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ABSTRACT

We consider systems or items which are placed on test at time zero, function for a period, and die at some random time. Failure may be due to one of several causes or modes. A model for this situation is that at birth nature chooses a life time Y_i from a population of times until death from mode i. The time at which the item fails is $\min(Y_i)$ and is the only life actually observed. The parameters of the life distributions may depend upon the levels of various stress variables the item is subject to. Maximum likelihood estimation methods are discussed in general. Specific methods are discussed for the smallest extreme-value distributions of life. Monte-Carlo results indicate the methods to be promising. Under appropriate conditions, the location parameters are nearly unbiased, the scale parameter is slightly biased, and the asymptotic covariances are rapidly approached.

1.0 INTRODUCTION

Suppose an item or system (a person, battery, computers, etc.) is brought to life at time zero, functions for a period of time, and then fails. We also suppose there are a finite number of possible causes (or modes) of failure labeled 1, 2, . . ., M. For example, people die of cancer, accident, heart attack, etc. A simplified conceptual model for actual item lifetime is that at birth nature chooses a failure time Y_m from a population of failure times due to mode m. (The Y_m may or may not be independent.) The observed life of the item is then $\min(Y_m)$ and it is known only that all the other lifetimes exceeded this value. Such a process is a form of progressive censoring. Cox [1] appears to have been the first to formulate this competing risk failure time model and applied it specifically in the instance of independent exponentially distributed failure times for two possible modes of failure.

Herman and Patell [4] extended the model to several independent competing risks, developed the maximum likelihood estimators for exponential and Weibull distributions of failure, and derived expressions for the asymptotic covariance matrices of the estimators. McCool [9] considers certain confidence interval techniques in this situation when life has a Weibull distribution. Moeschberger [10] extended the model to the case of dependent or correlated failure times. Other closely related developments are by Sampford [13], Moeschberger and David [11] and Hoel [5].

For any of the failure modes it may very well be that the population of failure times from which the item life is drawn may depend upon the level of various stress variables the item is subjected to during its life. For example, if we were to assume human life until death by cancer to follow an extreme value distribution with location parameter μ and scale parameter σ, then $\mu = \mu(x_1, \ldots, x_n)$ may be a function of exposure to pollution (x_1), amount smoked per day (x_2), etc. McCool [8] considers estimation, confidence interval, and multiple comparison methods when the life distribution is Weibull, failure modes act independently, and n observations are obtained from each population. (He essentially provides an ANOVA generalization.) Nelson [12] sketched a general approach to maximum likelihood estimation in the extension of the multiple regression situation. He applies the method in the case of a single stress variable (temperature), independent and lognormally distributed lifetimes and a number of failures by each mode for each level of the stress variable.

In this report we first present the general model for the competing failure modes assuming the location parameters for each mode are expressible as linear functions of the stress variables and the failure modes act independently. We then present the general form of the likelihood function and the likelihood equations are derived for the extreme value distributions. Solving these equations using nonlinear least squares techniques provides an estimate of the asymptotic covariance matrix of the estimators. Several Monte-Carlo experiments were performed. The results of these indicate that, under appropriate conditions, the estimators approach unbiasedness and their asymptotic covariances at reasonably small sample sizes.

2.0 THE GENERAL MODEL

Assume that an experiment is performed in which I items are life tested at varying combinations of J stress variables and that the design is specified by the design matrix

$$Z = \begin{bmatrix} z_{11} & \cdots & z_{1J} \\ \vdots & & \vdots \\ z_{I1} & \cdots & z_{IJ} \end{bmatrix} \tag{2-1}$$

The response observed for each item is a lifetime y_i $(i = 1, I)$ and the mode by which the item failed, m_i. The number of modes by which the item might fail is denoted M.

We assume that for observation i and mode m the cumulative distribution function of time until failure is defined by

$$F_i^{(m)} \left[y;\ \mu_i^{(m)},\ \sigma^{(m)}\right]$$

and the density function by

$$f_i^{(m)} \left[y;\ \mu_i^{(m)},\ \sigma^{(m)}\right]$$

where $\mu_i^{(m)}$ is a location parameter and $\sigma^{(m)}$ is a scale parameter. We thus assume that $\sigma^{(m)}$ is constant for all observations. We also assume that the location parameter is a function of the stress variables, in particular

$$\mu_i^{(m)} = \beta_1^{(m)} z_{i1} + \ldots + \beta_J^{(m)} z_{iJ} \tag{2-2}$$

In general we will not necessarily have

$$\sigma^{(m)} = \sigma^{(m')} \qquad \text{for } m \neq m' \tag{2-3}$$

nor

$$\beta_j^{(m)} = \beta_j^{(m')} \qquad \text{for } m \neq m' \tag{2-4}$$

That is, we do not a priori assume any of the failure modes to have parameter values in common.

We further assume that lifetimes $Y_i^{(m)}$ are drawn at random and independently from $F_i^{(m)}$ and that the observed lifetime is the smallest of these, min (y_i). This latter assumption implies an item can fail by at most one mode. Nelson [12] indicates a situation in which items may fail by one mode, be repaired, and then fail again by another mode. It is also possible that testing of an item may be terminated before it fails. These situations may be easily included in the model but will not be considered in this report.

Clearly, the profusion of arguments, subscripts and superscripts will be quite unwieldly. We will thus delete them whenever context clearly indicates the appropriate values.

3.0 THE LIKELIHOOD FUNCTION

We first consider the likelihood function corresponding to the i^{th} observation. If that observation is a failure by mode m, then the likelihood is

$$\ell_i = \left[1 - F_i^{(1)}\right] \cdots \left[1 - F_i^{(m-1)}\right] \left[f_i^{(m)}\right] \left[1 - F_i^{(m+1)}\right] \cdots \left[1 - F^{(M)}\right] \tag{3-1}$$

where

$$f_i^{(m)} = f_i^{(m)} \left[y;\ \mu_i^{(m)},\ \sigma^{(m)}\right] \tag{3-2}$$

and

$$\begin{aligned} 1 - F_i^{(n)} &= 1 - F_i^{(n)} \left[y;\ \mu_i^{(n)},\ \sigma^{(n)}\right] \\ &= \Pr\left[Y_i^{(n)} > y\right] \end{aligned} \tag{3-3}$$

That is, we multiply the density function corresponding to the mode of failure by the probability the item did not fail by any of the other modes. The likelihood is a simple product due to the independence assumption. If testing is terminated without an observed failure, the likelihood is given by

$$\ell_i = \prod_{1}^{M} \left[1 - F_i^{(m)}\right] \tag{3-4}$$

The overall likelihood is then the product of the likelihood for each observation and given by

$$L = \prod \ell_i \tag{3-5}$$

Taking natural logs of eq. (3-5)

$$\ln L = \sum_{i=1}^{I} \ln \left\{ f_i^{(m_i)} \prod_{m \neq m_i} \left[1 - F_i^{(m)}\right] \right\} \tag{3-6}$$

is obtained where

$$\ln L = \ln L\left[\beta_1^{(1)}, \ldots, \beta_J^{(1)}, \beta_1^{(2)}, \ldots, \beta_J^{(2)}, \ldots, \beta_1^{(M)}, \ldots, \beta_J^{(M)}, \sigma^{(1)}, \ldots, \sigma^{(M)}; y_1, \ldots, y_I\right] \tag{3-7}$$

To obtain maximum likelihood estimators, the standard method (assuming it works) is to solve the system of JM (nonlinear) equations

$$\left.\begin{aligned} &\frac{\partial \ln L}{\partial \beta_1^{(m)}} = 0 && (m = 1, M) \\ &\quad\vdots \\ &\frac{\partial \ln L}{\partial \beta_J^{(m)}} = 0 && (m = 1, M) \\ &\frac{\partial \ln L}{\partial \sigma^{(m)}} = 0 && (m = 1, M) \end{aligned}\right\} \tag{3-8}$$

