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NONLINEAR FOURIER ALGORITHM APPLIED TO SOLVING EQUATIONS OF GRAVITATIONAL GAS DYNAMICS

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16. Abstract

A nonlinear Fourier algorithm is applied to solving the axisymmetric stationary equations of gravitational gas dynamics. It permits the two dimensional problem of gas flow to be reduced to an approximating system of common differential equations, which are solved by a standard procedure of the Runge-Kutt type. For the adiabatic index $1<\gamma\leq 5/3$, a theorem of the existence of stationary conical shock waves with the cone vertex in the gravitating center is proved.

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NONLINEAR FOURIER ALGORITHM APPLIED TO SOLVING EQUATIONS OF GRAVITATIONAL GAS DYNAMICS

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Introduction

The development of efficient numerical methods of solution /3* of the equations of gravitational gas dynamics has become particularly urgent, in connection with the fact that the correct solution of this problem models the behavior of a gas flow in the vicinity of a "black hole." In turn, this can be of direct interest in observational astrophysics. The strong nonlinearity of the problem complicates the possibility of detailed numerical investigation of the two dimensional flow pattern, and the necessity of introduction of artificial viscosity and the subsequent incorrect approximation of the boundary conditions in a certain finite vicinity of a gravitating center in a rough difference network leads to a qualitative difference in the results of the works of various authors [1-3]. An attempt to overcome these difficulties was made in work [4], where some self modeling solutions of the gas dynamics equations were investigated, which could represent the asymptotics of flow in the vicinity of a gravitating center, on the assumption of uniformity of the flow at infinity. In the present paper, the formulation of a nonlinear algorithm with the use of a Fourier series is given, which permits solution of the two dimensional stationary problem to be reduced to a system of common differential equations, with subsequent solution of the latter by computer. Within the framework of the approach developed below, a proof of the existence of conical shock waves, with the cone vertex in the gravitating center, is successfully constructed for the adiabatic index $1<\gamma\leq 5/3$.

* Numbers in the margin indicate pagination in the foreign text.

1. Formulation of the Problem

Let, in region $L(0 \leq r < \infty,\ \theta \in [-\pi, \pi])$, axisymmetric, stationary gas flow with a gravitating center of mass M be described in a spherical coordinate system by the system of equations [4]

$$\left.\begin{aligned} &u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0, \\ &u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + \frac{1}{\rho}\frac{\partial P}{r\,\partial \theta} = 0, \\ &\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \rho u_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) = 0, \\ &u_r \frac{\partial S}{\partial r} + \frac{u_\theta}{r}\frac{\partial S}{\partial \theta} = 0 \end{aligned}\right\} \tag{1}$$

where the following notations are used: $u_r(r, \theta)$, $u_\theta(r, \theta)$ are /4
the radial and tangential components of the velocity; $\rho(r, \theta)$ is the gas density; $P(r, \theta)$ is the pressure; $S(r, \theta)$ is the entropy function; G is the gravitational constant; r, θ is the radius and angle of inclination of the radius vector to the axis of symmetry, respectively. It is assumed that functions ρ, S and P are connected by the equation of state

$$P(r,\theta) = S(r,\theta)\,\rho^{\gamma}, \tag{2}$$

where γ is the adiabatic index.

It is required to find the solution of (1)-(2), with supplementary conditions at infinity

$$u_r^{\infty}(\theta) = -U_0(\theta)\cos\theta,\ u_\theta^{\infty}(\theta) = U_0(\theta)\sin\theta,\ \rho^{\infty} = \rho_0(\theta),\ S^{\infty} = S_0(\theta) \tag{3}$$

and the axial symmetry condition with $\theta=0$ and $\theta=\pi$

$$\left.\frac{\partial^{2\nu} u_\theta}{\partial \theta^{2\nu}}\right|_{\theta=0,\pi} = 0,\ \frac{\partial^{2\nu+1} u_r}{\partial \theta^{2\nu+1}} = \frac{\partial^{2\nu+1} u_0}{\partial \theta^{2\nu+1}} = \frac{\partial^{2\nu+1} P}{\partial \theta^{2\nu+1}} = \frac{\partial^{2\nu+1} S}{\partial \theta^{2\nu+1}} = 0,\ \nu = 0, 1, \ldots \tag{4}$$

Further, instead of pressure $P(r, \theta)$, it is advisible to consider the enthalpy, after determining it by the relationship

$$H(r,\theta) = \frac{\gamma}{\gamma-1} S \rho^{\gamma-1}. \tag{5}$$

Now, in accordance with (2) and (5), system (1) takes the form

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}E(r,\theta)=\frac{u_\theta}{r}\left(\frac{\partial}{\partial r}ru_\theta-\frac{\partial}{\partial\theta}u_r\right)+\frac{H}{\gamma}\frac{\partial}{\partial r}\tilde{S},\\
&\frac{\partial}{\partial\theta}E(r,\theta)=-u_r\left(\frac{\partial}{\partial r}ru_\theta-\frac{\partial}{\partial\theta}u_r\right)+\frac{H}{\gamma}\frac{\partial}{\partial\theta}\tilde{S},\\
&r\frac{\sin\theta}{\gamma-1}\left(ru_r\frac{\partial H}{\partial r}+u_\theta\frac{\partial H}{\partial\theta}\right)+H\left(\sin\theta\frac{\partial}{\partial r}r^2u_r+r\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right)=0,\\
&ru_r\frac{\partial\tilde{S}}{\partial r}+u_\theta\frac{\partial}{\partial\theta}\tilde{S}=0,
\end{aligned}\right\}\tag{6}$$

where it is assumed that

$$E(r,\theta)=\frac{u^2(r,\theta)}{2}+H(r,\theta)-\frac{GM}{r},$$
$$u^2(r,\theta)=u_r^2+u_\theta^2,\quad \tilde{S}=\ln S(r,\theta).$$

