Linear and Nonlinear Interactions Between the Earth Tide and a Tectonically Stressed Earth

Christopher Beaumont
Oceanography Department, Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J1

Abstract. The earth tide provides a spatially and temporally predictable force that deforms the Earth and can be measured as changes in gravity, tilt, and strain at the Earth's surface. All things being equal, tidal constituent amplitudes and phases will not change with time. However, in the vicinity of earthquake focal regions conditions may not be equal. Crustal rocks stressed to more than ~0.6 of their failure strength exhibit material properties over and above that of linear elasticity. These effects, including dilatancy, are known from laboratory measurements but have not been proven *in situ*.

Interactions between the earth tide and crustal rocks that are under high tectonic stress are discussed in terms of simple phenomenological models. In particular, the difference between a nonlinear elastic model of dilatancy and a dilatancy model that exhibits hysteresis is noted. It is concluded that the small changes in stress produced by the earth tide act as a 'probe' of the properties of crustal rocks. Observations of earth tide tilts and strains in such high stress zones may, therefore, provide keys to the constitutive properties and the tectonic stress rate tensor of these zones.

Introduction

Research concerning earth tides has concentrated almost exclusively on the elastic behaviour of the Earth's crust and mantle, and the boundaries between elastic materials that give rise to the so called geologic, topographic and cavity effects. There are, however, zones of high tectonic stress within the Earth's crust that may have a more complex rheology which can be investigated with the earth tide. Small changes in stress associated with the tide act as a 'probe' both of the state of tectonic stress and of the constitutive properties of these zones. This concept, which has yet to be observed in *in situ* experiments, is based on results from laboratory measurements of the continuum properties of intact rock samples subject to deviatoric stress in excess of ~0.6 of their failure strength. These ideas are discussed in terms of simple phenomenological models. A rigorous treatment, including other possible rheologies, is not warranted until observations exhibiting anomalous tidal variations have been recorded.

Beaumont and Berger (1974) rather naively suggested that temporal variations in the Earth's admittance to the earth tide should accompany V_p/V_s seismic velocity anomalies, if the seismic velocity anomalies result from changes in the elastic properties of the crust. Their theoretical predictions suggested significant tidal admittance anomalies in the vicinity of 'elastic dilatant' crustal inclusions. These results have been criticized from the standpoint that the dilatant behaviour of rocks as observed in the laboratory (see, for example, Brace *et al*, 1966) is not merely a change in linear elastic properties. Consequently, an elastic inclusion model for tidal response represents an oversimplification. This paper provides an opportunity to extend Beaumont and Berger's results to more realistic rheologies.

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Tidal Interactions with a Nonlinearly Elastic Crust

In this paper the normal definition of dilatancy, a volumetric change induced by a deviatoric stress, is used. Infinitesimal linear elasticity theory is represented by the tensor equation

$$\varepsilon_{ij} = AI_1\,\delta_{ij} + G\sigma_{ij} \qquad i,j = 1,2,3\ ,$$

where ε_{ij} and σ_{ij} are the strain and stress tensors, δ_{ij} is the Kronecker delta, A and G are constants and I_1 is the first invariant of the stress tensor = σ_{ii} (summation convention over repeated indices implied). This equation is separable into isotropic and deviatoric parts,

$$\varepsilon_{ii} = 3AI_1 + G\sigma_{ii}$$

and

$$\varepsilon_{ij} = G\sigma_{ij}\ , \quad i \neq j\ ,$$

which demonstrates that deviatoric stress, $\sigma^D_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{ii}$, can induce only deviatoric strain and no volumetric change. Therefore, linearly elastic materials undergoing infinitesimal strain do not exhibit dilatancy.

Stuart and Dieterich (1974) considered a model of nonlinear elasticity complete to second order in stress (Reiner, 1945),

$$\varepsilon_{ij} = \phi_0\,\delta_{ij} + \phi_1\sigma_{ij} + \phi_2\sigma_{ij}\ , \qquad i,j = 1,2,3\ ,$$

where $\phi_k = \phi_k(I_1, I_2, I_3) \qquad k = 0,1,2\ ,$

and I_1, I_2 and I_3 are the scalar invariants of the stress tensor. Such a model requires seven instead of two elastic constants, that is α, B, C, H and M in addition to A and G.

$$\varepsilon_{ij} = (\alpha + AI_1 + BI_1^2 + CI_2)\,\delta_{ij} + (G + HI_1)\,\sigma_{ij} + M\sigma_{ik}\sigma_{kj}\ ,$$

$i,j,k = 1,2,3$.