If the different failure modes have no parameters in common, as will generally be true by eqs. (2-3) and (2-4), this system of JM equations splits into M separate systems of J equations which may be independently solved. In particular, the

m^{th} system reduces to

$$\left.\begin{aligned}
0 &= \sum_{\substack{\text{mode m}\\ \text{failures}}} \frac{\partial}{\partial \beta_1^{(m)}} \ln f_i^{(m)} + \sum_{\substack{\text{other mode}\\ \text{failures}}} \frac{\partial}{\partial \beta_J^{(m)}} \ln\left[1 - F_i^{(m)}\right] \\
&\vdots \\
0 &= \sum_{\substack{\text{mode m}\\ \text{failures}}} \frac{\partial}{\partial \beta_J^{(m)}} \ln f_i^{(m)} + \sum_{\substack{\text{other mode}\\ \text{failures}}} \frac{\partial}{\partial \beta_J^{(m)}} \ln\left[1 - F_i^{(m)}\right] \\
0 &= \sum_{\substack{\text{mode m}\\ \text{failures}}} \frac{\partial}{\partial \sigma^{(m)}} \ln f_i^{(m)} + \sum_{\substack{\text{other mode}\\ \text{failures}}} \frac{\partial}{\partial \sigma^{(m)}} \ln\left[1 - F_i^{(m)}\right]
\end{aligned}\right\} \quad (3\text{-}9)$$

4.0 WEIBULL AND EXTREME-VALUE (SMALLEST) DISTRIBUTIONS

Two commonly assumed life distributions are the Weibull and the smallest-extreme value distributions.

The cumulative distribution function for the Weibull distribution is given by

$$F(t;\alpha,\beta) = 1 - \exp[-(t/\alpha)^{\beta}] \tag{4-1}$$

where

α "scale" parameter

β "shape" parameter

It is easily verified that if T follows this Weibull distribution, then the application of the transformation $Y = \ln T$ yields the smallest extreme value distribution with cumulative distribution function

$$F(y;\mu,\sigma) = 1 - \exp\left\{-\exp[(y-\mu)/\sigma]\right\}$$

with

$$\sigma = 1/\beta$$

$$\mu = \ln(\alpha) \tag{4-2}$$

It is thus sufficient to consider this extreme value distribution. In particular, if we apply this distribution to the model previously developed we assume that for mode m failure and the i^{th} observation the density function for life is

$$f_i^{(m)}\left[y; \mu_i^{(m)}, \sigma^{(m)}\right] = \frac{1}{\sigma^{(m)}} \exp\left\{\frac{y - \mu_i^{(m)}}{\sigma^{(m)}} - \exp\left[\frac{y - \mu_i^{(m)}}{\sigma^{(m)}}\right]\right\} \tag{4-3}$$

and the distribution function is

$$F_i^{(m)}\left[y; \mu_i^{(m)}, \sigma^{(m)}\right] = 1 - \exp\left\{-\exp\left[\frac{y - \mu_i^{(m)}}{\sigma^{(m)}}\right]\right\} \tag{4-4}$$

where

$$\mu_i^{(m)} = \beta_1^{(m)} z_{i1} + \ldots + \beta_J^{(m)} z_{iJ} \tag{4-5}$$

Hence, we obtain the partial derivatives required for the likelihood equations as (deleting m superscript)

$$\left.\begin{aligned}
\frac{\partial \ln f_i}{\partial \sigma} &= -\frac{1}{\sigma} - \frac{y - \mu_i}{\sigma^2} + \frac{y - \mu_i}{\sigma^2} \exp\left(\frac{y - \mu_i}{\sigma}\right) \\
\frac{\partial \ln f_i}{\partial \beta_j} &= \frac{-z_{ij}}{\sigma} + \frac{z_{ij}}{\sigma} \exp\left(\frac{y - \mu_i}{\sigma}\right) \\
\frac{\partial \ln (1 - F_i)}{\partial \sigma} &= \frac{y - \mu_i}{\sigma^2} \exp\left(\frac{y - \mu_i}{\sigma}\right) \\
\frac{\partial \ln (1 - F_i)}{\partial \beta_j} &= \frac{z_{ij}}{\sigma} \exp\left(\frac{y - \mu_i}{\sigma}\right)
\end{aligned}\right\} \tag{4-6}$$

Substituting these into eqs. (3-9) we obtain the likelihood equations for any specified mode as

$$\left.\begin{aligned}
&\frac{1}{\sigma} \sum_{\substack{\text{failed by}\\ \text{mode}}} z_{ij} - \frac{1}{\sigma} \sum_{\substack{\text{all}\\ \text{failures}}} z_{ij} \exp\left(\frac{y_i - \mu_i}{\sigma}\right) = 0 \qquad (j = 1, J)\\
&\frac{1}{\sigma^2} \sum_{\substack{\text{failed by}\\ \text{mode}}} (y_i - \mu_i + \sigma) - \frac{1}{\sigma^2} \sum_{\substack{\text{all}\\ \text{failures}}} (y_i - \mu_i) \exp\left(\frac{y_i - \mu_i}{\sigma}\right) = 0
\end{aligned}\right\} \tag{4-7}$$

These nonlinear equations have no closed-form solution and need to be solved by some iterative method.

It is well-known that, under appropriate regularity conditions, maximum likelihood estimators (MLE) are consistent and asymptotically normally distributed. It is also known for the Weibull and extreme value distributions that the minimal sufficient statistics are the trivial ones consisting of the order statistics. It thus becomes of considerable interest to determine how rapidly the MLE's approach their asymptotic unbiasedness and covariance structure. The asymptotic covariance structure and the Cramer-Rao lower bound are defined by the inverse of the Fisher information matrix (Zacks [14, p. 194] and Kendall and Stuart [7, p. 28]).

For the model considered in this report the Fisher information matrix is the block-diagonal matrix

$$\begin{bmatrix} I^{(1)} & 0 & \cdots & 0 \\ 0 & I^{(2)} & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I^{(M)} \end{bmatrix} \tag{4-8}$$

where

$$-I^{(m)} = E \left[\begin{array}{cccc:c} \frac{\partial^2 \ln L}{\partial [\beta_1^{(m)}]^2} & \cdots & & \cdots & \frac{\partial^2 \ln L}{\partial \beta_1^{(m)} \partial \sigma^{(m)}} \\ \vdots & & & \ddots & \vdots \\ \cdots & & \cdots & \frac{\partial^2 \ln L}{\partial [\beta_J^{(m)}]^2} & \frac{\partial^2 \ln L}{\partial \beta_J^{(m)} \partial \sigma^{(m)}} \\ \hdashline \frac{\partial^2 \ln L}{\partial \beta_1^{(m)} \partial \sigma^{(m)}} & & \cdots & \frac{\partial^2 \ln L}{\partial \beta_J^{(m)} \partial \sigma^{(m)}} & \frac{\partial^2 \ln L}{\partial [\sigma^{(m)}]^2} \end{array} \right] \qquad (4\text{-}9)$$

In the following section it will be shown how an approximation to this expectation occurs naturally when solving the likelihood equations via nonlinear least squares.