In concluding the formulation of the problem, we make the assumption that the solution of system (6) is smooth, namely, we will assume that the shock waves break down the entire region of determination $L(r\in[0,\infty],\theta\in[-\pi,\pi])$ into $\ell\,(\ell\geq 1)$ subregions L_ℓ, within which the functions $u(r,\theta)$, $u_\theta(r,\theta)$, $u_r(r,\theta)$, $\tilde{s}(r,\theta)$ and $H(r,\theta)$ are infinitely differentiable with respect to angle θ, with the exception of the boundary $\mathcal{L}_S^\ell(r,\theta_S)$, where the Hugoniot /5
condition is valid

$$\left.\begin{aligned}
&(\rho u_n)_+=(\rho u_n)_-,\\
&(P+\rho u_n^2)_+=(P+\rho u_n^2)_-,\\
&E(r,\theta)_+=E(r,\theta)_-
\end{aligned}\right\}\tag{7}$$

2. Formulation of Fourier Algorithm

System of gravitational gas dynamics equations (1) and system (6) equivalent to it are systems of nonlinear equations, the numerical solution of which presents great difficulties. As is known [5], to construct efficient algorithms for solution of nonlinear equations, the fortunate choice of the form of presentation of the desired function is of decisive importance. Therefore, we dwell in greater detail on the procedure of representation of the solution and its justification. For this purpose, we define velocity components $u_r(r, \theta)$ and $u_\theta(r,\theta)$ by the expression

$$\left.\begin{aligned} u_r(r,\theta) &= -u(r,\theta)\cos\Phi(r,\theta), \\ u_\theta(r,\theta) &= u(r,\theta)\sin\Phi(r,\theta), \end{aligned}\right\} \tag{8}$$

where $u(r, \theta)$ is the velocity modulus, $\Phi(r, \theta)$ is the angle of inclination of $u_r(r, \theta)$ to vector $\vec{u}(r, \theta)$.

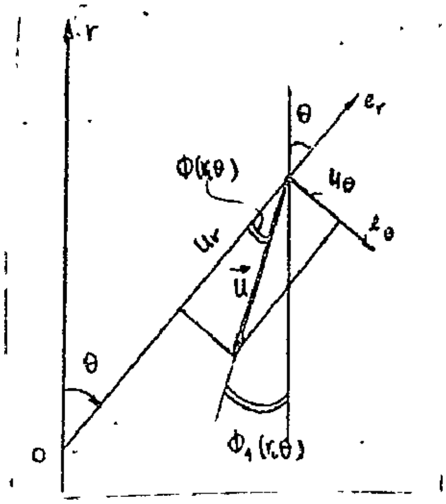

Fig. 1.

We establish the form of function $\Phi(r, \theta)$. Available information in the formulation of the problem (1)-(4) permits this to be done unambiguously. It fact, we analyze the geometric image of vector $\vec{u}$. It follows from Fig. 1 that /6

$$\Phi(r,\theta) = \theta + \Phi_1(r,\theta). \tag{9}$$

Further, from boundary conditions (3) and symmetry conditions (4), the following properties of the functions flow

$$\left.\begin{aligned} &1.\ \Phi_1(r,\theta) = -\Phi_1(r,-\theta),\ \theta\in[-\pi,\pi], \\ &2.\ \Phi_1(r,\theta) = \Phi_1(r,\theta+2\pi k),\ k=0,1,\dots, \\ &3.\ \Phi_1(r,\theta)\big|_{\theta=0,\pi} = 0, \quad 4.\ \lim_{r\to\infty}\Phi_1(r,\theta) = 0 \end{aligned}\right\} \tag{10}$$

By virtue of the definition, angular function $\Phi_1(r, \theta)$ is everywhere unequivocal and finite ($|\Phi_1(r, \theta)| < A$), with limited variation of the function, piecewise differentiable in region L, with the exception of shock wave fronts $\mathcal{L}_S^{\ell}(r, \theta_S)$, on the boundaries of which, there are finite, unilateral derivatives

$$\lim_{t\to+0}\frac{\Phi_1(r,\theta_s+t)-\Phi_1(r,\theta_s+0)}{t}, \quad \lim_{t\to+0}\frac{\Phi_1(r,\theta_s-t)-\Phi_1(r,\theta_s-0)}{-t}.$$

In this manner, function $\Phi_1(r, \theta)$ satisfies the Lipshits continuity conditions [9, 10]

$$|\Phi_1(r,\theta')-\Phi_1(r,\theta)| \le C|\theta'-\theta|^{\alpha},\ C>0,\ 0<\alpha\le 1,$$

by virtue of which, there is the following proposition.

Theorem 1

Function $\Phi_1(r, \theta)$, everywhere in region L, excepting points $\theta=\theta_S$, can be represented by the uniformly converging Fourier series

$$\Phi_1(r,\theta) = \sum_{K=1}^{\infty} \mu_K(r) \sin K\theta, \quad \mu_K(\infty) = 0, \tag{11}$$

and this representation is unique.

In fact, conditions (10) satisfy only functions of the type

$$\tilde{\Phi}_K(r,\theta) = \mu_K(r) \sin K\theta,$$

and, since there is a countable set of such functions, (11) follows from it, which proves the uniqueness of the representation. In turn, the Lipshits continuity conditions ensure uniform convergence of series (11) for $\theta \neq \theta_S$. We also propose that termwise differentiability of the series is possible.