These constants determine the quadratic behaviour of nonlinear elasticity. Isotropic and deviatoric stress-strain relationships are cross-coupled, therefore, deviatoric stress induces volumetric changes. In addition, the behaviour of such nonlinear elastic materials is anisotropic. This anisotropy, as emphasized by Stuart and Dieterich, is not intrinsic. It is a physical anisotropy that is caused by the stresses, but not the stress-strain relationship itself.

A possible nonlinear stress-strain relationship (figure 1) illustrates the interaction among a nonlinear elastic material, the tectonic stress, σ, and the earth tide stress, σ_E. This simplified approach ignores the fact that the σ versus ε or θ curves are functions of the stress invariants. No matter what the state of tectonic stress, strain in any direction will follow a curve of the type shown in figure 1, either concave or convex. The tidal stress (~0.01 bar), being much smaller than the range of tectonic stress (~5 kbar) or earthquake stress drops (~100 bar), will sample an essentially linear elastic behaviour proportional to the elastic tangent moduli of the nonlinear tectonic stress-strain curve. Therefore, the tidal admittance remains linearly elastic, but almost certainly anisotropic, and changes with increasing tectonic stress. Figure 1 suggests a decrease in tidal strain with increasing tectonic stress, but the exact form of the change will be more complex, some components of the strain tensor may increase while others decrease. Such changes can in theory be predicted in detail from the constitutive law and a knowledge of the tectonic and tidal stress tensors. At present, mere identification of temporal variations in the tidal admittance would be sufficient to demonstrate both a rheology more complex than linear elasticity and a changing tectonic stress. Heaton (1975) has outlined some of these ideas and has also pointed out that anisotropy will in general produce phase changes in the tidal admittance in addition to the amplitude variations.

Model of Dilatancy

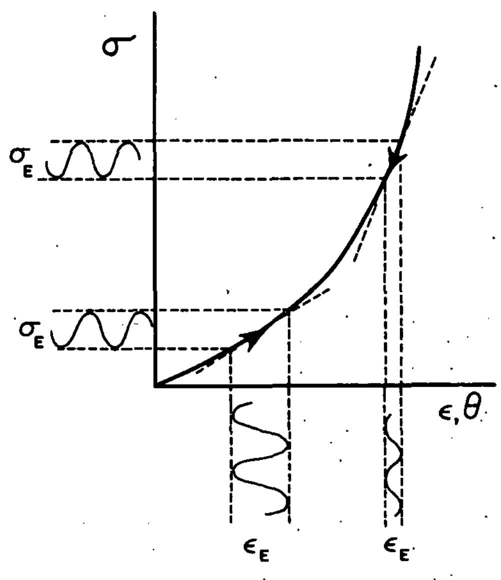

Fig. 1. A typical stress (σ) versus strain (ε) or dilatation (θ) graph for nonlinearly elastic crustal rocks. Superimposed tidal stress (σ_E) induces a tidal strain (ε_E) proportional to the tangent to the tectonic stress-strain curve.

Tidal Interactions with a Crust that Exhibits Stress Hysteresis

Intact rock samples, when subject to cyclical stress that approaches the failure strength of the rock, exhibit dilatancy in the manner predicted by the Stuart and Dieterich model. That is, the stress-strain relationship is no longer linear for stresses in excess of ~0.6 of the failure strength and deviatoric stress induces volumetric strain. However, such samples exhibit stress hysteresis in addition to a nonlinear behaviour. Strain is a double valued function of stress, the value depending on whether stress is increasing or decreasing. Results from Scholz and Kranz (1974) (figure 2) on intact Westerly granite illustrate this behaviour. Repeated stress cycles (1, 18, 19) reduce the minimum deviatoric stress required for the onset of dilatancy to ~300 bars, but in each stress cycle a broadly similar hysteresis loop is followed. A widely accepted explanation is that the energy loss, measured by the area of the hysteresis loop, is the work done against friction in opening and closing pre-existing microcracks within the rock sample (figure 2). An idealized representation of the behaviour of the θ (dilatational), ϕ and z components of strain in the cylindrical rock sample (figure 2, lower

