NASA Contractor Report 3138

Nonlinear Effects on Sound Propagation Through High Subsonic Mach Number Flows in Variable Area Ducts

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SUMMARY

A nonlinear theory for sound propagation in variable area
ducts carrying a nearly sonic flow is presented. Linear acoustic
theory is shown to be singular and the detailed nature of the
singularity is used to develop the correct nonlinear theory.
The theory is based on a quasi-one dimensional model. It is
derived by the method of matched asymptotic expansions.

In a nearly choked flow the theory indicates the following
processes to be acting: A transonic trapping of upstream
propagating sound causing an intensification of this sound in
the throat region of the duct; generation of superharmonics and
an acoustic streaming effect; development of shocks in the
acoustic quantities near the throat.

Several specific problems are solved analytically and
numerical parameter studies are carried out. Results indicate
that appreciable acoustic power is shifted to higher harmonics
as shocked conditions are approached. The effect of the throat
Mach number on the attenuation of upstream propagating sound
excited by a fixed source is also determined.

I. INTRODUCTION

The acoustic behavior of variable area ducts has recently been the subject of much attention in connection with sonic engine inlets. Observations of a correlation between axial Mach number and attenuation of sound radiated upstream from such inlets has led to a realization that nearly choked inlets can be effective in suppressing upstream propagating sound Ref. 1 - 3. However until recently, Ref. 4,5 , there has been no parallel theoretical study to explain the physical processes that are responsible for this attenuation. In Refs. 4,5 a new nonlinear theory describing sound propagation in ducts having a near sonic throat section has been derived and some numerical results given. The present report will give a detailed derivation of the nonlinear theory and present analytical and numerical studies to illustrate the predictions of the theory.

The theory provides a physical and quantitative understanding of the propagation process in the throat region. The nonlinear interaction between the sound field and the basic flow in the duct is shown to give rise to:

    (1) An intensification of the upstream propagating sound in the throat region due to a transonic type trapping;

    (2) The generation of higher harmonics of the fundamental (source) frequency;

    (3) An acoustic streaming effect;

    (4) The development of shock waves in the perturbation quantities in general.

A condition on the source strength and frequency, and the throat Mach number which determines whether a shock will occur has been derived and numerical results have been obtained which illustrate the development of such "acoustic" shocks. All numerical results presented in this report are for shock-free solutions. These results show that a large fraction of the acoustic power transmitted through the duct can be shifted into the superharmonics even in the absence of shock formation. The attenuation for upstream transmitted acoustic power as a function of the throat Mach number for a fixed source strength is also studied. The nonlinear theory predicts that the acoustic power should be of the order of ε^4 where ε is the deviation of the throat Mach number from unity while a naive linear theory predicts an ε^3 dependence.

The nonlinear theory is based on a quasi-one dimensional model which yields results of practical interest and provides basic understanding of the physical phenomena involved while remaining relatively simple to treat analytically and numerically. The theory is derived by observing that linearized acoustic theory fails to descirbe propagation of sound in ducts containing a near sonic throat section; the failure being manifested in singular behavior (infinite amplitudes) of the acoustic quantities in the throat region. A discussion of the failure of linear

theory is presented in Section IIB. A more complete treatment
is given in Ref. 6.   In Section IIIA the localized nature of
the singularity is exploited in deriving the correct nonlinear
acoustic theory by use of the method of matched asymptotic
expansions. The theory is applied to yield numerical results
for several cases of sound propagation from an acoustic source.
located at and upstream or downstream of a near sonic throat.
These results are discussed in Section IIIE.

The qualitative basis of the theory is relatively easily
understood in terms of the Riemann invariants of the acoustic
field. If $P(x,t)$ and $Q(x,t)$ represent the downstream and
upstream propagating acoustic waves, respectively, then the
results of the theory are as indicated in Figure 1. Nonlinear
transonic effects occur in the lowest order acoustic perturba-
tion quantities whenever the perturbation level in the throat
region is of the order of magnitude of the deviation of the
throat Mach number from unity. This can occur in two
ways: If the acoustic source is at the throat and has a
strength of $O(\epsilon)$, or if the acoustic source is outside the
throat and has a strength of $O(\epsilon^2)$ where ϵ is the Mach number
deviation. To fix ideas consider the latter case, illustrated
in Figure 1. Directly upstream of the source a continuous
reflection process occurs because of the variable area, so
that both P and Q waves are present at $O(\epsilon^2)$ and are described
by linear theory. However, the asymptotic inner solution shows
that the Q wave is strengthened to $O(\epsilon)$ by nonlinear interactions
with the steady flow which occur when $x = O(\epsilon)$. Since the inner
region is small compared to the scale of variations of the steady
flow, no reflection of the Q wave occurs at $O(\epsilon)$: The P wave
remains at $O(\epsilon^2)$, the relationship between them being ultimately
determined by an acoustic boundary condition applied at some
$x < 0$. Thus, as might be anticipated on physical grounds, the
downstream wave can be considered to propagate through the throat
essentially unaffected by the near-sonic nonlinearity, while the
upstream wave is intensified in the throat region due to a trans-
sonic trapping effect. The trapping of the upstream wave occurs
because its velocity relative to a fixed observer is practically
zero. It thus has a long residence time in the throat region
of the duct.

Other recent work on acoustic propagation in high subsonic
Mach number flow has been numerical,Ref. 7,8.   In both cases
a quasi-one dimensional model is treated.   In Ref. 7 an
expansion in harmonics has been used and the resulting coupled
equations integrated numerically. Some results are presented
only for unshocked solutions and for a special reflection
coefficient. They support the theoretical conclusions of the
present paper. In Ref. 8 the initial value problem is inte-
grated to a steady state and some numerical results are presented
for the acoustic velocity potential. There is no discussion of

the harmonic content of the solutions and the question of shocks is not considered.

Many earlier authors have studied linear quasi-one dimensional duct acoustics, but, in general, these studies have been concerned neither with the behavior nor with the validity of the linearized theory as the mean flow approaches sonic speed. A comprehensive review of earlier work can be found in Ref. 9. Tsien, and later Crocco Ref. 10, derived an exact solution which is discussed in Appendix D of the present report. Eisenberg and Kao Ref. 11 found another exact solution corresponding to a special, but practically unrealistic, area variation. Numerical evaluation of their analytical results clearly exhibits singular behavior as the axial Mach number approaches unity. Davis and Johnson Ref. 12 performed numerical computations of the linearized solution for certain acoustic boundary conditions, while King and Karamcheti Ref. 13 employed the method of characteristics to obtain similar numerical results.

The earlier numerical studies cited above yield results for sound propagation through a throat so long as the Mach number there is sufficiently far from unity. They include no attempt to resolve the specific behavior of the linear solution in the event that the throat flow approaches sonic speed, although they are certainly useful in indicating that difficulties exist with the theory in such a circumstance. Several numerical solutions illustrating the development of the acoustic singularity were presented in Ref. 6.

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II. SYMBOLS AND NOMENCLATURE

a   parameter measuring throat curvature

$A(x)$ area of duct, m^2

$A_{1n}^{\pm}$, $B_{1n}^{\pm}$ see Eq. (2.41)

$A_n^{\pm}$ magnitude of $A_{1n}^{\pm}$

A_1^*, A_2^*, B_1^*, B_2^* see Eq. (A7)

B   see Eq. (2.1)

$B(x)$, B_k see Eq. (2.34)

c_s stagnation sound speed, ms^{-1}

$c(x)$ steady flow sound speed

C_k see Eq. (2.33)

$c_{02}^{\pm}$, $c_{11}^{\pm}$, $c_{12}^{\pm}$ see Eq. (2.42)

$C_n^{\pm}(x)$, $D_n^{\pm}(x)$ inner Fourier coefficients

d, e   geometric parameters in area variation

$E_{1n}^{\pm}$, $E_{2n}^{\pm}$ see Eq. (D9)

$E_n^{\pm}$, $H_n^{\pm}$ real and imaginary parts of $I_n^{\pm}$

F_1, F_2, F_3, F_4 Hypergeometric functions

$g(x;\varepsilon)$ see Eq. (3.29)

$G(x;\varepsilon)$ see Eq. (2.5)

G_0 $G(x;\varepsilon = 0)$

$H(x)$ see Eq. (2.30)

$I_n^{\pm}$ see Eq. (3.32)

j, k, ℓ parameters in Crocco-Tsien solution

J   see Eq. (3.19)

K   parameter in Crocco-Tsien Mach number distribution

L   characteristic length associated with area variation, m

$M(x;\varepsilon)$ steady flow Mach number

$M_0(x)$ $M(x;0)$; leading order Mach number

$M_1(x)$, $M_2(x)$ see Eq. (2.12)

m_{10}, m_{20} local expansion Mach number terms

$m_0(X)$, $m_1(X)$ inner expansion Mach number terms

N_0, N   source amplitudes

$P(x,t)$ downstream perturbation Riemann invariant

P_1^i, P_2^i first and second order downstream perturbation
Riemann invariants (inner region)

$\bar{P}(x,t)$ total fluid pressure, $kg m^{-2}$

$Q(x,t)$ upstream perturbation Riemann invariant

Q_1^i, Q_2^i first and second order upstream perturbation
Riemann invariants (inner region)

$r(x,t,\varepsilon)$ outer, linearized density

$r_0(x,t)$, $r_1(x,t)$ first and second order terms of outer
density

$r_{0n}^{\pm}(x)$ complex amplitude of density

r_1^i, r_2^i first and second order terms of inner density

R_s stagnation density

s   see Eq. (2.11)

S_0 acoustic source strength

t   time, second

T   see Eqs. (2.26)

$\bar{u}(x,t)$ total fluid velocity, ms^{-1}

$U(x)$ basic steady flow, velocity, ms^{-1}

$u(x,t)$ perturbation velocity

$\hat{U}_0$, U   source amplitudes, ms^{-1}

V see Eqs. (2.26)

W see Eq. (2.29)

$W^{\pm}$, $W_1^{\pm}$, $W_2^{\pm}$ see Eq. (2.41)

x axial distance along duct

X stretched distance (inner variable)

x_e boundary location

z Crocco-Tsien axial distance variable

$\bar{\alpha}$ see Eq. (A7)

$\alpha(x)$ duct area ratio

β dimensionless Crocco-Tsien frequency

γ ratio of specific heats

Γ_n see Eq. (D.10)

Γ see Eq. (A7)

δ outer acoustic disturbance level

Δ Crocco-Tsien parameter

ε throat Mach number deviation from unity

$\theta_n^{\pm}$ phase angle inner solution

θ_{rn}, $\theta_{\mu n}$ phase angles outer solution

λ, λ_1, λ_2 eigenvalue of linearized acoustics local solution

$\mu(x,t;\varepsilon)$ outer linearized perturbation velocity

$\mu_0(x,t)$, $\mu_1(x,t)$ see Eq. (2.22)

$\mu_{0n}(x)$ complex velocity amplitude of n-th harmonic

$\mu_1^i(X,t)$, $\mu_2^i(X,t)$ see Eqs. (3.9)

ν see Eq. (3.32)

π see Eq. (3.15)

$\rho(x,t)$ total fluid density

$\rho(x,t)$ perturbation density

τ characteristic parameter inner region

$\phi(t)$ throat source function

ω radian source frequency (radians per sec.)

Ω dimensionless frequency

Special Notation

$|\;|$ indicates absolute value of a real quantity or magnitude of a complex quantity

subscript n   indicates a harmonic

superscript $\pm$ indicates $x > 0$, $x < 0$, respectively

superscript i   indicates inner variable

superscript 0   indicates outer variable

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III. FORMULATION AND OUTER EXPANSIONS

A. Perturbation Equations

Consider the propagation of sound in a variable area duct carrying a homentropic inviscid ideal gas flow. The acoustic wave length is assumed sufficiently large and the area variation sufficiently slow, that the field can be described by the equations of quasi-one dimensional gas dynamics (Ref. 14)

$$c_s \bar{\rho}_t + \bar{u}\bar{\rho}_x + \bar{\rho}\bar{u}_x + \bar{\rho}\bar{u}(\frac{A'}{A}) = 0 \quad ,$$

$$c_s \bar{u}_t + \bar{u}\bar{u}_x + \frac{1}{\bar{\rho}} \bar{p}_x = 0 \quad , \tag{2.1}$$

$$\frac{\bar{p}}{\bar{\rho}^\gamma} = \text{constant} = B \quad .$$

In equations (2.1) $\bar{p}$, $\bar{\rho}$, and $\bar{u}$ are the total fluid pressure, density, and axial velocity, and $A(x)$ is the duct cross sectional area. The dimensionless independent variables x and t are measured in units of L and L/c_s respectively, where L is a characteristic length associated with the area variation and c_s is the stagnation value of sound speed in the gas. The geometry of the problem is as indicated in Figure 2 where the origin of x corresponds to a throat: $A'(0) = 0$.

If the velocity and density in the basic steady flow in the duct are denoted by $U(x)$ and $R(x)$ respectively, then from (2.1),

$$UR' + RU' + RU(\frac{A'}{A}) = 0 \quad ,$$

$$\tag{2.2}$$

$$UU' + \gamma BR^{\gamma-2}R' = 0 \quad ,$$

where the energy relation in Eqs. (2.1) has been used to eliminate the pressure from the system. In order to study solutions to the system (2.1) which are small perturbations about the steady values U and R, it is convenient at the outset to define dimensionless variables $u(x,t)$ and $\rho(x,t)$ by

$$\bar{u}(x,t) = U(x)[1+u(x,t)] \quad , \quad \bar{\rho}(x,t) = R(x)[1+\rho(x,t)] \quad , \tag{2.3}$$

where it is assumed that $U(x) \neq 0$. It should be noted that the usual acoustic velocity and density are given by $U(x)u(x,t)$ and $R(x)\rho(x,t)$, respectively.

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Substituting Eq. (2.3) into Eq. (2.1) and employing the steady relations (2.2) yields the system of equations on u and ρ in the form

$$G^{1/2}\rho_t + M[(1+u)\rho_x + (1+\rho)u_x] = 0 \quad ,$$

$$G^{1/2}u_t + M(1+u)u_x + \frac{1}{M}(1+\rho)^{\gamma-2}\rho_x \qquad (2.4)$$

$$+ \frac{M'}{G}[(1+u)^2 - (1+\rho)^{\gamma-1}] = 0 \quad .$$

In Eqs. (2.4), $M(x)$ is the flow Mach number $U(x)/c(x)$, $c(x)$ is the speed of sound in the steady flow ($c^2 = \gamma BR^{\gamma-1}$), and

$$G(x) = (\frac{c_s}{c})^2 = 1 + \frac{(\gamma-1)M^2}{2} \quad , \qquad (2.5)$$

the latter expression following from the Bernoulli relation implied by the second of Eqs. (2.2).

For the purposes of physical interpretation and ease of asymptotic matching, the system (2.4) is recast in Riemann invariant form (Ref. 14). The upstream and downstream propagating perturbation Riemann invariats are defined by

$$P(x,t) = Mu + \frac{2}{\gamma-1}\left[(1+\rho)^{\frac{\gamma-1}{2}} - 1\right] \quad ,$$

$$Q(x,t) = Mu - \frac{2}{\gamma-1}\left[(1+\rho)^{\frac{\gamma-1}{2}} - 1\right] \quad . \qquad (2.6)$$

If the first equation in system (2.4) is multiplied by $(1+\rho)^{\frac{\gamma-3}{2}}$ and the second by $M(x)$, then adding and subtracting the resulting two equations and using Eq. (2.6) and analogous expressions for the partial derivatives of P and Q yields

$$P_t + \frac{1}{G^{1/2}}\left[M(1+u) + (1+\rho)^{\frac{\gamma-1}{2}}\right]P_x + \frac{MM'}{G^{3/2}}[(1+u)^2 - (1+\rho)^{\gamma-1}]$$

$$- \frac{M'}{G^{1/2}}u\left[M(1+u) + (1+\rho)^{\frac{\gamma-1}{2}}\right] = 0 \quad , \qquad (2.7)$$

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$$Q_t + \frac{1}{G^{1/2}} \left[M(1+u) - (1+\rho)^{\frac{\gamma-1}{2}} \right] Q_x + \frac{MM'}{G^{3/2}} \left[(1+u)^2 - (1+\rho)^{\gamma-1} \right]$$

$$- \frac{M'}{G^{1/2}} u \left[M(1+u) - (1+\rho)^{\frac{\gamma-1}{2}} \right] = 0 \quad . \quad (2.7)$$

To complete the elimination of u and ρ the relations

$$(1+u) = 1 + \frac{P+Q}{2M} \quad ,$$

$$(1+\rho)^{\frac{\gamma-1}{2}} = 1 + \frac{\gamma-1}{4} (P-Q) \quad ,$$

are substituted in Eqs. (2.7) to give

$$P_t + \frac{1}{G^{1/2}} \left[M + 1 + \frac{\gamma+1}{4} P + \frac{3-\gamma}{4} Q \right] P_x$$

$$+ \frac{MM'}{G^{3/2}} \left[\frac{P+Q}{M} + \frac{(P+Q)^2}{4M^2} - \frac{(\gamma-1)}{2} (P-Q) - \frac{(\gamma-1)^2}{16} (P-Q)^2 \right]$$

$$- \frac{M'}{G^{1/2}} \left(\frac{P+Q}{2M} \right) \left[M + 1 + \frac{\gamma+1}{4} P + \frac{3-\gamma}{4} Q \right] = 0 \quad , \quad (2.8)$$

$$Q_t + \frac{1}{G^{1/2}} \left[M - 1 + \frac{3-\gamma}{4} P + \frac{\gamma+1}{4} Q \right] Q_x$$

$$+ \frac{MM'}{G^{3/2}} \left[\frac{P+Q}{M} + \frac{(P+Q)^2}{4M^2} - \frac{(\gamma-1)}{2} (P-Q) - \frac{(\gamma-1)^2}{16} (P-Q)^2 \right]$$

$$- \frac{M'}{G^{1/2}} \left(\frac{P+Q}{2M} \right) \left[M - 1 + \frac{3-\gamma}{4} P + \frac{\gamma+1}{4} Q \right] = 0 \quad .$$

It is assumed that the basic steady flow quantities U and R are known from solving the system (2.2) subject to suitable boundary conditions applied far upstream and downstream of the throat of the duct, x = 0. Then Eqs. (2.4) or equivalently (2.8) are a quasi-linear system of partial differential equations with strong spatial dependence in the coefficients (due to the steady flow Mach number terms). These equations describe perturbation solutions of any size about the known steady flow.

Since propagation of sound in near sonic flows is the problem of interest, it will be further assumed that the boundary conditions on the basic flow yield a throat Mach number which is near unity. Hence, a parameter $\varepsilon = 1 - M(0)$ is introduced into the analysis, and the Mach number M is regarded as a function of ε as well as of x. In the analysis to follow, ε will be assumed to be positive, i.e., the basic flow is subsonic throughout the duct. The coefficients in the system (2.4) or (2.8) then all become functions of ε, and the nonlinear theory will result from considering these perturbation equations for small ε. In order to analyze these equations the detailed dependence of the basic steady flow on the parameter ε must be found. This is discussed in the next section.

B. Basic Steady Flow

The elementary equations of quasi-one dimensional steady flow (2.2) are discussed in detail in numerous texts; for example, a particulary comprehensive treatment has been given by Crocco (Ref. 15). It is straightforward to express any of the fluid quantities in terms of the duct area A(x) or, equivalently, in terms of the Mach number $M(x;\varepsilon)$. This was first analyzed in Ref. 6. It is the purpose of the present section to determine the behavior of M explicitly as a function of x and ε.

To begin consider the well-known relation implied by Eq. (2.2),

$$M' = \frac{-MGA'}{(1 - M^2)A} \quad , \qquad (2.9)$$

which becomes, after integration,

$$\alpha^s M^s(x)[1 + \frac{(\gamma - 1)M^2(0)}{2}] = M^s(0)[1 + \frac{(\gamma - 1)M^2(x)}{2}], \qquad (2.10)$$

where in Eq. (2.10) the definitions

$$\frac{A(x)}{A(0)} = \alpha(x) \qquad ,$$

and (2.11)

$$s = \frac{2(\gamma - 1)}{(\gamma + 1)}$$

have been used. Equation (2.10) is an implicit solution of the differential equation (2.9) and hence an algebraic equation for $M(x)$. Figure 3 shows sketches of typical integral curves of Eq. (2.9) for the type of area variation to be used here. Curves such as AB in the figure, for which M remains less than unity for all x are of interest. Since $1 - M(0) = \varepsilon$ is assumed small, it is natural to seek an expansion of $M(x;\varepsilon)$ in the form

$$M(x;\varepsilon) = M_0(x) + \varepsilon M_1(x) + \varepsilon^2 M_2(x) + \ldots \qquad , \quad (2.12)$$

where $M_0(0) = 1$. Substituting Eq. (2.12) into Eq. (2.10) and equating like powers of ε one finds that $M_0(0)$ must satisfy

$$\frac{(\gamma + 1)\alpha^s M_0^s}{2} = 1 + \frac{(\gamma - 1)M_0^2}{2} \qquad , \quad (2.13)$$

while $M_1(x) = 0$, and

$$M_2(x) = \frac{-2M_0[1 + \frac{(\gamma - 1)M_0^2}{2}]}{(\gamma + 1)(1 - M_0^2)} \qquad (2.14)$$

Obviously the expansion (2.12) is not uniformly valid near x = 0: The third term is of order ε whenever $1 - M_0(x)$ is as small as ε, or equivalently $M_2(x)$ becomes infinite as $x \to 0$.