5.0 SOLUTION BY LEAST SQUARES

Using the general nonlinear regression model

$$W = h(\xi, \theta) + \epsilon \qquad (5\text{-}1)$$

where $\xi^T = (\xi_1, \ldots, \xi_J)$ is a vector of independent variables, $\epsilon^T = (\epsilon_1, \ldots, \epsilon_n)$ is a vector of random errors, $\theta^T = (\theta_1, \ldots, \theta_p)$ is a vector of parameters to estimate, and the maxtrix of observed data is

$$\begin{bmatrix} W_1 & \xi_{11} & \cdots & \xi_{1J} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ W_n & \xi_{n1} & \cdots & \xi_{nJ} \end{bmatrix} \qquad (5\text{-}2)$$

Approximating the function h as its truncated Taylor series gives (Draper and Smith [2, Ch. 10])

$$h(\xi, \theta) = h(\xi, \theta_o) + \sum (\theta_k - \theta_{ko}) \frac{\partial h(\xi, \theta_o)}{\partial \theta_k} . \qquad (5\text{-}3)$$

Also, let

θ_o = prior guess for θ

γ = correction vector

$$\left.\begin{aligned} h^o &= h(\xi, \theta_o) \\ \gamma_k &= \theta_k - \theta_{ko} \end{aligned}\right\} \qquad (5\text{-}4)$$

and X be the matrix of partial derivatives of h evaluated at $\theta = \theta_o$. That is,

$$(X_{u,k}) = \left[\frac{\partial\, h(\xi_u, \theta)}{\partial\, \theta_k} \Bigg|_{\theta=\theta_o} \right] \qquad (5\text{-}5)$$

With this notation eq. (5-1) can be approximated as

$$W - h^o = X\gamma + \epsilon \qquad (5\text{-}6)$$

which has (least squares) solution

$$\hat{\gamma} = \widehat{(\theta - \theta_o)} = (X^T X)^{-1} X^T (W - h^o) \qquad (5\text{-}7)$$

Thus $\hat{\gamma}$ is the estimated correction vector. This method may be applied to the solution of the likelihood equations if the W's, θ's, and h of eq. (5-1) are defined as

$$\left.\begin{aligned} &W_1 = 0;\ \theta_1 = \beta_1;\ h(\xi_1, \theta) = \frac{\partial \ln L}{\partial \beta_1} \\ &\quad\vdots \\ &W_J = 0;\ \theta_J = \beta_J;\ h(\xi_J, \theta) = \frac{\partial \ln L}{\partial \beta_J} \\ &W_{j+1} = 0;\ \theta_{J+1} = \sigma;\ h(\xi_{J+1}, \theta) = \frac{\partial \ln L}{\partial \sigma} \end{aligned}\right\} \qquad (5\text{-}8)$$

With this notation we see that the partial derivatives of $h(\xi, \theta)$ required in eqs. (5-3) and (5-5) are actually the second partial derivatives of the log-likelihood function required in the expectation of eq. (4-9). Thus when the X matrix is evaluated one has an estimate of the covariance matrix of the parameters. When used together with Monte-Carlo simulation experiments, substitution of the known parameters into the second derivatives and the observed sample lifetimes for each trial may be done. These results are then averaged to obtain an estimate of the expectation of eq. (4-9). The application of least squares methods to Maximum Likelihood estimation has been treated by Jennrich and Moore [6].

6.0 SIMULATION EXPERIMENTS

Three series of sampling experiments were performed. In each of these experiments only two failure modes were considered. The first experiment investigates the effects of censoring and sample size when the two modes have constant location parameters and identical scale parameters. The second experiment investigates the effects of various experiment designs in the case where mode 1 location parameter is a linear function of a single stress parameter and mode 2 location parameter is constant. Both modes had identical scale parameters. The third experiment investigates the effects of ill-conditioning on the estimators.

For each experiment the true values of the parameters were known. Uniformly distributed random variables were generated and observations for each mode were obtained by inverting the cumulative distribution functions. For each observation one lifetime for each mode was generated and the observed lifetime chosen as the smaller of the two. Only estimates of mode 1 parameters were obtained and recorded. The starting values for the iterative solutions were the true values used to generate the observations.

For each simulation, several pieces of information were obtained. The means, mean squared errors, and covariance matrix of the estimates were recorded. The average amount of censoring by mode 2 was recorded along with a separate mean value of the estimates of the scale parameter for each amount of censoring. Also the Fisher information matrix of eq. (4-9) was evaluated for each sample and its mean value recorded. This was then inverted to obtain an estimate of the asymptotic covariance matrix of the parameter estimates.

The goal of these experiments was to obtain information concerning the means and variances of the parameter estimates and to investigate the approach to asymptotic behavior. Due to the iterative nature of the solutions required some sample sizes were rather small in order to avoid excessive computer times.

The following three sections describe each of the experiments and the results obtained.

6.1 EXPERIMENT A

This series of simulations investigates the performance of the MLE's for two populations with identical scale parameters and location parameters of varying separation. In particular the parameters chosen are

$$\sigma^{(1)} = \sigma^{(2)} = 0.1$$

$$\mu^{(1)} = 2.2$$

$$\mu^{(2)} = \mu^{(1)} + \Delta, \quad \Delta = (-0.1,\ 0,\ 0.1,\ 0.3)$$

(These are highly skewed and distinctly non-normal distributions. Figure 1 shows one such extreme value density function and the normal density with the corresponding moments.)

Sample sizes of $n = 5$, 10, and 20 were considered. Not all combinations of Δ and n were investigated. Those combinations investigated are given in table I. There were 1000 simulations performed for each case. Any sample which had less than two mode 1 failures was rejected and another sample chosen in its place.

The results of these experiments are presented in table I. The first two columns specify the combination of Δ and n. Column three indicates the parameter and column four the mean of the 1000 estimates for that simulation. Column five presents the estimated standard deviation of the mean reported in column four and is obtained by dividing the observed standard deviation of that parameter by the square root of the number of simulations.

The major points are that the location parameter is slightly biased toward lower values and that the scale parameter can be strongly biased. The bias in the location parameter is worst for the smallest sample sizes and the most censoring. The bias in the scale parameter is close to $-1/n$ where n is the sample size. The bias in $\hat{\sigma}$ could also depend upon Δ but this is not perfectly clear. Figure 2 plots the means of $\hat{\sigma}$ and of $\hat{\mu}$ as a function of Δ for the three sample sizes. Also included is a bar indicating plus and minus one standard error of the mean. The bias may decrease as Δ decreases.

Harter and Moore [3] report on similar experiments. They were not concerned with competing failure modes but rather with the case where the r_1 smallest and/or r_2 largest observations are censored. For this case they report rapid approach of the estimators to their asymptotic properties (if censoring is not severe) and that the scale parameter estimate is biased downward by a factor of $1/m$ where $m = n - r_1 - r_2$ is the number of uncensored observations. Similar results were found in the competing failure mode type of censoring.

The last four columns of table I provide the observed covariance matrices of the estimates $\hat{\mu}^{(1)}$ and $\hat{\sigma}^{(1)}$ and the corresponding estimate based upon asymptotic theory.

The observed covariance matrix was computed as

$$\left.\begin{aligned} v\left[\hat{\mu}^{(1)}\right] &= \frac{1}{k}\sum\left[\hat{\mu}_i^{(1)} - M\ \hat{\mu}^{(1)}\right]^2 \\ \operatorname{cov}\left[\hat{\mu}^{(1)}, \hat{\sigma}^{(1)}\right] &= \frac{1}{k}\sum\left[\hat{\mu}_i^{(1)} - M(\hat{\mu})\right]\left[\hat{\sigma}^{(1)} - M(\hat{\sigma})\right] \\ v\left[\hat{\sigma}^{(1)}\right] &= \frac{1}{k}\sum\left\{\hat{\sigma}^{(1)} - M\left[\hat{\sigma}^{(1)}\right]\right\}^2 \end{aligned}\right\} \tag{6-1}$$

where

$k = 1000$ trials

and

$$\left.\begin{aligned} M\left[\hat{\mu}^{(1)}\right] &= \frac{1}{k}\sum\hat{\mu}_i^{(1)} \\ M\left[\hat{\sigma}^{(1)}\right] &= \frac{1}{k}\sum\hat{\sigma}_i^{(1)} \end{aligned}\right\} \tag{6-2}$$