Further, by virtue of the uniform convergence, limited to the finite segment

$$\Phi(r,\theta) = \theta + \sum_{K=1}^{N} \mu_K(r) \sin K\theta, \tag{12}$$

for expression (9), we obtain the desired approximate Fourier representation /7

$$\left.\begin{aligned} u_r(r,\theta) &= -u(r,\theta)\cos\left(\theta + \sum_{K=1}^{N} \mu_K \sin K\theta\right), \\ u_\theta(r,\theta) &= u(r,\theta)\sin\left(\theta + \sum_{K=1}^{N} \mu_K \sin K\theta\right) \end{aligned}\right\} \tag{13}$$

3. Fourier Approximation of Two Dimensional Problem (1)-(6) of System of Common Differential Equations

We note that numerical solution of system of stationary gas dynamics equations (6) with a gravitating center assumes a transition through the nonstationary problem of the method of establishment and difference approximation of the initial equations. By virtue of the fact that the equations to be solved have first order differential operators, the difference equations which approximate them, because of limitations on stability, have the same first order of approximation in difference network $\ell_h(r_h, \theta_h)$.

Fourier representation (11)-(13), established above, permits the development of a fundamentally different approach to solution of the problem, which consists of reduction of the system of two dimensional equations in partial derivatives to a system of common differential equations in normal form which approximate them, solved relative to the derivatives. Let the difference network with respect to angle θ have N_θ nodes. It is known [7] that any periodic function in such a difference network can be approximated by a discrete Fourier series, with number of terms $N \leq N_\theta$. Thus, the Fourier representation is the analog of a difference description of the problem with respect to angular variable θ. Moreover, the uniform convergence of series (11) allows it to be hoped that the initial problem can be solved satisfactorily, with the minimum possible number of terms of representations (12)-(13).

We also note a characteristic feature of equations (6), which is that they include functions of different parity with respect to the angular variable, and that this circumstance leads to difficulties in obtaining a consistent order of approximation, with respect to angle, of the approximating equations. In fact, it is easy to show that the application of an expansion in a Taylor series to system (6) gives rise to a meshing chain of equations. The mathematical technique developed below, within the framework of the Fourier representation under consideration, permits these difficulties to avoided and construction of a sufficiently general theory of reduction of gravitational gas dynamics equations (6) /8
to a closed system of common differential equations, which can be solved with respect to the derivatives of the initial functions.

We now proceed to description of the algorithm. For this purpose, we first substitute expression (8) in system (6), and we carry out identical transformations, with the use of the notations

$$
\left.\begin{array}{l}
E(r,\theta) = \frac{m}{2} + H - \frac{1}{r}, \quad m(r,\theta) = u^2(r,\theta), \\
R(r,\theta) = u\frac{\partial}{\partial r} r u_\theta - u\frac{\partial}{\partial\theta} u_r, \quad \widetilde{S} = \ln S,
\end{array}\right\} \tag{14}
$$

where, following the work of Hunt [2], the Bondi radius ($R_B = GM/c_\infty^2$) is adopted as the unit of distance, the asymptotic velocity of sound at infinity c_∞ is adopted as the unit of velocity and, in this manner, we will assume further that system (6) is made dimensionless.

Then, with (14) taken into account, from (6) there flows

$$
r\cos\phi\frac{\partial E}{\partial r} - \sin\phi\frac{\partial E}{\partial\theta} = 0, \tag{15}
$$

$$
r\cos\phi\frac{\partial\widetilde{S}}{\partial r} - \sin\phi\frac{\partial\widetilde{S}}{\partial\theta} = 0, \tag{16}
$$

$$
\cos\phi\, R(r,\theta) = K_1(r,\theta), \quad K_1(r,\theta) = \frac{\partial E}{\partial\theta} - \frac{H}{\gamma}\frac{\partial\widetilde{S}}{\partial\theta}, \tag{17}
$$

$$
R(r,\theta) = \sin\phi\frac{r}{2}\frac{\partial m}{\partial r} + \cos\phi\, m r\frac{\partial\phi}{\partial r} + \sin\phi\left(1 - \frac{\partial\phi}{\partial\theta}\right)m + \frac{\cos\phi}{2}\frac{\partial m}{\partial\theta}, \tag{18}
$$

$$
\frac{ru\sin\theta}{\gamma-1}\left(-r\cos\phi\frac{\partial H}{\partial r} + \sin\phi\frac{\partial H}{\partial\theta}\right) + H\left(-\sin\theta\frac{\partial}{\partial r}r^2 u\cos\phi + r\frac{\partial}{\partial\theta}u\sin\phi\sin\theta\right) = 0. \tag{19}
$$

Equation (19), in turn, is reduced to the form

$$
\begin{aligned}
&\sin\theta\left[\cos\phi\left(\frac{m}{\gamma-1} - H\right)\frac{r}{2}\frac{\partial H}{\partial r} + \sin\phi\, H m r\frac{\partial\phi}{\partial r} + \cos\phi\frac{m}{r(\gamma-1)} - \frac{\sin\phi}{2}\left[\frac{m}{\gamma-1} - H\right]\frac{\partial m}{\partial\theta} - \right. \\
&\left. - \cos\phi\, H m\left(2 - \frac{\partial\phi}{\partial\theta}\right)\right] + \cos\theta\sin\phi\, H m = 0.
\end{aligned} \tag{20}
$$

We introduce the notation

$$
\phi_\nu = \frac{\partial^\nu\phi}{\partial\theta^\nu}, \quad E_\nu = \frac{\partial^\nu E}{\partial\theta^\nu}, \quad m_\nu = \frac{\partial^\nu m}{\partial\theta^\nu}; \quad H_\nu = \frac{\partial^\nu H}{\partial\theta^\nu}, \quad \tau = \ln r. \tag{21}
$$

