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AN APPROXIMATION METHOD FOR ELECTROSTATIC VLASOV TURBULENCE

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ABSTRACT

Electrostatic Vlasov turbulence in a bounded spatial region is considered. An iterative approximation method with a proof of convergence is constructed. The method is non-linear and applicable to strong turbulence.

I. INTRODUCTION

Consider the one dimensional Vlasov-Maxwell system of equations,

1. $$\frac{\partial F}{\partial \tau} + v\frac{\partial F}{\partial x} - E(x,\tau)\frac{\partial F}{\partial v} = 0$$

2. $$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{\infty} dv\, F(x,v,\tau) = 1 - n(x,\tau)$$

3. $$\frac{\partial E}{\partial \tau} = \int_{-\infty}^{\infty} dv\, v\, F(x,v,\tau) = u(x,\tau)$$

for the electron distribution function, $F(x,v,\tau)$, and the electric field, $E(x,\tau)$, with a stationary and uniform ion background. Assume the existence of a solution of this system of equations for $-1 \le x \le 1$ (x is dimensionless, and measured in units of an arbitrarily length scale, L), $-\infty < v < \infty$, and $\tau > 0$. Then a method for constructing approximations to this solution can be developed as follows:

The Fourier-Fourier transform method[1] can be used to transform equations 1-3 into an infinite system of first order hyperbolic partial differential equations. This system has been studied before.[1] One further aspect of this system will be considered here; it is that, on truncating the infinite system, the resulting finite system is of a standard form which has been used to produce constructive proofs of the existence of solutions to a wide class of such systems.[2] Here, the existence of a solution to the finite system will be assumed. The methods used for the proofs of existence will be applied, nevertheless, to produce approximations to the exact solution of the finite system. The existence of the finite system solution and the issue of how well it approximates the solution of the infinite system will be addressed elsewhere.

II. THE FOURIER-FOURIER TRANSFORM

Let,

4. $$f(x,v,\gamma) = \sum_{m=-\infty}^{\infty} e^{im\pi x} \frac{1}{2}\int_{-\infty}^{\infty} dr\, e^{-i\pi r v} f_m(r,\gamma)$$

and

$$\epsilon(x,\gamma) = \sum_{m=-\infty}^{\infty} e^{im\pi x} \epsilon_m(\gamma)$$

in which

$$f_m(r,\gamma) = \frac{1}{2}\int_{-1}^{1} dx\, e^{-im\pi x} \int_{-\infty}^{\infty} dv\, e^{i\pi r v} F(x,v,\gamma)$$

and

$$\epsilon_m(\tau) = \frac{1}{2}\int_{-1}^{1} dx\ e^{-im\pi x}\ E(x,\tau)$$

then, $f = F$ and $\epsilon = E$ for $-1 < x < 1$, and on the boundaries $f(\pm 1, v,\tau) = (1/2\ [F(1,v,\tau) + F(-1,v,\tau)]$ and $\epsilon(\pm 1,\tau) = (1/2)\ [E(1,\tau) + E(-1,\tau)]$. Both f and ϵ are periodic in x with period, 2.

From equations 1 through 3,

6. $$\frac{\partial f_m}{\partial \tau} + m\frac{\partial f_m}{\partial \gamma} + i\pi\gamma\sum_{n=-\infty}^{\infty}\epsilon_n f_{m-n} = J_m$$

7. $$im\pi\epsilon_m = \delta_{m,0} - f_m(0,\tau) + (-1)^{m+1}(1 - f_0(0,\tau))$$

and,

8. $$\frac{d\epsilon_m}{d\tau} = -\frac{i}{\pi}\left.\frac{\partial f_m}{\partial \gamma}\right|_{\gamma=0}$$

in which $J_m = (-1)^m J_o$, and

$$J_o(\gamma,\tau) = \frac{i}{2\pi}\frac{\partial}{\partial\gamma}\left[F(1,\gamma,\tau) - F(-1,\gamma,\tau)\right]$$

where,

$$F(\pm 1,\gamma,\tau) = \int_{-\infty}^{\infty} dv\ e^{i\pi\gamma v}\ F(\pm 1, v,\tau)$$

Equations 7 and 8 are redundant when $m \neq 0$, and equation 7 yields no information when $m = 0$. In the following equation 7 will be used to determine the $\epsilon_m(\tau)$ when $m \neq 0$, and equation 8 will be used to determine $\epsilon_o(\tau)$.

It will be assumed in the following that $f_m = 0$ for $m > M$ where M is arbitrarily large, but finite. Since it is expected that those modes which have wavelengths comparable or shorter than the Debye length in the plasma will be strongly damped,[3] there is perhaps some a priori justification for the truncation.

In Appendix A it is argued that the solution of equations 1-3 is determined by a choice of $F(x,v,0)$ on the initial plane ($-1 \lesssim x \lesssim 1$, $-\infty < v < \infty$, $\tau = 0$), of $\Delta(v,\tau) = F(-1,v,\tau) - F(1,v,\tau)$ in terms of F on the boundaries ($x = \pm 1$, $-\infty < v < \infty$, $\tau > 0$), and of $\epsilon_o(0)$. That argument does not depend on the assumption of a neutral plasma (space-averaged). In the following a neutral plasma will be assumed and then it will be shown that the choice of the equivalent quantities, $f_m(\nu,0)$, $J_o(\nu,\tau)$ and $\epsilon_o(0)$ uniquely determines the solution of the truncated equations 6-8.

III. THE NEUTRAL PLASMA

The space-averaged electron density is $f_o(0,\tau)$. The restriction to a neutral plasma is affected by setting $f_o(0,\tau) = 1$. This restriction is consistent with equation 6 only for a limited class of $J_o(\nu,\tau)$. From equation 6,

$$\frac{\partial f_o(0,\tau)}{\partial \tau} = J_o(0,\tau) = \frac{1}{2}\,\delta u(\tau)$$

where $\delta u(\tau) = u(-1,\tau) - u(1,\tau)$. In the following it will be assumed that $J_o(0,\tau) = 0$, and that $f_o(0,\tau,) = f_o(0,0) = 1$.

Fyfe and Montgomery* have noted that $\epsilon_0(\tau)$ cannot be chosen freely. They have produced an exact solution for $\epsilon_0(\tau)$ and $u_0(\tau)$ (the space average of $u(x,\tau)$) from their model of the one dimensional Vlasov-Maxwell plasma. Their results apply to the periodic plasma ($\Delta(v,\tau) = 0$). A generalization to the non-periodic neutral plasma being considered here is possible.