The estimate of the asymptotic covariance matrix was obtained as follows. Suppose the sample size is n where $y_1, \ldots, y_m$ are mode 1 failures and $y_{m+1}, \ldots, y_n$ are mode 2 failures. Then

$$\frac{\partial^2 \ln L}{(\partial \sigma)^2} = -\frac{m}{\sigma^2} - \frac{2}{\sigma^3} \sum_{i=1}^{m} (y_i - \mu)$$

$$+ \frac{1}{\sigma^3} \sum_{i=1}^{n} \left[\exp\left(\frac{y_i - \mu}{\sigma}\right)\right] \left[\frac{1}{\sigma}\left(y_i - \mu\right)^2 + 2(y_i - \mu)\right]$$

$$\frac{\partial^2 \ln L}{\partial \sigma \partial \mu} = -\frac{m}{\sigma^2} + \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(1 + \frac{y_i - \mu}{\sigma}\right) \exp\left(\frac{y_i - \mu}{\sigma}\right)$$

$$\frac{\partial^2 \ln L}{(\partial \mu)^2} = \frac{-1}{\sigma^2} \sum_{i=1}^{n} \exp\left(\frac{y_i - \mu}{\sigma}\right) \qquad (6\text{-}3)$$

where the μ and σ are the known values of $\mu^{(1)}$ and $\sigma^{(1)}$ used in generating the observations. It is thus possible to estimate

$$E_y \begin{Bmatrix} \dfrac{\partial^2 \ln L}{\partial \mu^2} & \dfrac{\partial^2 \ln L}{\partial \mu \partial \sigma} \\ \dfrac{\partial^2 \ln L}{\partial \mu \partial \sigma} & \dfrac{\partial^2 \ln L}{\partial \sigma^2} \end{Bmatrix} \qquad (6\text{-}4)$$

by obtaining the average of each expression in 6-3 over all the simulated samples. It should be noted that for small Δ and n this approximation may be biased to an unknown degree due to rejection of samples with insufficient mode 1 failures. The covariances are estimated by calculating the inverse of this matrix.

Examination of table I shows that for $\Delta = 0.3$ the asymptotic and observed covariances are quite close for all n's. For $\Delta = 0.1$ the diagonal elements are all quite close while the covariance is somewhat different for $n = 5$. Even for $\Delta = 0$ there is good agreement except for the $n = 5$ case. For $\Delta = -0.1$ it appears the asymptotically based approximation is a considerable underestimate.

The general conclusion is that if $\Delta \geq 0$ the estimators approach the asymptotic covariance structure very rapidly. Large Δ implies little censoring by mode 2. These results are in qualitative agreement with Harter and Moore [3].

6.2 EXPERIMENT B

This series of simulations is the simplest nontrivial situation in which the location parameter of one of the modes is dependent upon a stress variable. The models chosen for simulation are graphed in figure 3 and defined by

$$\mu^{(1)} = 2.2 + 0.2\, z_2$$

$$\mu^{(2)} = 2.5$$

$$\sigma^{(1)} = \sigma^{(2)} = 0.1$$

The goal of the experiment was to investigate further the bias in the estimates and the effect of different experiment designs with respect to z_2. In particular, three different sample sizes were used ($n = 7$, 12, and 22). For each sample size, three distributions of these observations with respect to z_2 were used. For each, z_2 was restricted to the values 0, ± 1. One design concentrated all the replication at $z_2 = 0$, the second design spread the replication as evenly as possible among the three points, and the third design spread the replication evenly between the two extreme values. The results are based on 100 simulations of each case.

Table II presents the means and the standard deviations of the means of each parameter. For the $n = 7$ sample size it appears that $\hat{\beta}_1^{(1)}$ is biased slightly toward lower values, $\hat{\beta}_2^{(1)}$ is apparently unbiased, and $\hat{\sigma}^{(1)}$ is biased to lower values. The amount of bias of $\hat{\sigma}$ does not appear to depend upon the design. The standard deviation of $\hat{\beta}_2$ decreases as the design places more replication at the extremes of the range.

Table III compares the observed covariance matrices, the estimated asymptotic covariances, and the covariance matrix of the location parameters assuming naive normal theory. The observed and estimated asymptotic matrices were calculated as in experiment A. The normal theory matrix is obtained by assuming that the estimation process is equivalent to multiple linear regression with normally distributed error term and ignoring the fact of competing failure modes. That is, the matrix is given by

$$(1.645)\ \sigma^2\ (Z^T Z)^{-1}$$

where Z is the design matrix and $1.645\ \sigma^2$ is the variance of the smallest extreme value distribution with scale parameter σ. This describes the covariance structure of the location parameters only. Because of the limited number of simulations, these results are only indicative of the true behavior and only cursory mention of these results is made here.

For all cases, the observed and asymptotic covariances between $\hat{\beta}_1$ and $\hat{\beta}_2$ are near zero as in the normal situation. The covariances between $\hat{\sigma}$ and both $\hat{\beta}_1$ and $\hat{\beta}_2$ are also small as in the normal case. Inspection shows the normal approximation to be somewhat too large in general. The asymptotic approximation appears to be closer for $v(\hat{\beta}_1)$ but tending to under estimate $v(\hat{\beta}_2)$. In order to provide more accurate comparisons, considerably more simulations should be performed. The results obtained here indicate that the approach to asymptotic covariance structure may be reasonably rapid.

6.3 EXPERIMENT C

This series of experiments was intended to examine what might be termed estimability conditions. This is most easily described with reference to figure 4 where $\mu^{(1)} = \beta_1 + \beta_2 z + \beta_3 z^2$ are plotted. Five different combinations of the β's were used and are presented in table IV and illustrated in figure 4. The scale parameters in all cases are

$$\sigma^{(1)} = \sigma^{(2)} = 0.1$$

For each simulation the design consisted of n observations each at $z = (-2, -1, 0, 1, 2)$.

With model A, there will be very few mode 2 failures at $z = \pm 2$ and it will be relatively clear that a second order polynominal is required to fit the data. In a more extreme situation such as with model E, however, there will be few if any mode 1 failures at $z = 1$, $z = 2$. It will then be considerably more difficult to determine if the model should be as in E or as in F. In essence, the censoring caused by mode 2 failures makes estimation of all three β parameters difficult much as if in the ordinary linear regression situation observations were not made at $z = 1$, $z = 2$. That is, as if we were faced with ill-conditioned normal equations.

Tables IV and V(a) to (d) present the results of 100 simulations for various combinations of model and sample size. Table IV presents the means of the parameter estimates while table V(a) to (d) presents the observed covariance matrices of the estimates, the estimated asymptotic covariance matrices and naive normal theory approximation.

Considering first the means of the parameter estimates it is seen that $\hat{\sigma}^{(1)}$ is biased low for each sample size and the amount of bias appears to depend only upon the sample size, not upon the amount of censoring.

The locus of the location parameter estimates is more biased as there is more censoring by mode 2 failures as is evident in figure 5(a) to (e). In these figures a solid line indicates the true value of $\mu^{(1)}$. The graphs of the mean values of $\hat{\mu}^{(1)}$ are indicated with dashed lines. For model A where there is minimal censoring, there is negligible bias. For model C where significant censoring begins to occur there is a minor but noticeable upward bias where the censoring occurs and similar bias downward where negligible censoring occurs. For model E where almost all of the failures for positive z are mode 2 failures, there is considerable upward bias at the right end of the function but minimal bias at the left end. These results are quite reasonable of course. Where failures are not observed, the maximum likelihood procedure pushes the location parameter toward as large a value as consistent with the data where there are failures observed.

For each simulation series, the observed covariances, asymptotic covariances, and normal theory covariances of the scale parameters are given in table V(a) to (d) and certain plots of the results given in figures 6 to 8.