Then, by solving equations (17)-(20) with respect to derivative $\partial\phi/\partial\tau$, we obtain

$$
\begin{aligned}
&\operatorname{tg}\theta\, A + \operatorname{tg}\phi\, \Gamma = 0, \quad \Gamma = mH, \\
&A(r,\theta) = \widetilde{A}(r,\theta) - B(r,\theta), \quad \widetilde{A}(r,\theta) = \frac{1}{2}C(r,\theta)\frac{\partial m}{\partial\tau}; \\
&B(r,\theta) \doteq (1+\operatorname{tg}^2\phi)(1-\phi_1)\Gamma + \operatorname{tg}\phi\frac{m m_1}{2(\gamma-1)} - \left(\frac{m}{r(\gamma-1)} - \Gamma\right) - \operatorname{tg}\phi(1+\operatorname{tg}^2\phi)HK_1, \\
&C(r,\theta) = \frac{m}{\gamma-1} - H(1+\operatorname{tg}^2\phi),
\end{aligned} \tag{22}
$$

$$\frac{\partial\Phi}{\partial\tau} = \mathcal{D}, \quad \mathcal{D} = (1+tg^2\Phi)\frac{K_1}{m} - tg\Phi\frac{1}{2m}\frac{\partial m}{\partial\tau} - (1-\Phi_1)tg\Phi - \frac{m_1}{2m}. \tag{23}$$

Here, we also write equations (15) and (16), which, with notation (21) taken into account, take the form /9

$$\frac{\partial E}{\partial\tau} - tg\Phi E_1 = 0, \quad \frac{\partial\tilde{S}}{\partial\tau} - tg\Phi\tilde{S}_1 = 0. \tag{24}$$

As the following analysis shows, function $\tilde{s}(r, \theta)$ should be determined by the relationship

$$\tilde{S}(r,\theta) = \frac{\lambda(r,\theta)}{E(r,\theta)}, \quad K_1(r,\theta) = E_1\left(1+\frac{\lambda H}{\gamma E^2}\right) - \frac{H\lambda_1}{\gamma E}, \tag{25}$$

where $\lambda(r, \theta)$ satisfies equation (24).

Further, the algorithm consists of the following. We substitute representation (12) in equations (22)-(24). Now, with the use of symmetry conditions (4), and the smoothness and infinite differentiability of the solution with respect tothe angle everywhere outside the boundaries of the shock wave front, we differentiate equations (22)-(24) in sequence, until, as $\theta\to 0$ or $\theta\to\pi$, a closed system of common differential equations is obtained from variable $r\in(0, \infty)$, for determination of coefficients m_{2u}, E_{2u}, λ_{2u}, $u=0, 1, \ldots, N$, μ_ℓ, $\ell=1, 2, \ldots, N$. For this purpose, the rule of Liebnitz [8] of differentiation of products of functions will be useful subsequently. It consists of the following

where

$$\left.\begin{aligned} \frac{d^\nu(ZY)}{d\theta^\nu} &= \sum_{k=0}^{\nu}\binom{\nu}{k} Z_k(\theta) Y_{\nu-k}(\theta), \\ q_k^\nu &= \binom{\nu}{k} = \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!}, \quad q_0^\nu = 1, \\ Z_0(\theta) &= Z(\theta), \quad Y_0(\theta) = Y(\theta) \end{aligned}\right\} \tag{26}$$

Since the products of functions of different parities with respect to angle are included in equations (22)-(25), it is advisible to consider modifications of relationship (26) separately below, for the cases of even and odd derivatives.

1. Let $u=2n+1$, $n=0, 1, \ldots$, and simultaneously, $z_k(0)=0 \forall k=2\ell$, $\ell=0, 1, \ldots$. Then,

$$(zY)_{2n+1} = \sum_{\ell=0}^{n} q_{2\ell+1}^{2n+1} z_{2\ell+1} Y_{2(n-\ell)} |_{\theta=0}. \tag{27}$$

2. Now, let $u=\ell n$, $n=0, 1, \ldots$, and $z_k(0)=0 \forall k=2\ell+1$, $\ell=0, 1,$ /10
... From this, we obtain

$$(zY)_{2n} = \sum_{\ell=0}^{n} q_{2\ell}^{2n} z_{2\ell} Y_{2(n-\ell)} |_{\theta=0}. \tag{28}$$

We proceed to calculation of the derivatives of function $\Phi(r, \theta)$, for time being, bounded only by symmetry conditions (4) at $\theta=0$. By virtue of definition (12), we find

$$\left.\begin{aligned} &\Phi_{2n} = \Phi_{2n}(0) = 0, \quad n = 0, 1, \ldots N, \\ &\Phi_{2n+1} = \delta_{1,2n+1} + (-1)^n \sum_{k=1}^{N} \mu_k(r)\, k^{2n+1}. \end{aligned}\right\} \tag{29}$$

Subsequently, the sequences of higher derivatives of functions $\varphi=\mathrm{tg}\Phi$, $F=1+\mathrm{tg}^2\Phi$, $T=\mathrm{tg}\theta$ also are required, the values of which are determined by the following expressions

$$\left.\begin{aligned} &\varphi_{2n} = T_{2n} = 0, \quad F_{2n+1} = 0, \quad n = 0, 1, \ldots N, \\ &\varphi_1 = \Phi_1, \quad \varphi_3 = \Phi_3 + 2\Phi_1^3, \quad \varphi_5 = \Phi_5 + 20\Phi_3\Phi_1^2 + 16\Phi_1^5, \\ &\varphi_7 = \Phi_7 + 140\Phi_1\Phi_3^2 + 42\Phi_1^2\Phi_5 + 30\cdot16\Phi_3\Phi_1^4 + 17\cdot16\,\Phi_1^7, \\ &F_0 = 1, \quad F_2 = 2\Phi_1^2, \quad F_4 = 8\Phi_1\Phi_3 + 12\Phi_1^4, \\ &F_6 = 10\Phi_1\Phi_5 + 64\Phi_1\Phi_3F_2 + 14\Phi_1^2F_2^2 + 8F_1^2F_4, \\ &T_1 = 1, \quad T_3 = 2, \quad T_5 = 16, \quad T_7 = 17\cdot16, \ldots \end{aligned}\right\} \tag{30}$$