It remains to find $M_0(x) = M(x;0)$ in terms of x: i.e., to solve Eq. (2.13). It is assumed for simplicity that $\alpha(x)$ is an analytic function near x = 0 with $\alpha'(0) = 0$, although the analysis to be carried out is easily extended to cover cases where α may have jump discontinuities in its higher derivatives at x = 0. Thus one may write $\alpha(x)$ in the form

$$\alpha(x) = 1 + ax^2 + bx^3 + dx^4 + ex^5 + \ldots \qquad , \quad (2.15)$$

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The nonlinear acoustic theory to be studied below is derived
for ducts with area variations satisfying the condition
$a = A''(0)/2A(0) > 0$. This is not a significant restriction
since if $a < 0$ the duct cannot have a throat and the case
$a = 0$ is obviously special for an actual duct. Expanding
$M_0(x)$ in a power series of the form

$$M_0(x) = 1 + m_{10}x + m_{20}x^2 + \ldots , \qquad (2.16)$$

substituting Eqs. (2.15) and (2.16) into Eq. (2.13) and
equating to zero corresponding powers of x, yields for the
branch $M_0(x) \leq 1$,

$$M_0(x) = 1 - [\frac{(\gamma + 1)}{2}]^{1/2} |x| + m_{20}x^2 . \qquad (2.17)$$

Hence, the leading term of $M(x;\varepsilon)$ behaves as a piecewise
linear function of x near the throat so long as $a \neq 0$.
The actual value of m_{20} in Eq. (2.17) is

$$m_{20} = \pm [\frac{(\gamma + 1)}{2}]^{1/2} \frac{b}{2} + \frac{(5\gamma - 3)a}{12} ,$$

the $\pm$ sign holding for $x \lessgtr 0$ respectively. All the basic
steady flow quantities can be expressed in terms of the Mach
number

$$M(x;\varepsilon) = 1 - [\frac{(\gamma+1)a}{2}]^{1/2}|x| + m_{20}x^2 + \ldots + \varepsilon^2 M_2(x) + \ldots , \quad (2.18)$$

where m_{20}, $M_2(x)$, etc., depend on $M_0(x)$ and $\alpha(x)$. This expansion
is to be considered asymptotically valid as $\varepsilon \to 0$ for fixed x.
It is nonuniform near $x = 0$; the appropriate Mach number dis-
tribution near $x = 0$ will be given in Section III-A.

C. Singularity in Linear Acoustics

    In this section linearized acoustic theory shall be
analyzed. In particular the breakdown of the theory for pro-
pagation in high subsonic flow will be demonstrated and the
detailed nature of the singularity in acoustic quantities
shall be determined. These results are essential for the
derivation of the correct nonlinear theory.
    If it is assumed that the perturbation quantities u and
ρ are small then the system (2.4) can be linearized. Let δ
be a small parameter measuring the order of magnitude of the

perturbations. The physical interpretation of δ in terms of an acoustic distrubance level will be discussed in Section III-C. Expanding u and ρ in powers of δ as

$$u(x,t;\varepsilon,\delta) = \delta\mu(x,t;\varepsilon) + O(\delta^2) + \ldots \quad ,$$

$$\rho(x,t;\varepsilon,\delta) = \delta r(x,t;\varepsilon) + O(\delta^2) + \ldots \quad , \tag{2.19}$$

substituting Eqs. (2.19) into Eqs. (2.4) and neglecting all but first order terms in δ yields the linearized acoustic equations

$$G^{1/2} r_t + M(r_x + \mu_x) = 0 \quad ,$$

$$G^{1/2}\mu_t + M\mu_x + \frac{1}{M} r_x + \frac{M'}{G} [2\mu - (\gamma-1)r] = 0 \quad . \tag{2.20}$$

Similarly, utilizing Eqs. (2.19) in Eqs. (2.6), the Riemann invariants are written in terms of the first-order perturbation quantities as

$$P(x,t;\varepsilon,\delta) = \delta(M\mu + r) + O(\delta^2) \quad ,$$

$$Q(x,t;\varepsilon,\delta) = \delta(M\mu - r) + O(\delta^2) \quad . \tag{2.21}$$

The linearized version of the system (2.8) is simple to derive, but it will not be necessary to use it in the analysis which follows.

Equations (2.20), subject to appropriate boundary conditions, generally must be solved numerically because of their variable coefficients. It is necessary for present purposes to analyze the behavior of solutions to Eqs. (2.20) in the vicinity of the throat of the duct when the throat Mach number M(0) is close to unity. It is well known that the system (2.20) is singular at any point where M(x) = 1. This can be seen most simply by subtracting the two equations; the resulting equation has no μ_x term, and the coefficient r_x becomes $(M^2 - 1)/M$, which vanishes as $M \to 1$. This can only occur at x = 0 for the duct of Figure 2. The singularity at x = 0 implies that, in general, the acoustic quantities r and μ will be singular when the flow is sonic there. Thus, as will be seen in what follows, r and μ generally become arbitrarily large near x = 0 as M(0) approaches unity, thereby violating the assumptions made in deriving Eqs. (2.20)

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that μ, μ_x, r and r_x all remain bounded.

As a result of the singular behavior of the system (2.20) for high subsonic Mach numbers in the throat region, linearized acoustic theory fails for any δ, no matter how small, if the throat Mach number is sufficiently close to unity, i.e., for ε small enough. In order to describe sound propagation in the duct in this event, one must reformulate the perturbation scheme to take into account non-linear terms in the system (2.4) which were neglected in Eqs. (2.20). However, in order to proceed in this direction it is necessary to know precisely the nature of the singular behavior of the solutions μ and r to Eqs. (2.20). This behavior has been recognized, but never resolved, in previous treatments of the system (2.20) Ref. 9.

Since the steady flow depends on the parameter $\varepsilon = 1 - M(0)$, the coefficients in the acoustic equations and hence the acoustic quantities μ and r are functions of ε. For $\varepsilon << 1$ we look for solutions of the acoustic equations in the form

$$r = r_0(x,t) + \varepsilon r_1(x,t) + \ldots \quad ,$$

$$\mu = \mu_0(x,t) + \varepsilon \mu_1(x,t) + \ldots \quad . \tag{2.22}$$

Inserting Eq. (2.22) into Eq. (2.20) and using expansion (2.12) for the coefficients, one obtains, after neglecting higher order terms in ε

$$G_0^{1/2} r_{0_t} + M_0 (r_{0_x} + \mu_{0_x}) = 0 \quad ,$$

$$G_0^{1/2} \mu_{0_t} + M_0 \mu_{0_x} + \frac{1}{M_0} r_{0_x} + \frac{M_0'}{G_0} (2\mu_0 - (\gamma-1) r_0) = 0 \quad , \tag{2.23}$$

where G_0 is defined by Eq. (2.5) with $M = M_0$. $M_0(x)$ is the leading order term corresponding to any Mach number distribution which yields a sonic velocity at the throat and a subsonic velocity throughout the remainder of the duct. These equations are singular at $x = 0$ since $M_0(0) = 1$. Analytical solutions of the system for arbitrary $M_0(x)$ and arbitrary time dependence are not readily found. However, for harmonic time dependence the system can be reduced to a system of ordinary differential equations with a singular point at $x = 0$ and no other singular points within the duct. The nature of this singular point determines the singular behavior in the time

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harmonic acoustic quantities. Explicit analytical results concerning the exact nature of the singular point and the dependence on $M_0(x)$ are now derived by use of series solution methods for linear ordinary differential equations.

Thus solutions of Eqs. (2.23) in the form

$$r_0(x,t) = r_{0n}(x)e^{in\Omega t} \quad ,$$

$$\mu_0(x,t) = \mu_{0n}(x)e^{in\Omega t} \quad , \qquad (2.24)$$

are sought, where $\Omega = \omega L/c_s$ is the dimensionless fundamental frequency and ω is the radian frequency (fundamental) of the harmonic solution. The parameter n is introduced since superharmonics of the fundamental will be shown to occur due to nonlinear effects. Substitution of Eq. (2.24) into Eqs. (2.23) and rewriting the resulting system in standard form yields

$$r'_{0n} = \frac{M_0}{1 - M_0^2}[Tr_{0n} - S\mu_{0n}] \quad ,$$

$$\qquad (2.25)$$

$$\mu'_{0n} = \frac{M_0}{1 - M_0^2}[-Vr_{0n} + S\mu_{0n}] \quad ,$$

where

$$S(x) = in\Omega G_0^{1/2} + \frac{2M_0'}{G_0} \quad ,$$

$$T(x) = in\Omega G_0^{1/2} + \frac{(\gamma - 1)M_0'}{G_0} \quad , \qquad (2.26)$$

$$V(x) = \frac{in\Omega G_0^{1/2}}{M_0^2} + \frac{(\gamma - 1)M_0'}{G_0} \quad .$$

As has been seen in Section II-B the behavior of $1 - M_0(x)$ near $x = 0$, and hence the nature of the singular point at $x = 0$, depends upon the corresponding behavior near $x = 0$ of the area $A(x)$. The method of Frobenius (Ref. 16) will now be used to analyze the behavior of the general solution

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to Eqs. (2.25) near x = 0 for any area variation with A"(0) ≠ 0. The case of A"(0) = 0 is discussed in detail in Ref. 6.

Since A"(0) ≠ 0 the corresponding Mach number distribution is given locally by Eq. (2.18). The Mach number is a piecewise linear function of x; thus one can analyze the system separately for x < 0 and x > 0. General solutions valid for x < 0 and x > 0 can be constructed by analytically extending the coefficients of the system to x > 0 and x < 0, respectively. This is done by extending $M_0(x)$ as follows: To extend to x > 0 take

$$M_0(x) = 1 + [\frac{(\gamma + 1) a}{2}]^{1/2} x + \ldots \quad , \quad (2.27)$$

and to extend to x < 0 take

$$M_0(x) = 1 - [\frac{(\gamma + 1) a}{2}]^{1/2} x + \ldots \quad . \quad (2.28)$$

Consider the system (2.25) with $M_0(x)$ given by either of Eqs. (2.27) or (2.28). In each case, S, T, and V are extended across x = 0 as analytic functions, and the only singularity is easily seen to be a simple pole by using Eq. (2.27) or (2.28). Hence both of the extended systems have a singular point of the first kind at x = 0, and solutions can be found by the method of Frobenius.

One accordingly rewrites the system (2.25), using matrix notation, as

$$W' = \frac{M_0}{1 - M_0^2} \begin{bmatrix} T & -S \\ -V & S \end{bmatrix} W \quad , \qquad W = \begin{bmatrix} r_{0n} \\ \mu_{0n} \end{bmatrix} \quad . \quad (2.29)$$

As is customary in dealing with such singular points, one exhibits explicitly the simple pole by writing

$$1 - M_0^2(x) = xH(x) \quad , \quad (2.30)$$

where H(x) is an analytic function of x. Then using Eq. (2.30) and defining a coefficient matrix by

$$B(x) = \frac{M_0}{H} \begin{bmatrix} T & -S \\ -V & S \end{bmatrix} \quad , \quad (2.31)$$

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the first of Eqs. (2.29) is rewritten as

$$W' = \frac{1}{x} BW \quad .$$ (2.32)

Next one looks for a solution of the system (2.32) in the Frobenius form

$$W = x^\lambda \sum_{k=0}^{\infty} C_k x^k \quad ,$$ (2.33)

where λ is an unknown constant and the C_k are unknown 2×2 constant matrices with $C_0 \neq 0$. Since $B(x)$ is an analytic function of x, it can be written as

$$B(x) = \sum_{k=0}^{\infty} B_k x^k \quad ,$$ (2.34)

where the constant matrices B_k can be found by expanding each of the elements of $B(x)$ given in Eq. (2.31) in power series. Thus, for example, upon noting that $V(0) = T(0)$,

$$B_0 = \frac{M_0(0)}{H(0)} \begin{bmatrix} T(0) & -S(0) \\ -T(0) & S(0) \end{bmatrix} \quad .$$ (2.35)

Substituting expressions (2.33) and (2.34) into Eqs. (2.32) and equating corresponding powers of x yields

$$(B_0 - \lambda I) C_0 = 0$$ (2.36)

and the recurrence relation

$$(B_0 - (\lambda+k) I) C_k = - \sum_{j=0}^{k-1} B_{k-j} C_j \quad .$$ (2.37)

Since $C_0 \neq 0$, Eq. (2.36) implies that λ must be an eigenvalue of the matrix B_0. By using the definitions of T and S from Eq. (2.26) and the fact that

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$$\lim_{x \to 0} H(x) = \lim_{x \to 0} \left[\frac{1 - M_0^2(x)}{x}\right] = -2M_0'(0)$$

the eigenvalues of B_0 are easily found to be

$$\lambda_1 = 0 \quad ,$$

$$\lambda_2 = -1 - \frac{i\Omega\left[\frac{(\gamma + 1)}{2}\right]^{1/2}}{M_0'(0)} \qquad (2.38)$$

Since λ_1 and λ_2 are distinct and do not differ by an integer, then corresponding to each λ the eigenvector C_0 can be determined from Eq. (2.36) and the other C_k follow from the recurrence relation (2.37). In addition, standard theorems (Ref. 16) insure that the resulting infinite series converge and are actually solutions of the system (2.23). Thus, two linearly independent solutions, and hence a general solution, can be found for the system.

To determine the form of the general solution for $x < 0$ it is assumed that $M_0(x)$ is given by the extension (2.27). Then $M_0'(0) = [(\gamma+1)a/2]^{1/2}$, and two linearly independent solutions of the system are the analytic solution $W_1^-(x)$ corresponding to $\lambda_1 = 0$, and, corresponding to λ_2,

$$W_2^-(x) = x^{-(1+i n\Omega/a^{1/2})} \hat{W}_2^-(x) \quad . \qquad (2.39)$$

Here the matrix $\hat{W}_2^-(x)$ is analytic at $x = 0$ and $\hat{W}_2^-(0) \neq 0$. Similarly for $x > 0$ the extension given by Eq. (2.28) is used to obtain two linearly independent solutions: Namely, $W_1^+(x)$, which is analytic, and

$$W_2^+(x) = x^{-(1-i n\Omega/a^{1/2})} \hat{W}_2^+(x) \quad . \qquad (2.40)$$

Thus the general solution to system (2.32) can be written as the linear combination

$$W^{\pm} = \begin{bmatrix} r_{0n} \\ \mu_{0n} \end{bmatrix} = \frac{A_{1n}^{\pm}}{|x|} \hat{W}_2^{\pm}(x) e^{\left[\frac{\pm i n \ln |x|}{a^{1/2}}\right]} + B_{1n}^{\pm} W_1^{\pm}(x) \quad , \quad (2.41)$$

where $A_{1n}^{\pm}$ and $B_{1n}^{\pm}$ are arbitrary complex constants, and the + and - signs are taken for $x > 0$ and $x < 0$, respectively. In order to satisfy general acoustic boundary conditions $A_{1n}^{\pm}$ will not vanish, and thus the amplitude of the acoustic quantities will grow like x^{-1} as $x \to 0$. In addition, their phases are logarithmically infinite at $x = 0$. One can observe that, even if $A_{1n}^{\pm} = 0$ jump discontinuities in the derivatives of r_{0n} and μ_{0n} at $x = 0$ will exist in general because of the jump discontinuity in $M_0'(x)$.

In the following development, it will be required to have an explicit representation of two terms of the power series for $W_2^{\pm}$ and one term of the series for $W_1^{\pm}$. Thus

$$W_2^{\pm} = \sum_{k=0}^{\infty} \hat{C}_k^{\pm} |x|^k = \begin{bmatrix} 1 \\ c_{02}^{\pm} \end{bmatrix} + \begin{bmatrix} c_{11}^{\pm} \\ c_{12}^{\pm} \end{bmatrix} |x| + \ldots \quad ,$$

$$(2.42)$$

$$W_1^{\pm} = \sum_{k=0}^{\infty} \hat{D}_k^{\pm} |x|^k = \begin{bmatrix} 1 \\ d_{02}^{\pm} \end{bmatrix} + \ldots$$

where unified expressions for $W_1^{\pm}$ and $W_2^{\pm}$ have been introduced by use of the absolute value. The coefficient matrices in these expressions can be determined by solving Eqs. (2.36) and (2.37) for the $x < 0$ case and the analogous equations for $x > 0$. Equivalently both cases can be treated simultaneously by solving the matrix equations

$$(B_0 - \lambda I)\hat{C}_0^{\pm} = 0 \quad ,$$

$$[B_0 - (\lambda+1)I]\hat{C}_1^{\pm} = -B_1\hat{C}_0^{\pm} \quad , \tag{2.43}$$

$$B_0\hat{D}_0^{\pm} = 0 \quad .$$

In Eqs. (2.43), $\lambda = -1 - (in\,\Omega\,\text{sgn}\,x)/a^{1/2}$, $B_0 = B(0)$, and $B_1 = B'(0)$ where $B(x)$ is given in Eq. (2.31).

It is a matter of straightforward but lengthy algebra to compute the coefficients B_0 and B_1 and to solve the algebraic Eqs. (2.43) for the three required matrices. An outline of this computation is provided in Appendix A. The results of interest come out to be:

$$c_{02}^{\pm} = -1 \quad ,$$

$$c_{12}^{\pm} + c_{11}^{\pm} = -\left[\frac{(\gamma+1)a}{2}\right]^{1/2} \text{sgn}\,x \quad ,$$

$$\tag{2.44}$$

$$c_{12}^{\pm} - c_{11}^{\pm} = \left(1 + \frac{in\Omega}{a^{1/2}}\text{sgn}\,x\right)\left[\frac{b}{a} - \frac{(3-\gamma)}{6}\left(\frac{2a}{\gamma+1}\right)^{1/2}\text{sgn}\,x\right] \quad ,$$

$$d_{02}^{\pm} = \frac{in\Omega(\gamma+1) - 2(\gamma-1)a^{1/2}\text{sgn}\,x}{in\Omega(\gamma+1) - 4a^{1/2}\text{sgn}\,x} \quad ,$$

in which a and b are the coefficients in the expansion (2.15) for the duct area ratio.

Using the results of Eqs. (2.41), (2.22) and (2.24) the local form of the acoustic solution for any duct described by Eq. (2.15) can be written as

$$r_n = A_{1n}^{\pm}\left(\frac{1}{|x|} + c_{11}^{\pm}\right)\exp in\Omega\left(t + \frac{\text{sgn}\,x}{a^{1/2}}\ln|x|\right) + B_{1n}^{\pm}\exp in\Omega t + O(\varepsilon) \quad ,$$

$$\tag{2.45}$$

$$\mu_n = A_{1n}^{\pm}\left(-\frac{1}{|x|} + c_{12}^{\pm}\right)\exp in\Omega\left(t + \frac{\text{sgn}\,x}{a^{1/2}}\ln|x|\right) + B_{1n}^{\pm}d_{02}^{\pm}\exp in\Omega t + O(\varepsilon) \quad ,$$

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where the values of $c_{11}^{\pm}$, $c_{12}^{\pm}$, and $d_{02}^{\pm}$ are given in Eqs. (2.44).

It is the local expansion Eq. (2.45) which provides the key to the analysis of the acoustic behavior of the converging-diverging duct. For any fixed δ, no matter how small, the linearized acoustic theory fails to approximate the perturbation quantities near the throat of the duct as the flow Mach number there approaches unity. In general, both ρ and u become arbitrarily large in this circumstance, thereby violating the assumption made in deriving Eqs. (2.23) that they remain small. Numerical solutions of the lienarized acoustic equations illustrating the build-up of these arbitrarily large amplitudes were presented in Ref. 6.

The expansion (2.19) will be termed the outer expansion in what follows. It is assumed to be asymptotically valid as an approximation to the solution to Eq. (2.4) as $\delta \rightarrow 0$ with x, t and ε fixed. The result (2.45) shows that it is not uniformly valid as $x \rightarrow 0$ and $\varepsilon \rightarrow 0$. In the vicinity of $x = 0$ nonlinear effects become important. The correct nonlinear theory which must be applied to describe the sound field in the region of near sonic flow near the throat will be presented in the next chapter.

IV. NONLINEAR THEORY

A. Inner Expansion and Basic Equations

The results of the previous section lead to the conclusion that linearized acoustic theory fails to properly describe sound propagation in the vicinity of the throat for $\varepsilon \rightarrow 0$. Both the singular nature of the outer expansion of the basic steady flow Eq. (2.14) and of the acoustic solution Eq. (2.45) indicate that the size of the region around $x = 0$ in which the outer expansions fail is of $O(\varepsilon)$. Thus we introduce a "stretched" or inner variable

$$X = [\frac{\gamma+1}{2}]^{1/2} \frac{x}{\varepsilon} \qquad (3.1)$$

and examine the basic flow and the perturbation solution in the inner region, $X = O(1)$.