Table V(a) presents the results for $n = 1$ and cases A and B only since it was difficult to get batch simulation runs to complete without numerical difficulties in the other cases. Tables V(b) to (d) present all the results for $n = 2, 4$, and 8. The naive normal theory approximation would predict that $\hat{\beta}_2$ and $\hat{\sigma}$ are mutually independent and independent of $\hat{\beta}_1$ and $\hat{\beta}_3$. Examination of the tables indicates that for case A (with minimal censoring) the corresponding observed and asymptotic covariances are indeed relatively small. The covariances become larger as the degree of censoring for positive z values increases. Normal theory and intuition also indicate significant covariance between $\hat{\beta}_1$ and $\hat{\beta}_3$. This is also borne out qualitatively.

Since one picture is worth a thousand words, plots of the variances of $\hat{\beta}_1$ are indicated in figure 6, variances of $\hat{\beta}_2$ in figure 7 and variances of $\hat{\beta}_3$ in figure 8. In each, the horizontal scale is n, the number of replications of the design, and the vertical scale is the variance of the estimator. These figures indicate that when censoring is infrequent (model A) both the normal theory and asymptotic variances behave qualitatively and quantitatively similar to the observed variances for even the smallest sample sizes. As the censoring becomes more severe the observed variances are much higher than either normal theory or asymptotic variances for small n while they become comparable for larger n.

It is easy to understand why the normal theory variance is an underestimate. It is due to the (effective) inestimability for many samples because many observations for positive z are mode 2 failures. Hence slope and curvature are difficult to define. The naive normal theory approximation ignores this. It is not clear why the asymptotic estimate should tend to be small.

These results indicate that the approach to asymptotic theory is rapid if the model is "estimable" but slow otherwise. They also indicate that rapid approach to normal theory approximation is quite possible. Much further simulation is required to substantiate these conjectures.

7.0 CONCLUSIONS AND RECOMMENDATIONS

Previous studies have concentrated on the competing failure mode estimation problem where many samples have been obtained from one single population or at most where the location parameter depends on one stress variable and repeated observations are made at each level of the stress variable. This paper is oriented toward generalizing the multiple linear regression situation where more than one stress variable exists and not many replications are performed at each combination of stress variables.

This study has indicated the maximum likelihood approach to be feasible. For moderate sample sizes the estimators approach the asymptotic covariance structure and asymptotic unbiasedness under certain conditions. When estimating parameters for mode i when most of the failures are by mode i the location parameters are nearly unbiased while the scale parameter can be substantially biased for small sample sizes. Both normal theory and asymptotic theory covariance structures appear to hold for small samples. More extensive simulations are required to concretely verify and quantify these results.

In the situation where there is substantial censoring of mode i failures by other modes, care must be taken to insure that the proposed model is estimable. Otherwise serious biasing and much increased covariances result.

Based on the results obtained, the following procedure is recommended.

(1) Perform an ordinary multiple linear regression analysis of all observed lifetimes regardless of mode of failure. This will provide rough estimates of the terms needed in the response function and an initial guess for the scale parameters.

(2) Fit the model obtained from all the data to each mode separately using as observations only the failures by that mode. This will indicate whether or not all terms are estimable. If there is serious ill-conditioning, some terms will have to be dropped to provide a model for which all terms are estimable. The ordinary least squares regression analysis of this model will then provide initial estimates of the location parameter values and the scale parameter.

(3) Refine these estimates by obtaining the maximum likelihood estimators.

This report has not addressed the questions of hypothesis testing and confidence limits. If the approach to asymptotic behavior is indeed as rapid as indicated here, then perhaps tests and confidence limits based on the estimate of the Fisher Information matrix provided by the sample will perform reasonably. Additional simulation studies to investigate this should prove valuable.

REFERENCES

1. Cox, D. R. (1959). The Analysis of Exponentially Distributed Life-Times with Two Types of Failure. Journal of the Royal Statistical Society, Ser. B 21: 411-421.

2. Draper, Norman R. and Smith, Harry (1966). Applied Regression Analysis. Wiley, New York.

3. Harter, H. Leon and Moore, Albert H. (1968). Maximum-Likelihood Estimation, From Doubly Censored Samples, of the Parameters of the First Asymptotic Distribution of Extreme Values. Journal of the American Statistical Association 63: No. 323, 889-901.

4. Herman, R. J. and Patell, R. K. N. (1971). Maximum Likelihood Estimation for Multi-Risk Model. Technometrics 13: No. 2, 385-396.

5. Hoel, D. G. (1972). A Representation of Mortality Data by Competing Risks. Biometrics 28: 475-488.

6. Jennrich, R. I. and Moore, R. H. (1975). Maximum Likelihood Estimation by Means of Nonlinear Least Squares. Technical Report No. 9, University of California, Los Angeles.

7. Kendall, Maurice G. and Stuart, Alan (1967). The Advanced Theory of Statistics. Vol. 2, 2nd edition, Hafner, New York.

8. McCool, J. I. (1978). Competing Risk and Multiple Comparison Analysis for Bearing Fatigue Tests. ASLE Transactions 21: No. 4, 271-284.

9. McCool, J. I. (1976). Estimation of Weibull Parameters with Competing-Mode Censoring. IEEE Transactions on Reliability R-25: No. 1, 25-31.

10. Moeschberger, M. L. (1974). Life Tests Under Dependent Competing Causes of Failure. Technometrics 16: No. 1, 39-47.

11. Moeschberger, M. L. and David, H. A. (1971). Life Tests Under Competing Causes of Failure and the Theory of Competing Risks. Biometrics 27: 909-933.

12. Nelson, W. B. (1974). Analysis of Accelerated Life Tests Data with a Mix of Failure Modes by Maximum Likelihood. Report 74CRD160, General Electric Co., Schenectady, New York.

13. Sampford, M. R. (1952). The Estimation of Response-Time Distributions. II. Multi-Stimulus Distributions. Biometrics 8: 307-369.

TABLE I. - SUMMARY OF RESULTS OF SIMULATION EXPERIMENT A

[Results are based on 1000 simulations. Standard deviation of mean is estimated as observed standard deviation divided by $\sqrt{1000}$.]

Δ	n	Parameter	Mean of estimates	Standard deviations of mean	Observed covariance ($\times 10^3$)		Asymptotic covariance ($\times 10^3$)	
-0.1	20	μ	2.205	0.0021	4.31		2.73	
		σ	.099	.0012	1.61	1.33	1.00	1.13
0	5	μ	2.177	0.0019	3.45		4.39	
		σ	.087	.0016	1.02	2.70	.96	2.29
	10	μ	2.194	0.0015	2.22		2.20	
		σ	.092	.0011	.47	1.25	.43	1.27
	20	μ	2.200	0.0012	1.36		1.04	
		σ	.097	.0008	.31	0.71	.16	0.60
0.1	5	μ	2.191	0.0016	2.65		2.72	
		σ	.084	.0013	.13	1.79	-.15	1.78
	10	μ	2.195	0.0012	1.43		1.38	
		σ	.093	.0010	.03	0.94	-.07	0.84
	10	μ	2.195	0.0012	1.54		(a)	
		σ	.093	.0009	.08	0.89	(a)	(a)
	20	μ	2.196	0.0009	0.76		0.69	
		σ	.096	.0006	-.03	0.41	-.01	0.43
.3	5	μ	2.194	0.0015	2.22		2.28	
		σ	.083	.0011	-.33	1.21	-.51	1.40
	10	μ	2.196	0.0011	1.16		1.14	
		σ	092	.0008	-.23	0.62	-.23	0.64
	10	μ	2.195	0.0011	1.19		(a)	
		σ	.092	.0008	-.15	.65	(a)	(a)
	20	μ	2.198	0.0008	0.66		0.57	
		σ	.096	.0006	-.10	.32	-.12	.32

(a) Not recorded.