Now, since, with even derivatives of equation (22) with respect to angle at $\theta=0$, the identity satisfies 0, we examine the odd derivatives which, by virtue of (27), take the recurrent form

(31)

where

$$A_{2(n-\ell)} = \tilde{A}_{2(n-\ell)} - B_{2(n-\ell)},$$
$$\tilde{A}_{2(n-\ell)} = \frac{1}{2}\sum_{k=0}^{(n-\ell)} q_{2k}^{2(n-\ell)} C_{2k} \frac{d}{d\tau} m_{2(n-\ell-k)},$$
$$C_{2k} = \frac{\Gamma_{2k}}{\gamma-1} - \sum_{\nu=0}^{k} q_{2\nu}^{2k} H_{2\nu} F_{2(n-\nu)},$$
$$B_{2(n-\ell)} = \sum_{\nu=0}^{n-\ell} q_{2\nu}^{2(n-\ell)} \Gamma_{2\nu} \sum_{\ell'=0}^{\nu} q_{2\ell'}^{2\nu} L_{2\nu} F_{2(\nu-\ell')} + \sum_{\nu=0}^{n-\ell} q_{2\nu+1}^{2(n-\ell)} \varphi_{2\nu+1} \sum_{\ell'=0}^{\nu} q_{2\ell'}^{2\nu+1} m_{2\ell'} m_{2(\nu-\ell')}$$
$$- \frac{m_{2(n-\ell)}}{\gamma(\gamma-1)} + \Gamma_{2(n-\ell)} - \sum_{\ell'=0}^{n-\ell} q_{2\ell'+1}^{2(n-\ell)+1} K_{2(\ell'+1)} F_{2(n-\ell-\ell')} \varphi_{2\ell'+1}, \quad L_{2\nu} = \delta_{0,\nu} - \Phi_{2\nu+1}$$

From equation (31), /11

$$A_{2n} = -\varphi_1 \Gamma_{2n} - \frac{1}{q_1^{2n+1}} \sum_{\ell=1}^{n} q_{2\ell+1}^{2n+1} \left(T_{2\ell+1} A_{2(n-\ell)} + \varphi_{2\ell+1} \Gamma_{2(n-\ell)}\right), \tag{32}$$

follows, from which, in accordance with (22), we obtain

$$\left.\begin{aligned} &\frac{dm_{2n}}{d\tau} = f_{2n}, \quad n=0,1,\dots,N, \\ &f_{2n} = \left(B_{2n} + Q_{2n} - \varphi_1 \Gamma_{2n} - \sum_{k=1}^{n} q_{2k}^{2n} C_{2k} f_{2(n-k)}\right)/C_0, \\ &Q_{2n} = -\frac{1}{q_1^{2n+1}} \sum_{\ell=1}^{n} q_{2\ell+1}^{2n+1} \left(T_{2\ell+1} A_{2(n-\ell)} + \varphi_{2\ell+1} \Gamma_{2(n-\ell)}\right) \end{aligned}\right\} \tag{33}$$

Thus, it was shown that system of common differential equations (31) has a triangular matrix, which permits its trivial solution with respect to the derivatives of variable r of desired functions m_{2n}, n=0. 1, ..., N . Similarly, a system is found for determination of moments E_{2n}

$$\frac{dE_{2n}}{d\tau} - \sum_{\ell=0}^{n} q_{2\ell+1}^{2n} \varphi_{2\ell+1} E_{2(n-\ell)} = 0; \quad n=0,1,\dots N. \tag{34}$$

Finally, we proceed to determination of the coefficients of Fourier representation (12) in equation (23). Here, in accordance with (29), we find

$$(-1)^n \sum_{k=1}^{N} \frac{d\mu_k}{d\tau} k^{2n+1} = \Phi_{2n+1}, \quad n=0,1, \quad N,$$
$$\Phi_{2n+1} = \sum_{\ell=0}^{n} q_{2\ell+1}^{2n+1} K_{2(\ell+1)} F_{2(n-\ell)} \sum_{\ell'=0}^{\ell} q_{2\ell'}^{2\ell} \xi_{2(n-\ell-\ell')} - \sum_{\ell=0}^{n} q_{2\ell+1}^{2n+1} m_{2(\ell+1)} \xi_{2(n-\ell)} \tag{35}$$
$$- \sum_{\ell=0}^{n} q_{2\ell+1}^{2n+1} \xi_{2(n-\ell)} \sum_{\nu=0}^{\ell} q_{2\nu+1}^{2\ell+1} \varphi_{2\nu+1} f_{2(\ell-\nu)} - \sum_{\ell=0}^{n} q_{2\ell+1}^{2n+1} \varphi_{2\ell+1} L_{2(n-\ell)},$$

$$\xi_{2\nu} = \left(1/m\right)_{2\nu}. \tag{36}$$

Further, we write (35) in vector matrix form. For this purpose, we introduce the notations

$$\vec{\mu}(r) = (\mu_1(r), \mu_2(r), \ldots \mu_N(r)), \quad \vec{\mathfrak{D}} = (\mathfrak{D}_1, \mathfrak{D}_2, \quad \mathfrak{D}_{2N+1}),$$

$$\hat{P}_{nk} = (-1)^n k^{2n+1}, \quad n \in [0, N-1], \quad k = 1, 2, \ldots N,$$

from which, finally, we obtain /12

$$\frac{d\vec{\mu}}{d\tau} = \hat{P}^{-1}\vec{\mathfrak{D}}, \tag{37}$$

where inverse matrix $\hat{P}^{-1}$ exists everywhere, by virtue of the good dependence of $\hat{P}$, and it is calculated once for the given order of approximation N of the problem.