Equations (2.2) can be used to derive the well-known equation for the flow Mach number in terms of the area ratio of the duct:

$$\frac{dM}{dx} = - \frac{MG}{1-M^2} \frac{1}{\alpha} \frac{d\alpha}{dx} \qquad . \qquad (3.2)$$

The solution to this equation is given by Eq. (2.10), but it is convenient here to solve Eq. (3.2) over again in the inner region. An inner expansion for M of the form

$$M(x;\varepsilon) = M^i(X;\varepsilon) = m_0(X) + \varepsilon m_1(X) + \ldots \qquad (3.3)$$

is assumed. Transforming Eq. (3.2) to the independent variable X of Eq. (3.1), substituting the expansion (3.3) into Eqs. (2.5) and (3.2) and using Eq. (2.15) for α, then yields the sequence of equations on the m_i. The first two of these are

$$\frac{dm_0}{dX} = 0 \quad , \quad \text{and} \quad m_1 \frac{dm_1}{dX} = aX \qquad .$$

Thus,

$$m_0 = b_0 \quad , \quad \text{and} \quad m_1 = \pm (aX^2 + b_1)^{1/2} \quad ,$$

where b_0 and b_1 are constants which in general would be

determined by asymptotic matching of the expansion (3.3)
to the outer expansion (2.18). However, in this case the
matching process can be avoided because of the fact that
the small parameter ε is defined in terms of M^i itself.
Thus, $M^i(0;\varepsilon) \equiv 1 - \varepsilon$, which implies that $b_0 = 1$, $b_1 = 1$,
and that the minus sign on m_1 is appropriate. Hence, the
inner expansion of the basic flow Mach number becomes

$$M^i(X;\varepsilon) = 1 - \varepsilon(aX^2 + 1)^{1/2} + O(\varepsilon^2) \quad . \qquad (3.4)$$

Using Eq. (3.4), corresponding inner expansions of any of
the basic flow quantities can be constructed. For example
from Eq. (2.5)

$$G(X;\varepsilon) = G^i(X;\varepsilon) = \frac{\gamma+1}{2} - \varepsilon(\gamma+1)m_1(X) + O(\varepsilon^2) \quad (3.5)$$

and

$$M'(x;\varepsilon) = \frac{dM^i}{dX}\frac{dX}{dx} = -(\frac{\gamma+1}{2})^{1/2} m_1'(X) + O(\varepsilon)$$

where

$$m_1(X) = (aX^2 + 1)^{1/2} \quad . \qquad (3.6)$$

Similarly

$$U^i(X;\varepsilon) = c(X)M^i(X;\varepsilon) = c_s \frac{M^i(X;\varepsilon)}{[G^i(X;\varepsilon)]^{1/2}} \quad . \quad (3.7)$$

The above results for the inner expansion of the basic
steady flow will now be used to derive the non-
linear inner equations for acoustic propagation. The
Riemann Invariant form of the perturbation equation will be
treated in detail while analogous results for density and
velocity will be sketched in Appendix B.
    If the relations (3.4) and (3.5) and the change of
variable (3.1) are substituted into Eqs. (2.7), the
equations become

$$P_t^i + (\frac{1}{\varepsilon} + \frac{\gamma+1}{\gamma-1} m_1 + \ldots)[(1-\varepsilon m_1 + \ldots)(1+u^i) + (1+\rho^i)^{\frac{\gamma-1}{2}}]P_X^i \qquad (3.8a)$$

$$- \frac{2m_1'}{\gamma+1}[1 + \frac{2(\gamma-2)}{\gamma+1}\varepsilon m_1 + \ldots][(1+u^i)^2 - (1+\rho^i)^{\gamma-1}]$$

$$+ m_1'(1 + \frac{\gamma-1}{\gamma+1}\varepsilon m_1 + \ldots)u^i[(1-\varepsilon m_1 + \ldots)(1+u^i) + (1+\rho^i)^{\frac{\gamma-1}{2}}] = 0,$$

and

$$Q_t^i + (\frac{1}{\varepsilon} + \frac{\gamma+1}{\gamma-1} m_1 + \ldots)[(1-\varepsilon m_1 + \ldots)(1+u^i) - (1+\rho^i)^{\frac{\gamma-1}{2}}]Q_X^i \qquad (3.8b)$$

$$-\frac{2m_1'}{\gamma+1}[1 + \frac{2(\gamma-2)}{\gamma+1}\varepsilon m_1 + \ldots][(1+u^i)^2 - (1+\rho^i)^{\gamma-1}]$$

$$+ m_1'(1 + \frac{\gamma-1}{\gamma+1}\varepsilon m_1 + \ldots)u^i[(1-\varepsilon m_1 + \ldots)(1+u^i) - (1+\rho^i)^{\frac{\gamma-1}{2}}] = 0$$

respectively. In Eqs. (3.8) the notation u^i, ρ^i, P^i and Q^i has been used to denote the expressions for the perturbation quantities as functions of the inner variables X and t. Now it is assumed that u^i and ρ^i can be expanded in powers of ε as

$$u^i(X,t;\varepsilon) = \varepsilon\mu_1^i(X,t) + \varepsilon^2\mu_2^i(X,t) + \ldots \quad ,$$

$$\rho^i(X,t;\varepsilon) = \varepsilon r_1^i(X,t) + \varepsilon^2 r_2^i(X,t) + \ldots \quad . \qquad (3.9)$$

These expansions of u and ρ yield the corresponding expressions for P and Q from Eq. (2.6):

$$P^i(X,t;\varepsilon) = \varepsilon P_1(X,t) + \varepsilon^2 P_2(X,t) + \ldots \quad ,$$

$$Q^i(X,t;\varepsilon) = \varepsilon Q_1(X,t) + \varepsilon^2 Q_2(X,t) + \ldots \quad . \qquad (3.10)$$

where

$$P_1 = \mu_1^i + r_1^i \quad , \qquad Q_1 = \mu_1^i - r_1^i \quad ,$$

$$P_2 = \mu_2^i + r_2^i - \frac{\gamma-3}{4}(r_1^i)^2 - m_1\mu_1^i \quad , \qquad (3.11)$$

$$Q_2 = \mu_2^i - r_2^i - \frac{\gamma-3}{4}(r_1^i)^2 - m_1\mu_1^i \quad .$$

The expansion for P^i from Eq. (3.10) and the series of Eq. (3.9) are substituted into Eq. (3.8a). Equating the coefficients of powers of ε separately equal to zero then yields

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$$O(1): \quad P_{1_X} = 0 \quad , \tag{3.12a}$$

$$O(\epsilon): \quad P_{2_X} = -\frac{1}{2} P_{1_t} - \frac{\gamma-1}{\gamma+1} m_1' P_1 \quad . \tag{3.12b}$$

Similarly, if the expansion for Q^i from Eq. (3.10) and the series of Eq. (3.9) are substituted into Eq. (3.8b), the lowest order terms yield

$$Q_{1_t} + (\frac{3-\gamma}{4} P_1 + \frac{\gamma+1}{4} Q_1 - m_1)Q_{1_X} - \frac{2m_1'}{\gamma+1} [\frac{3-\gamma}{2} P_1 + \frac{\gamma+1}{2} Q_1] = 0. \tag{3.13}$$

The inner equation of motion on Q_2 follows in a similar manner. However, because it is complicated and will not be needed in the following, it is omitted here.

Equation (3.12a) and Eq. (3.13) consitute the leading order nonlinear theory governing sound propagation through a near-sonic-throat of the form of Eq. (2.15) with $a \neq 0$. At this point, considerable progress in treating these equations can be made by taking advantage of certain facts which are apparent. First, both the first and second order equations on P are linear. Second, the combination $\mu + r$ from Eq. (2.45) in non-infinite as $x \to 0$, because the terms in $|x|^{-1}$ cancel, whereas $\mu - r$ in the outer (linear acoustic) solution is infinite as $x \to 0$. These considerations suggest that in reality P is of smaller order than Q in the inner region, and that the downstream wave passes through the throat essentially unaffected by nonlinear interactions with the near-sonic flow. This suggestion is supported by physical reasoning as well. The Q wave, which propagates against the flow, is intensified relative to the downstream wave P by a transonic trapping near the throat. Thus, it will be assumed here, and verified later by asymptotic matching, that $P_1 \equiv 0$. Then Eqs. (3.12) and (3.13) imply

$$Q_{1_t} + (\frac{\gamma+1}{4} Q_1 - m_1)Q_{1_X} - m_1' Q_1 = 0 \quad , \tag{3.14}$$

and that P_2 is independent of x.

If the definition

$$\pi(X,t) = \frac{\gamma+1}{4} Q_1 - m_1 \quad , \tag{3.15}$$

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where $m_1 = (aX^2 + 1)^{1/2}$, is introduced into Eq. (3.13) it becomes

$$\pi_t + \pi\pi_X = aX \quad . \tag{3.16}$$

The solution of this equation will be discussed in the next section.

B. Solution of the Nonlinear Inner Equation

The nonlinear partial differential equation

$$\pi_t + \pi\pi_X = aX \quad , \qquad -\infty < X < \infty \tag{3.17}$$

which describes the upstream propagating sound in the throat region will now be solved by the method of characteristics Ref. 17. It follows directly by this method that the characteristic curves in the t, X plane are solutions to the differential equation

$$\frac{dt}{dX} = \frac{1}{\pi} \quad , \tag{3.18}$$

and that along these curves,

$$\frac{d\pi}{dX} = \left(\frac{d\pi}{dt} \right) \left(\frac{dt}{dX} \right) = \frac{aX}{\pi} \quad .$$

Integrating this ordinary differential equation along the characteristics yields

$$\pi = \pm \ (aX^2 + J)^{1/2}$$

where J is a constant along a characteristic. It can be shown using matching considerations similar to those given in Section III-C below that the solution to the above equation corresponding to the plus sign must be discarded. Thus for the remainder of this report the solution to be used is

$$\pi = - \ (aX^2 + J)^{1/2} \quad . \tag{3.19}$$

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If positive solutions for π are assumed then it can be shown that the total axial velocity $\bar{u}$ will be supersonic throughout the inner region. Hence, unless shock waves are introduced in the inner region, matching with the subsonic outer solution is not possible. [It is expected that positive solutions will be necessary to treat cases with shocks present.]

It is convenient in solving the specific problems considered in Section III-D to introduce the parameter τ, where $t = \tau$ along the line $X = 0$ and to call the value of π at $X = 0$, $\phi(t)$. Then using

$$\pi(0,t) = -J^{1/2} = \phi(t) = \phi(\tau) \quad ,$$

the solution π can be written in the parametric form

$$\pi(X, \tau(X,t)) = -[aX^2 + \phi^2(\tau)]^{1/2} . \tag{3.20}$$

Using Eq. (3.20) in Eq. (3.18) and integrating the resulting separable ordinary differential equation yields the one parameter family of charasteristic curves

$$t = \tau - \frac{1}{a^{1/2}} \ln\left[\frac{a^{1/2}X + [aX^2 + \phi^2(\tau)]^{1/2}}{|\phi(\tau)|}\right] \tag{3.21}$$

If the function ϕ is known then Eqs. (3.20) and (3.21) provide a solution to Eq. (3.17) for $-\infty < X < \infty$. This will be the case for an acoustic source of $0(\varepsilon)$ located at the throat of the duct; hence, the inner solution is completely determined to leading order and plays the role of a boundary excitation on the acoustic field in the outer region. On the other hand, if the source of sound is located in the outer region, then $\phi(t)$ cannot be determined without simultaneously solving the outer equations and asymptotically matching their solutions with the inner solution π. Regardless of the location of the source $\phi(t)$ must be a periodic function, with period $2\pi/\Omega$ equal to that of the source, since the overall acoustic field and hence π must be periodic functions. In the remainder of this section and in the next section on asymptotic matching ϕ shall be assumed to be a periodic function. When ϕ is periodic then $\pi(X,\tau)$ is periodic in τ with the same period. In addition Eq. (3.21) implies that $t - \tau$ is a periodic function of τ with this period. Thus

π is periodic in t with the period of ϕ.

Before carrying out the matching several observations are in order concerning the solution π and the characteristic curve. In particular a condition will be derived which insures the existence or nonexistence of acoustic shocks.

If π is allowed to vanish, then a careful analysis would show that $\partial\pi/\partial X$ will be infinite, and hence shocks in the solution π are to be expected. In addition it follows from Eq. (3.20) that

$$\frac{\partial\pi}{\partial X} = \left(\frac{\partial\pi}{\partial X}\right)_\tau + \left(\frac{\partial\pi}{\partial\tau}\right)_X \frac{\partial\tau}{\partial X} = \frac{aX}{\pi} + \frac{\phi\phi'}{\pi}\frac{\partial\tau}{\partial X} \qquad (3.22)$$

where

$$\phi' = \frac{d\phi}{d\tau} \quad .$$

Rewriting Eq. (3.21) as

$$F(X,t,\pi) = t - \tau - \frac{1}{a^{1/2}}\ln\left[\frac{|a^{1/2}X + (aX^2 + \phi^2)^{1/2}|}{|\phi|}\right] = 0$$

and using the implicit function theorem yields

$$\frac{\partial\tau}{\partial X} = \frac{-F_X}{F_\tau} = \frac{\phi}{\pi\phi - X\phi'} \qquad (3.23)$$

where

$$F_X = -\frac{1}{\pi} \quad \text{and} \quad F_\tau = -\frac{1 + X\phi'}{\pi\phi} \quad . \qquad (3.24)$$

It is clear from Eq. (3.22) that if $\partial\tau/\partial X$ is finite then $\partial\pi/\partial X$ will remain finite (and hence no shocks will appear in the inner solution). It follows from Eq. (3.23) that $\partial\tau/\partial X$ remains finite if

$$S_0 = \pi\phi - X\phi' = -(aX^2+\phi^2)\phi - X\phi' \neq 0 \quad . \qquad (3.25)$$

Since at $X = 0$ $S_0 = -|\phi|\phi > 0$, $S_0 > 0$ for all $X \le 0$, $t > 0$. Using Eq. (3.25) the condition $S_0 > 0$ is equivalent to

$$(aX^2 + \phi^2)^{1/2} > -\frac{X\phi'}{\phi} \quad .$$

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This condition is always satisfied if $\phi' > 0$ since $-X/\phi < 0$. If however, $\phi' < 0$ for any t which is the case for $\phi(t)$ periodic then $S_0 > 0$ if and only if

$$a\phi^2 - (\phi')^2 \geq 0 \quad . \tag{3.26}$$

If condition (3.26) is not satisfied, then it follows from Eq. (3.24) the F_τ vanishes at some X,t. Hence, the characteristic curves intersect. Therefore, violation of condition (3.26) implies that shocks must appear in the inner perturbation quantities. The physical interpretation of Eq. (3.26) will be discussed in Section III-D.

C. Asymptotic Matching

The solution in the inner region has been expressed in a form suitable for matching in the previous section. This solution will now be matched to the outer solution in order to relate the overall perturbation level δ away from the throat region to the perturbation level ε in the throat or inner region and to relate the acoustic field in the outer region to the acoustic field near the throat. The matching process will thus involve a distinguished limit and is intricate. Since the basic steady flow quantities have been matched in Section IIIB, it is only necessary to match the perturbation velocity and density u and ρ or equivalently, the perturbation Riemann invariants P and Q.

Here the matching will be carried out for P and Q. The matching of u and ρ is assured by the matching of P and Q. For details of the former see Ref. 4. In order to completely determine the leading order inner and outer solutions matching of the leading order P and Q waves will be required.

The one-term inner expansion of Q, denoted here by Q^{il}, is given by Eqs. (3.10), (3.15) and (3.20) in the form

$$Q^{il} = \varepsilon Q_1(X,t) = \frac{4\varepsilon}{\gamma+1} [(aX^2 + 1)^{1/2} - (aX^2 + \phi^2(\tau))^{1/2}]. \tag{3.27}$$

To apply the matching principle of Van Dyke, Ref. 18, to relate the inner to the outer solutions a one-term outer expansion of the expression (3.27) is required. This is obtained by using Eq. (3.1) to express the inner quantity in outer variables

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$$\varepsilon Q_1 \left[\sqrt{\frac{\gamma+1}{2}} \frac{x}{\varepsilon}, \ t \right]$$

$$= \frac{4}{\gamma+1} \left[\frac{a(\gamma+1)}{2} \right]^{1/2} |x| \left[\left(1 + \frac{2\varepsilon^2}{a(\gamma+1)x^2} \right)^{1/2} - \left(1 + \frac{2\varepsilon^2\phi^2}{a(\gamma+1)x^2} \right)^{1/2} \right] .$$

Expanding this expression for fixed x and small ε using the series

$$(1 + y)^{1/2} = 1 + \frac{1}{2}y + \ldots \quad ,$$

and retaining the leading term in ε yields

$$Q_{o1}^{i1} = \left(\frac{2}{\gamma+1} \right)^{3/2} \frac{\varepsilon^2}{a^{1/2}|x|} [1 - \phi^2(\tau)] \quad . \tag{3.28}$$

Here the notation Q_{o1}^{i1} is used to indicate the one-term outer expansion of the one-term inner expansion.

The parameter τ in Eq. (3.28) is determined from the characteristics of Eq. (3.21), also by expanding in ε with x fixed and retaining only the leading term:

$$t = \tau + \frac{\text{sgn } x}{a^{1/2}} \ln|\phi(\tau)| + g(x;\varepsilon) \quad , \tag{3.29}$$

where

$$g(x;\varepsilon) = - \frac{\text{sgn } x}{a^{1/2}} \ln \left[2 \left(\frac{\gamma+1}{2} a \right)^{1/2} \frac{|x|}{\varepsilon} \right] \quad .$$

To utilize the results (3.28) and (3.29) requires displaying the explicit dependence of Q_{o1}^{i1} on x and t. Because $\phi^2(\tau)$ is a periodic function of t with period $2\pi/\Omega$ it can be expanded in a Fourier series as

$$\phi^2(\tau) = \frac{1}{2} C_0^{\pm} + \sum_{n=1}^{\infty} \left[C_n^{\pm}(x) \cos n\Omega t + D_n^{\pm}(x) \sin n\Omega t \right] \quad , \tag{3.30}$$

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in which the + and − signs refer to $x > 0$ and $x < 0$, respectively. The evaluation of the coefficients in Eq. (3.30) is carried out in detail in Appendix C. The results are

$$C_n^\pm = \frac{\Omega}{\pi}\left(1 + \frac{n^2\Omega^2}{4a}\right)^{-1/2} |I_n^\pm| \cos n\Omega(g \pm \theta_n^\pm)$$

$$D_n^\pm = \frac{\Omega}{\pi}\left(1 + \frac{n^2\Omega^2}{4a}\right)^{-1/2} |I_n^\pm| \cos n\Omega(g \pm \theta_n^\pm) \tag{3.31}$$

In Eq. (3.31) the $I_n^\pm$ are defined by

$$I_n^\pm = \int_0^{2\pi/\Omega} |\phi|^\nu \exp(\pm in\Omega\tau)\, d\tau \qquad , \tag{3.32}$$

and the phase angles by

$$\theta_n^\pm = -\frac{1}{n\Omega}\left[\tan^{-1}\frac{n\Omega}{2a^{1/2}} + \tan^{-1}\frac{H_n^\pm}{E_n^\pm}\right] \qquad . \tag{3.33}$$

in which $E_n^\pm$ and $H_n^\pm$ are the real and imaginary parts of $I_n^\pm$. The exponent ν in Eq. (3.32) is

$$\nu = 2 - \frac{in\Omega}{a^{1/2}} \qquad .$$

Therefore, combining Eqs. (3.28), (3.29), (3.30), and (3.31) the explicit representation of Q_{ol}^{il} is

$$Q_{ol}^{il} = \left(\frac{2}{\gamma+1}\right)^{3/2} \frac{\varepsilon^2}{a^{1/2}|x|}\left[1 - \frac{\Omega}{2\pi}|I_0^\pm|\right.$$

$$\left. - \frac{\Omega}{\pi}\sum_{n=1}^{\infty}\left(1 + \frac{n^2\Omega^2}{4a}\right)^{-1/2} |I_n^\pm| \cos n\Omega(t - g \mp \theta^\pm)\right]. \tag{3.34}$$

The outer expansion of the complex solution Q for a periodic wave of period $2\pi/\Omega$ can be constructed as a Fourier series from Eqs. (2.21), (2.22) and (2.24) in the form

$$Q^O = \delta \sum_{n=1}^{\infty} (M_0 \mu_{0n} - r_{0n}) \exp(in\Omega t) + O(\varepsilon\delta) \quad . \quad (3.35)$$

The one-term inner expansion of the leading term of this outer expansion, denoted here by Q_{il}^{ol}, is formed by expanding each harmonic component for small ε, keeping X fixed, and retaining only the leading term. This is equivalent to expanding each harmonic for small x, which is precisely the result given for μ_{0n} and r_{0n} by Eq. (2.45). If Eq. (2.17) is used for $M_0(x)$, then to leading order for small x Eq. (2.45) yields

$$M_0 \mu_{0n} - r_{0n} \stackrel{\sim}{=} \frac{-2A_{1n}^{\pm}}{|x|} \exp in\Omega (t + \frac{\text{sgn } x}{a^{1/2}} \ln|x|) \quad .$$

The physical value of Q^O is found by taking the real part of Eq. (3.35). Thus, if $A_{1n}^{\pm}$, $n \geq 1$, is written as $A_n^{\pm} \exp(in\Omega\Psi_n^{\pm})$ with $A_n^{\pm}$ real, the one-term inner expansion of the physical Q follows as

$$Q_{il}^{ol} = -\frac{2\delta}{|x|} [A_{10}^{\pm} + \sum_{n=1}^{\infty} A_n^{\pm} \cos n\Omega (1 + \frac{\text{sgn } x}{a^{1/2}} \ln|x| + \Psi_n^{\pm})] \quad . (3.36)$$

Now, by the matching principle, Ref. 18 $Q_{ol}^{il} = Q_{il}^{ol}$. Equating the results of Eqs. (3.34) and (3.36) gives

$$\delta = \varepsilon^2 \quad .$$

$$A_{10}^{\pm} = -\frac{1}{2} (\frac{2}{\gamma+1})^{3/2} \frac{1}{a^{1/2}} (1 - \frac{\Omega}{2\pi} |I_0^{\pm}|)$$

$$A_n^{\pm} = \frac{\Omega}{2\pi} (\frac{2}{\gamma+1})^{3/2} \frac{1}{a^{1/2}} (1 + \frac{n^2\Omega^2}{4a})^{-1/2} |I_n^{\pm}| \quad (3.37)$$

$$\Psi_n^{\pm} = \left\{ \frac{1}{a^{1/2}} \ln[\frac{2}{\varepsilon} (\frac{\gamma+1}{2} a)^{1/2}] - \theta_n^{\pm} \right\} \text{sgn } x$$

Eqs. (3.37) yield the relationships between the complex constants $A_{1n}^{\pm}$ of Eq. (2.45) in the linearized solution on either side of

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the throat region. In addition, the result shows that the upstream wave Q, which is $O(\delta)$ in the outer region, is strengthened to $O(\varepsilon) = O(\delta^{1/2})$ in the inner region as a result of nonlinear interactions with the near-sonic flow.