TABLE II. - MEANS AND ESTIMATED STANDARD DEVIATIONS OF MEANS OF PARAMETER ESTIMATES IN SIMULATION EXPERIMENT B

[Results are based on 100 simulations for each case.]

Design		Mean			Standard deviation mean		
		(1, 5, 1)	(2, 3, 2)	(3, 1, 3)	(1, 5, 1)	(2, 3, 2)	(3, 1, 3)
n = 7	β_1	2.183	2.181	2.191	0.0040	0.0039	0.0047
	β_2	.203	.193	.206	.0097	.0052	.0048
	σ	.083	.081	.083	.0029	.0030	.0034

Design		Mean			Standard deviation mean		
		(1, 10, 1)	(4, 4, 4)	(5, 2, 5)	(1, 10, 1)	(4, 4, 4)	(5, 2, 5)
n = 12	β_1	2.192	2.198	2.197	0.0030	0.0031	0.0031
	β_2	.212	.199	.197	.0114	.0046	.0038
	σ	.089	.093	.093	.0022	.0025	.0026

Design		Mean			Standard deviation mean		
		(1, 20, 1)	(7, 8, 7)	(10, 2, 10)	(1, 20, 1)	(7, 8, 7)	(10, 2, 10)
n = 22	β_1	2.200	2.200	2.199	0.0022	0.0022	0.0021
	β_2	.210	.202	.198	.0101	.0028	.0026
	σ	.095	.092	.093	.0018	.0027	.0019

TABLE III. - OBSERVED, ASYMPTOTIC, AND NORMAL THEORY COVARIANCES

OBTAINED FROM EXPERIMENT B

[Results are based on 100 simulations of each case.]

Design		Observed covariance ($\times 10^3$)			Asymptotic covariance ($\times 10^3$)			Normal theory covariance ($\times 10^3$)	
1,5,1	β_1	1.58			1.71			2.35	8
	β_2	-.34	9.50		.29	5.76		.00	8.23
	σ	-.38	-.84	0.81	-.17	.29	0.93		
2,3,2	β_1	1.52			1.74			2.35	
	β_2	-.19	2.71		.28	3.08		.00	4.11
	σ	-.05	-.13	0.88	-.16	.27	0.93		
3,1,3	β_1	2.22			1.71			2.35	
	β_2	.45	2.34		.13	1.88		.00	2.74
	σ	-.25	.23	1.14	-.34	.02	0.92		
1,10,1	β_1	0.92			0.97			1.37	
	β_2	.08	13.07		.15	7.04		.00	8.23
	σ	-.11	-.38	0.49	-.14	.08	0.56		
4,4,4	β_1	0.93			1.00			1.37	
	β_2	.18	1.99		.13	1.45		.00	2.06
	σ	-.21	.16	0.65	-.21	.13	0.52		
5,2,5	β_1	0.98			1.01			1.37	
	β_2	.14	1.44		.13	1.17		.00	1.64
	σ	-.18	.14	0.68	-.20	.12	0.53		
1,20,1	β_1	0.47			0.52			0.74	
	β_2	.32	10.19		.04	5.18		.00	8.23
	σ	-.12	-.11	0.33	-.12	.02	0.29		
7,8,7	β_1	0.50			0.55			0.74	
	β_2	.04	0.77		.06	0.85		.00	1.18
	σ	-.09	.04	0.28	-.10	.03	0.34		
10,2,10	β_1	0.46			0.35			0.74	
	β_2	.09	.67		.07	0.59		.00	0.82
	σ	-.07	.12	0.36	-.10	.06	0.32		

TABLE IV. - SIMULATION RESULTS FOR EXPERIMENT C

[Results are based upon 100 simulations of each case.]

		True parameters			Means of estimates			
		β_1	β_2	β_3	β_1	β_2	β_3	σ
n = 1	A	2.000	0.000	-0.100	1.966	-0.004	-0.099	0.049
	B	2.000	.050	-.075	1.970	.048	-.077	.051
	C	-----	-----	------	-----	------	------	-----
	D	-----	-----	------	-----	------	------	-----
	E	-----	-----	------	-----	------	------	-----
n = 2	A	2.000	0.000	-0.100	1.991	-0.002	-0.103	0.078
	B		.050	-.075	1.992	.050	-.080	.073
	C		.100	-.050	1.996	.104	-.049	.076
	D		.150	-.025	2.024	.278	.023	.080
	E		.200	.000	2.053	.323	.045	.067
n = 4	A	2.000	0.000	-0.100	2.002	-0.002	-0.104	0.091
	B		.050	-.075	1.995	.049	-.078	.089
	C		.100	-.050	1.993	.100	-.051	.089
	D		.150	-.025	2.008	.174	-.017	.090
	E		.200	.000	2.024	.253	.018	.089
n = 8	A	2.000	0.000	-0.100	2.004	0.001	-0.103	0.095
	B		.050	-.075	2.002	.050	-.078	.095
	C		.100	-.050	2.002	.101	-.052	.095
	D		.150	-.025	1.997	.165	-.016	.096
	E	-----	-----	------	-----	-----	-----	-----

TABLE V. - OBSERVED, ASYMPTOTIC, AND NORMAL THEORY COVARIANCES FOR EXPERIMENT C

(a) (n = 1)

[Results based on 100 simulations of each case.]

Model		Observed covariance ($\times 10^3$)				Asymptotic covariance ($\times 10^3$)				Normal theory covariance ($\times 10^3$)		
A	β_1	6.89				7.86				7.99		
	β_2	.00	1.51			-.05	0.95			0	1.64	
	β_3	-2.09	.10	1.05		-2.31	.03	0.98		-2.35	0	1.18
	σ	-.77	.00	.16	1.12	.74	.00	-.45	1.68			
B	β_1	7.09				7.98						
	β_2	.10	1.75			.09	1.02					
	β_3	-2.32	-.02	1.23		-2.33	-.01	0.99				
	σ		.20	.00	1.07	.66	.07	-.39	1.59			

TABLE V. - Continued.

(b) (n = 2)

Model		Observed covariance ($\times 10^3$)				Asymptotic covariance ($\times 10^3$)				Normal theory covariance ($\times 10^3$)		
A	β_1	5.38				3.99				4.00		
	β_2	.00	0.82			.01	.56			.00	0.82	
	β_3	-1.32	-.03	0.44		-1.14	.02	0.48		-1.18	.00	0.59
	σ	.28	.03	-.09	0.68	.12	.09	-.10	0.77			
B	β_1	4.03				4.05						
	β_2	-.19	0.87			.06	0.59					
	β_3	-1.05	.09	0.52		-1.16	-.01	0.50				
	σ	-.17	.12	-.04	0.68	.19	.00	-.11	0.88			
C	β_1	5.89				4.21						
	β_2	2.18	4.85			.14	0.77					
	β_3	-.57	1.61	1.12		-1.18	.08	0.54				
	σ	.00	.31	.20	0.82	.10	.31	.20	0.82			
D	β_1	27.59				5.01						
	β_2	37.56	75.57			.37	2.06					
	β_3	11.39	27.70	11.35		-1.35	.66	0.95				
	σ	-.19	.03	.02	0.95	.31	.36	.05	0.91			
E	β_1	29.98				4.95						
	β_2	40.38	74.35			1.58	7.97					
	β_3	12.86	27.70	11.25		-.69	3.39	2.13				
	σ	1.25	1.52	.50	0.70	.38	1.34	.45	1.33			

TABLE V. - Continued.