From equation 8 (or equation 3),

9.
$$\frac{d\epsilon_0}{d\tau} = -\frac{i}{\pi}\left.\frac{\partial f_0}{\partial r}\right|_{r=0} \equiv -\frac{i}{\pi} f_0'(0,\tau) = u_0$$

and from equation 6 (or equation 1),

$$-\frac{i}{\pi}\frac{\partial f_0'(0,\tau)}{\partial \tau} + \epsilon_0(\tau) = J_0'(0,\tau)$$

or,

10.
$$\frac{du_0}{d\tau} + \epsilon_0 = \frac{1}{2}\delta P$$

where $\delta P(\tau) = P(-1,\tau) - P(1,\tau)$ and

$$P(x,\tau) = \int_{-\infty}^{\infty} dv\, v^2 F(x,v,\tau)$$

(Notice that since, for the neutral plasma, $u(-1,\tau) = u(1,\tau)$, δP is actually just the difference in electron plasma temperature at the boundaries.) An exact solution of equations 9 and 10 is available; it is

$$\begin{pmatrix} \epsilon_o(\tau) \\ u_o(\tau) \end{pmatrix} = \begin{pmatrix} \epsilon_o(0)\cos\tau + u_o(0)\sin\tau \\ u_o(0)\cos\tau - \epsilon_o(0)\sin\tau \end{pmatrix} + \frac{1}{2}\int_0^\tau d\lambda \; \delta P(\lambda) \begin{pmatrix} \sin(\tau-\lambda) \\ \cos(\tau-\lambda) \end{pmatrix} \qquad 11.$$

The result of Fyfe and Montgomery is regained when $\delta P = 0$.

Notice that it is possible to obtain large $\epsilon_o(\tau)$ and $u_o(\tau)$ due to an approximately linear growth of the integral in equation 11 with increasing τ if $\delta P(\tau)$ contains harmonic oscillations with period 1 (the inverse plasma frequency). It is not possible for $\epsilon_o(\tau)$ to be constant in time unless δP is also constant, $\epsilon_o(0) = 1/2\ \delta P$ and $u_o(0) = 0$. Under these conditions $u_o(\tau) = u_o(0) = 0$. Since $\epsilon_o(\tau)$ is a measure of the potential difference on the boundaries, it should be noted that the preceding statements concerning $\epsilon_o(\tau)$ apply also to that potential difference. All of the above, and any other consequence of equation 11, apply exactly for the neutral plasma no matter what else is occuring in the plasma.

IV. BASIC INTEGRAL EQUATION

Given the solution for $\epsilon_o(\tau)$ and $u_o(\tau)$, a major reduction in the complexity of equations 6-8 can be made by introducing a new dependent variable through,

$$f_m(v,\tau) = K_m(v,\tau)\exp\left[-i\pi\int_0^\tau d\lambda\,[v - m(\tau-\lambda)]\,\epsilon_o(\lambda)\right]$$

then,

$$12. \quad \frac{\partial K_m}{\partial \tau} + m\frac{\partial K_m}{\partial \nu} - \nu \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) K_n(0,\tau)\, K_{m-n} = \sigma_m$$

where,

$$\sigma_m(\nu,\tau) = J_m(\nu,\tau)\exp\left[i\pi\int_0^\tau d\lambda\,[\nu - m(\tau-\lambda)]\,\epsilon_o(\lambda)\right]$$

and the prime on the summation symbol indicates that the n = 0 term is omitted. Since $\epsilon_o(\tau)$ can be considered a known function of time, a solution of equation 12 for K_m is equivalent to a solution of equations 6-8 for f_m. Notice that in the special case of a periodic plasma ($J_m = 0$) equation 12 becomes independent of $\epsilon_o(\tau)$. Thus, a single solution of equation 12, which will be shown to be determined solely by $K_m(\nu,0)$, is equivalent to the entire class of solutions for f_m which contains all possible choices of $\epsilon_o(0)$.

Using the method of characteristics[5], equation 12 can be integrated once to obtain,

$$K_m(\nu,\tau) = K_m(\nu - m\tau, 0) + \int_0^\tau d\lambda\,\sigma_m(\nu - m(\tau-\lambda),\lambda)$$

13.

$$+ \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right)\int_0^\tau d\lambda\,[\nu - m(\tau-\lambda)]\,K_n(0,\lambda)\,K_{m-n}(\nu - m(\tau-\lambda),\lambda)$$

Equation 13 will play a central role in the following development of approximations to K_m.

V. THE APPROXIMATION METHOD

The result which will be obtained in this section can be simply stated as follows:

A sequence of functions, $K_m(\nu,\tau;\alpha)$, will be introduced with

$$14.\quad K_m(\nu,\tau;0) = K_m(\nu - m\tau, 0) + \int_0^{\tau} d\lambda\, \sigma_m(\nu - m(\tau-\lambda), \lambda)$$

Since $F(x,v,0)$ will be assumed given, $K_m(\nu,0)$ can be considered a known function of ν which is uniquely related to the initial F; σ_m can be determined from $\Delta(v,\tau)$. The other members of the sequence are to be related to each other through,

$$15.\quad \frac{\partial K_m(\alpha+1)}{\partial \tau} + m\,\frac{\partial K_m(\alpha+1)}{\partial \nu} - \nu \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) K_n(0,\tau;\alpha)\, K_{m-n}(\alpha) = \sigma_m$$

or,

16. $$K_m(\nu, \tau; \alpha+1) = K_m(\nu, \tau; 0) + \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^{\tau} d\lambda \, [\nu - m(\tau-\lambda)] \, K_n(0, \lambda; \alpha) \, K_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

It will be shown that $\lim_{\alpha \to \infty} K_m(\nu, \tau; \alpha) = K_m(\nu, \tau)$; i.e., it will be shown that the sequence of approximations must converge to the exact solution. This convergence will not depend on the presence of a small parameter for expansion purposes and will apply for any finite value of M. Thus, this approximation method applies to strong turbulence with any finite number of wave modes, no matter how large. Notice that the character of the method is to place any member of the sequence in quadrature (through equation 16); it does not produce equations which must be solved.

1. Preliminaries to Proof of Convergence

To facilitate the proof of convergence the K_m will be asssumed vector components of a (2M + 1) - dimensional vector function, $K(\nu,\tau) = (K_{-M}(\nu,\tau), K_{-M+1}(\nu,\tau) \ldots K_M(\nu,\tau))$. Equation 16 can be considered an integral transformation which relates members of the sequence through $K(\alpha + 1) = T\, K(\alpha)$.

The proof of convergence will be given on the closed domain, O, pictured in figure 1. An examination of equation 16 will show that knowledge of $K(\alpha)$ on O is necessary and sufficient to determine $K(\alpha + 1)$ on O. The domain, O, is centered on the line $\nu = 0$ since on that line all of the quantities of physical interest (the various moments of F as well as the Fourier components of the electric field) can be found. The time, T, is any finite time of interest; the solution will be obtained for all $0 < \tau \leq T$.

A further subdivision of O is necessary. Imagine the strips on the (ν,τ) - plane defined by $r\delta < \tau \leq (r+1)\delta$ where $r = 0,1,2...$ and $\delta > 0$ is to be determined. Then, let $O_{r\delta}$ be the intersection of the r'th strip with O. The transformation, T, will be shown to be a contraction* on each of the $O_{r\delta}$ and then the results for the stationary element of T on each strip will be pieced together to yield equation 16.