The results of Eqs. (3.37) indicate types of nonlinear effects predicted by the theory. They include in addition to the intensification of the Q-wave a generation of superharmonics and an acoustic streaming effect.

It remains to determine the relation between the complex constants $B_{1n}^{\pm}$ of Eq. (2.45), which is done by asymptotically matching the downstream wave P. In the inner region the fact that $P_1 \equiv 0$, and Eq. (3.12b) imply that $P_2 = P_2(t)$ and thus using Eq. (3.10) implies

$$P^{il} = \varepsilon^2 P_2(t)$$

which is equal to P_{ol}^{il} because P_2 is independent of X. On the other hand, the complex P in the outer region is

$$P^{ol} = \delta \sum_{n=1}^{\infty} (M_0 \mu_{0n} + r_{0n}) \exp(in\Omega) \qquad (3.38)$$

from Eq. (2.21). Use of Eq. (2.17) for $M_0(x)$ and expansion of the components of the series (3.38) for small x using Eq. (2.45) yields for the leading term

$$M_0 \mu_{0n} + r_{0n} \cong A_{1n}^{\pm} [c_{11}^{\pm} + c_{12}^{\pm} + (\tfrac{\gamma+1}{2}) a)^{1/2} \text{ sgn } x]$$

$$\times \exp in\Omega(t + \frac{\text{sgn } x}{a^{1/2}} \ln|x|) + B_{1n}^{\pm} (1 + d_{02}^{\pm}) \ .$$

Then Eq. (2.44) shows that the term multiplying $A_{1n}^{\pm}$ vanishes in general so that

$$P_{il}^{ol} = \delta \sum_{n=0}^{\infty} B_{1n}^{\pm} (1 + d_{02}^{\pm}) \exp(in\Omega t) \qquad , \qquad (3.39)$$

in which $d_{02}^{\pm}$ is given in Eq. (2.44). This result verifies the earlier assumption that $P_1(X,t) \equiv 0$, as the leading order outer

term for P is not singular as $x \to 0$. Finally, since P_{o1}^{i1} is independent of X, the matching can be accomplished by simply equating the complex expressions (3.39) on either side of $x = 0$. Thus

$$B_{1n}^+ (1 + d_{02}^+) = B_{1n}^- (1 + d_{02}^-)$$

and solving for the ratio of B_{1n}^- to B_{1n}^+ using Eq. (2.44) for $d_0^\pm$ yields

$$\frac{B_{1n}^-}{B_{1n}^+} = \frac{[in\Omega(\gamma+1) + 4a^{1/2}]}{[in\Omega(\gamma-1) - 4a^{1/2}]} \frac{(in\Omega - a^{1/2})}{(in\Omega + a^{1/2})} \quad . \quad (3.40)$$

Equations (3.37) and (3.40) represent the primary results of the inner solution process. If the function $\phi(t)$ is known, then they provide two complex equations for the four complex constant $A_{1n}^\pm$ and $B_{1n}^\pm$ for each n. The remaining two equations to determine these constants are the acoustic boundary and source conditions which will be stated for specific cases in the following section. In general, of course, $\phi(t)$ is not known, and must be determined along with the constants $A_{1n}^\pm$ and $B_{1n}^\pm$ taking account of the complicated coupling between the inner and the outer solutions. The general case can be reduced to a coupled system of transcendental equations for A_{1n}^+ and Ψ_n^+. It will not be discussed further in this report.

D. Solution of Specific Boundary Value Problems

In order to complete the specification of a boundary value problem two boundary conditions must be prescribed for the perturbation quantities. These could both be applied in the outer region, on either side of the throat. A typical case would be an anechoic termination upstream of the throat and a single harmonic sinusoidal velocity source downstream of the throat. Another possibility is to have one boundary condition applied at the throat of the duct and the other either upstream or downstream of the throat. Several examples of the latter case will be discussed in this section.

As was discussed earlier, the function $\phi(t)$, upon which the asymptotic matching is based is known *a priori* only if the acoustic source is located at the throat of the duct. In order to illustrate the theory in the present report, it will be assumed that ϕ is known. This allows solution of two specific

problems: transmission of sound upstream or downstream from a source at the throat and transmission through the throat from a reconstructed source in the outer region which yields the specified ϕ at the throat.

In order to solve these problems it will be assumed here that the function $\phi(t)$ is given by

$$\phi(t) = (1 - N_0)(-1 + N \sin \Omega t) \qquad (3.41)$$

where N_0 and N are specified constants. It will now be verified that this ϕ corresponds to a perturbation velocity at the throat having both a steady and a single unsteady sinusoidal component at frequency Ω. To show this consider a source condition on the total perturbed velocity field of the form

$$U(x;\varepsilon)u(x,t;\varepsilon)\Big|_{x=0} = U^i(X;\varepsilon)u^i(X,t;\varepsilon)\Big|_{X=0}$$

$$= \varepsilon\left[\hat{U}_0 + \hat{U} \sin \Omega t\right] \qquad (3.42)$$

where $\hat{U}$ and $\hat{U}_0$ are constants. Here the source strength is taken to be of order ε to be consistent with the condition that the acoustic perturbation quantities are of order ε in the inner region. Using Eqs. (3.7) and (3.9) this condition becomes to first order in ε

$$\mu^i(0,t) = \frac{1}{c_s}(\frac{\gamma+1}{2})^{1/2}(\hat{U}_0 + \hat{U} \sin \Omega t) \qquad (3.43)$$

Since $P_1 = \mu_1^i + r_1^i = 0$ and $Q_1 = \mu_1^i - r_1^i = 2\mu_1^i$, Eq. (3.43) implies

$$Q_1(0,t) = \frac{2}{c_s}(\frac{\gamma+1}{2})^{1/2}(\hat{U}_0 + \hat{U} \sin \Omega t)$$

and from Eq. (3.15)

$$\pi(0,t) = \frac{\gamma+1}{4} Q_1(0,t) - m_1(0)$$

$$= \frac{1}{c_s}(\frac{\gamma+1}{2})^{3/2}(\hat{U}_0 + \hat{U} \sin \Omega t) - 1 \quad . \qquad (3.44)$$

However $\pi(0,t) = \phi(t)$ when $\phi(t) < 0$ and then equating Eqs. (3.44) and (3.41) gives

$$N_0 = \left[\frac{1}{c_s} (\frac{\gamma+1}{2})^{3/2} \hat{U}_0 \right] \qquad (3.45)$$

and

$$N = \frac{(\frac{\gamma+1}{2})^{3/2} \hat{U}_1}{c_s (1 - N_0)} \quad . \qquad (3.46)$$

Thus prescribing a velocity source of the form Eq. (3.42) at the throat of the duct completely determines $\phi(t)$ in the form of Eq. (3.41). It should be noted that if $N_0 = 0$, the applied source has no steady part.

Since $\phi(t)$ is known, use of Eq. (3.41) in Eq. (3.32) gives

$$I_0^{\pm} = \frac{\pi}{\Omega} (1 - N_0)^2 (2 + N^2)$$

So that Eq. (3.37) yield for $A_{10}^{\pm}$

$$A_{10}^{\pm} = - \frac{1}{2} (\frac{2}{\gamma+1})^{3/2} \frac{1}{a^{1/2}} [1 - (1-N_0)^2 (1 + \frac{N^2}{2})] \quad . \qquad (3.47)$$

Now the coefficient $A_{10}^{\pm}$ represents a steady component in the acoustic perturbation quantities in the outer region, which is a form of acoustic streaming. If $N_0 = 0$ then there is no such steady component at the throat. In this case $A_{10}^{\pm} \neq 0$. However, by choosing

$$N_0 = 1 - (1 + \frac{N^2}{2})^{-1/2} \qquad (3.48)$$

the streaming term $A_{10}^{\pm}$ can be suppressed in the outer region. One case to be considered here will require that condition (3.48) holds with $N_0 \neq 0$. Thus no streaming will occur in the outer region. The other case to be considered in detail will have a nonzero streaming term.

If $N_0 = 0$ then $A_{10}^{\pm} \neq 0$ for $\phi(t)$ given by Eq. (3.41) and a streaming term will occur in the outer solution. Here an acoustic source at the throat of the duct will cause an acoustic stream effect in the outer region.

Since solutions without shocks are being considered in the report the function $\phi(t)$ must satisfy condition (3.26) namely

$$a(\phi)^2 - (\phi')^2 \geq 0 \quad .$$

If ϕ is given by Eq. (3.41), then, this condition becomes

$$\frac{a}{\Omega^2 N^2} \geq [\frac{\cos \Omega t}{1 - N \sin \Omega t}]^2 \quad .$$

Maximizing the right side of this inequality with respect to t yields

$$\frac{a}{\Omega^2 N^2} \geq \frac{1}{1-N^2} \quad , \quad \text{or} \quad N^2 \leq \frac{1}{1 + \frac{\Omega^2}{a}} \qquad (3.49)$$

Thus in order to have no shocks in the inner region the conditions $\phi < 0$ (since $\pi < 0$) and Eq. (3.49) must be satisfied. The condition $\phi < 0$ implies that if $N_0 = 0$ then $N < 1$ while if $N_0 \neq 0$ then $0 < N_0 < 1$ and $0 < N < 1$. It is interesting to note in the case $N_0 = 0$ that if the parameter ε is reintroduced then the source strength is in effect εN. If this product is denoted by S_0 then Eq. (3.49) can be rewritten as

$$S_0^2 \leq \frac{\varepsilon^2}{1 + \frac{\Omega^2}{a}} \quad . \qquad (3.50)$$

In this form the shocking condition relates the source strength and frequency, the curvature of the throat and the deviation of the throat Mach number from unity. As is to be expected shocks will occur at lower source strengths for higher Mach number flows. Similarly a high frequency source or a flat throat region (small a) will lead to earlier shocking.

To completely prescribe the boundary value problem and determine $A_{1n}^{\pm}$ and $B_{1n}^{\pm}$, two more conditions are required. These are the acoustic boundary and source conditions. If the boundary conditions are applied well away from $x = 0$ then it is

necessary to solve the outer equations (2.25) before completing the solution. This can be done numerically. To illustrate the theory here, however, it is more convenient to choose a special dust shape due to Crocco and Tsien for which an analytical solution to the system (2.25) is known. This solution is discussed in Appendix D.

For the case of upstream propagation from the throat it will be assumed that the plane $x = -x_e$ of the Crocco-Tsien duct is anechoic so that the acoustic boundary condition there is

$$p^{o1}(-x_e, t) = 0 \quad . \tag{3.51}$$

Use of Eqs. (3.38) in Eq. (3.51) as discussed in Appendix D then gives

$$\frac{E_{1n}^-}{E_{2n}^-} = (2K)^{\ell-1} \frac{A_{1n}^-}{B_{1n}^-} = \Gamma_n(-x_e) \tag{3.52}$$

in which $\Gamma_n(x)$ is a ratio of combinations of hypergeometric functions whose specific form is given in Appendix D. Eq. (3.52) is the third relation used in the determination of $A_{1n}^\pm$ and $B_{1n}^\pm$, in the case of anechoic conditions upstream of the throat.

The final equation to determine $A_{1n}^\pm$ and $B_{1n}^\pm$ is the source condition. For the source at the throat, prescribing the acoustic perturbation velocity results in Eq. (3.41) for $\phi(t)$, which completely solves the problem of proagation out through the exit upstream. A_{1n}^- is given by Eq. (3.37) directly, B_{1n}^- is determined using Eq. (3.47), and the sound field is constructed using Eqs. (D7) and (D8).

A similar procedure would solve the problem of downstream propagation from a throat source if the exit condition were applied at a positive x. In this case Eq. (D11) is used instead of Eq. (3.47) and the source condition determines A_{1n}^+ instead of A_{1n}^-.

The procedure just described was implemented numerically and results were obtained for three different types of boundary value problems. The details of the numerics are discussed in the next section.

For the special case of transmission from a source downstream of the throat which can be treated using the above approach, $\phi(t)$ is still assumed to be given by Eq. (3.41). In this case Eq. (3.37) yields $A_{1n}^{\pm}$, Eq. (3.52) gives B_{1n}^{-} and Eq. (3.40) determines B_{1n}^{+}. The acoustic field on either side of the throat is then given by Eqs. (D7) and (D8). This result can be viewed as the sound field transmitted upstream from a periodic velocity source located at some $x > 0$, where the strengths of each of the harmonic source components are constructed using Eq. (D8) at the source location. This procedure is special in that the source strength and harmonic content is not to be specified but rather is part of the solution. An iterative procedure could be set up to control the harmonic content of the source by changing the form of $\phi(x)$ (by increasing its harmonic content). This case will not be discussed further in the present report.

E. Numerical Results and Discussion

The results to be presented here were obtained by numerically implementing the procedure described in the last section. That is:

(1) A choice of the parameters N_0, N_1 and Ω in $\phi(t)$ (Eq. (3.41)) and the parameter a was made. Then the integrals $I_n^{\pm}$ were evaluated numerically by Simpson's rule. This determines, using Eq. (3.37), the constants $A_{10}^{\pm}$, $A_n^{\pm}$ and upon choosing a value of ε the phase angles $\Psi_n^{\pm}$.

(2) Either Eq. (3.47) or Eq. (D11) was then employed to determine either B_{1n}^{+} or B_{1n}^{-} respectively. The former being used when the anechoic condition is applied upstream of the throat and the latter for a downstream anechoic boundary. The hypergeometric functions in these expressions were evaluated by use of the standard power series representation for these functions.

First although the results of the asymptotic matching are sufficient to transfer the acoustic field through the duct without further consideration of the inner solution, it is of interest to display this solution to illustrate the approach to shock formation discussed earlier. Figures 5 - 8 show the inner acoustic velocity $\mu_1^i(X,t)$ evaluated by use of Eqs. (B7) and (3.21) at one fixed time for $|X| \leq 10$ and for the indicated values of the parameters N, N_0, $\overline{\Omega}$ and a. In Figure 5 the value of $N_0 = 0$ and hence the source at $x = 0$ has no

steady part. The steepening to a shock is clearly indicated and condition (3.49) yields the value N = .7746 for shock formation. Figure 6 yield similar results for the indicated values of N, with N_0 related to N by Eq. (3.48). Similarly in Figures 7 and 8 the variation of $\mu^i(X,t)$ with the source frequency Ω and the geometric parameter a respectively is shown. In both cases the shock formation is evident.

In Figure 9 the acoustic velocity is shown for six equally spaced times over a period $2\pi/\Omega$. The upstream propagating nature of the inner acoustic velocity is clearly evident.

The remaining discussion will require an expression for the time averaged acoustic power transmitted out through the exit of the duct which in each case is taken at either x = +.06 or x = -.06. The acoustic power, calculated according to the expression given in Ref. 19 can be written in terms of the dimensionless variables of this report as

$$\frac{W}{R_s c_s^3} = \frac{\alpha(x) M_0}{2} G^{\left(\frac{1-3\gamma}{2(\gamma-1)}\right)} \left[2\left((r_{00})^2 + M_0^2 (\mu_{00})^2 - (1+M_0^2)(r_{00})(\mu_{00}) \right) \right.$$

$$\left. + \sum_{n=1}^{\infty} \left\{ |r_{0n}|^2 + M_0^2 |\mu_{0n}|^2 + (1+M_0^2)|r_{0n}||\mu_{0n}|\cos n\Omega(\theta_{rn}-\theta_{\mu n}) \right\} \right]$$

in which θ_{rn} and $\theta_{\mu n}$ are the phase angles of the complex r_{0n} and μ_{0n}, respectively, and R_s is the stagnation density of the steady flow. Here r_{00} and μ_{00} correspond to steady perturbations or acoustic streaming type terms.

Figures 10 - 13 give results for a source located at the throat of the duct, with $N_0 = 0$ and an anechoic upstream boundary. In Figure (10) results for two cases with the source strength fixed at $S_0 = \varepsilon N = .03$ and .06 are presented. The effect of increasing the throat Mach number (decreasing ε) on the upstream transmitted acoustic power is determined. In both cases there is marked attenuation with increasing Mach number. It is of interest to note that shocks will occur for $\varepsilon = .08484$ when $S_0 = .06$ and for $\varepsilon = .04242$ when $S_0 = .03$ (Eq. (3.50)). In Figures 11 and 12 the upstream transmitted power is presented for varying Ω and varying a respectively, with S_0 and ε fixed in both cases. Results indicate that the transmitted acoustic power increases with increasing source frequency and decreases with increasing a.

In order to study the harmonic content of the solution in the outer region upstream of the throat the ratio W_n/W was calculated where W_n is the acoustic power carried by the n-th harmonic. In Figure 13 this ratio is plotted against the deviation of the throat Mach number from unity (ε) for the fixed source strength $S_0 = \varepsilon N = .03$. Curves are given for the first three harmonics. It is seen that as ε decreases the non-linear interaction causes an increasing fraction of the total power to be distributed among the higher harmonics. For $\varepsilon = .05$ approximately 86% of the power is carried out in the fundamental harmonic. This nonlinear effect occurs even though no shock has formed in the inner region. It is quite important to emphasize that if linearized theory had been applied in the inner region all of the acoustic power would have remained in the fundamental harmonic. This result is of obvious signficance in the design of experiments to measure sound radiation out of near-sonic inlets.

The remaining results discussed in this report correspond to cases where $N_0 \neq 0$ and the parameters N and N_0 are related by Eq. (3.48). Hence no streaming will occur in the outer region. in these cases. In Figures 14 - 17 results are presented for an upstream anechoic boundary. The results of Figure 14 are similar to those of Figure 13 in that about 15% of the energy is carried by higher harmonics for values of N below the value where shocks will occur. In Figure 15 , $\varepsilon = .1$, $N_0 = .01$ and $a = 1.0$. Hence the throat Mach number and the source strength are fixed and the variation of W_n/W with frequency is given for the first three harmonics. The graphs show that the nonlinear effect due to increasing source frequency leads to multiple harmonics and a spreading of part of the power to higher harmonics.

A similar result is shown for variable throat curvature parameter a in Figure (16). In this case the flatter the throat the more pronounced the nonlinear effect. This is as should be expected since for a fixed throat Mach number ($\varepsilon = .1$ in this case) the duct with the flatter throat region will have a greater portion of high subsonic flow around the throat and hence a more pronounced nonlinear effect. Figure 17 gives the dimensionless acoustic power $W/R_s c_s^3$ radiated upstream as a function of N_0 for ε fixed. As is to be expected the acoustic power transmitted increases as the value of N_0 is increased.

In Figures 18 - 20 results are given for the case where the anechoic boundary was placed downstream of the throat and thus the power flow will be downstream. There is however an upstream propagating Q-wave in the duct because of reflections of the downstream propagating P-wave. These reflections are due to flow gradients (not the boundary) and occur in the outer region (downstream from the throat). The presence of the Q-wave can be expected to lead to nonlinear results.

This is in fact the case as is indicated in Figures 18 - 20 where results similar to Figures 14 - 16 respectively are given. It is interesting to note that the total power is larger in the downstream case than in the upstream case for the same source at the throat. This is to be expected since in the former case the power flow is in the fluid flow direction. The spread of power to the higher harmonics is also more pronounced in this case. It is interesting to note in Figure 19 that as Ω increases the second and then the third harmonics have maximum values for W_n/W. This indicates as do other numerical results that as shocking conditions are approached a great many harmonics are carrying the acoustic power.

V. CONCLUSIONS

I. The study has shown, using a quasi-one dimensional model, that linear acoustic theory is not valid in general for sound propagation in near sonic duct flow. Linear acoustic quantities become infinite in such flows at the throat of a converging diverging duct.

II. A new nonlinear theory to describe sound propagation in such situations is derived by the method of matched asymptotic expansions. The nonlinear theory must be used in the vicinity of the throat while linear theory still holds away from the throat. Asymptotic matching is used to connect the nonlinear and linear regions and to completely determine the acoustic field in the duct.

III. Analytical solutions are obtained to the nonlinear equations, thereby illustrating the physical propagation process. These are:

(1) Intensification of upstream propagating sound due to a transonic trapping.

(2) To leading order, downstream propagation obeys linear theory.

(3) An infinite number of superharmonics are generated by a single frequency acoustic source.

(4) Acoustic streaming effects are present in general.

(5) Shocks will occur in the acoustic quantities when condition (3.26) or (3.50) is not satisfied. This condition relates source amplitude and frequency, steady flow Mach number and the curvature of the throat of the duct.

(6) The acoustic power radiated out of the duct depends on the ε^4 and thus decreases rapidly as the throat Mach number approaches unity.

IV. Numerical studies are carried out for several simple boundary value problems to illustrate the predictions of the theory. These results include:

(1) Appreciable percentages (over 15% in many cases) of the acoustic power transmitted upstream out of the duct will be carried by superharmonics of the source frequency. This can occur before acoustic shocks appear.