Thus, the solution of two dimensional gravitational gas dynamics equation (6), in the case of random flow in regions of smoothness of the solution with respect to angle θ, is reduced to system (3N+1) of common differential equations in normal form, solved with respect to the derivatives of π of functions m_{2u}, E_{2u}, μ_{1u+1}, u=0, 1, ..., by means of which, at each point $(r, \theta) \in \mathcal{L}_S^{\ell}(r, \theta)$, the two dimensional flow pattern can be regenerated

$$\left.\begin{aligned} & m(r,\theta) \cong \sum_{\nu=0}^{N} m_{2\nu}(r)\frac{\theta^{2\nu}}{(2\nu)!}, \quad E(r,\theta) \cong \sum_{\nu=0}^{N} E_{2\nu}(r)\frac{\theta^{2\nu}}{(2\nu)!}, \\ & u_r(r,\theta) = -m^{1/2}(r,\theta)\cos\left(\theta + \sum_{k=1}^{N}\mu_k(r)\sin k\theta\right), \\ & u_\theta(r,\theta) = m^{1/2}(r,\theta)\sin\left(\theta + \sum_{k=1}\mu_k(r)\sin k\theta\right) \end{aligned}\right\} \tag{38}$$

To complete formulation of the algorithm for system (33)-(37), the initial values of the functions determined, which are produced, either by boundary conditions (3) at infinity, or by the asymptotic conditions in the vicinity of the center ($r \to 0$), must be added. We note that the problem under consideration is solved most simply, in the case of potential flow with uniform boundary conditions, when $R(r, \theta)=0$ and $E(r, \theta)=E_\infty(\theta)$. Below, as an example which illustrates the approach stated, a system of unidimensional differential equations is presented, to which equations (6) are reduced in the potential case, with N=1, and

$$\frac{dm_0}{d\tau} = -\frac{2m_0(\bar{e}^{\tau} + 2\mu c_s^2)}{m_0 - c_s^2},$$

$$\frac{d\mu}{d\tau} = \frac{(1+\mu)(\bar{e}^{\tau} + 2\mu c_s^2)}{m_0 - c_s^2} + \mu + \mu^2 - \frac{1}{2}\frac{m_2}{m_0}; \tag{39}$$

$$\frac{dm_2}{d\tau} = \left[\frac{(\gamma+1)(e^{-\tau}+2\mu c_s^2)}{(m_0-c_s^2)^2} + \frac{\frac{3}{2}(1+\mu)m_0' + 2(\gamma-1)\mu m_0 - 4\mu c_s^2 - 2e^{-\tau}}{m_0 - c_s^2}\right] m_2 -$$

$$-2\frac{c_s^2}{m_0-c_s^2}\left[\mu\left(1+\frac{9}{2}\mu+2\mu^2\right)+\frac{3}{2}(1+\mu)^2\frac{(e^{-\tau}+2\mu c_s^2)}{m_0-c_s^2}\right] m_0 , \qquad (39)$$

where /13

$$c_s^2 = (\gamma-1)\left(\frac{M_\infty}{2}+\frac{1}{r}-\frac{1}{\gamma-1}-\frac{m_0}{2}\right),\ m_0(\infty)=M_\infty^2,\ m_2(\infty)=0,\ \mu(\infty)=0,$$

c_s^2 – квадрат скорости звука, $c_s^2(\infty) = 1$; $M = m_0^{\frac{1}{2}}/c_s$.

4. Existence of Shock Waves

A characteristic feature of the gravitational gas dynamics equations under consideration is stationary shock waves, at the boundaries of which disruption of the continuity of the solution occurs. The nonlinear Fourier algorithm (12)-(13), described above, permits analysis and proof of the existence of shock waves by quite simple means. In fact, as has been pointed out, from the axial symmetry of the initial problem, the existence of two lines of symmetry flows, with $\theta=0$ and $\theta=\pi$. By using the latter situation, we now obtain from system (6) a system of differential equations approximating it, with the use of the $\theta=\pi$ line as the axis of symmetry. As a result, the same system of equations (33)-(37) is obtained, with the exception of functions $\Phi^-_{2n+1}=\Phi_{2n+1}$ included in it, determined by relationships (29), which should be substituted here by the expression

$$\Phi^+_{2n+1} = \delta_{0,n} + \sum_{k=1}^{N}(-1)^{n+k}\mu_k^+(\nu)k^{2n+1}, \quad \Phi^+_{2n}=0,\ n=0,1,\ldots \qquad (40)$$

Now, let there be a solution of system (33)-(37) and continuity in region L, both with $\Phi^-(r, \theta, \mu^-)$, and with $\Phi^+(r, \theta, \mu^+)$. But then, by virtue of the equality of the lines of symmetry at $\theta=0$ and that $\theta=\pi$, an angle θ_S should exist, which divides the region of the determination of L into two subregions of smoothness $L^-(r, \theta\leq\theta_S)$ and $L^+(r, \theta\gtrless\pi-\theta_S)$, at the boundaries of which joining of the solutions occurs from angular functions $\Phi^-(r, \theta, \mu^-)$ and $\Phi^+(r,\theta, \mu^+)$, which are not certainly equal to each other. In turn, angular functions Φ^- and Φ^+ are directly connected with the characteristics of equations (6). Actually, by definition, the characteristics (or flow lines) are expressed by the relationships

$$\frac{dr}{r\,u_r(r,\theta)} = \frac{d\theta}{u_\theta(r,\theta)},$$

from which we obtain /14

$$\frac{dr}{r\,d\theta} = -\operatorname{ctg}\Phi(r,\theta). \qquad (41)$$

Thus, each subregion L^- and L^+ should be described by its family of characteristics $r^-(\theta)$ and $r^+(\theta)$, for which, in the limit of point $r(\theta)=0$, there is a common point of both families. Since, subsequently, analysis of the behavior of the solution in the vicinity of the center is of basic interest, we first note that there is asymptotic estimate for it [4]

$$m(r,\theta) = \frac{\tilde{m}(\theta)}{r} + m_\infty + O(r^\beta),\ r\to 0,\ \beta > 0. \qquad (42)$$

On the other hand, by virtue of the fact that $\Phi_1 = \sum_{k=1}^{N} \mu_k(r)\sin k\theta$ is the change in angle of movement of the gas as a result of the gravitational force, which has to be finite everywhere, the existence of the finite limit

$$\lim_{r\to 0} |(\Phi(r,\theta)-\theta)| = \left| \sum_{k=1}^{N} \mu_k(0)\sin k\theta \right| \leq \alpha < \infty. \qquad (43)$$

follows from this. Finally, since, by definition, the shock wave front is connected with an abrupt change in the direction of movement of the gas, further, there is the following proposition about shock waves.