The following definition of a norm will be used. Let,

$$\|K\|_O = \sup_{(\text{On domain } O)} \left(\max \left\{ |K_{-M}(\nu,\tau)|, \ldots, |K_M(\nu,\tau)| \right\} \right)$$

At each point, (ν,τ), the absolute values of all of the $K_m(\nu,\tau)$ are to be taken, and then the maximum of these chosen. Then, the supremum, on O, of the resulting function is to be found. The vector, K, and the domain, O, have been used here for illustrative purposes. Other vectors and domains will appear, but in each case, the symbol, $|| \ ||_O$, has the analogous meaning.

In Appendix B it is shown that, in the limit $M = \infty$, $||K||_O = 1$. In the following it will be assumed that $||K||_O$ exists for finite M. This is not actually a new assumption; on O, it is totally equivalent to the earlier assumption of the existence of a solution to the truncated system. It will also be assumed that the boundary conditions are chosen so that $||\sigma||_O$ exists. The number, $N = 2(||K||_O + ||\sigma||_O T)$, will be used.

2. Convergence on a Narrow Strip

Equation 13 can be used to show that

$$K_m(\nu,\tau) = K_m(\nu - m(\tau - r\delta), r\delta) + \int_{r\delta}^{\tau} d\lambda\, \sigma_m(\nu - m(\tau-\lambda), \lambda)$$

17.

$$+ \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_{r\delta}^{\tau} d\lambda [\nu - m(\tau-\lambda)]\, K_n(0,\lambda) K_{m-n}(\nu - m(\tau-\lambda), \lambda)$$

for any $\tau \geq r\delta$. The proof that, on $O_{r\sigma}$,

$$K_m(\nu,\tau) = K_m(\nu - m(\tau - r\delta), r\delta) + \int_{r\delta}^{\tau} d\lambda\, \sigma_m(\nu - m(\tau-\lambda), \lambda)$$

18.

$$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_{r\delta}^{\tau} d\lambda [\nu - m(\tau-\lambda)]\, K_n(0,\lambda;\alpha) K_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

will now be given with $K_m(\nu,\tau;\ 0)$ determined by equation 14. The symbol, $T_{r\delta}$, will be used for the integral transformation in equation 17 on $O_{r\delta}$. The proof is in three parts.

3. <u>Convergence at large</u>

The proof of convergence on a narrow strip given in the preceding section will be used here to construct a proof of convergence at large (on the domain, O). It will be shown that, on O, $\lim_{\alpha\to\infty} K_m(\nu,\tau;\alpha) = K_m(\nu,\tau)$ where $K_m(\nu,\tau;0)$ is given by equation 14 and $K_m(\nu,\tau;\alpha)$ is given by equation 16. An inductive argument will be given which assumes equation 18 on $O_{r\delta}$ as a starting point.

Proof:

Notice from equation 18 that, when r = 0,

19.

$$K_m(\nu,\tau) = K_m(\nu - m\tau, 0) + \int_0^{\tau} d\lambda\, \sigma_m(\nu - m(\tau-\lambda), \lambda)$$
$$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^{\tau} d\lambda\, [\nu - m(\tau-\lambda)]\, K_n(0,\lambda;\alpha)\, K_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

for $0 < \tau \leq \delta$. (It will be assumed that the values of ν under consideration here are always on O.) Assume, for some value of r, that,

20.

$$K_m(\nu, r\delta) = K_m(\nu - mr\delta, 0) + \int_0^{r\delta} d\lambda\, \sigma_m(\nu - m(r\delta-\lambda), \lambda)$$
$$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left|\frac{1}{n}\right| \int_0^{r\delta} d\lambda\, [\nu - m(r\delta-\lambda)]\, K_n(0,\lambda;\alpha)\, K_{m-n}(\nu - m(r\delta-\lambda), \lambda; \alpha)$$

b). Part Two

Let $V(\nu,\tau)$ and $W(\nu,\tau)$ be any $(2M + 1)$-dimensional vector fields on $O_{r\delta}$ such that both $||V||_{O_{r\delta}} \leq N$ and $||W||_{O_{r\delta}} \leq N$. Define $V' = T_{r\delta} V$ and $W' = T_{r\delta} W$. Then, $||V' - W'||_{O_{r\delta}} \leq 1/2\ ||V - W||_{O_{r\delta}}$ for δ small enough.

Proof:

$$|V_m'(\nu,\tau) - W_m'(\nu,\tau)|$$

$$\leq \sum_{n=-M}^{M}{}' \left|\frac{1}{n}\right| \int_{r\delta}^{\tau} d\lambda \, |\nu - m(\tau-\lambda)| \, |V_n(0,\lambda) V_{m-n}(\nu - m(\tau-\lambda),\lambda)$$

$$- W_n(0,\lambda) W_{m-n}(\nu - m(\tau-\lambda),\lambda)|$$

$$\leq \delta \left([MT + \tfrac{1}{2} M\delta] 2N \sum_{n=-M}^{M}{}' \left|\frac{1}{n}\right| \right) ||V - W||_{O_{r\delta}}$$

Thu $||V'-W'||_{O_{r\delta}} \leq 1/2\ ||V-W||_{O_{r\delta}}$ if δ is small enough.

Discussion

This part of the proof shows that $T_{r\delta}$ is a contraction. When $T_{r\delta}$ is applied to the difference of two normed vector fields, the resulting difference is reduced. In view of part a). of this proof, $T_{r\delta}$ can be applied an arbitrary number of times to a pair of suitably chosen initial vector fields with the difference between the resulting vector fields reduced each time. In the following this basic property will be used to find the stationary element of $T_{r\delta}$.

c). Part Three

Let $K_m(\nu,\tau;\ 0)$ be defined by equation 14. Define $K_m(\nu,\tau;\ \alpha) = T_{r\delta}\ K_m(\nu,\tau;\ \alpha-1)$ on $O_{r\delta}$. Then, $K_m(\nu,\tau) = \lim_{\alpha\to\infty} K_m(\nu,\tau;\ \alpha)$ on $O_{r\delta}$.

Proof

$$|K_m(\nu,\tau;0)| \leq ||K||_0 + ||\sigma||_0 T = \tfrac{1}{2} N$$

Therefore, $||K(0)||_{0_{r\delta}} \leq (1/2)\,N < N$. From part a). then, $||K(\alpha)||_{0_{r\delta}} \leq N$ for all α. Now, in the result of part b). of this proof, let $K(\alpha) = V$ and $K(\beta) = W$. Then,

$$\|K(\alpha) - K(\beta)\|_{O_{r\delta}} \leq \frac{1}{2}\|K(\alpha-1) - K(\beta-1)\|_{O_{r\delta}}$$

Suppose, $\alpha = \beta$. Then, $||K(\alpha) - K(\beta)||_{0_{r\delta}} = 0$ for all α. Suppose, $\alpha > \beta \geq 1$. Then, $||K(\alpha) - K(\beta)||_{0_{r\delta}} \leq (1/2)^{\beta}\ ||K(\alpha - \beta) - K(0)||_{0_{r\delta}} \leq 4N(1/2)^{\beta}$. Thus, $||K(\alpha) - K(\beta)||_{0_{r\delta}} \to 0$ as $(\alpha,\beta)\to\infty$. Similarly $||K(\alpha) - K(\beta)||_{0_{r\delta}} \to 0$ as $(\alpha,\beta)\to\infty$ when $\beta > \alpha \geq 1$. Thus, $K(\alpha)$ is a Cauchy sequence[7] on $0_{r\delta}$. $K(\alpha)$ converges uniformly to K^{*} on $0_{r\delta}$ where $K^{*} = \lim_{\alpha\to\infty} K(\alpha)$ $= \lim_{\alpha\to\infty} T_{r\delta}\, K(\alpha-1) = T_{r\delta}K^{*}$. Since $K^{*} = T_{r\delta}\, K^{*}$ it is a stationary element of $T_{r\delta}$. But, it is easy to see that the stationary element of $T_{r\delta}$ is unique.