(2) Increasing the source strength or frequency or decreasing the geometric parameter a increases the nonlinear effect.

(3) Shocks will occur at typical acoustic source frequencies and amplitudes and thus their effect must be studied further.

V. Numerical and analytical results indicate that superharmonics and shocks will be observed in near sonic duct flows. Experimental work to detect such effects in actual flows would be of much interest and benefit in understanding the acoustic processes at work.

Appendix A. LINEAR ACOUSTIC THEORY

In this appendix a sketch of the derivation of the constants in the local expansion of linear acoustic theory Eqs. (2.44) will be given. To determine $d_{02}^{\pm}$ the last of Eqs. (2.43)

$$B_0 \begin{bmatrix} 1 \\ d_{02}^{\pm} \end{bmatrix} = \frac{M_0(0)}{H(0)} \begin{bmatrix} T(0) & -S(0) \\ -T(0) & S(0) \end{bmatrix} \begin{bmatrix} 1 \\ d_{02}^{\pm} \end{bmatrix} = 0 \quad (A1)$$

must be solved. Here T and S were defined in Eqs. (2.26). This is just two linear homogeneous equations in two unknowns and thus eigenvectors of the B_0 matrix corresponding to the zero eigenvalue $\lambda = 0$, are being found. Thus

$$d_{02}^{\pm} = \frac{T(0)}{S(0)} = \begin{bmatrix} in\Omega G_0^{1/2} + \dfrac{2M_0'}{G_0} \\ \hline in\Omega G_0^{1/2} + \dfrac{(\gamma-1)M_0'}{G_0} \end{bmatrix} x = 0 \quad (A2)$$

Using the results

$$M_0'(0) = -[(\gamma+1)\tfrac{a}{2}]^{1/2} \operatorname{sgn} x$$

$$G_0(0) = \frac{(\gamma+1)}{2} \qquad (A3)$$

Eq. (A2) reduces to the desired result (Eq. (2.44)).

The constants $c_{02}^{\pm}$ are found by solving the first of Eqs. (2.43):

$$[B_0 - \lambda I] \begin{bmatrix} 1 \\ c_{02}^{\pm} \end{bmatrix} = 0$$

where

$$\lambda = -1 - \frac{(in\Omega \text{ sgn } x)}{a^{1/2}} \tag{A4}$$

and B_0 is as given in Eq. (A1). The solution is easily found to be $c_{02}^{\pm} = -1$ by using the definitions of $T(0)$, $S(0)$ and Eqs. (A3).

To find $c_{11}^{\pm}$ and $c_{12}^{\pm}$ the second of Eqs. (2.43)

$$[B_0 - (\lambda+1)I] \begin{bmatrix} c_{11}^{\pm} \\ c_{12}^{\pm} \end{bmatrix} = -B_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{A5}$$

must be solved. Here the eigenvalue λ is given by Eq. (A4), B_0 is given in Eq. (A1) and hence the matrix $B_1 = B'(0)$ must be found. The computation of B_1 is long but straightforward. From the definition of $B(x)$ (Eq. (2.31)) it follows that

$$B'(x) = \left(\frac{M_0}{H}\right)' \begin{bmatrix} T & -S \\ -V & S \end{bmatrix} + \frac{M_0}{H} \begin{bmatrix} T' & -T' \\ -V' & S' \end{bmatrix} \quad .$$

or

$$B'(0) = \begin{bmatrix} F'(0)T(0) + F(0)T'(0) & -(S(0)F'(0) + S'(0)F(0)) \\ \\ -(F'(0)V(0) + F(0)V'(0)) & S(0)F'(0) + S'(0)F(0) \end{bmatrix} \tag{A6}$$

where $F(x) = M_0(x)/H(x)$. Using Eq. (A6) in Eq. (A5) and solving the latter by inverting the coefficient matrix yields

$$\begin{bmatrix} c_{11}^{\pm} \\ c_{12}^{\pm} \end{bmatrix} = \Gamma \begin{bmatrix} A_1^* + \bar{\alpha}A_2^* \\ \\ B^* - \bar{\alpha}B_2^* \end{bmatrix} \tag{A7}$$

where

APPENDIX A

$$\bar{\alpha} = (1+\lambda_2)H(0) \ , \qquad \Gamma = \bar{\alpha}H(0)[\bar{\alpha} - T(0) - S(0)]$$

$$A_1^* = S(0)F(0)(V'(0)-T'(0)) \ , \quad B_1^* = T(0)F(0)(V'(0)-T'(0))$$

$$A_2^* = F'(0)[T(0) + S(0)] + F(0)[T'(0) + S'(0)]$$

$$B_2^* = F'(0)[V(0) + S(0)] + F(0)[V'(0) + S'(0)] \quad .$$

To arrive at the final result (Eqs. (2.44)) the quantities F, T, S, V and their derivatives must be evaluated at x = 0. This is tedious and the details will not be presented. It should be noted that it is easier to do the actual computations by rewriting Eq. (A7) to obtain expressions for $c_{12}^\pm + c_{11}^\pm$ and $c_{12}^\pm - c_{11}^\pm$ and hence this is the form of the results presented in Eqs. (2.44).

Appendix B. INNER EQUATIONS FOR VELOCITY AND DENSITY

In order to derive the inner equations for the acoustic density and velocity Eqs. (3.11) could be solved for μ_1 and r_1 in terms of P_1 and Q_1 and the result substituted in Eqs. (3.12) and (3.13), or equivalently the expansions (3.9) and the steady flow expansions (3.4), (3.5) could be substituted into Eqs. (2.4) and coefficients of each power of ε are set equal to zero. The latter procedure yields for the first of Eqs. (2.4)

$$O(\varepsilon): \quad r_{1X}^i + \mu_{1X}^i = 0$$

$$O(\varepsilon^2): \quad r_{1t}^i + \mu_1^i r_{1X}^i + r_1^i \mu_{1X}^i + r_{2X}^i + \mu_{2X}^i = 0$$

(B1)

and for the second of Eqs. (2.4)

$$O(\varepsilon): \quad r_{1X}^i + \mu_{1X}^i = 0$$

$$O(\varepsilon^2): \quad \mu_{1t}^i + \mu_{2X}^i + r_{2X}^i + (\mu_1^i - m_1)\mu_{1X}^i$$

(B2)

$$+ [(\gamma-2)r_1^i + m_1]r_{1X}^i - \frac{2aX}{(\gamma+1)m_1}[2\mu_1^i - (\gamma-1)r_1^i] = 0 .$$

The continuity and momentum equations yield redundant equations for the leading order density and velocity. Thus the $O(\varepsilon^2)$ equations will be needed to derive two independent equations for r_1^i and μ_1^i. The redundancy is repeated at $O(\varepsilon^2)$, and thus subtraction of the second of Eqs. (B1) from the second of Eqs. (B2) results in a second independent equation on the first order quantities. Therefore, the first-order inner equations for the perturbation quantities can be taken to be the set consisting of the first of Eqs. (B1) and (B2) and the difference between the two $O(\varepsilon^2)$ equations:

$$\mu_{1t}^i - r_{1t}^i + (\mu_1^i - m_1)\mu_{1X}^i + [(\gamma-2)r_1^i + m_1]r_{1X}^i - r_1^i \mu_{1X}^i$$

$$- \mu_1^i r_{1X}^i - \frac{2aX}{(\gamma+1)m_1}[2\mu_1^i - (\gamma-1)r_1^i] = 0 .$$

(B3)

Finally, the first equation is used to eliminate r_{1X}^i from Eq. (B3) to give the first-order inner equations in the form

$$r_{1X}^i + \mu_{1X}^i = 0 \ ,$$ (B4)

$$\mu_{1t}^i - r_{1t}^i + [2\mu_1^i - (\gamma-1)r_1^i - 2m_1]\mu_{1X}^i - \frac{2aX}{(\gamma+1)m_1}[2\mu_1^i - (\gamma-1)r_1^i] = 0 \ .$$

Equations (B4) are the lowest order nonlinear equations of motion which govern sound propagation in the vicinity of a near-sonic throat.

A second order equation can be found by forming the second order Riemann invariant (Eqs. (3.11))

$$P_2 = \mu_2^i + r_2^i - \frac{\gamma-3}{4}(r_1^i)^2 - m_1\mu_1^i \ .$$ (B5)

Using the second of Eqs. (B1) and the relation $\mu_1^i + r_1^i = 0$, which is implied by Eq. (B4) yields

$$\frac{\partial P_2}{\partial X} = \frac{\partial}{\partial X}(\mu_2^i + r_2^i - \frac{\gamma-3}{4}(r_1^i)^2 - m_1\mu_1^i) = 0$$

and hence $P_2 = P_2(t)$. The relations $P_1 \equiv 0$, $P_2 = P_2(t)$ and the second of Eqs. (B4) are equivalent to the equations derived in Section IIIB. In fact the second of Eqs. (B4) can be transformed to Eq. (3.14) by letting $Q_1 = \mu_1^i - r_1^i$.

The solution for μ_1^i and r_1^i can be expressed in terms of π by use of the relations $Q_1 = 2\mu_1^i$, $\mu_1^i = -r_1^i$ and Eqs. (3.15). This yields

$$\mu_1^i = - r_1^i = \frac{2}{\gamma+1}[\pi(X,t) + m_1(X)] \ .$$ (B6)

Using Eq. (3.20) for π gives the leading order inner solution

$$\mu_1^i = - r_1^i = \frac{2}{\gamma+1}[-(aX^2 + \phi^2(\tau))^{1/2} + (aX^2 + 1)^{1/2}] \ ,$$ (B7)

where $\tau(X,t)$ is given implicitly by the characteristic equation (3.21).

Appendix C. FOURIER EXPANSION OF THE INNER SOLUTION

In Section IIIC the Fourier series for the outer expansion of the inner solution was found assuming the Fourier expansion of the function $\phi^2(\tau)$ was known. In this appendix the details of finding this expansion of $\phi^2[\tau(x,t)]$ will be provided. Since ϕ^2 is a period function of period $2\pi/\Omega$ in t a series in the form

$$\phi^2 = \frac{1}{2} C_0^\pm + \sum_{n=1}^{\infty} \left[C_n^\pm(x) \cos n\Omega t + D_n^\pm(x) \sin n\Omega t \right] \qquad (C1)$$

will be sought, with

$$\left\{ \begin{array}{c} C_n^\pm \\ \\ D_n^\pm \end{array} \right\} = \frac{\Omega}{\pi} \int_0^{2\pi/\Omega} \phi^2(\tau(x,t)) \left\{ \begin{array}{c} \cos \Omega t \\ \\ \sin \Omega t \end{array} \right\} dt \qquad . \qquad (C2)$$

It should be noted that $C_n^\pm$ and $D_n^\pm$ are functions of the outer variable x since the outer limit of the inner solution is being considered. The expression (C2) for the coefficient will now be simplified and written in the form suitable for matching by explicitly exhibiting the x dependence implicit in Eq. (C2). Differentiating the characteristic relation (Eq. (3.29))

$$t = \tau + \frac{\text{sgn } x}{a^{1/2}} \ln |\phi(\tau)| + g(x;\epsilon) \qquad (C3)$$

yields

$$\frac{dt}{d\tau} = 1 + \frac{\text{sgn } x}{a^{1/2}} \frac{\phi'}{\phi} \qquad . \qquad (C4)$$

The variable of integration in Eq. (C2) is changed from t to τ by using Eqs. (C3) and (C4) yielding

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$$
\begin{Bmatrix} C_n^{\pm} \\ \\ D_n^{\pm} \end{Bmatrix} = \frac{\Omega}{\pi} \int_0^{2\pi/\Omega} \phi^2 \left[1 + \frac{\text{sgn } x \ \phi'}{a^{1/2}} \right]
$$

$$
\times \begin{Bmatrix} \cos n\Omega \\ \\ \sin n\Omega \end{Bmatrix} \left[\tau + g(x;\varepsilon) + \frac{\text{sgn } x}{a^{1/2}} \ln |\phi(\tau)| \right] d\tau \quad .
$$

Here the fact that $\phi(\tau)$ is periodic with period $2\pi/\Omega$ has been used to determine the limits of integration. Integrating the ϕ' term in $C_n^{\pm}$ and $D_n^{\pm}$ by parts yields

$$
C_n^{\pm} - \frac{n\Omega}{2a^{1/2}} \text{sgn } x \ D_n^{\pm}
$$

$$
= \frac{\Omega}{\pi} \int_0^{2\pi/\Omega} \phi^2(\tau) \cos n\Omega \left[\tau + g(x;\varepsilon) + \frac{\text{sgn } x}{a^{1/2}} \ln |\phi| \right] d\tau \quad \text{(C5)}
$$

and

$$
D_n^{\pm} + \frac{n\Omega}{2a^{1/2}} \text{sgn } x \ C_n^{\pm}
$$

$$
= \frac{\Omega}{\pi} \int_0^{2\pi/\Omega} \phi^2(\tau) \sin n\Omega \left[\tau + g(x;\varepsilon) + \frac{\text{sgn } x}{a^{1/2}} \ln |\phi| \right] d\tau \quad \text{(C6)}
$$

respectively. Multiplying Eq. (C6) by i and adding it to Eq. (C5) yields

$$
\left(C_n^{\pm} - \frac{n\Omega}{2a^{1/2}} \text{sgn } n \ D_n^{\pm} \right) + i \left(D_n^{\pm} + \frac{n\Omega}{2a^{1/2}} \text{sgn } x \ C_n^{\pm} \right) = \frac{\Omega}{\pi} \exp(in\Omega g) I_n^{\pm} \quad \text{(C7)}
$$

where $I_n^{\pm}$ was defined in Eq. (3.32). Since $I_n^{\pm}$ is independent of x the x dependence of $C_n^{\pm}$ and $D_n^{\pm}$ has been explicitly found in the right side of Eq. (C7).

APPENDIX C

In order to solve Eqs. (C6) and (C7) for the Fourier coefficients $C_n^{\pm}$ and $D_n^{\pm}$ the term $I_n^{\pm}$ is written as

$$I_n^{\pm} = E_n^{\pm} + iH_n^{\pm} = |I_n^{\pm}| \; e^{i \tan^{-1} \frac{H_n^{\pm}}{E_n^{\pm}}} \quad .$$

Then after a straightforward calculation the expressions (3.31) are found for $C_n^{\pm}$ and $D_n^{\pm}$.

Appendix D. THE CROCCO-TSIEN SOLUTION

An exact analytical solution of the outer acoustic equations (2.25) was discovered by Tsien and applied extensively by Crocco to problems of combustion instability in rocket engines (Ref. 10). Their solution was for a basic steady flow that was subsonic upstream of the throat and supersonic downstream of the throat. This solution was generalized in Ref. 6 to totally subsonic steady flow. This generalized solution is presented here.

Suppose the Mach number distribution $M_0(x)$ of Eq. (2.17) is of the form

$$M_0 = (1 - K|x|) [\frac{\gamma+1}{2} - \frac{\gamma-1}{2} (1 - K|x|)^2]^{-1/2} \qquad |x| < \frac{1}{K} \qquad (D1)$$

where K is a constant. Expansion of Eq. (D1) for small x yields

$$M_0(x) = 1 - \frac{K(\gamma+1)}{2} |x| + \frac{3}{8} (\gamma^2 - 1)K^2 x^2 + \dots . \qquad (D2)$$

This is expansion of the form of Eq. (2.17) with

$$K^2 = \frac{2a}{(\gamma+1)}$$

and

$$m_{20} = \frac{3}{8} (\gamma^2 - 1) K^2 . \qquad (D3)$$

The area ratio $\alpha(x)$ corresponding to Eq. (D1) can be determined from Eq. (2.13); it is shown in Figure 4 for three values of the parameter K over a range $0 \le x \le 0.5$. Tsien and Crocco showed, by introducing a new independent variable $z = (1 - K|x|)^2$ into Eqs. (2.25) that these equations were equivalent to

$$z(1-z) \frac{d^2 r_{0n}}{dz^2} - 2[1 + \frac{i\beta}{\gamma+1}] z \frac{dr_{0n}}{dz} - [\frac{i\beta(2+2i)}{2(\gamma+1)}] r_{0n} = 0 \qquad (D4)$$

$$\mu_{0n} = [(\gamma-1+i\beta) r_{0n} - (\gamma+1)(1-z) \frac{dr_{0n}}{dz}] [2+i\beta]^{-1} \qquad (D5)$$

where

$$\beta = -(\frac{\gamma+1}{2})^{1/2} \frac{n\Omega}{K} \, \text{sgn} \, x \qquad (D6)$$

Eq. (D4) a hypergeometric equation with complex coefficients Ref. 20 and thus its general solution can be written as

$$r_{0n} = E_{1n}^{\pm}(1-z)^{1-\ell}F(-k,-j,2-\ell;1-z) + E_{2n}^{\pm}F(j,k,\ell;1-z) \, , \qquad (D7)$$

where

$$j = \frac{\ell-1}{2} - \frac{1}{2}\Delta \, , \quad k = \frac{\ell-1}{2} + \frac{1}{2}\Delta \, ,$$

$$\ell = 2 + \frac{2i\beta}{\gamma+1} \, , \quad \Delta = [1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\beta^2]^{1/2} \, ,$$

and F is the standard hypergeometric function of the indicated complex arguments. Calculating r_0' from Eq. (D7) by using the well known properties of the hypergeometric function gives μ_0 from Eq. (D5) in the form

$$\mu_{0n} = \frac{1}{2+i\beta} \left\{ E_{2n}^{\pm}[(\gamma-1+i\beta)F_1 + \frac{(\gamma+1)jk}{\ell}(1-z)F_2] \right.$$

$$+ E_{1n}^{\pm}[(2\gamma+i\beta-(\gamma+1)\ell)(1-z)^{1-\ell}F_3 \qquad (D8)$$

$$\left. + \frac{(\gamma+1)jk}{2-\ell}(1-z)^{2-\ell}F_4] \right\}$$

where

$$F_1 = F(j,k,\ell;1-z) \, , \quad F_2 = F(j+1;k+1,\ell+1;1-z) \, ,$$

$$F_3 = F(-k,-j,2-\ell;1-z) \, , \quad F_4 = F(-k+1,-j+1,3-\ell;1-z) \, .$$

Finally, if r_{0n} in Eq. (D7) is multiplied by exp $in\Omega t$ and expanded for small values x (i.e., z-1 → 0) it must be of

the form given in Eq. (2.45), which determines $E_{1n}^{\pm}$ and $E_{2n}^{\pm}$ in terms of $A_{1n}^{\pm}$ and $B_{1n}^{\pm}$:

$$E_{1n}^{\pm} = (2K)^{\ell-1} A_{1n}^{\pm} \quad \text{and} \quad E_{2n}^{\pm} = B_{1n}^{\pm} \quad . \qquad (D9)$$

If the anechoic boundary condition

$$P^{01}(-x_e,t) = \delta \sum_{n=0}^{\infty} (M_0 \mu_{0n} + r_{0n}) e^{in\Omega t} = 0$$

is applied then using Eqs. (D1), (D7), (D8) and (D9) yields Eq. (3.47) namely

$$B_{1n}^{-} = (2K)^{\ell-1} (\Gamma_n(-x_e))^{-1} A_{1n}^{-}$$

where

$$\Gamma_n(-x_e)$$

$$= -\left[(2+i\beta)^{-1} \left[(\gamma-1+i\beta)F_1 + \frac{(\gamma+1)jk}{\ell} (1-z_e)F_2 \right] + \frac{F_1}{M_0(-x_e)} \right]$$

$$\times \left[(2+i\beta^{-1}) \left[(2\gamma+i\beta-(\gamma+1)\ell)(1-z_e)^{1-\ell}F_3 + \frac{(\gamma+1)jk}{2-\ell} (1-z_e)^{2-\ell}F_4 \right] \right.$$

$$\left. + \frac{(1-z_e)^{1-\ell}F_3}{M_0(-x_e)} \right] \quad , \qquad (D10)$$

and

$$z_e = [1 - K|-x_e|]^2 \quad .$$

In this expression the hypergeometric functions are evaluated at $z = z_e$.

If the anechoic boundary condition

$$Q^{01}(x_e,t) = \delta \sum_{n=0}^{\infty} (M_0 \mu_{0n} - r_{0n}) e^{in\Omega t} = 0$$

is applied then a similar procedure to that given above yields

$$B_{1n}^+ = (2K)^{\ell-1}(\Gamma_n(x_e))^{-1}A_{1n}^+$$

where $\Gamma_n(x_e)$ is given by an expression similar to Eq. (D10), with the following changes: the signs of terms divided by $1/M_0$ should be changed and all quantities in (D10) should be evaluated at $x = x_e > 0$.