Dilatancy of Intact Westerly Granite (Scholz and Kranz, 1974)

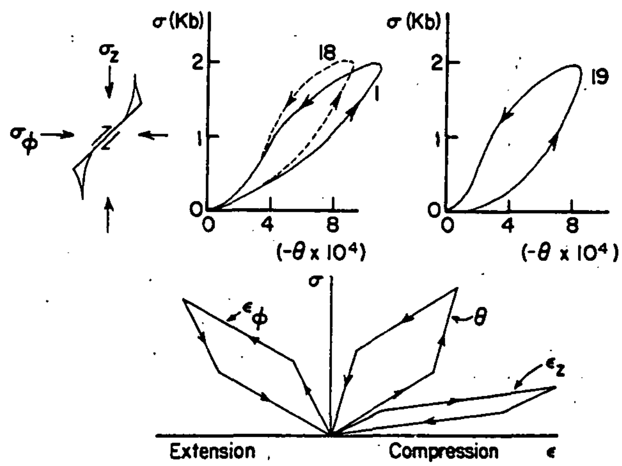

Fig. 2. Dilatancy of intact Westerly granite (from Scholz and Kranz, 1974). The panels (left to right, top to bottom) illustrate: 1) the opening of a microcrack under applied stress; 2) and 3) the changes in dilatation (θ) during stress cycles 1, 18 and 19; 4) idealized stress-strain relationships for the ε_ϕ, θ, and ε_z components of strain.

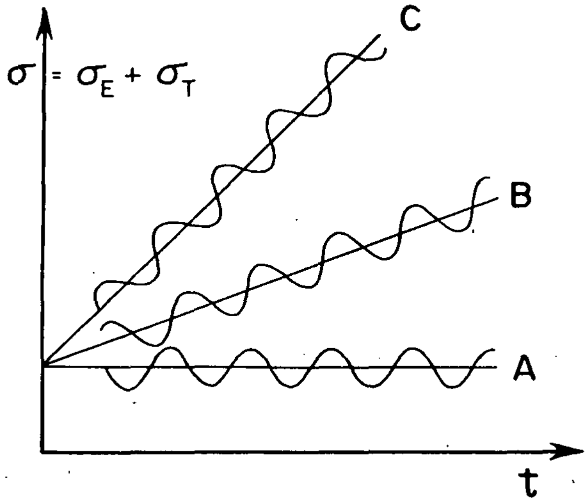

Fig. 3. The variation of superimposed tidal and tectonic stress (σ) with time (t). This simplified graph illustrates a linearly increasing tectonic stress and a tidal stress comprised of a single sinusoid. The significance of A, B and C is discussed in the text.

panel) demonstrates that all strain components are double valued. This model assumes that all cracks start to open and close at the same state of stress. It provides a first approximation to the true behaviour of the rock sample and can be represented mathematically in terms of an elastic-plastic material with internal back stresses. This description is discussed in detail in a later section.

The main concern is the interaction between the earth tide and crustal zones that exhibit such a dilatant behaviour with hysteresis. It will be seen that the interaction is different from that predicted for a nonlinearly elastic zone. Consider three simple cases (figure 3) in which the total stress is represented by a superposition of a linearly changing tectonic stress and a simple sinusoidally varying tidal stress. In A, the tectonic stress rate is either zero or very slowly varying ($d\sigma_T/dt \ll (d\sigma_E/dt)_{mean}$). In B, the tectonic stress rate is approximately equal to the mean tidal stress rate ($d\sigma_T/dt \sim (d\sigma_E/dt)_{mean}$). In C, the tectonic stress rate is sufficiently high that there is no decrease in total stress ($d\sigma_T/dt > (d\sigma_E/dt)_{max}$).

The interaction of these three possible stress-time regimes with the idealized hysteresis curve (figure 4) suggests that the character of the tidal response (strain or tilt) is peculiar to each of the three tectonic stress rate regimes. Consequently, observations of tidal strain can, in principle, be used to measure tectonic stress rates.