Suppose there are two stationary elements, K^{*} and L^{*}. Then, $K^{*} - L^{*} = T_{r\delta}\, K^{*} - T_{r\delta}\, L^{*}$, and

$$\|K^{*} - L^{*}\|_{O_{r\delta}} = \|T_{r\delta}K^{*} - T_{r\delta}L^{*}\|_{O_{r\delta}} \leq \frac{1}{2}\|K^{*} - L^{*}\|_{O_{r\delta}}$$

Thus, $||K^{*} - L^{*}||_{0_{r\delta}} = 0$.

Since $K(\nu,\tau)$ is a stationary element of $T_{r\delta}$ on $0_{r\delta}$, it is the stationary element given by $K(\nu,\tau) = \lim_{\alpha\to\infty} K(\nu,\tau;\,\alpha)$. Thus, equation 18 follows.

3. Convergence at large

The proof of convergence on a narrow strip given in the preceding section will be used here to construct a proof of convergence at large (on the domain, O). It will be shown that, on O, $\lim_{\alpha \to \infty} K_m(\nu,\tau;\, \alpha) = K_m(\nu,\tau)$ where $K_m(\nu,\tau;\, 0)$ is given by equation 14 and $K_m(\nu,\tau;\, \alpha)$ is given by equation 16. An inductive argument will be given which assumes equation 18 on $O_{r\delta}$ as a starting point.

Proof:

Notice from equation 18 that, when $r = 0$,

$$K_m(\nu,\tau) = K_m(\nu - m\tau, 0) + \int_0^{\tau} d\lambda\, \sigma_m(\nu - m(\tau-\lambda), \lambda)$$

19.

$$+ \lim_{\alpha\to\infty} {\sum_{n=-M}^{M}}' \left(\frac{1}{n}\right) \int_0^{\tau} d\lambda \,[\nu - m(\tau-\lambda)]\, K_n(0,\lambda;\alpha)\, K_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

for $0 < \tau \leq \delta$. (It will be assumed that the values of ν under consideration here are always on O.) Assume, for some value of r, that,

$$K_m(\nu, r\delta) = K_m(\nu - mr\delta, 0) + \int_0^{r\delta} d\lambda\, \sigma_m(\nu - m(r\delta-\lambda), \lambda)$$

20.

$$+ \lim_{\alpha\to\infty} {\sum_{n=-M}^{M}}' \left(\frac{1}{n}\right) \int_0^{r\delta} d\lambda \,[\nu - m(r\delta-\lambda)]\, K_n(0,\lambda;\alpha)\, K_{m-n}(\nu - m(r\delta-\lambda), \lambda; \alpha)$$

Notice from equation 19 that equation 20 is true when r = 1. By the induction hypothesis (equation 20),

$$K_m(v - m(\tau - r\delta), r\delta) = K_m(v - m\tau, 0) + \int_0^{r\delta} d\lambda\, \sigma_m(v - m(\tau-\lambda), \lambda)$$

21.

$$+ \lim_{\alpha \to \infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^{r\delta} d\lambda\, [v - m(\tau-\lambda)]\, K_n(0, \lambda; \alpha)\, K_{m-n}(v - m(\tau-\lambda), \lambda; \alpha)$$

Equation 21 can be substituted into equation 18 to obtain,

$$K_m(v, \tau) = K_m(v - m\tau, 0) + \int_0^{\tau} d\lambda\, \sigma_m(v - m(\tau-\lambda), \lambda)$$

22.

$$+ \lim_{\alpha \to \infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^{\tau} d\lambda\, [v - m(\tau-\lambda)]\, K_n(0, \lambda; \alpha)\, K_{m-n}(v - m(\tau-\lambda), \lambda; \alpha)$$

on $O_{r\delta}$; i.e., for $r\delta < \tau \leq (r + 1)\,\delta$. In particular, equation 22 is true for $\tau = (r + 1)\,\delta$. Thus, if equation 20 is true for any value of r, it is true for all larger values of r. Since equation 20 is true for r = 1, it is true for all values of r and equation 22 is true everywhere on O.

Discussion:

Equation 22 is the primary result of this paper. From equation 22 a sequence of functions can be computed with the understanding that the sequence will converge to the truncated Fourier series expansion of the Vlasov plasma distribution function.

VI. CONVERGING SEQUENCES FOR THE KINETIC AND FIELD ENERGIES

In Appendix B it is shown that, in the limit $M = \infty$, K'_m and K''_m ($K'_m = \partial K_m/\partial \nu$, etc.) can be uniformly bounded on O. In the following it will be assumed tha $||K'||_0$ amd $||K''||_0$ exist for finite M. In this case essentially the same procedure as given above can be carried out to prove,

$$K'_m(\nu,\tau) = K'_m(\nu - m\tau, 0) + \int_0^\tau d\lambda\, \sigma'_m(\nu - m(\tau-\lambda), \lambda)$$

23. $$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^\tau d\lambda\, K_n(0,\lambda;\alpha)\, K_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

$$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^\tau d\lambda [\nu - m(\tau-\lambda)]\, K_n(0,\lambda;\alpha)\, K'_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

and,

$$K''_m(\nu,\tau) = K''_m(\nu - m\tau, 0) + \int_0^\tau d\lambda\, \sigma''_m(\nu - m(\tau-\lambda), \lambda)$$

24. $$+ 2 \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^\tau d\lambda\, K_n(0,\lambda;\alpha)\, K'_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

$$+ \lim_{\alpha\to\infty} \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right) \int_0^\tau d\lambda [\nu - m(\tau-\lambda)]\, K_n(0,\lambda;\alpha)\, K''_{m-n}(\nu - m(\tau-\lambda), \lambda; \alpha)$$

as long as the boundary conditions are chosen so that $||\sigma'||_0$ and $||\sigma''||_0$ exist.