(c) (n = 4)

Model		Observed covariance ($\times 10^3$)	Asymptotic covariance ($\times 10^3$)	Normal theory covariance ($\times 10^3$)
A	β_1	2.24	1.94	2.00
	β_2	.15 0.33	.02 0.28	.00 0.41
	β_3	-.59 -.04 0.26	-.55 .00 0.24	-.59 .00 0.29
	σ	.01 -.01 -.01 0.35	.02 .03 -.04 0.38	
B	β_1	2.25	2.14	
	β_2	.01 0.31	.04 0.30	
	β_3	-.59 .04 0.25	-.61 -.01 0.26	
	σ	-.06 .04 .00 0.48	.09 .01 -.05 0.40	
C	β_1	2.28	2.30	
	β_2	-.09 0.51	.10 0.42	
	β_3	-.65 .14 0.31	-.65 .04 0.30	
	σ	-.16 .15 .06 0.57	.12 .09 -.02 0.49	
D	β_1	5.90	2.37	
	β_2	4.46 9.10	.23 1.25	
	β_3	.60 3.15 1.46	-.61 .43 0.49	
	σ	.37 4.27 .08 0.51	.08 .33 .09 0.55	
E	β_1	12.68	2.60	
	β_2	15.06 27.52	.90 4.09	
	β_3	4.38 10.06 4.10	-.34 1.73 1.10	
	σ	.48 .72 .22 0.53	.13 .61 .22 0.58	

TABLE V. - Concluded.

(d) (n = 8)

Model		Observed covariance ($\times 10^3$)				Asymptotic covariance ($\times 10^3$)				Normal theory covariance ($\times 10^3$)		
A	β_1	1.20				0.96				1.00		
	β_2	.00	0.15			.00	0.14			.00	0.20	
	β_3	-.31	.00	0.13		-.27	.00	0.12		-.29	.00	0.15
	σ	.09	.00	-.04	0.17	.00	.00	-.02	0.19			
B	β_1	1.35				1.32						
	β_2	.00	0.14			.02	0.15					
	β_3	-.43	.02	0.19		-.29	.00	0.12				
	σ	.04	.00	-.03	0.20	.02	.01	-.02	0.20			
C	β_1	1.49				1.12						
	β_2	.09	0.22			.04	0.21					
	β_3	-.43	.02	0.19		-.31	.02	0.15				
	σ	.15	.10	-.02	0.26	.05	.05	-.01	0.24			
D	β_1	1.69				1.24						
	β_2	.27	2.92			.12	0.62					
	β_3	-.39	1.32	0.84		-.32	.21	0.24				
	σ	.19	.33	.07	0.33	.09	.15	.03	0.27			

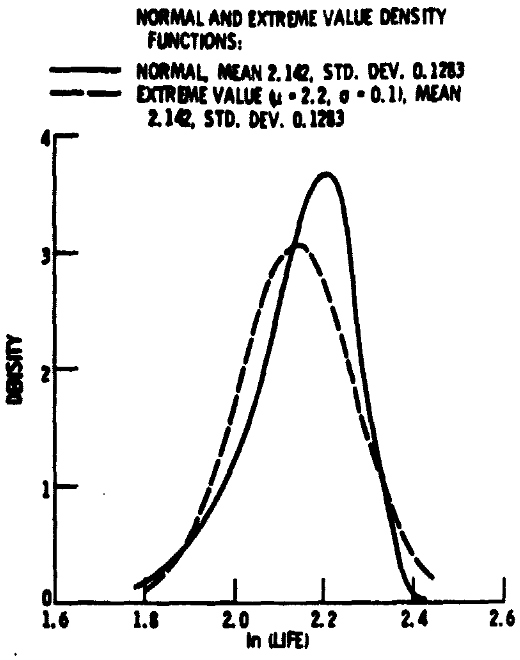


Figure 1.

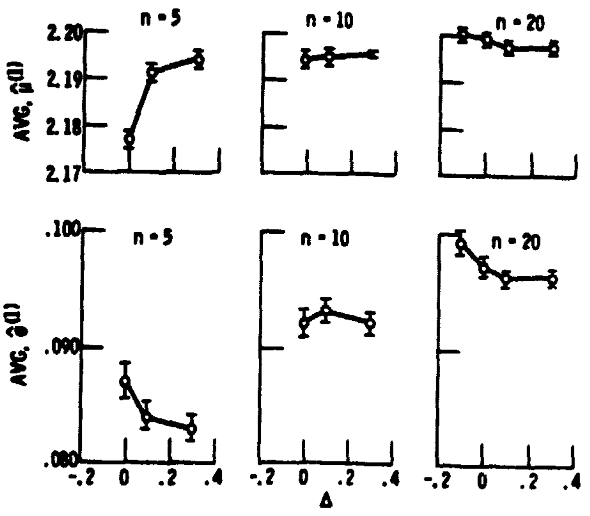


Figure 2. - The means of $\hat{\mu}$ and $\hat{\sigma}$ as function of Δ for the three sample sizes n = 5, 10, 20. Bars indicate one standard error of the mean.

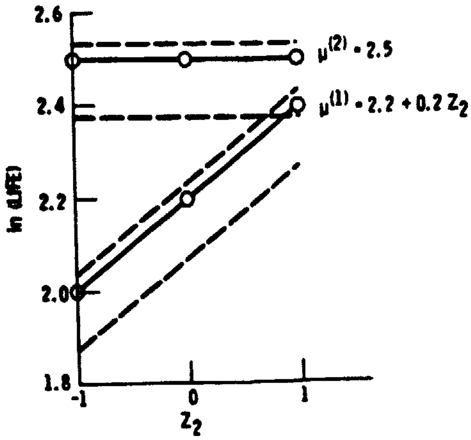


Figure 3. - The location parameters of modes one and two as a function of Z_2. The dashed lines indicate the upper and lower quartiles of each distribution.

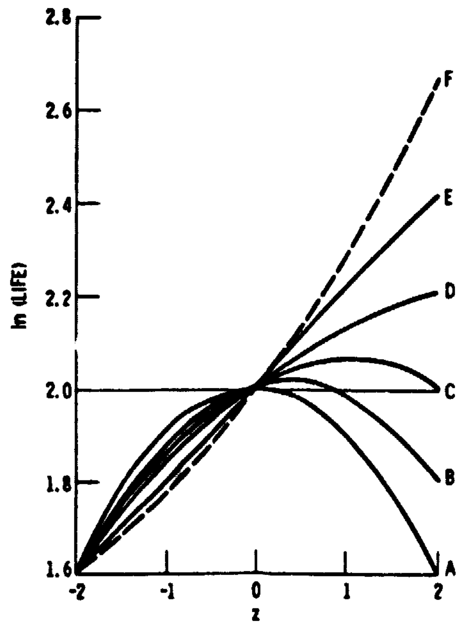
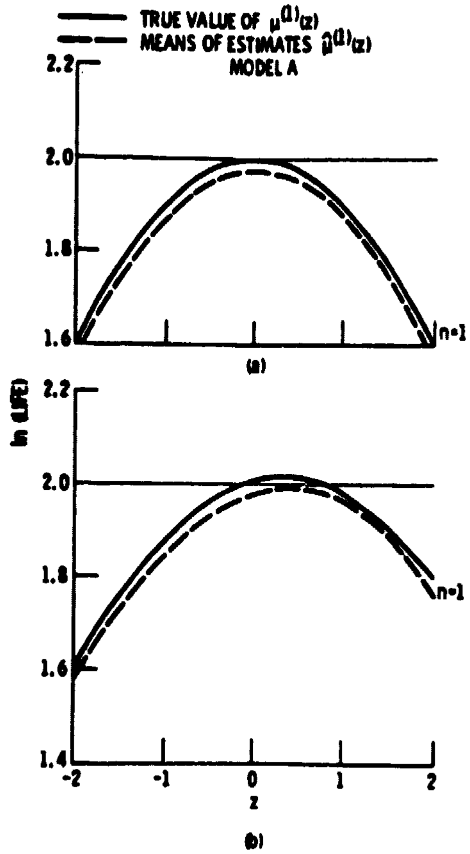


Figure 4. - The function $\mu^{(1)}(z)$ for models A-E.