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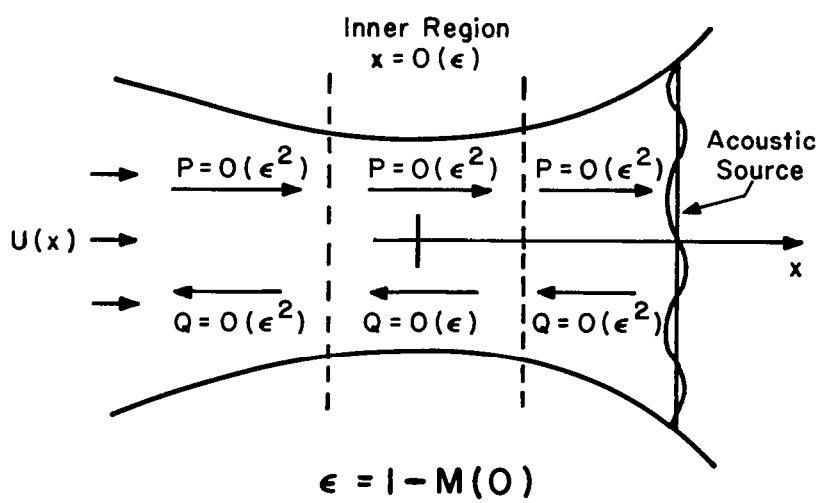
FIG. 1.   Riemann Invariants in a near
          sonic throat.
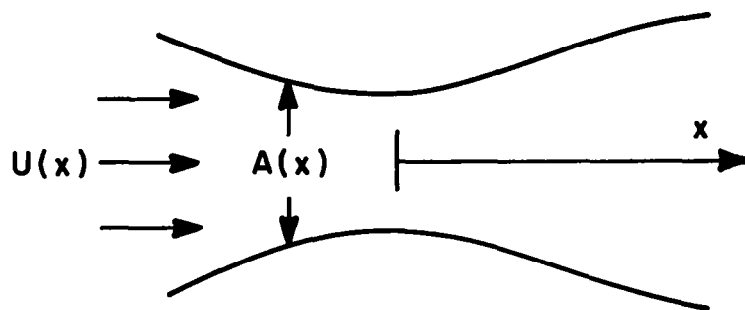
FIG. 2.   Typical duct geometry.
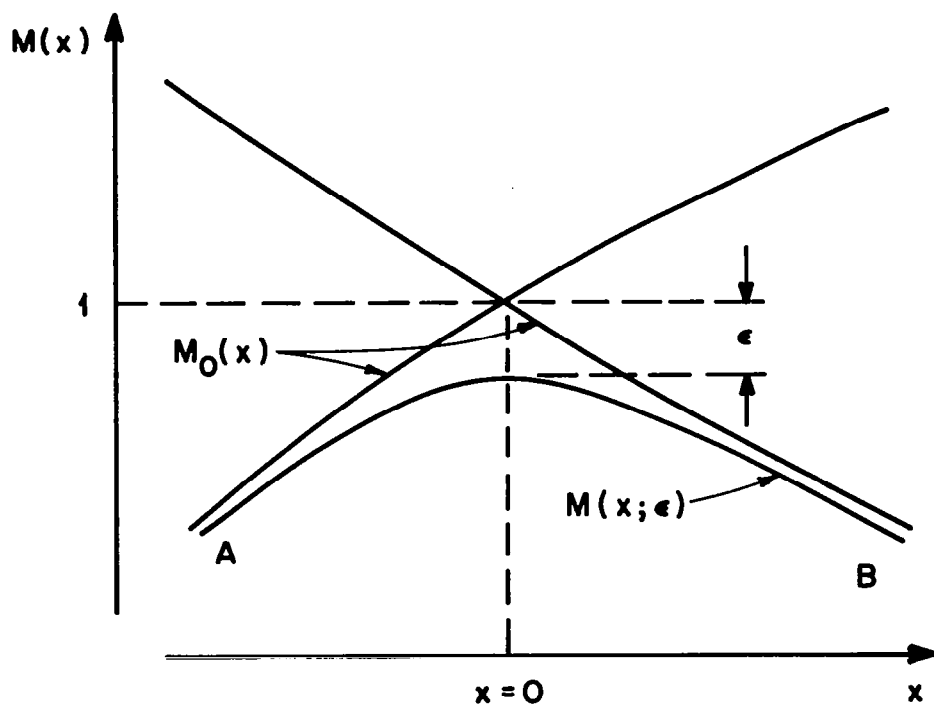
FIG. 3. Typical steady flow integral curve for
Mach number distribution.

63
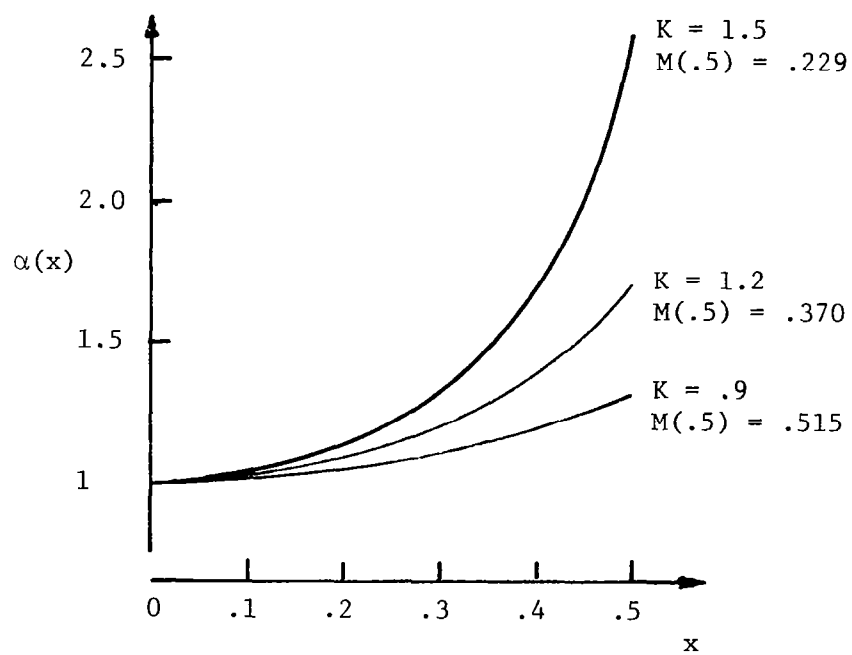
FIG. 4.   Area distribution for
          Crocco-Tsien duct.
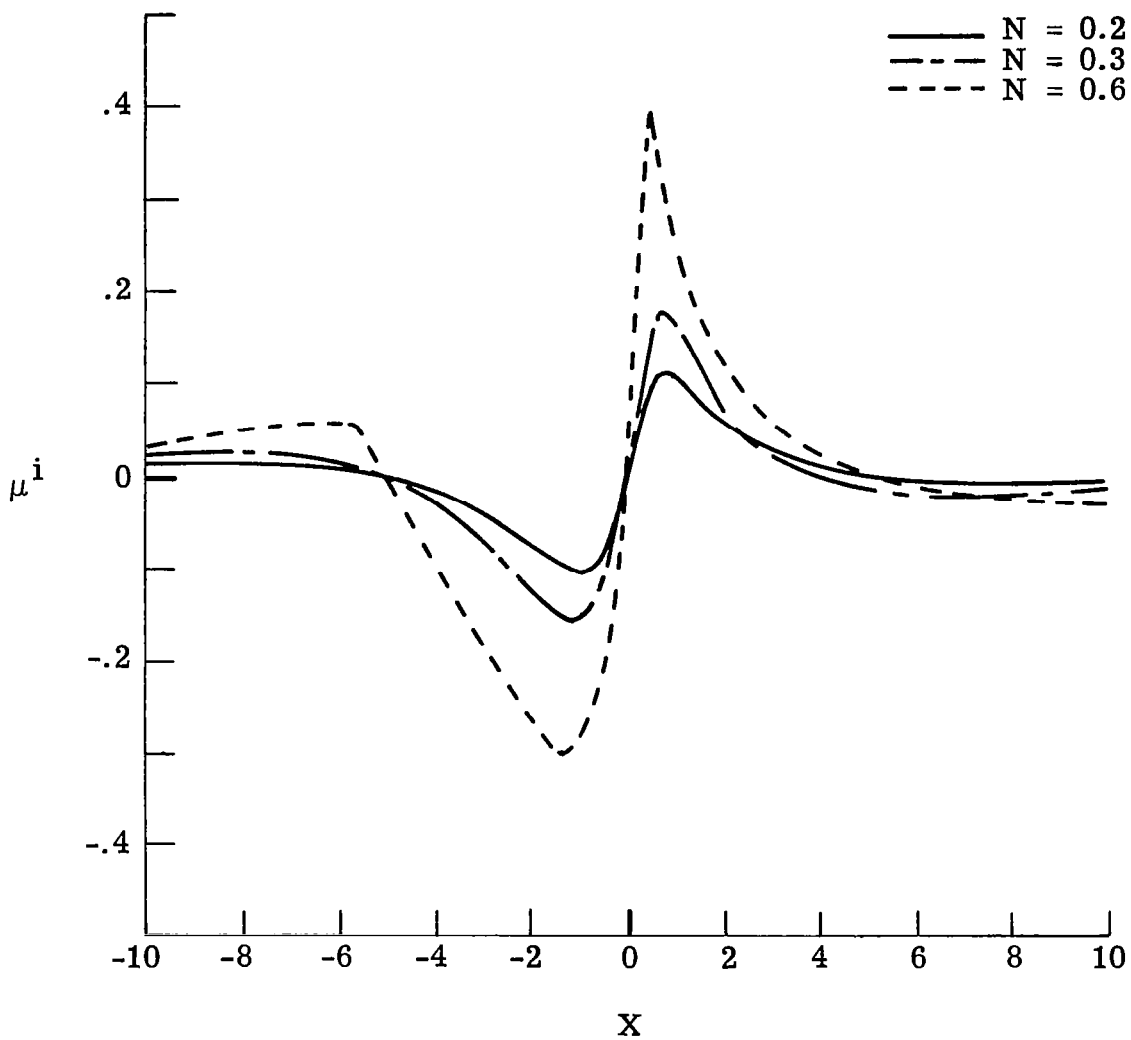
FIG. 5.   Acoustic velocity in the inner region for varying source strength; $\Omega = 4/3$, $a = 1$, $N_0 = 0$.
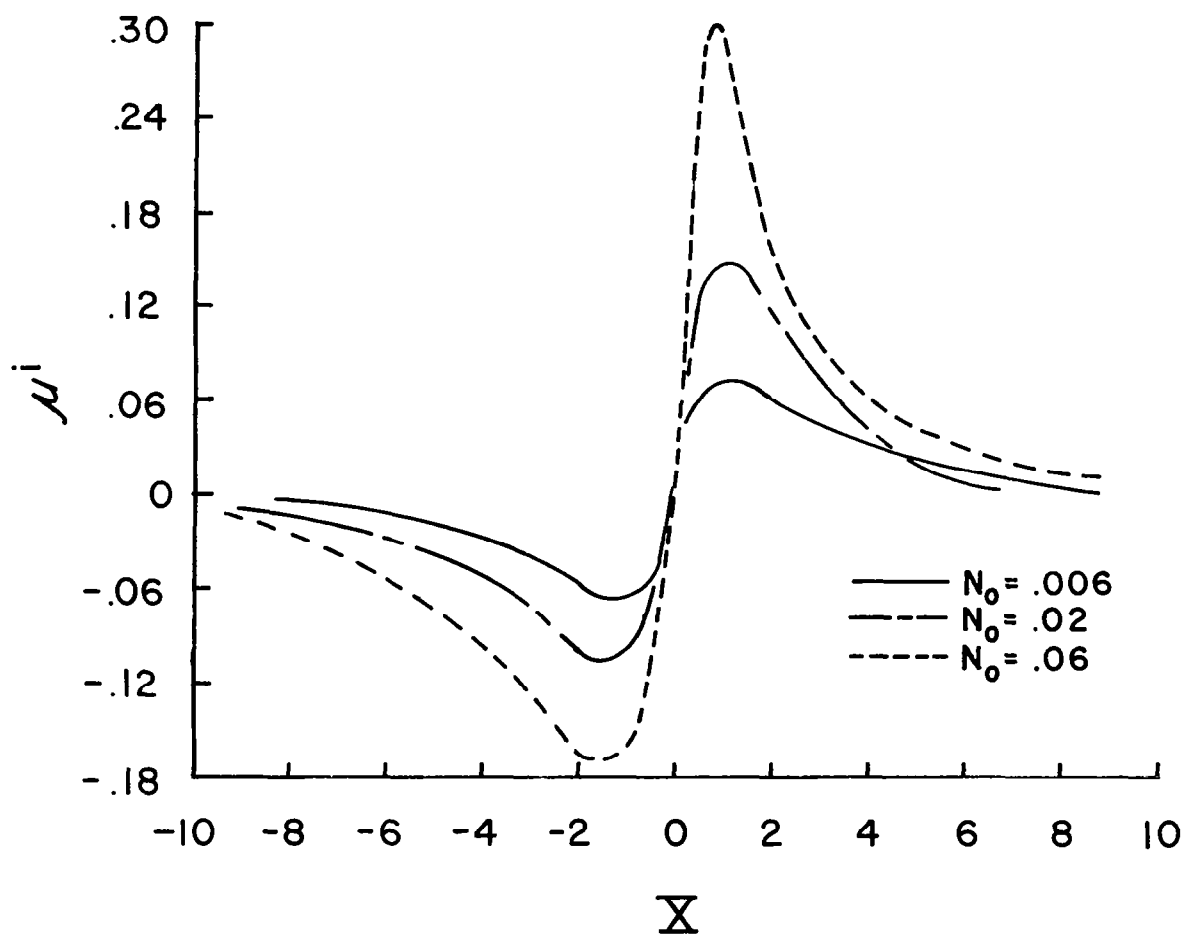
FIG. 6.   Acoustic velocity in the inner region for varying
source strength; a = 1, Ω = 1.
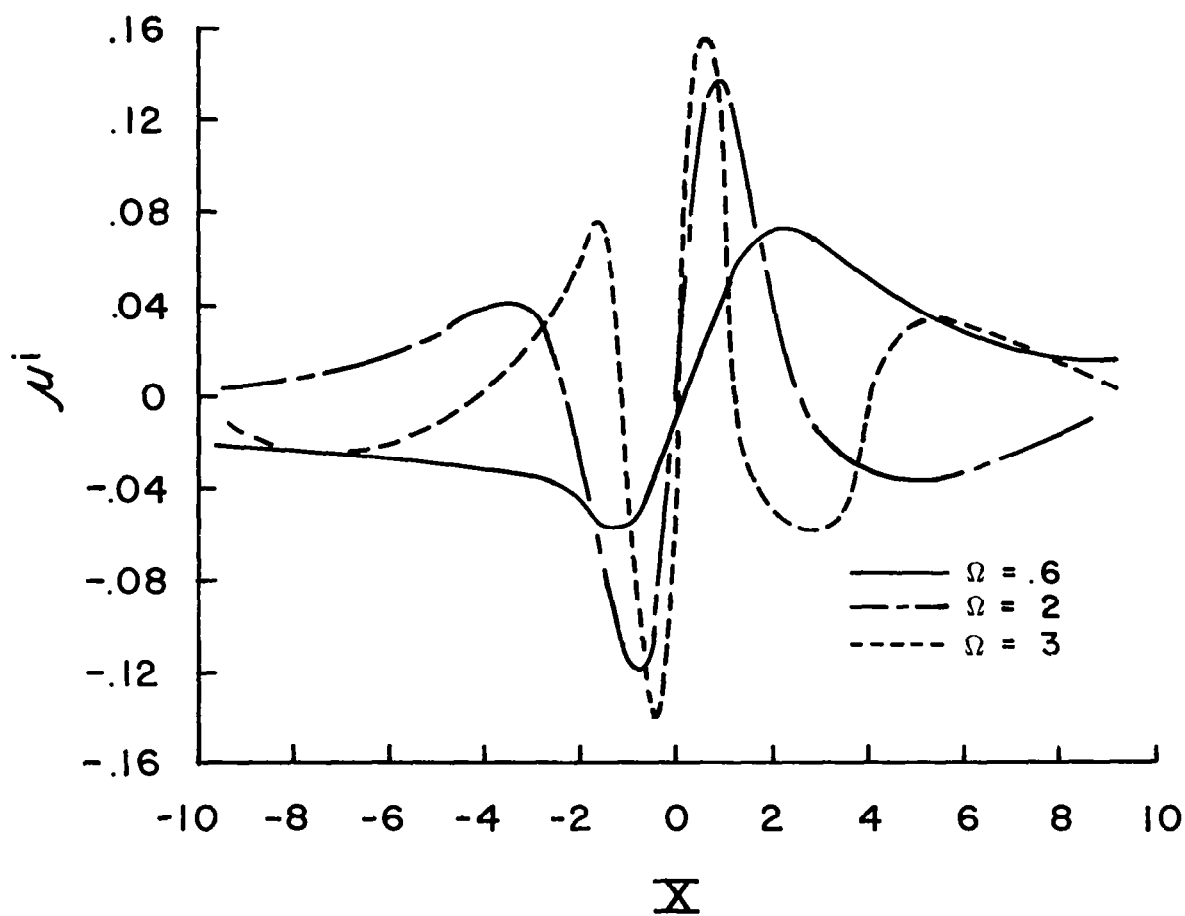
FIG. 7. Acoustic velocity in the inner region for varying source frequency; $a = 1$, $N_0 = .01$.
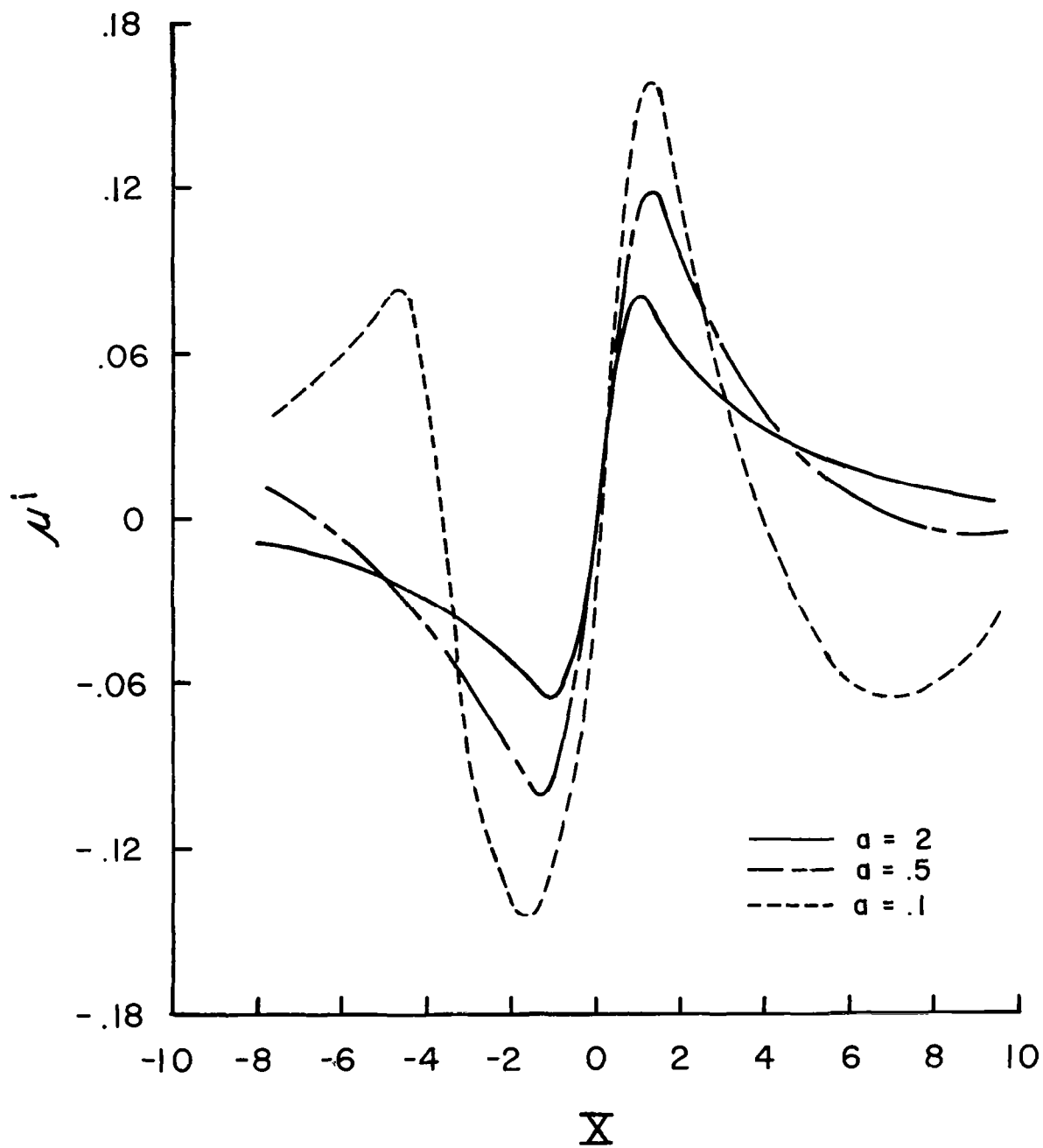
FIG. 8. Acoustic velocity in the inner region for varying
curvature of the throat; $\Omega = 1$, $N_0 = .01$.
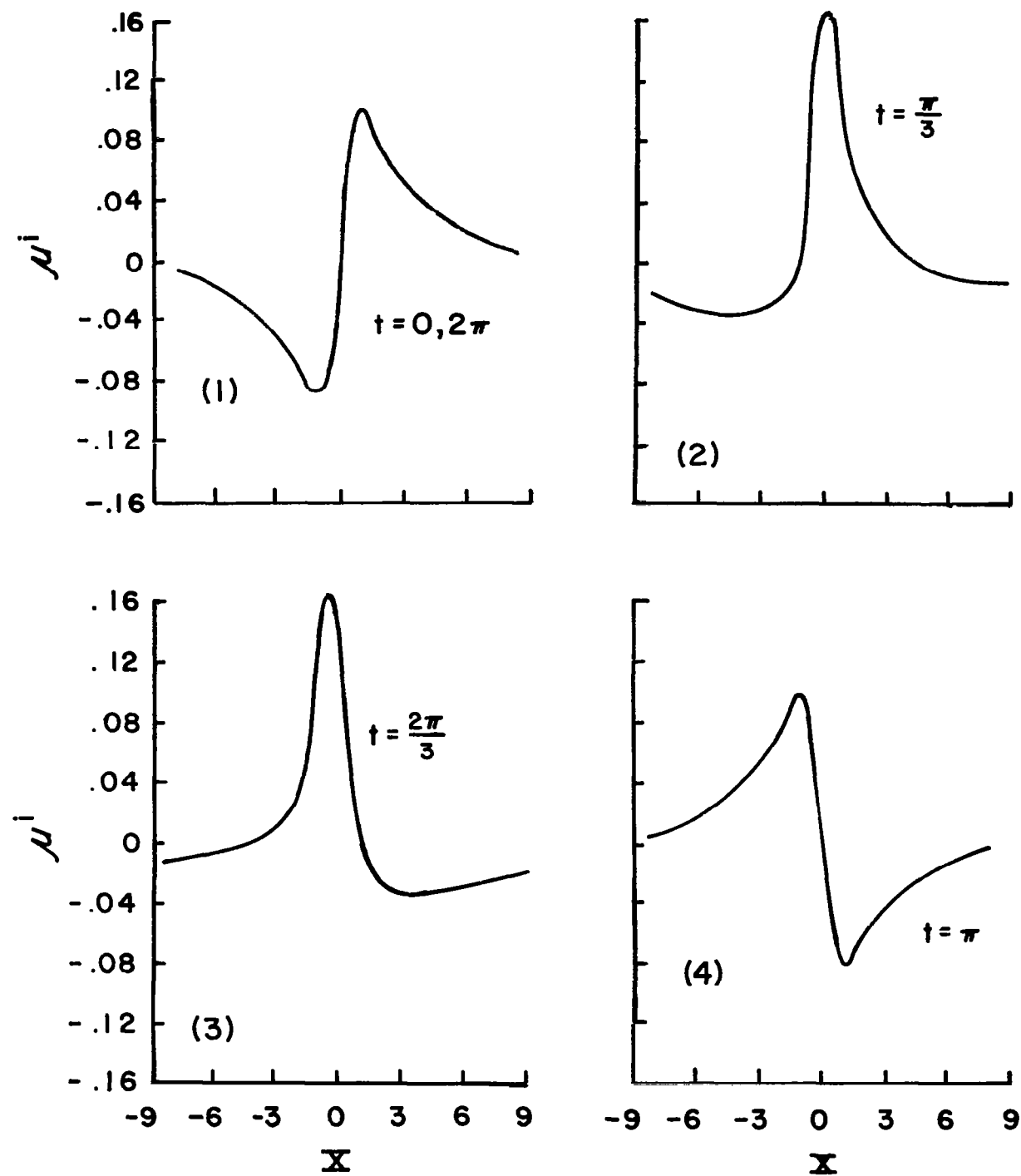
FIG. 9. Time history of acoustic velocity in the inner region over one period; $\Omega = 1$, $N_0 = .01$, $a = 1$.
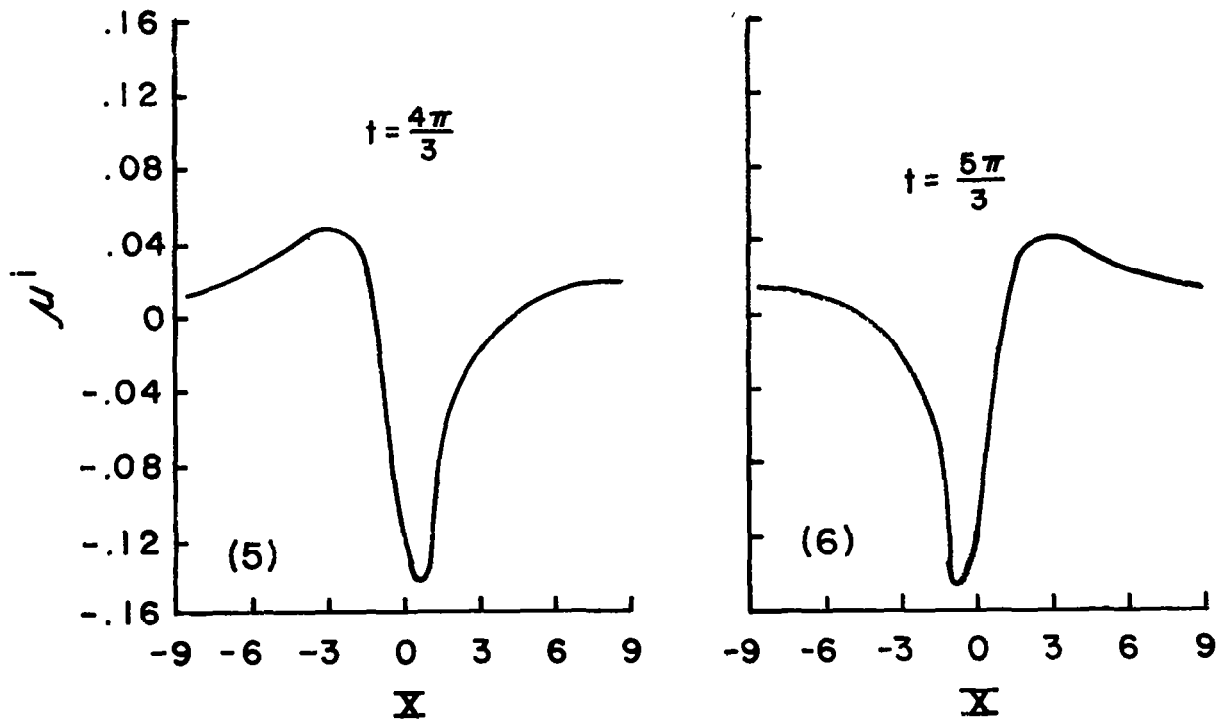
FIG. 9. (Continued) Time history of acoustic velocity in the inner region over one period; $\Omega = 1$, $N_0 = .01$, $a = 1$.