Equations 23 and 24 show that the sequences of functions which are obtained by differentiating the $K_m(\nu,\tau;\ \alpha)$ converge to the respective derivatives of $K_m(\nu,\tau)$. Thus, $K'_m(\alpha)$ and $K''_m(\alpha)$ can be expected to approximate the exact derivatives. The derivatives, with respect to ν, play an important role in the application of this theory. At $\nu = 0$, K_m and its derivatives are related to the coefficients in Fourier series expansions of the moments of the distribution function. For example,

$$n(x,\tau) = \sum_{m=-\infty}^{\infty} f_m(0,\tau)\, e^{im\pi x}$$

and,

$$u(x,\tau) = -\frac{i}{\pi} \sum_{m=-\infty}^{\infty} f'_m(0,\tau)\, e^{im\pi x}$$

in which f_m and f'_m can be computed from K_m and K'_m (see Appendix B).

From equations 23 and 24 it can be shown that,

25.

$$K''_0(0,\tau;\alpha) = K''_0(0,0) + \int_0^{\tau} d\lambda\, \sigma''_c(0,\lambda) - \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right)^2 \left[\, |K_n(0,0)|^2 - |K_n(0,\tau;\alpha-1)|^2 \,\right]$$

This equation bears on energy conservation in the plasma. Using

$$Q(x,\tau) = \int_{-\infty}^{\infty} dv\, v^3\, F(x,v,\tau)$$

and $P_o(\tau)$ for the space average of $P(x,\tau)$, equation 25 can be rewritten in terms of more familiar notation as,

$$
\begin{aligned}
P_o(\tau;\alpha) = {} & P_o(0) + \epsilon_o^2(0) + \left(\frac{1}{\pi}\right)^2 \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right)^2 \left|K_n(0,0)\right|^2 \\
& - \left[\epsilon_o^2(\tau) + \left(\frac{1}{\pi}\right)^2 \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right)^2 \left|K_n(0,\tau;\alpha-1)\right|^2\right] \qquad (26.) \\
& + \frac{1}{2}\int_0^{\tau} d\lambda \, \delta Q(\lambda)
\end{aligned}
$$

where $\delta Q(\lambda) = Q(-1,\lambda) - Q(1,\lambda)$. The α'th iterate to the space averaged kinetic energy density is given by $P_o(\tau;\alpha)$. The contributions to $P_o(\tau;\alpha)$ on the right side of equation 26 are as follows: The first line in equation 26 gives the total energy in the plasma at $\tau = 0$. $P_o(0)$ is the initial kinetic energy, $\epsilon^2_o(0)$ is the initial electric field energy in the space averaged part of the field, and the sum on this line represents the field energy in $E'(x,0)$ averaged over space. The second line of equation 26 gives the negative of the total field energy at any time. The sum in the second line is the space average of the field energy in the turbulent part $(E'(x,\tau))$ of the electric field, as given by the $(\alpha-1)$'th iterate. The last line of equation 26 gives the accumulated net transfer of energy into $-1 < x < 1$. Thus, energy is conserved at each iteration. Furthermore, sequences of approximations for the kinetic and electric field energies in the plasma can be computed from equation 26. From equations 23 and 24 it can be seen that these sequences must converge to, respectively, the kinetic energy in the truncated distribution function and the corresponding turbulent electric field energy.

VII. CONCLUSION

A one dimensional electrostatic Vlasov-Maxwell plasma model has been considered in a bounded spatial region. Consideration has been limited to a plasma with uniform and stationary ion background and with zero space-averaged charge. An iterative method has been constructed for computing a sequence of approximations to the probability distribution function for the initial-boundary value problem.

The probability distribution function has been Fourier transformed in its velocity variable, and Fourier series expanded in its spatial variable. The Fourier series expansion has been truncated at an arbitrarily large but finite value. It has been assumed that a solution exists to the finite system of partial differential equations which govern the truncated expansion of the distribution function. Under this assumption it has been shown that the sequence of approximations mentioned in the preceding paragraph must converge to the exact solution of the truncated system. Convergence does not depend on the presence of a small expansion parameter for expansion purposes. The degree to which the solution of the truncated system approximates the solution of the infinite system has not been considered, but, in view of the arbitrarily large number of Fourier modes that is allowed, a good approximation is anticipated in many applications.

The issue of the rate at which convergence occurs is under investigation at present. In those situations where convergence is rapid enough to make this iterative method useful, it can be viewed as an approximation technique for Vlasov turbulence which is non-linear and applicable to strong turbulence in a bounded region of space.

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APPENDIX A

Consider the one-dimensional Vlasov-Maxwell system of equations 1-3. Assume the existence of a solution to this system of equations on the domain, D, defined by $-1 < x < 1$, $-\infty < v < \infty$, and $\tau > 0$. Then, what combination of initial and boundary conditions on F and E uniquely, consistently, and conveniently determines that solution?

Given the existence of $E(x,\tau)$, the method of characteristics can be used to solve equation 1.[5] The solution is $F(x(s), v(s), \tau(s)) = F(x(0), v(0), \tau(0))$ where $x(s)$, $v(s)$, and $\tau(s)$ are the solutions of the system,

$$\frac{dx(s)}{ds} = v(s)$$

$$\frac{dv(s)}{ds} = -E(x(s), \tau(s))$$

$$\frac{d\tau(s)}{ds} = 1$$

subject to $x(0)$, $v(0)$, and $\tau(0)$ for initial conditions. In typical applications of this type of solution, the point, $(x(0), v(0), \tau(0))$, is assumed on one of the boundaries where $F(x(0), v(0), \tau(0))$ is known,. Then, F is known everywhere along the characteristic line given by $x(s)$, $v(s)$, and $\tau(s)$ for $s \geq 0$. The complete solution depends on filling all

(x,v,τ) of interest with characteristic lines which are connected to boundaries where $F(x(0), v(0), \tau(0))$ is known. But, this typical approach is self-contradictory for the initial-boundary value problem being considered here.

Figure 2 contains schematic representations of projections onto the plane, $v = 0$, of various types of possible characteristic lines. Progression along the characteristic lines, with s increasing, is indicated by arrows. The dashed lines indicate characteristics which enter D on the initial plane ($-1 \leq x \leq 1$, $-\infty < v < \infty$, $\tau = 0$) or on either of the boundaries ($x = \pm 1$, $-\infty < v < \infty$, $\tau > 0$) and then remain trapped in D. These are the only characteristics that can be treated as outlined in the preceding paragraph. All other characteristic lines (solid lines) enter D and then exit, with s increasing. The solution of equation 1 gives F = constant along each of these characteristic lines. Thus, F must have a single value at every pair of entry and exit points. The initial and boundary values for F are not independent and cannot be chosen arbitrarily.

There is no unique method for choosing the initial-boundary values for F such that the possible contradictions discussed above are avoided. The method used here has been chosen for both mathematical and observational convenience. In figure 3 it is demonstrated that the solution at an arbitrary point, (x,v,τ), is determined by specifying $F(x,v,0)$ on the initial plane, and $\Delta(v,\tau) = F(-1,v,\tau) - F(1,v,\tau)$ in terms of F on the boundaries. The solid lines indicate possible characteristic lines along which F = constant. The dashed lines connect exit and entry points at which the value of F must be related through the use of $\Delta(v,\tau)$. In the example presented in figure 3, $F(x,v,\tau) = F(x_o. v_o, 0)$ $+ \Delta(v_1, \tau_1) - \Delta(v_2, \tau_2)$. Given the existence of $E(x,\tau)$, this choice of

$F(x,v,0)$ and $\Delta(v,\tau)$ uniquely determines $F(x,v,\tau)$ through equation 1. The type of contradictions discussed above are avoided because $\Delta(v,\tau)$ always relates F on an exiting characteristic line to F on an entering characteristic line.