In figure 4A the tectonic stress rate is much less than the mean tidal stress rate. For states of total stress, σ, that are below that required for dilatancy (line 1-2) a superimposed tidal stress (σ_E) produces a normal amplitude elastic

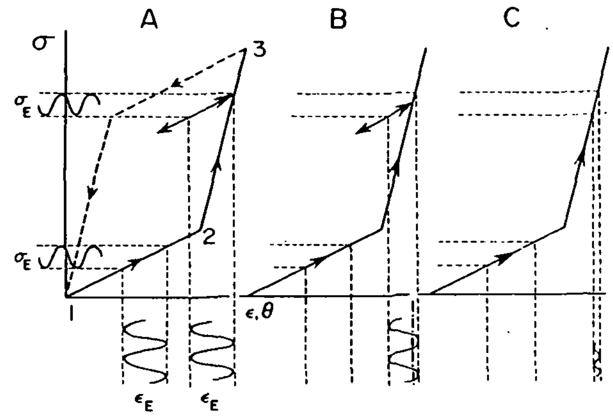

Fig. 4. Interaction between the earth tide (σ_E, ε_E) and an idealized stress hysteresis loop for the three tectonic stress rates, A, B, and C from figure 3.

tidal strain (ε_E). Moreover, the same response occurs even when the total stress is sufficiently high to produce a dilatant response (line 2-3). This result is a direct consequence of the general character of hysteresis curves. That is, any small stress decrease will be accompanied by an elastic strain recovery. The dilatant strain is not recovered until all the elastic strain has been recovered. Consequently, for small tectonic stress rates the crust retains its normal elastic admittance to the earth tide stress.

In figure 4C the tectonic stress rate is sufficiently high that there is never a decrease in total stress. In the pre-dilatancy region the response is identical to that of A. There will be no tidal anomaly. However, once the tectonic stress has moved into the dilatant region the strain is forced to change in proportion to the slope of the 2-3 line because there is no stress recovery. The tidal strain undergoes an anomalous decrease, or increase (see figure 2) but remains linearly related to the tidal potential.

The most interesting interaction is that shown in figure 4B. In the dilatant region stress reduction is accompanied by an elastic strain recovery proportional to the slope of line 1-2. The response remains elastic until the stress increases to the former stress maximum. Stress increases beyond the previous maximum induce strains proportional to the slope of line 2-3. The overall response (strain or tilt) is nonlinear. The magnitude of the nonlinearity depends on the relative slopes of lines 1-2 and 2-3, and on the relative size of the tectonic and mean tidal stress rates, the maximum nonlinearity occurring when these are equal. The main consequence of a nonlinear response will be the appearance of numerous additional lines in the tidal spectrum at sum and difference frequencies of the tidal constituents. Their detectability depends on the signal to noise ratio at the nonlinear interaction frequencies.

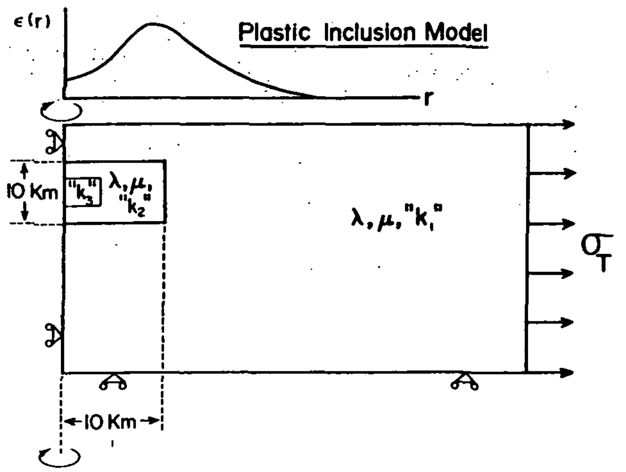

Fig. 5. Elastic-plastic model of a weak crustal inclusion. The model has uniform elastic properties (λ and μ) but the yield strength, characterized by k_1, k_2, and k_3, varies with position. Details of the precise yield criteria are discussed in the text.