TRUE VALUE OF $\mu^{(1)}(z)$
MEANS OF ESTIMATES $\hat{\mu}^{(1)}(z)$
MODEL A
2.2
2.0
1.8
1.6
n=1
(a)
ln (LIFE)
2.2
2.0
1.8
1.6
1.4
n=1
-2
-1
0
1
2
z
(b)

Figure 5.

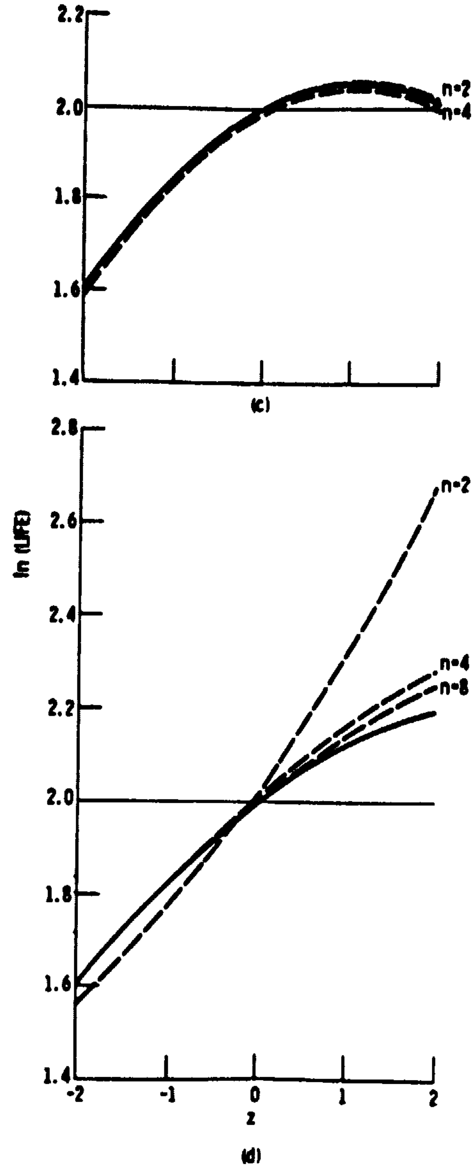
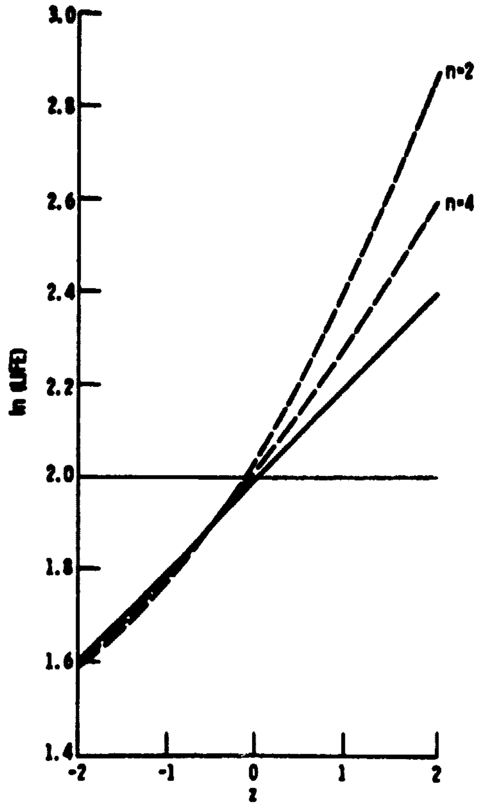


Figure 5. - Continued.



(e)

Figure 5. - Concluded.

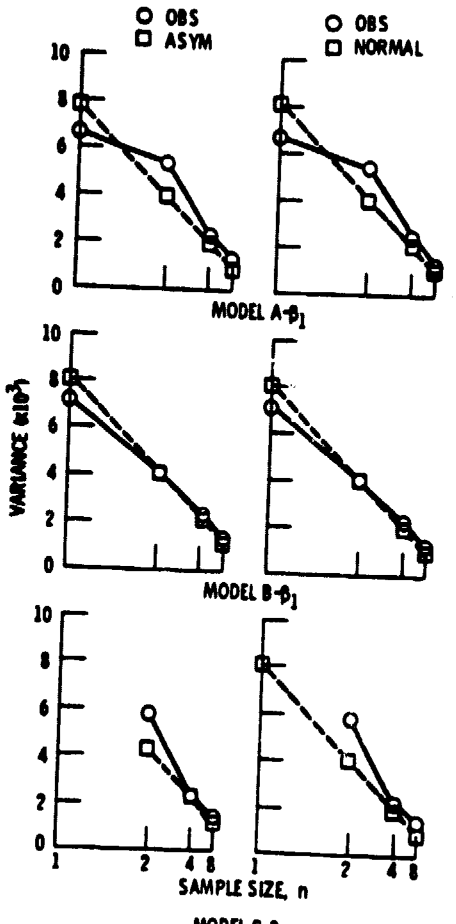


Figure 6. - Variance of β_1 ($\times10^3$) as a function of sample size for the different model models.

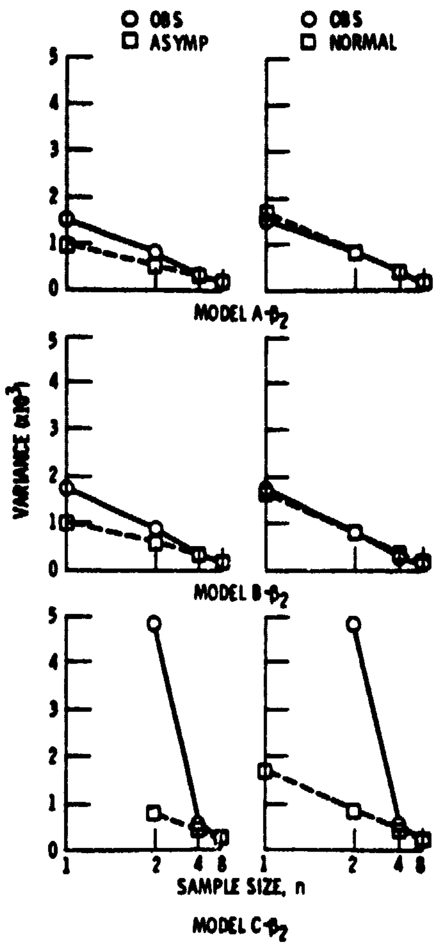
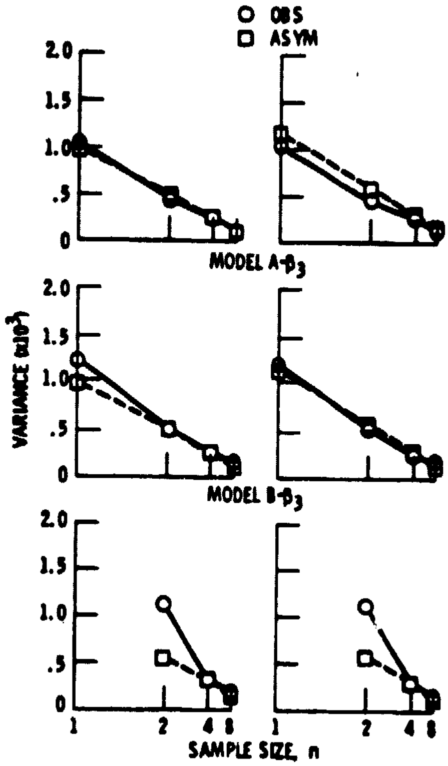


Figure 7. - Variance of β_2 (x10^{3}) as a function of sample size n for different mode 1 models.



MODEL C-β_3

Figure 8. - Variance of β_3 (x10^3) as a function of sample size for the different mode 1 models.