The Δ-method for specifying the boundary data allows for the expression of the data in terms of the measurable moments of the distribution function on the boundaries (e.g., the difference in density, current, temperature, etc.,) where others do not. (Mathematically, it is acceptable to specify F on the boundaries for incoming velocities only. It is also possible to express the boundary conditions in terms of the ratio, $F(x = 1)/F(x = -1)$. Neither of these boundary conditions can be expressed in terms of the moments on the distribution function on the boundaries.) Notice that the well studied "periodic plasma" in which $\Delta(v,\tau) = 0$ is a special case of the Δ-method. More generally, the Δ-method plays a natural role in the Fourier series expansion analysis of the non-periodic plasma. A choice of $F(x,v, 0)$ and $\Delta(v,\tau)$ leads to a unique set of $F_m(v,0)$ and $J_m(v,\tau)$. Thus, the Δ-method has been chosen as the basis for this study of the initial-boundary value problem for the Vlasov-Maxwell plasma.

Now, assume $F(x,v,\tau)$ is known. Then, what additional initial-boundary data must be specified to determine $E(x,\tau)$ from equations 2 and 3?

If $E(x,\tau)$ is separated into its space average, $\epsilon_0(\tau)$, plus an x-dependent part through $E(x,\tau) = \epsilon_0(\tau) + E'(x,\tau)$, then equation 2 determines $E'(x,\tau)$ only. From equation 2, A.1 $$E'(x,\tau) = J_0(\tau) + x\left(1 - n_0(\tau)\right) - \int_{-1}^{x} dx' \left[n(x',\tau) - n_0(\tau)\right]$$

where $n_0(\tau)$ is the space average of $n(x,\tau)$, and

A.2 $$J_0(\tau) = \frac{1}{2} \int_{-1}^{1} dx \; x \; n(x,\tau)$$

From equation 3,

A.3 $$\frac{d\epsilon_0}{d\tau} = u_0(\tau)$$

where $u_0(\tau)$ is the space average of $u(x,\tau)$. Notice, if F is known, then $E'(x,\tau)$ is given by equation A.1 with no freedom for choosing boundary or initial conditions. Further, given F, $u_0(\tau)$ can be calculated, and then, $\epsilon_0(\tau)$ can be calculated from equation A.3 if $\epsilon_0(0)$ is specified. Thus, only $\epsilon_0(0)$ need be specified to determine $E(x,\tau)$ given F.

By combining the argument given above for determining F given E and for determining E given F, it seems plausible that specification of $F(x,v,0)$, $\Delta(v,\tau)$ and $\epsilon_0(0)$ uniquely determines a solution of equations 1-3. This is _not_ a _proof_ of the existence, nor the uniqueness of that solution; it is, at best, a plausibility argument.

APPENDIX B

The Fourier series expansion of F has been truncated in section II, and the resulting truncated version of equations 6-8 has been solved in principle in section VI, not for F, but for

$$f(x,v,\tau) = \sum_{m=-M}^{M} e^{im\pi x} \frac{1}{2}\int_{-\infty}^{\infty} d\tilde{\nu}\, e^{-i\pi\tilde{\nu}v} f_m(\tilde{\nu},\tau) \tag{B.1}$$

The $f_m(\nu,\tau)$ are the solution of the truncated equations which are still related to $f(x,v,\tau)$ in the usual manner:

$$f_m(\tilde{\nu},\tau) = \frac{1}{2}\int_{-1}^{1} dx\, e^{-im\pi x} \int_{-\infty}^{\infty} dv\, e^{i\pi\tilde{\nu}v} f(x,v,\tau) \tag{B.2}$$

The degree to which f approximates F has not been investigated in this paper. However, since the theory developed in this paper applies to a truncation for arbitrarily large M it has been assumed that this approximation can be made as good as necessary. In particular, it has been assumed that for some finite M, $||K||_0$, $||K||_0$ and $||K''||_0$ exist. The conditions under which this assumption is valid are under investigation.

The goal of this appendix is to prove that $||K||_0 = 1$ and both $||K'||_0$ and $||K'||_0$ exist when $M = \infty$. For this purpose it is convenient to introduce

B.3 $$\phi(\tau) = \int_0^{\tau} d\lambda\, \epsilon_0(\lambda)$$

In the following it will be assumed that the boundary conditions are chosen such that, when the solution of equation 11 is substituted into equation B.3, the resulting $\phi(\tau)$ can be uniformly bounded on the interval, $0 < \tau \leq T$. From the definition of the K_m,

B.4 $$|K_m| = |f_m|$$

B.5 $$|K_m'| \leq \pi |\phi| |f_m| + |f_m'|$$

and

B.6 $$|K_m''| \leq \pi^2 \phi^2 |f_m| + 2\pi |\phi| |f_m'| + |f_m''|$$

If f_m, f'_m and f''_m can be uniformly bounded on 0, then the goal of this appendix will have been achieved.

From equation B.2,

$$|f_m(r,\tau)| \leq \frac{1}{2} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dv\, F(x,v,\tau) = f_0(0,\tau)$$

But, in section III it was shown that $f_0(0,\tau) = 1$. Thus $|f_m(\nu,\tau)| \leq 1$ and $||K||_0 = 1$.

By differentiating equation B.2 it can be shown that,

$$\left(\frac{1}{\pi}\right)^2 |f_m''(\nu,\tau)| \leq \frac{1}{2}\int_{-1}^{1} dx \int_{-\infty}^{\infty} dv\, v^2 F(x,v,\tau) = -\left(\frac{1}{\pi}\right)^2 f_0''(0,\tau)$$

where $f''_0(0,\tau) \leq 0$. But, $-(1/\pi)^2 f''_0(0,\tau)$ is actually the kinetic energy in the plasma. The equation which governs conservation of energy can be obtained from equations 6-8 and written,

$$-\left(\frac{1}{\pi}\right)^2 f_0''(0,\tau) = -\left(\frac{1}{\pi}\right)^2 f_0''(0,0) + \epsilon_c^2(0) + \left(\frac{1}{\pi}\right)^2 \sum_{n=-\infty}^{\infty}{}' \left(\frac{1}{n}\right)^2 |f_n(0,0)|^2$$

$$- \left[\epsilon_c^2(\tau) + \left(\frac{1}{\pi}\right)^2 \sum_{n=-M}^{M}{}' \left(\frac{1}{n}\right)^2 |f_n(0,\tau)|^2\right]$$

$$+ \frac{1}{2}\int_0^{\tau} d\lambda\, \delta Q(\lambda)$$

where $\delta Q(\lambda)$ is the net rate at which energy is entering the region in x of interest (see section VII). Thus,

$$\left(\frac{1}{\pi}\right)^2 |f_m''(\nu,\tau)| \leq -\left(\frac{1}{\pi}\right)^2 f_0''(0,0) + \epsilon_c^2(0) + \left(\frac{1}{\pi}\right)^2 \sum_{n=-\infty}^{\infty}{}' \left(\frac{1}{n}\right)^2 |f_n(0,0)|^2$$

$$+ \frac{1}{2}\int_0^{\tau} d\lambda\, \delta Q(\lambda)$$

The first line of the right side of this equation represents the total initial energy in the plasma. If this initial energy is chosen bounded, and if the rate at which energy is allowed to enter, $\delta Q(\tau)$, is assumed uniformly bounded on $0 \leq \tau \leq T$, then $|f''_m(\nu,\tau)|$ is uniformly bounded on 0.