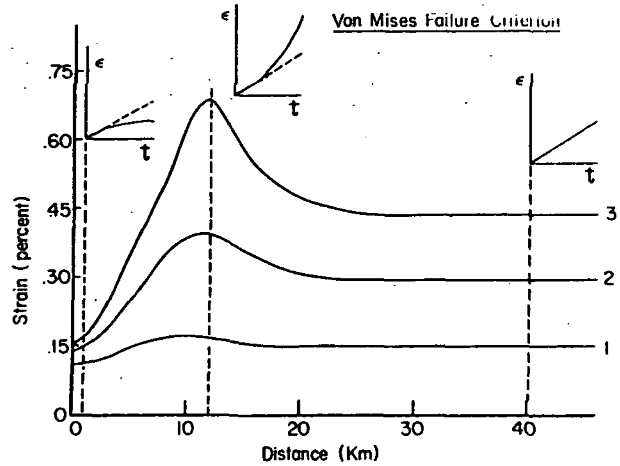

Fig. 6. Surface radial strain anomaly ($\varepsilon(r)$) as a function of three equal increases (1, 2, 3) of tectonic stress (σ_T) for the elastic-plastic model with inclusions that have a Von Mises yield criterion. See text for details.

Plastic Models of Crustal Inclusions that Exhibit Hysteresis

It is anticipated that only confined volumes (crustal inclusions) will experience high stress concentrations at any given time. Only these inclusions will have a rheology that deviates from the normal elastic response. Alternatively, inclusions that have repeatedly experienced failure will almost certianly appear 'weak' in the sense that they will exhibit a departure from linear elasticity at lower stresses than the surrounding intact crust. An inclusion model (figure 5) similar to that employed by Beaumont and Berger (1974) but with an elastic-plastic rheology is used to explore the latter alternative. The model, a part of an axisymmetric half space with a 10 km radius, 10 km deep disc shaped inclusion buried at a depth of 4 km, is intended to be very simple. The half space has uniform elastic properties (λ,μ) but the inclusion is characterized by yield strengths k_2 and k_3 that are less than that, k_1, for the surrounding crust. The result of particular interest is the character of the surficial strain, $\varepsilon(r)$, or tilt as a function of tectonic stress, σ_T. The results (figures 6 and 7) were obtained for two different yield criteria and associated plastic flow laws by finite element modelling (Bathe, Wilson and Iding, 1974, and Bathe, Ozdemir and Wilson, 1974).

The radial strain anomaly as a function of increasing stress (figure 6) is for a model with a Von Mises failure criterion (Prager and Hodge, 1951),

$$F(k,\sigma) = J_1 - k ,$$

where J_1 is the first invariant of the deviatoric stress tensor and $k = \sigma_y^2/3$, where σ_y is the yield stress in simple tension. Yielding occurs when $F \geq 0$. The curves 1, 2 and 3 illustrate the strain anomaly for three equal stress increments. The strain as a function of position is constant for stress states $F < 0$. However, once yielding has occurred (curve 1), the buried inclusion appears 'weaker' and a strain anomaly develops in response to increasing stress by virtue of the contrast in properties between the inclusion and the surrounding crust. The inset figures illustrate the time variation of surface radial strain at selected points under the assumption that stress increases linearly with time. These figures may be interpreted in a similar manner to figure 4 because tectonic stress is proportional to time. The transition from an elastic (dashed line) to an anomalous response is now smooth, unlike that of figure 4, because the surrounding elastic crust 'filters' the plastic response of the inclusion. Strain recovery (not shown) on stress reduction will follow a hysteresis curve very much like that observed by Scholz and Kranz (1974) (figure 2, smooth curves).

In fact, there is an almost perfect analogy between the behaviour of the elastic-plastic inclusion model and that of a rock sample in the laboratory. For the rock sample, yielding (slip on pre-existing micro-cracks) occurs over a range of stress because each micro-crack has its own yield strength. Similarly, the crust is inhomogeneous and includes many inclusions like that of figure 5, each of which has its own yield strength. The elastic-plastic inclusion model is equivalent to a rock sample with one or more micro-cracks concentrated in one region. The analogy between a rock sample and the crust may be made complete by choosing a plastic yield criterion which has the same form as that for slip across micro-cracks. A suitable criterion is the Mohr-Coulomb criterion, which when generalized to three dimensions becomes the Drucker-Prager yield criterion,