Theorem 2

Let, in region $L(r\in[0,\infty], \theta\in[-\pi,\pi])$, with $1\leq\gamma\leq 5/3$, there be a solution of system of axisymmetric, stationary, gravitational gas dynamics equations (6), which satisifies boundary conditions (3) at infinity and asymptotic behavior conditions (42) in the vicinity of the center where

$$m(r,\theta) = \frac{\widehat{m}(\theta)}{r} + m_\infty + O(r^\beta), \ \beta \geq 0, r \to 0,$$
$$m = u_r^2 + u_\theta^2, \ u_r = -m^{1/2}\cos\Phi(r,\theta), \ u_\theta = m^{\frac{1}{2}}\sin\Phi(r,\theta)$$

Let further under the same conditions there be a solution of system of differential equations (33)-(37) which approximate them, for Φ^-_{1n+1} and Φ^+_{9n+1} on the lines of symmetry $\theta=0$ and $\theta=\pi$ respectively, a

$$\Phi^-_{1n+1} = \delta_{0,n} + \sum_{k=1}^{N} (-1)^n \mu_k^- K^{2n+1},$$
$$\Phi^+_{2n+1} = \delta_{0,n} + \sum_{k=1}^{N} (-1)^{n+k} \mu_k^+ K^{2n+1}.$$

Then, with $\mu_k^{\pm}(v) \neq 0 \forall k=2\ell+1$, a curve $\mathcal{L}_S(r, \theta) \gtreqless 0$ is found, which goes out of the gravitating center at angle of inclination $\theta_S < \pi$, on /15 which, with arbitrary $N \geq 1$, angle $\Phi(r,\theta) = \theta + \sum_{k=1}^{N} \mu_k(v) \sin k\theta$ there is the discontinuity $|\Phi^+(r, \theta_S)_{+\varepsilon} - \Phi^-(r, \theta_S)_{-\varepsilon}| = \delta_S \neq 0$. In this case, in the regions $L^-(\theta \leq \theta_S)$ and $L^+(\theta < \theta_S)$, the solutions are described by the corresponding families of characteristics $r^-(r_0, \theta_0^\infty, \theta)$ and $r^+(0, \theta_0 < \pi, \theta)$, which has intersection points $r^-|\theta_S = r^+|\theta_S$ on $\mathcal{L}_S(r, \theta_S)$, which, by definition, means the existence of a stationary front of a conical shock wave, with the origin in the center $r=0$.

We proceed to proof of the statements, on the basis of which we propose establishment of the impossibility of unbroken continuation of the asymptotic solution with respect to angle θ, from region L^- to region L^+. For this purpose, it is sufficient here to use the first equation of system (33), which has the form

$$\frac{1}{2} r^2 \frac{dm_0}{dr} = \frac{m_0 \left(2H_0(1-\Phi_1) r - \frac{1}{\gamma-1}\right)}{m_0\left(\frac{1}{\gamma-1} - \frac{H_0}{m_0}\right)} = -\frac{\left(1 - 2(\gamma-1) r H_0 (1-\Phi_1)\right)}{1 - (\gamma-1)\frac{H_0}{m_0}}, \quad (44)$$

where

$$H_0(r) = E_0 + \frac{1}{r} - \frac{m_0(r)}{2}.$$

Now, since, as $r \to 0$, asymptote (42) occurs, we obtain the estimates

$$\lim_{r\to 0} r H_0(r) = 1 - \frac{\tilde{m}_0}{2}, \quad \lim_{r\to 0} \frac{H_0}{m_0} = \frac{1}{\tilde{m}_0} - \frac{1}{2},$$
$$\lim_{r\to 0} r^2 \frac{d}{dr} m_0(r) = \lim \left(r \frac{d\tilde{m}}{dr} - \tilde{m}_0 \right) = -\tilde{m}_0 .$$

We introduce the notation $\delta_\mu = 1 - \Phi_1(\mu(0))$.

Then, from equation (44), as $r \to 0$, we find

$$\left(1 - \frac{\tilde{m}_0}{2}\right)\left(\frac{\gamma-1}{2} + 1 - 2(\gamma-1)\delta_\mu\right) = 0, \tag{45}$$

from which two characteristic solutions follow

$$\left.\begin{array}{ll} 1. & \delta_\mu = \dfrac{\gamma+1}{4(\gamma-1)}, \quad \tilde{m}_0 < 2, \\ 2. & \tilde{m}_0 = 2, \quad \delta_\mu \neq \dfrac{\gamma+1}{4(\gamma-1)}. \end{array}\right\} \tag{46}$$

We initially investigate the first case. From the definition of Φ_1, as $r \to 0$, there flows

$$\left.\begin{array}{l} \sum_{K=1}^{N} \mu_K^-(0) K = -\delta_\mu, \\ \sum_{K=1}^{N} (-1)^K \mu_K^+(0) K = -\delta_\mu \end{array}\right\} \tag{47}$$

