresponding wavelengths from (21) are also included. We see that the approximation breaks down as expected when the wavelength is of the same order as the depth of compensation, i.e., as $\lambda/h \to 1$. For depths of compensation of 50 km or less the error in using (27) is 10% or less over a wide range of wavelengths. It should be emphasized that the case of two mass layers is an extreme case of compensation at depth. The realistic case of distributed mass with depth will lead to lower errors than those given in Figure 4 if the mass differences are limited to a depth h.

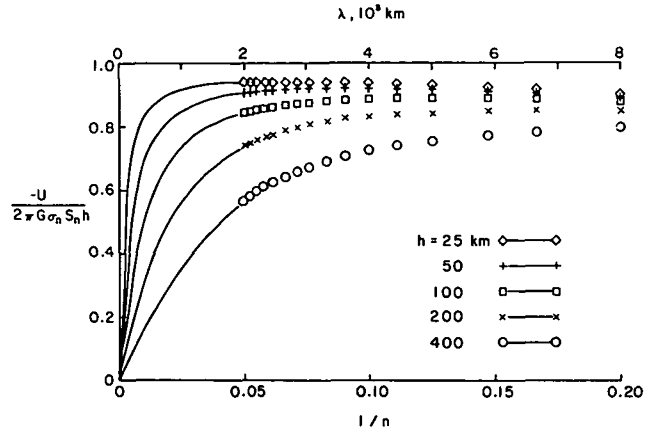

Fig. 4. The ratio of the surface potential U on the earth from (25) to the value for a surface dipole layer (27) as a function of 1/n for various values of the depth of compensation h. Also included are the values of the wavelength λ corresponding to the value of 1/n from (21).

Discussion

The gravitational field and potential outside a closed surface can be uniquely related to a surface distribution of mass. The Bouguer formula (1) relates the local gravity anomaly to the magnitude of the surface mass distribution σ. However, over horizontal distances greater than about 100 km on the earth's surface, the condition of isostasy is a good approximation and requires that the surface mass distribution of σ be zero.

If the surface mass distribution is zero the gravitational field and potential outside a closed surface can be uniquely related to a surface dipole distribution of mass. In this case (3) relates the local gravitational potential anomaly to the magnitude of the surface dipole layer σ. The measured distribution of surface potential anomalies can be directly used to obtain a surface distribution of the density dipole strength. This surface mass dipole distribution can be directly related to the density distribution in the crust and lithosphere, although there will also be other, deeper contributions to the external gravitational field and potential. We have shown that the local association of the potential anomaly with the dipole density distribution is a good approximation for the depths of compensation associated with the crust or lithosphere.

The gravitational potential anomaly is directly proportional to the geoid anomaly. The geoid anomaly is measured directly by radar altimetry from the GEOS-3 satellite. Haxby and Turcotte (1978) have shown that several measurements of geoid anomalies can be related to density variations in the crust and lithosphere using (3).

Acknowledgements. This research was supported by the National Aeronautics and Space Administration, NASA Grant NSG 5060. The authors would like to acknowledge stimulating discussions with A. Kuckes and W. Kaula. This is contribution 608 of the Department of Geological Sciences, Cornell University.

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