$$F(I_1,J_2,k') = \alpha I_1 + J_2^{1/2} - k'$$

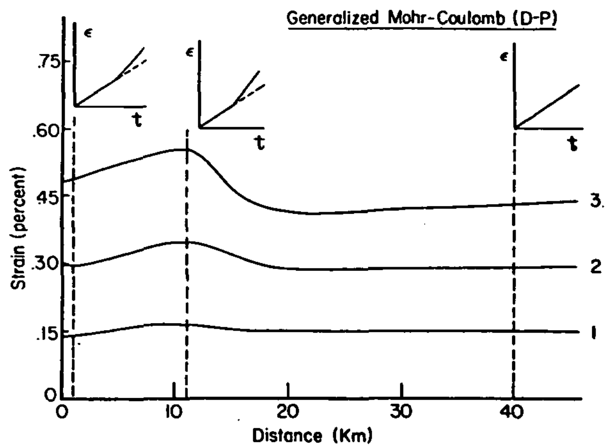

Fig. 7. Surface radial strain anomaly ($\varepsilon(r)$) as a function of three equal increases (1, 2, 3) of tectonic stress (σ_T) for the elastic-plastic model with inclusions that have a Drucker-Prager (generalized Mohr-Coulomb) yield criterion. See text for details.

(Drucker and Prager, 1952), where J_2 is the second invariant of the deviatoric stress tensor and α and k' are material properties related to the cohesion c and angle of friction ψ. $\alpha = 2\sin\psi/\sqrt{3}(3-\sin\psi)$, $k' = 6c\cos\psi/\sqrt{3}(3-\sin\psi)$. It is also interesting that the associated flow law is dilatant.

$$\dot{\theta}^P = 3\alpha\lambda \ ,$$

where $\dot{\theta}^P$ is the rate of plastic dilatation and λ is a constant proportional to the rate of plastic working, $\sigma_{ij}\dot{\varepsilon}^P_{ij}$, or defined in terms of the plastic strain rate and the yield function.

$$\dot{\varepsilon}^P_{ij} = \lambda\,\partial F/\partial\sigma_{ij}$$

(Drucker and Prager, 1952).

The results for the Drucker-Prager model (figure 7) suggest that the surface tectonic strain anomalies will not be as large as those for the Von Mises model. The difference is partly a result of the two yield crieteria, though values for the constants α, k' and k were chosen so that yielding occurs at the same tectonic stress for both models. A more important factor is the effect of dilatancy. The major difference between the strain for the two models at distances $\lesssim$10 km is due to strain induced by tectonic uplift of the zone over the dilatant inclusion. As the value of α is reduced toward zero, the results of the Drucker-Prager model trend smoothly to those of the Von Mises model.

When the tectonic stress is reduced, the strain behaviour of the inclusion models is very similar to that of rock samples. The plastic inclusions do not possess internal 'back stresses'; therefore, the plastic strain would be irrecoverable if the inclusions were not embedded in an elastic matrix. 'Back stresses' in the elastic matrix ensure that the plastic strain is recovered. That there is irrecoverable plastic work done during this process ensures that the stress-strain relation will exhibit hysteresis over a stress cycle. The situation in a stressed rock sample is very similar. The micro-cracks have no intrinsic 'back stresses.' It is the stress in the elastic matrix that ensures that dilatancy is recovered.

The only weakness of the plastic flow laws that have been used is that the rheology is elastic-perfectly plastic; that is, there is no strain hardening. Such a model suggests that once slip across a micro-crack has been initiated it will proceed at constant stress.

We are now in a position to predict the form of tidal interactions with the elastic-plastic inclusion models. The ε versus t inset graphs of figures 6 and 7 are the equivalents of the ε versus σ graphs of figure 4. The interpretation is exactly the same with the addition that the tidal anomalies will vary with position on the surface of the model. At large distances from the inclusion the response remains normally elastic; no anomalous tidal admittance is predicted. In the neighbourhood of the inclusion the character of the tidal admittance will depend on the relative tidal and tectonic stress rates.

Discussion and Conclusions

The intent of this paper has been a discussion of phenomenological models of linear and nonlinear variations in tidal admittance that result from stress dependent properties of crustal rocks. The proposed constitutive relations are those observed for intact laboratory samples at deviatoric stress levels in excess of ~0.6 of their failure strength. If crustal rocks exhibit the same properties in situ, it follows that nonlinear variations in tidal admittance of the kind predicted would indicate: 1) that tectonic stresses are sufficiently large to induce nonlinear behaviour, 2) that crustal dilatancy with hysteresis is occurring, and 3) that tectonic stress rates are comparable to the tidal stress rate. The absence of a nonlinear tidal admittance is not such a useful result for it merely indicates that one or more of the above conditions has not been met.