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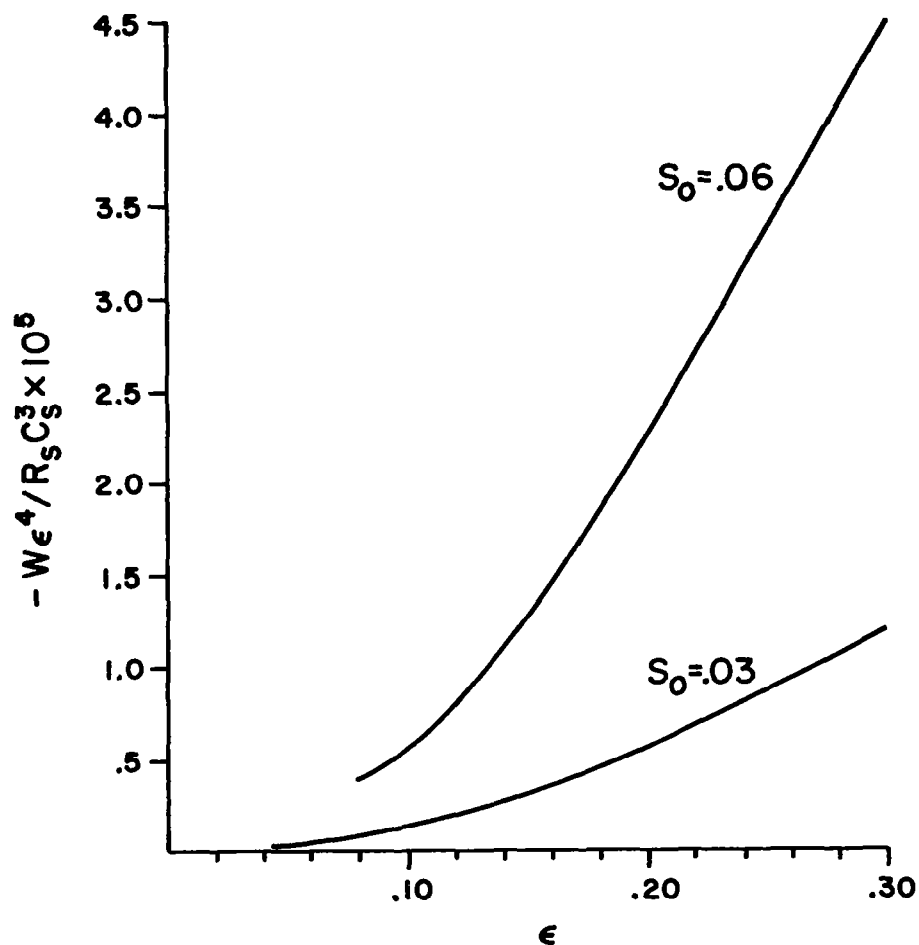
FIG. 10.   Acoustic power transmitted
            upstream for two fixed sources
            vs. the throat Mach number;
            $a = 1$, $\Omega = 1$.
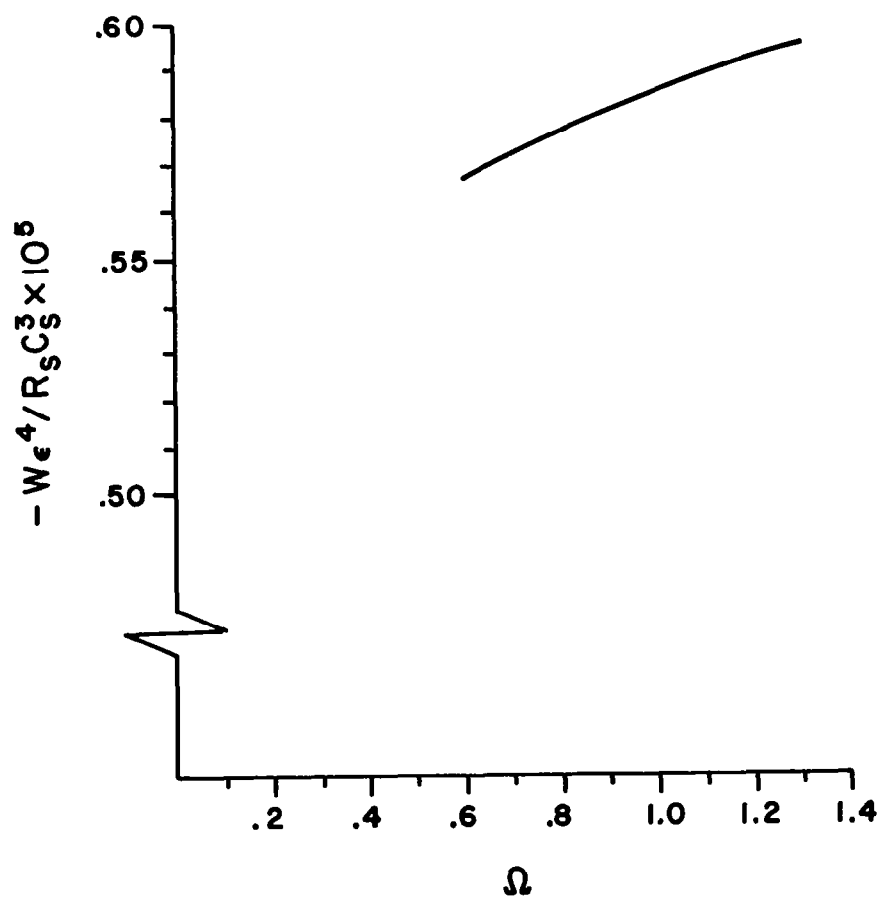
FIG. 11.   Acoustic power transmitted
            upstream as source frequency
            varies; a = 1, N = .6,
            ε = .1.
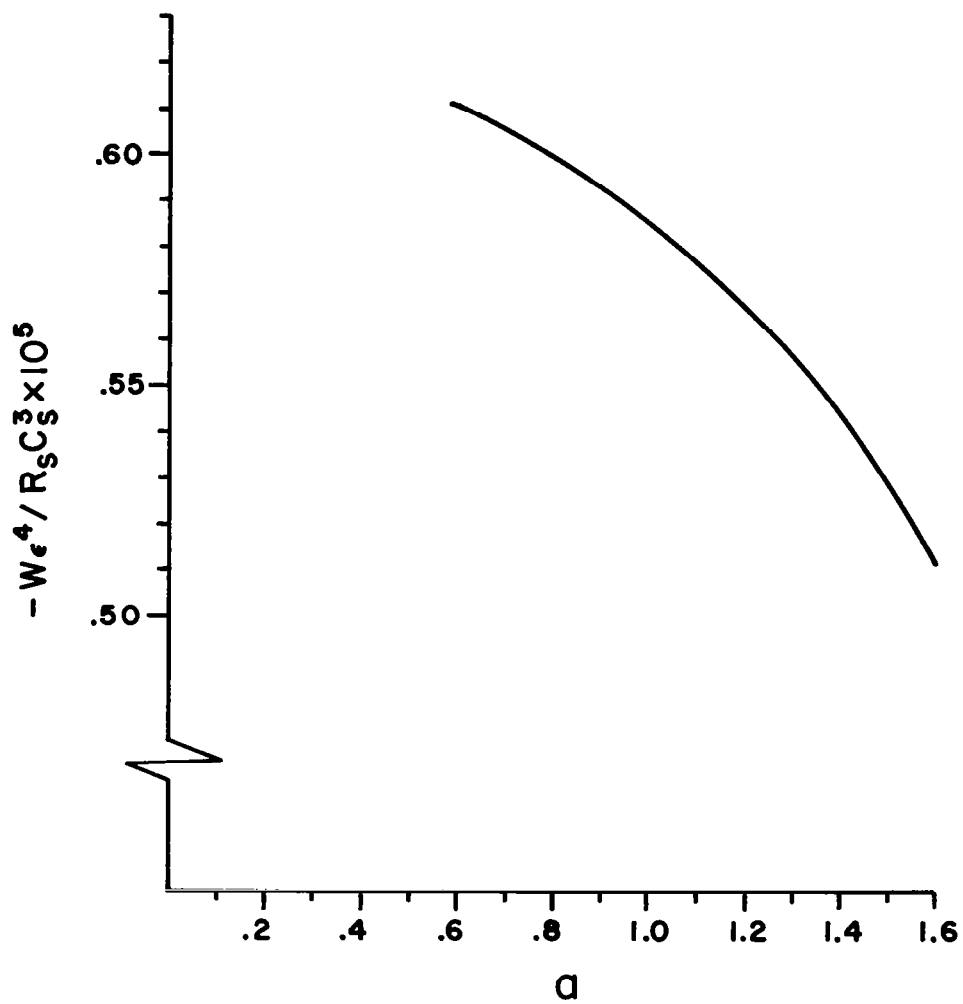
FIG. 12. Acoustic power transmitted upstream as
          curvature of the throat is varied;
          $\Omega = 1$, $N = .6$, $\varepsilon = .1$.
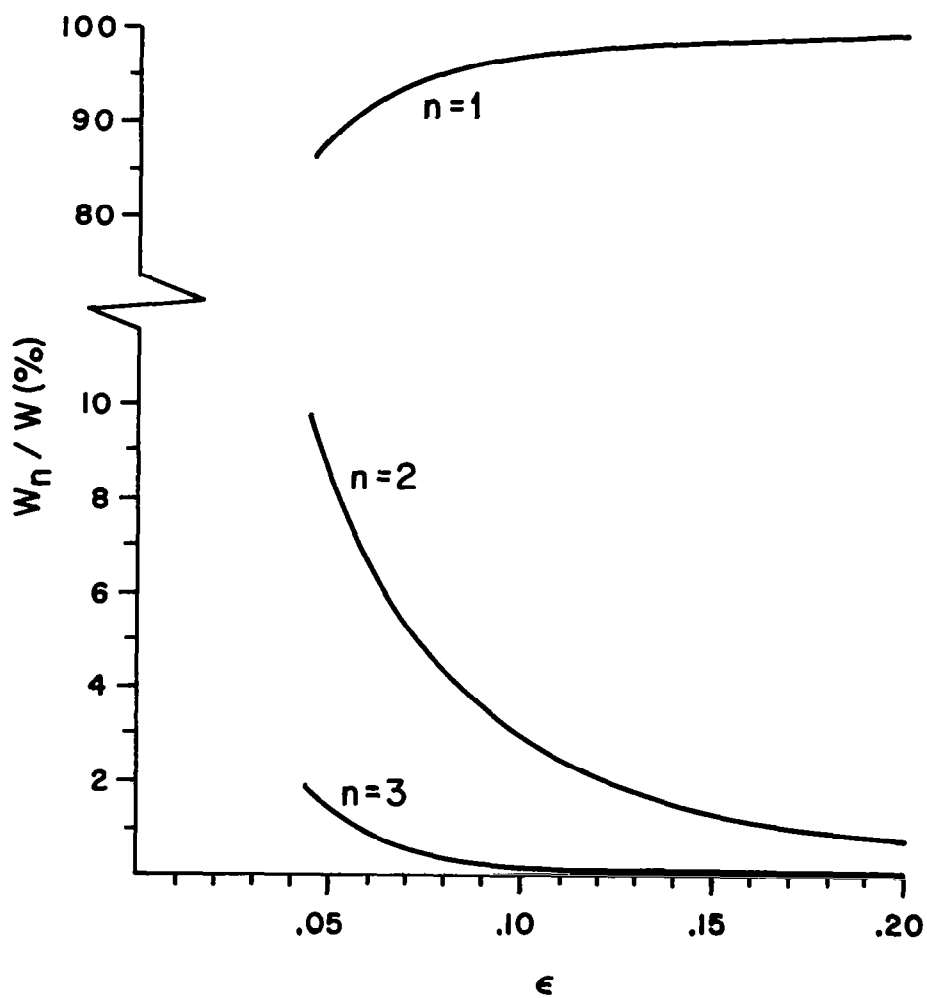
FIG. 13. Percentage distribution of
total power among harmonics
for varying Mach number;
$\Omega = 1$, $a = 1$, $N_0 = 0$,
$S_0 = .03$.
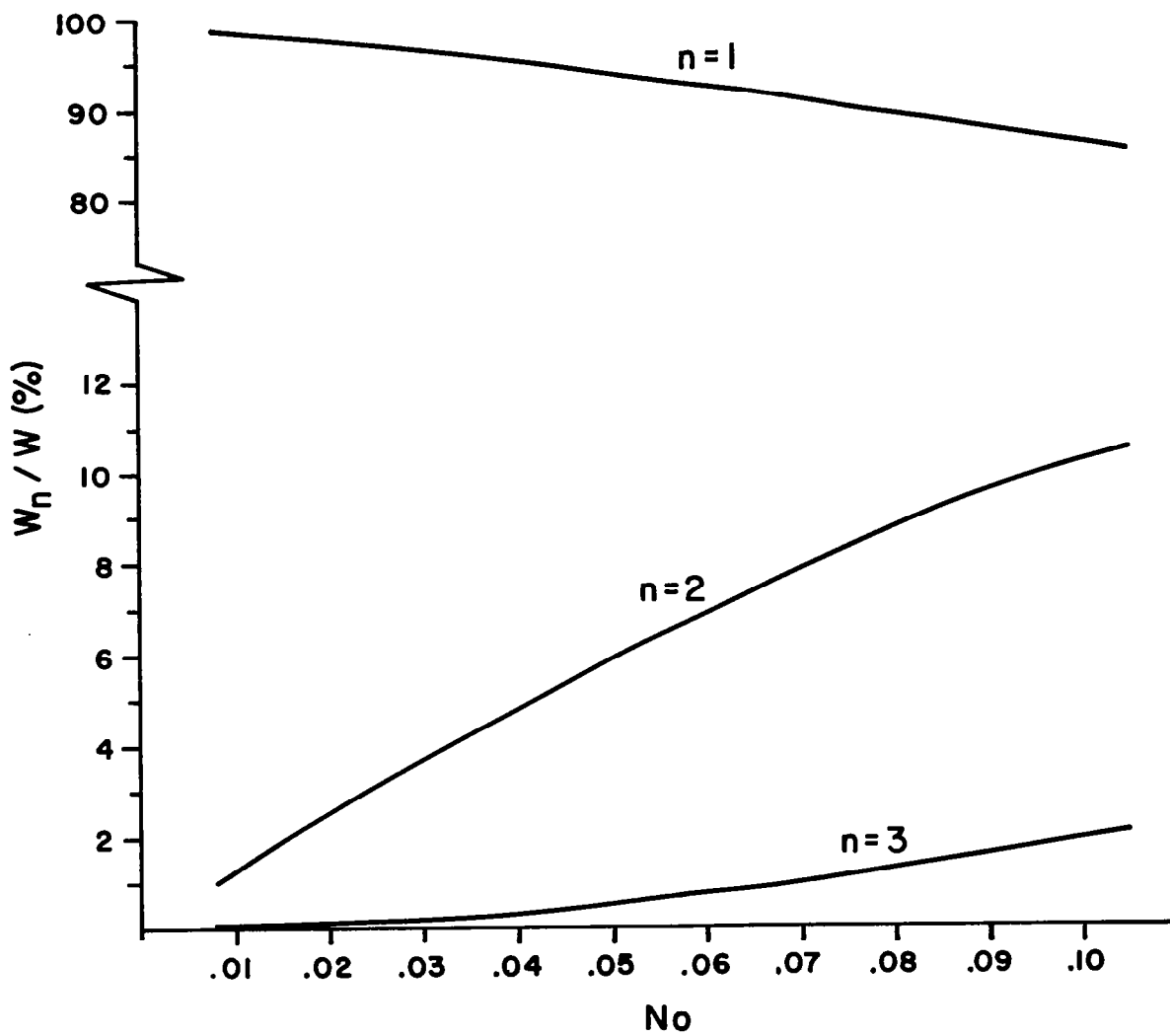
FIG. 14. Percentage distribution of total power among harmonics for varying source strength; $\Omega = 1.0$, $a = 1$, $\varepsilon = .1$.
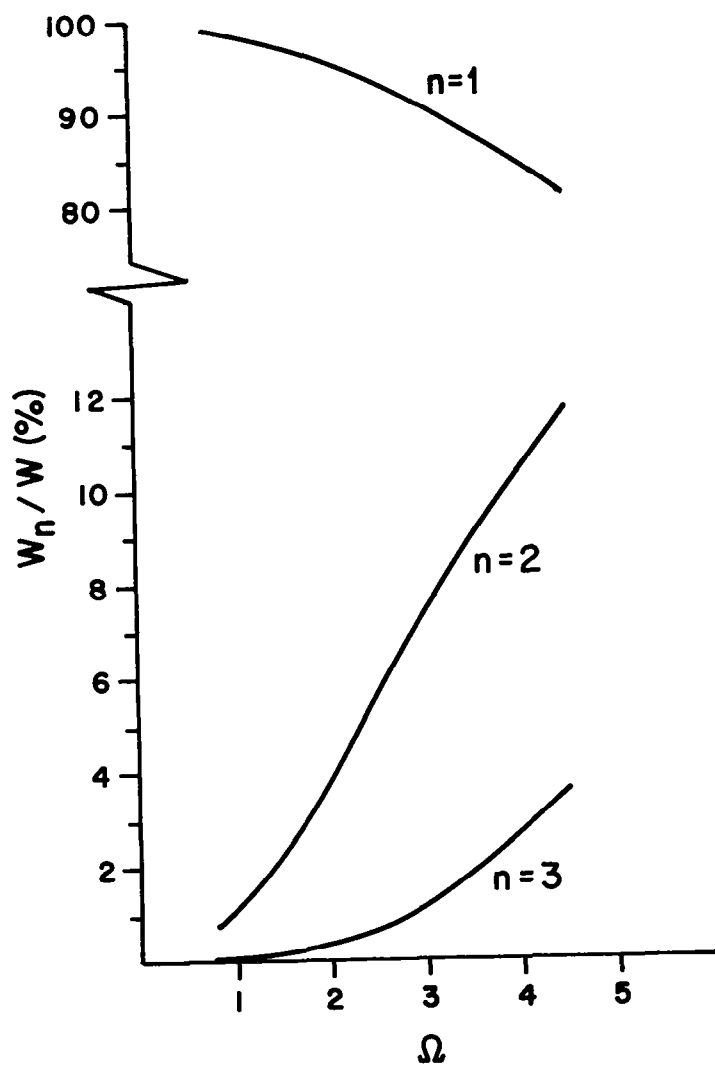
FIG. 15. Percentage distribution of
total power among harmonics
for varying source frequency;
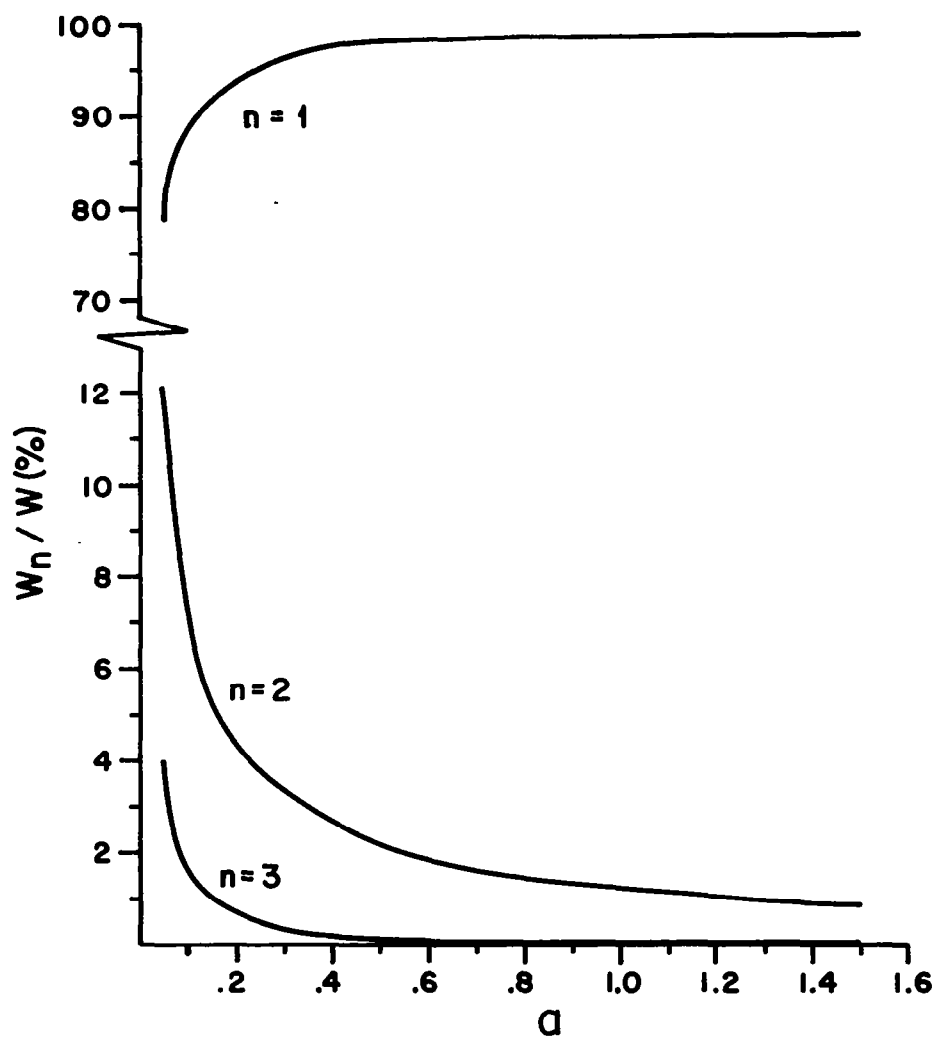$N_0 = .01$, $a = 1$, $\varepsilon = .1$.