Given the uniform bounds of $f_0(0,\tau)$ and $f''_0(0,\tau)$ which have been obtained above, it is, possible to prove a uniform bound on f'_m. From equation B.2,

$$|f'_m(\tau,\tau)| \le \frac{\pi}{2}\int_{-1}^{1}dx\int_{-\infty}^{\infty}dv\,|v|\;F(x,v,\tau)$$

$$= \frac{\pi}{2}\int_{-1}^{1}dx\left[\int_{-\infty}^{-1}dv + \int_{-1}^{1}dv + \int_{1}^{\infty}dv\right]|v|\,F(x,v,\tau)$$

$$\le \frac{\pi}{2}\int_{-1}^{1}dx\left[\int_{-\infty}^{-1}dv\,v^2 + \int_{-1}^{1}dv + \int_{1}^{\infty}dv\,v^2\right]F(x,v,\tau)$$

$$\le -\frac{1}{\pi}f''_0(0,\tau) + f_0(0,\tau)$$

Thus, f'_m is uniformly bounded on 0, as well as f_m and f''_m, and from equations B.5 and B.6 $||K'||_0$ and $||K''||_0$ must exist.

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LIST OF FIGURE CAPTIONS

FIGURE 1 The domain 0 on the (ν,τ)-plane.

FIGURE 2 A schematic representation of all possible characteristic curves on the domain D.

FIGURE 3 A possible construction of $F(x,v,\tau)$ using $F(x,v,0)$ and $\Delta(v,\tau)$.