The most interesting condition concerns the need for comparable tectonic and tidal stress rates. This condition does not arise for constitutive models of the type proposed by Stuart and Dieterich (1974), nor is a nonlinear tidal admittance predicted. Unfortunately, tidal stress rates are not optimal. Mean tectonic stress rates are of the order 1-20 bars/year (10^{-1} - 2 MPa/year), if it is assumed that earthquake stress drops are a measure of the tectonic stress accumulation between repeat earthquakes in the same area. The assumptions involved in this estimate are discussed by Dieterich (1978). Mean tidal stress rates for mid-latitudes are larger, from 50-100 bars/year (5-10 MPa/year), where the mean tidal stress rate is taken to mean $|\dot{\sigma}_E|$. Consequently, nonlinear interactions are predicted only under what might be termed an accelerated tectonic stress rate. Earthquake precursors suggest that accelerated tectonic stress rates are probable before earthquakes.

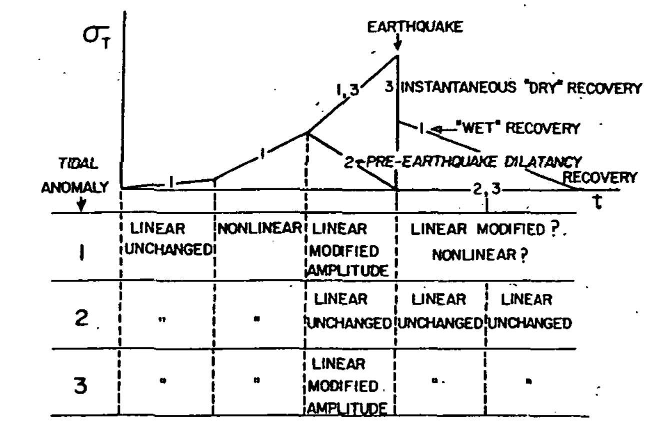

Fig. 8. Possible variations in tidal admittance during an earthquake cycle (σ_T v t) for three postulated earthquake models. 1 is the dilatancy-diffusion model in which post-earthquake dilatancy recovery is controlled by the diffusion of fluids. 2 is a model in which dilatancy is recovered before the earthquake occurs. 3 is a model that postulates instantaneous dilatancy recovery at the time of the earthquake. The details of the variation of tidal admittance will depend on *the relative tidal and tectonic stress rates*, and the tectonic stress required for dilatancy to occur.

The very appearance of a stress induced precursor and its disappearance before an earthquake indicates that there is stress redistribution during the period for which a precursor exists. If precursors are a manifestation of stress induced dilatancy that is recovered as a broad zone of cracks coalesce to form a fault zone, there must be significant stress redistribution during the process. The same conclusion is reached if the diffusion of fluids is also involved. Possible variations in tidal admittance for postulated pre-, co-, and post-seismic processes are summarized in figure 8.

A comment on tidal triggering of earthquakes is relevant at this point. Heaton (1975), among others, has noted that the probability that an earthquake will be tidally triggered is greatest when $|\dot{\sigma}_E| >> |\dot{\sigma}_T|$. This result assumes that the earthquake occurs through a Mohr-Coulomb type failure at that phase of the tide when shear stress on the fault plane is maximized and normal stress is minimized. This is exactly the condition for which a nonlinear tidal admittance is least likely. Conversely, tidal triggering of earthquakes is least likely when $|\dot{\sigma}_E| << |\dot{\sigma}_T|$, but this is the most favoured condition for an anomalous tidal admittance. The conclusion, which can be tested, is that earthquakes that appear to be tidally triggered are unlikely to have anomalous tidal precursors in the period immediately before the earthquake.

The overall conclusions may be summarized in the following manner. The Earth's admittance to the earth tide may be sensitive to tectonic stress if in situ crustal rocks exhibit the same stress dependent properties as those observed for intact laboratory samples. The earth tide tilt and strain in a region of high and variable tectonic stress will exhibit linear variations if the Earth is nonlinearly elastic. This result is similar to that predicted by Beaumont and Berger (1974) *with the additional effect that the tidal* admittance will be anisotropic. A more interesting result is predicted if crustal rocks exhibit stress hysteresis. At high tectonic stress the admittance of the Earth to the earth tide will be a function of the tectonic stress rate. In particular, a nonlinear admittance is predicted when the tidal and tectonic stress rates are approximately equal.

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