FIG. 16. Percentage distribution of total power
          among harmonics for varying throat
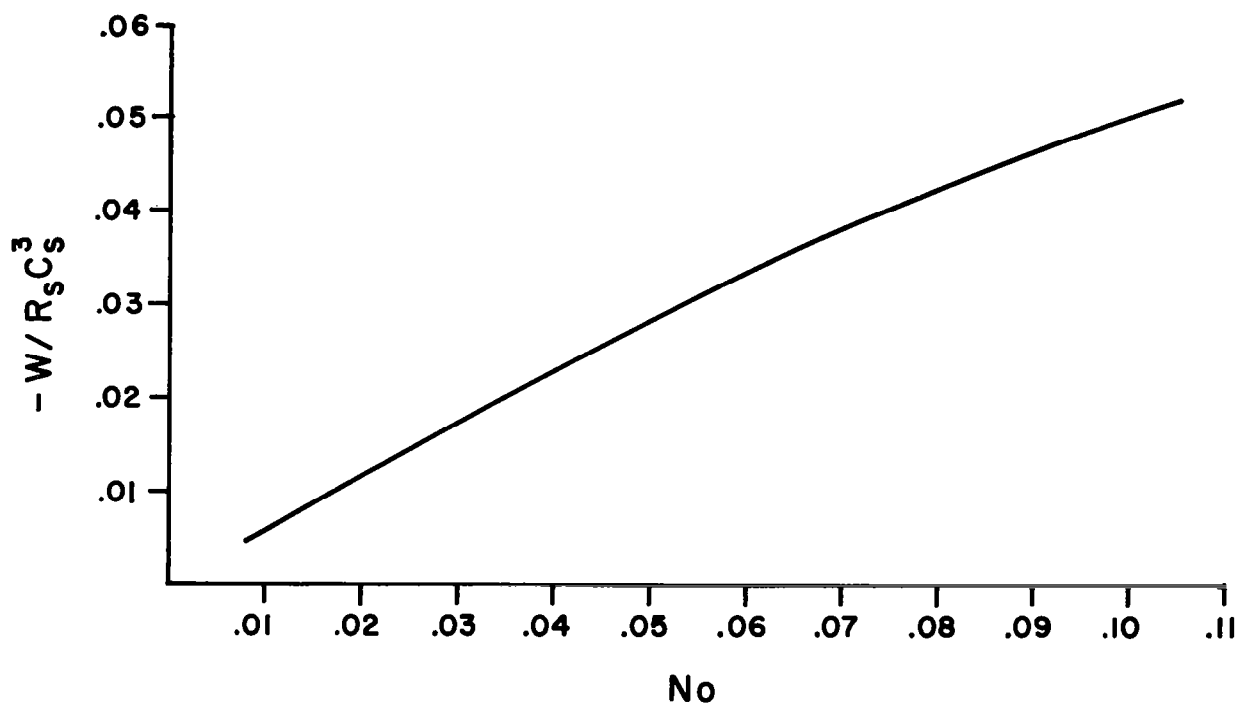   curvature; N_0 = .01, Ω = 1, ϵ = .1.

FIG. 17.   Total power transmitted; $\Omega = 1$, $a = 1$, $\varepsilon = .1$.
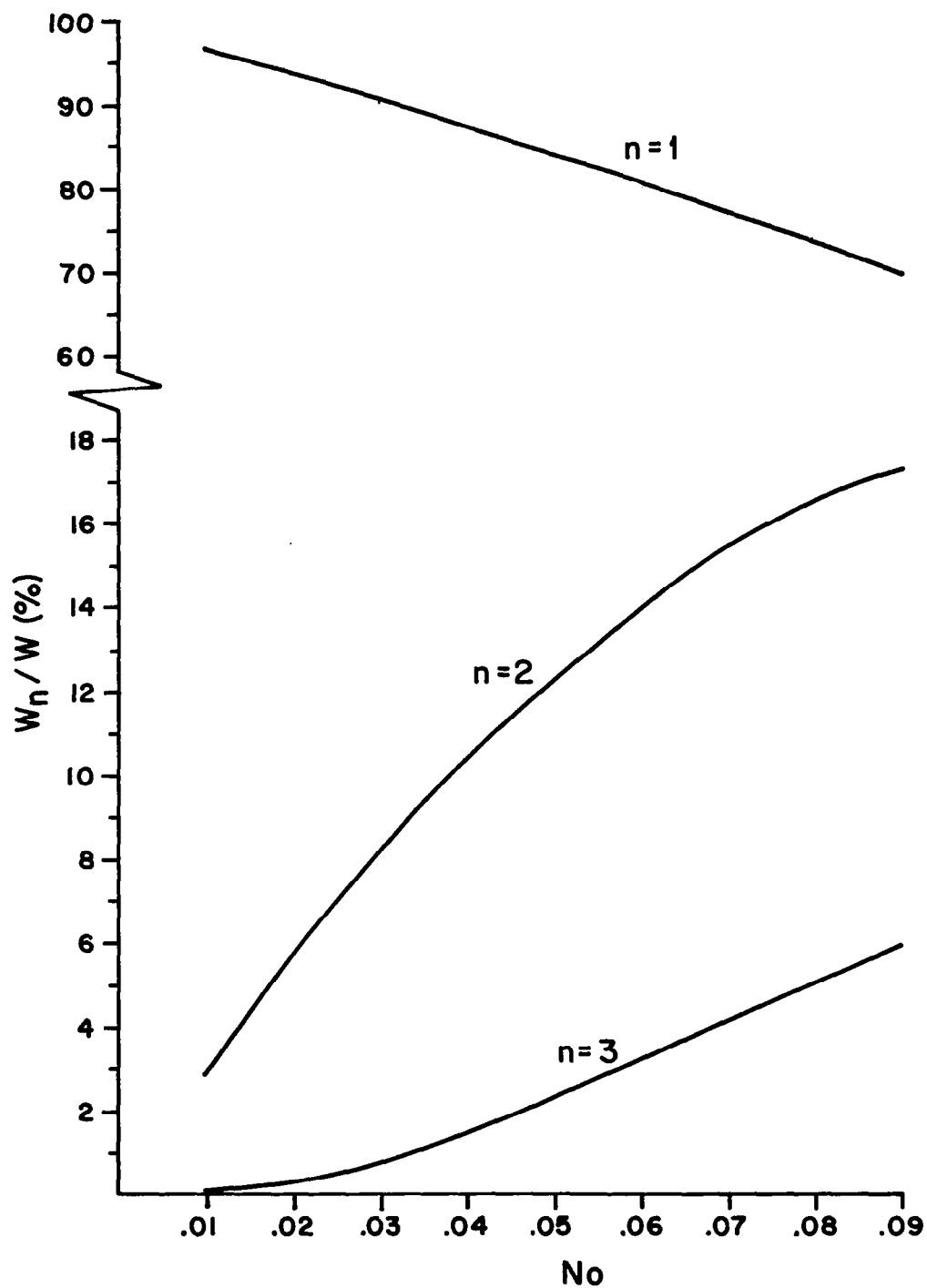
FIG. 18.   Percentage distribution of total power among harmonics
           for varying source strength (downstream power flow);
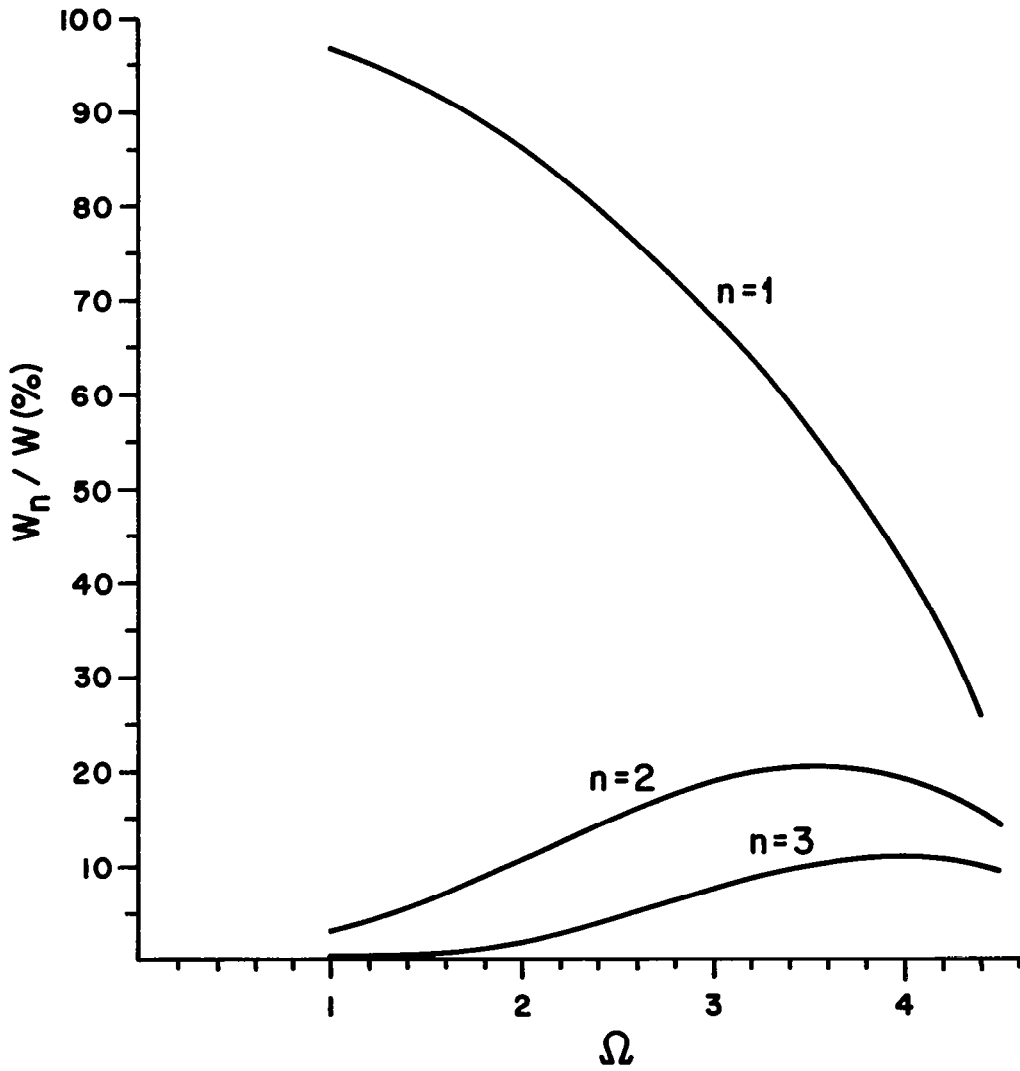 $\Omega = 1$, $a = 1$, $\varepsilon = .1$.

FIG. 19. Percentage distribution of total power
among harmonics for varying source
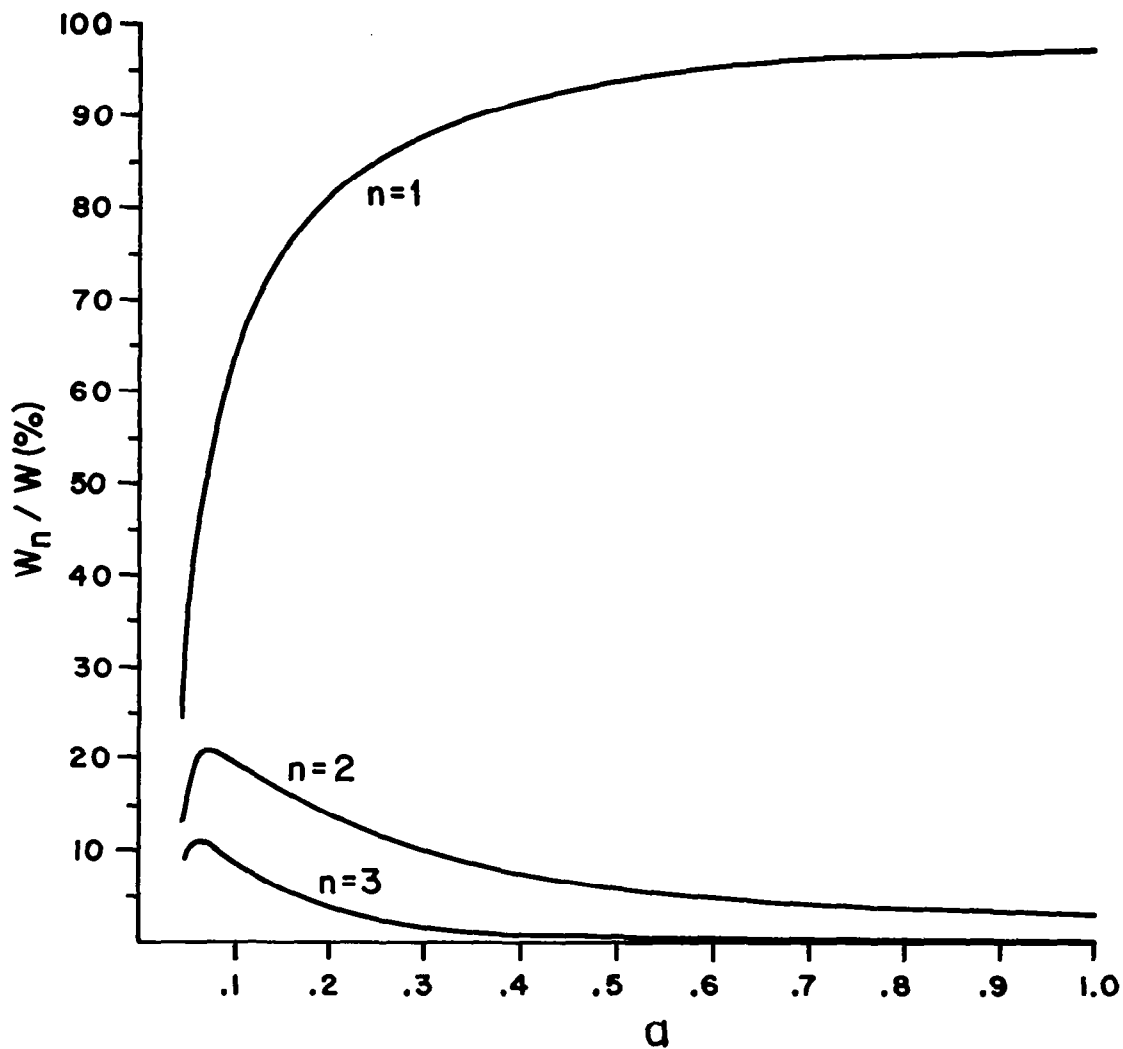frequency (downstream power flow);
$N_0 = .01$, $a = 1$, $\varepsilon = .1$.

FIG. 20. Percentage distribution of total power among harmonics for varying throat curvature (downstream power flow); $N_0 = .01$, $\Omega = 1$, $\varepsilon = .1$.

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15. Supplementary Notes

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Final Report

16. Abstract

A nonlinear theory for sound propagation in variable area ducts carrying a nearly sonic flow is presented. Linear acoustic theory is shown to be singular and the detailed nature of the singularity is used to develop the correct nonlinear theory. The theory is based on a quasi-one dimensional model. It is derived by the method of matched asymptotic expansions.

In a nearly choked flow the theory indicates the following processes to be acting: A transonic trapping of upstream propagating sound causing an intensification of this sound in the throat region of the duct; generation of superharmonics and an acoustic streaming effect; development of shocks in the acoustic quantities near the throat.

Several specific problems are solved analytically and numerical parameter studies are carried out. Results indicate that appreciable acoustic power is shifted to higher harmonics as shocked conditions are approached. The effect of the throat Mach number on the attenuation of upstream propagating sound excited by a fixed source is also determined.

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