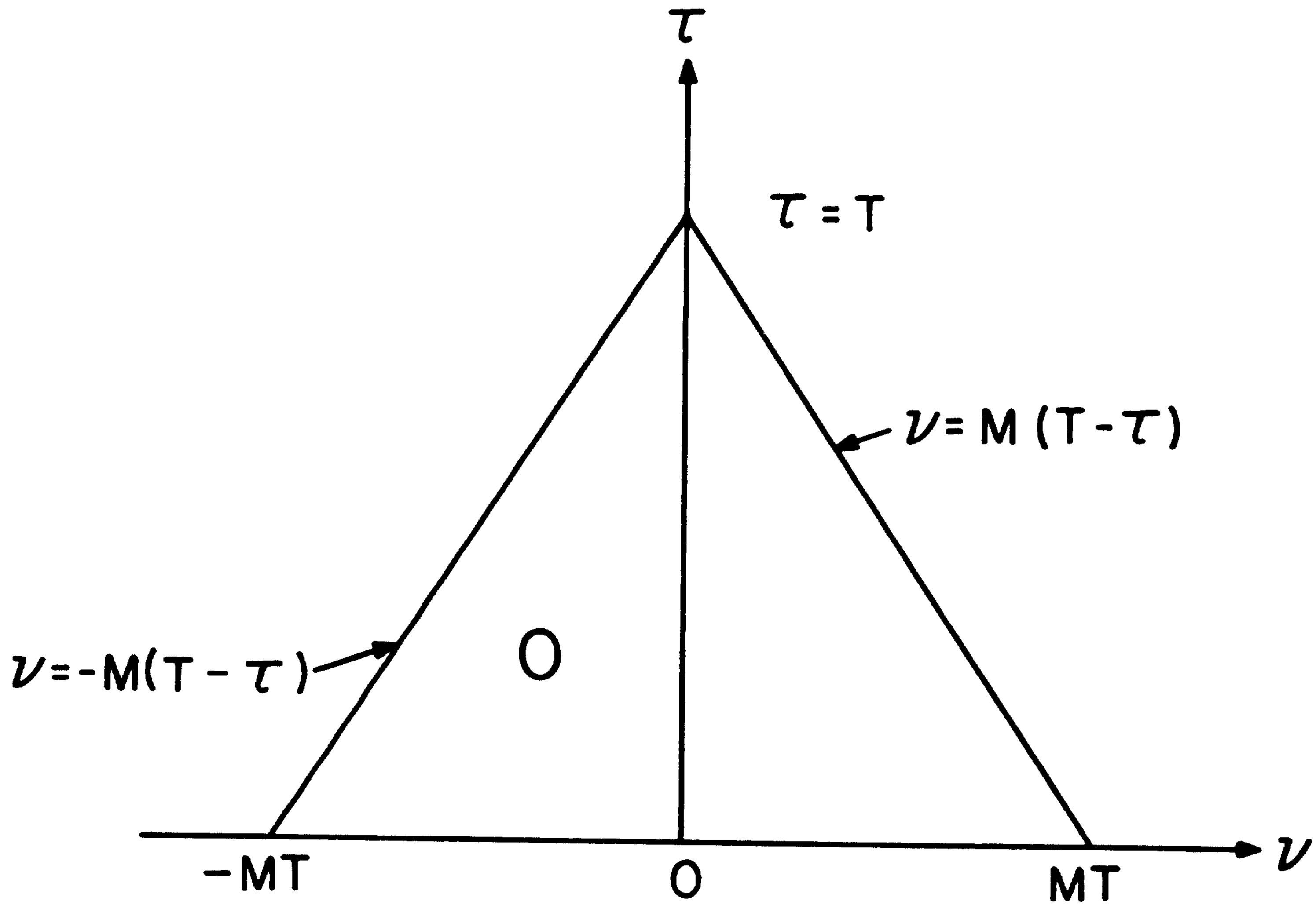

Figure 1

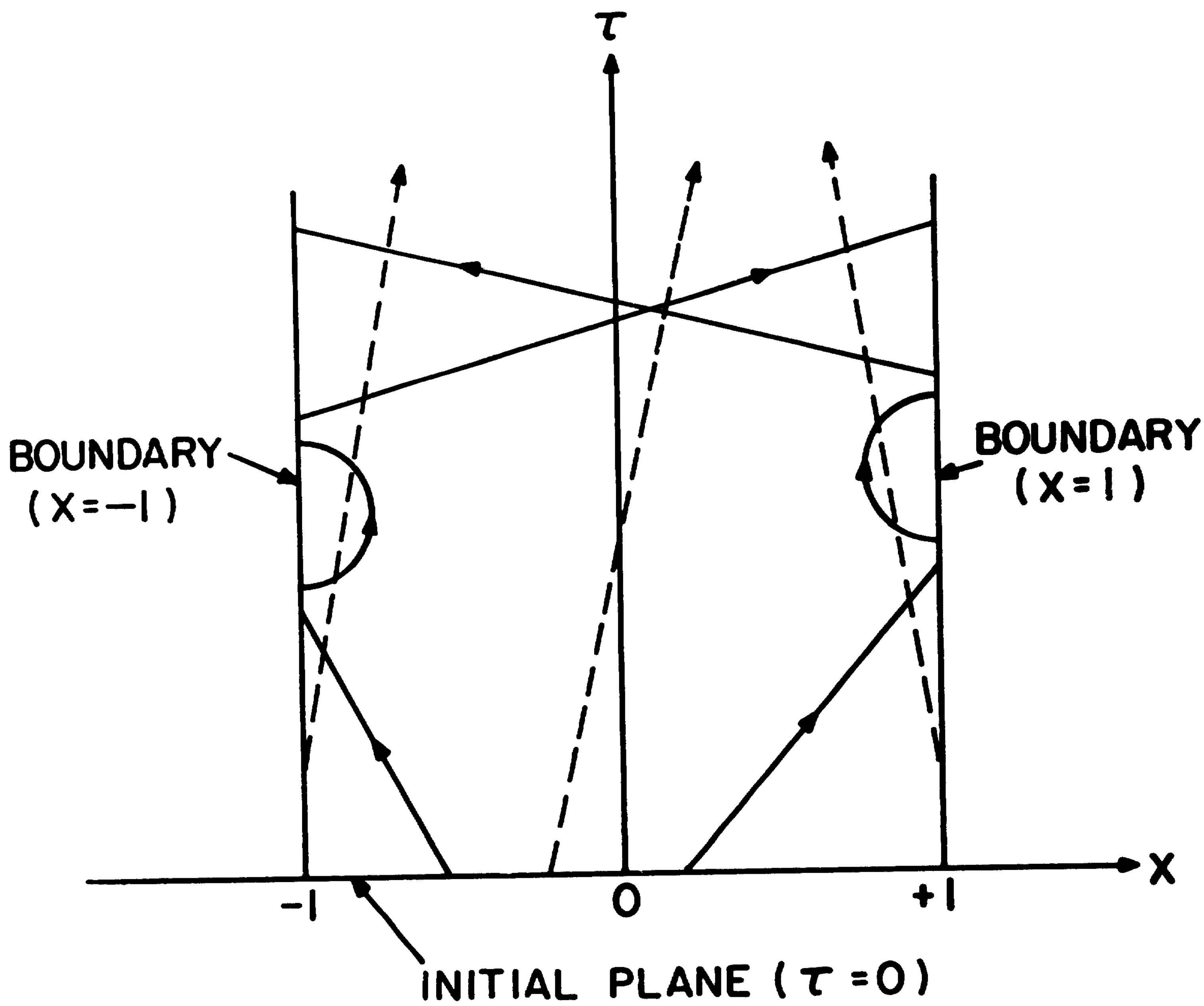

Figure 2

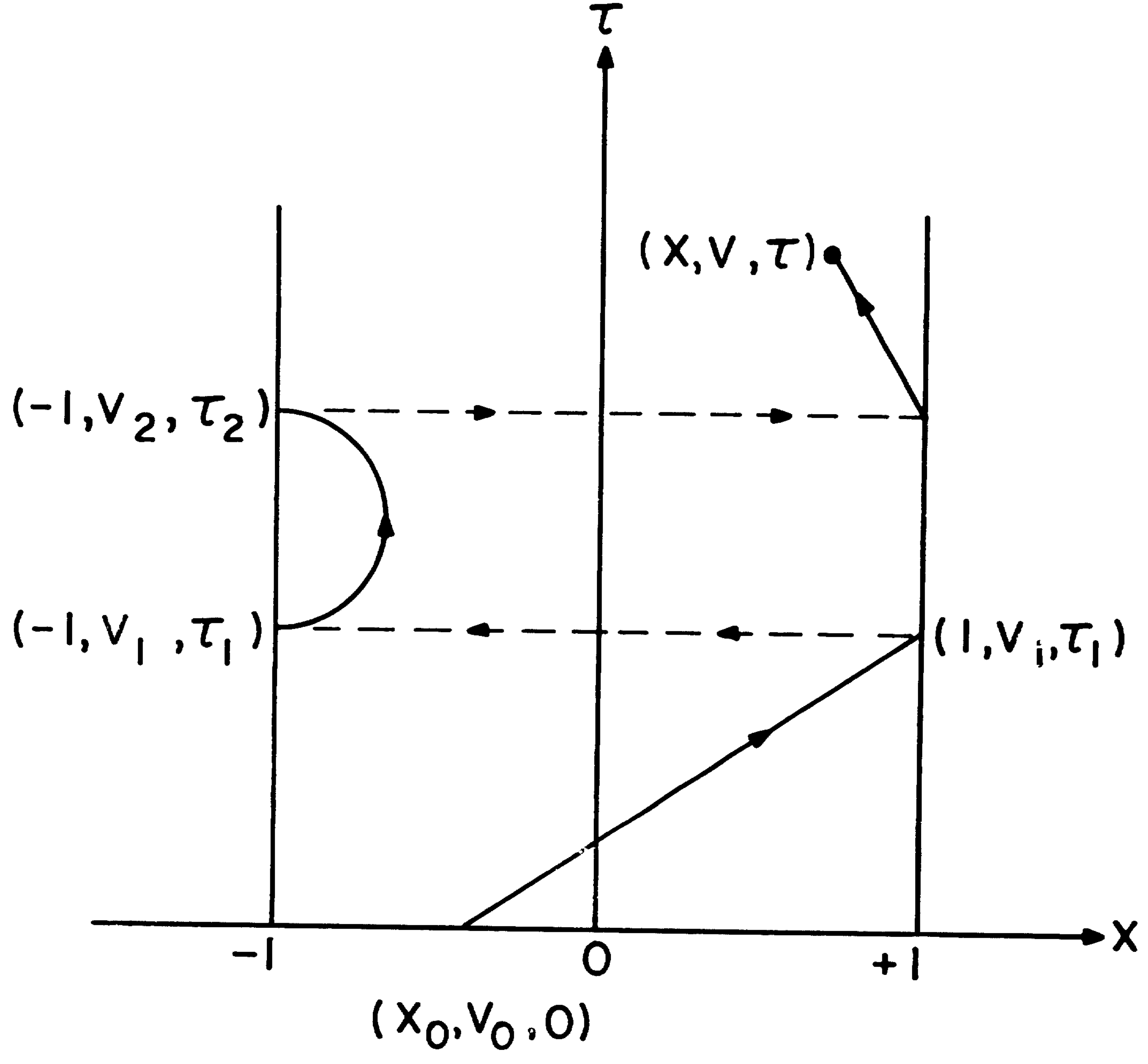

Figure 3