Further, we come back to determination of angles $\Phi(r, \theta)$ /16

$$\Phi^-(\theta) = \theta + \sum_{K=1}^{N} \mu_K^-(0) \sin K\theta,$$
$$\Phi^+(\theta) = \theta + \sum_{K=1}^{N} \mu_K^+(0) \sin K\theta.$$

By virtue of theorem 1, functions $\Phi^-(\theta)$ and $\Phi^+(\theta)$ are continuous everywhere in the region of determination $\theta \in [-\pi, \pi]$. Into consideration of the function, we introduce

$$\Phi(\theta) = \left\{ \begin{array}{ll} \Phi^-(\theta), & \theta \leq \theta_s \\ \Phi^+(\theta), & \theta > \theta_s \end{array} \right\} \tag{48}$$

and the notations

$$\beta^- = \sum_{K=1}^{N} (-1)^K \mu_K^-(0) K, \quad \beta^+ = \sum_{K=1}^{N} \mu_K^+(0) K.$$

Then, it is evident that, by virtue of (37), there are the inequalities $|\beta^+| \neq \delta\mu$, $|\beta^-| \neq \delta\mu$, if only $\mu_k^{\pm}(0) \neq 0$ for $\forall k=2\ell+1$, $\ell=0, 1, \ldots$. We now show that function (48) is continuous everywhere, with the exception of $\theta_S \in (-\pi, \pi)$. In fact, becaue of the continuity of $\Phi^-(\theta)$ and $\Phi^+(\theta)$, for small angles, the expansion

$$\Phi^-(\theta) \cong \theta\left(1+\sum_{k=1}^{N} \mu_k^-(0)k\right) = \theta(1-\delta_\mu);\ \Phi^+(\theta) \cong \theta\left(1+\sum_{k=1}^{N} \mu_k^+(0)k\right) = \theta(1+\beta^+).$$

is valid, from which there follows

$$\lim_{\varepsilon\to 0}\left(\Phi^+(\theta_{s+\varepsilon}) - \Phi^-(\theta_{s-\varepsilon})\right) = \theta_s(\beta^+ + \delta_\mu) = \delta_s \neq 0.$$

In a similar manner, for the vicinity $\theta=\pi-\theta'$, we obtain

$$\lim_{\varepsilon\to 0}\left(\Phi^+(\pi-\theta'_{s-\varepsilon}) - \Phi^-(\pi-\theta'_{s+\varepsilon})\right) = \theta_s(\beta^- + \delta_\mu) \neq 0.$$

Since functions $\Phi^{\pm}$ are continuous by pairs everywhere it follows that function $\Phi(r, \theta)$, determined by expression (48), also is continuous everywhere, with the exception of angle θ_S. Thus, it is has been shown that, in the vicinity of the gravitating center, the solution of system (33)-(37), which approximates the initial problem, cannot be unbrokenly continued by angle from one region of smoothness to the other and, consequently, for some angles, there is a discontinuous solution. In order to complete the analysis of this asymptote $(\tilde{m}_0<2)$, we return to equations (34) for the moments of energy. Since

$$E(r,\theta) = \frac{m}{2} + H - \frac{1}{r},$$

in the general case, for $E(r, \theta)$, as $r\to 0$, we obtain the representation

$$E(r,\theta) = E_0 + \sum_{\nu=1}^{N} \frac{\varepsilon_{2\nu}}{r} \frac{\theta^{2\nu}}{(2\nu)!},\quad E^+(r,\theta) = E_0^+ + \sum_{\nu=1}^{N} \frac{\varepsilon_{2\nu}^+}{r} \frac{\theta'^{2\nu}}{(2\nu)!},\quad \theta' = \pi-\theta,\ \theta' \in [0,\pi]. \tag{49}$$

We substitute (49) in system (34). By virtue of the triangularity of the latter, to carry out its nontrivial solutions, fulfillment of the following conditions is necessary /17

$$\left.\begin{aligned} &\Phi_1 = -\frac{1}{2},\quad \varepsilon_{2\nu} = P_{2\nu}(\mu)\,\varepsilon_2;\ \nu = 2,\ldots N,\\ &P_4 = q_3^4\varphi_3 \Big/ \left(\frac{q_1^4}{2} - 1\right),\\ &P_{2\nu}(\mu) = \left(\sum_{\ell=1}^{\nu-1} q_{2\ell+1}^{2\nu}\varphi_{2\ell+1}P_{2(\nu-\ell)}\right) \Big/ \left(\frac{q_1^{2\nu}}{2} - 1\right),\ \nu>2. \end{aligned}\right\} \tag{50}$$

We now find index γ, at which, together with $\delta\mu=(\gamma+1)/(4(\gamma-1))$, solvability condition (50) occurs. In fact, by definition, $\Phi_1=1-\delta\mu=-(5-3\gamma)/(4(\gamma-1))$. On the other hand, it follows from (50) that $\Phi_1=-1/2$. From this, we find $(5-3\gamma)/(2(\gamma-1))=1$ or $\gamma=7/5=1.4$.

Thus, it has been established that characteristic solution (46) for $\tilde{m}_0\neq 2$ provides the required asymptotic behavior, only with index $\gamma=1.4$. In this case, the flow in both regions of smoothness is nonpotential, and it cannot be continued unbrokenly from region L^- to L^+. Another confluence case is index $\gamma=5/3$, at which $\delta\mu(5/3)=1$ and, consequently, $\Phi_1=0$. This corresponds to potential flow conditions in both regions, also separated by a discontinuity boundary.

We now proceed to the asymptotic solutions with $\tilde{m}_0=2$, $\delta\mu\neq\frac{\gamma+1}{4(\gamma-1)}$. Here, for enthalpy $H(r, \theta)$, we obtain

$$H(r,\theta)=H_0^0+\frac{1}{\gamma}\left(1-\frac{\tilde{m}_0}{2}\right)+\sum_{\nu=1}^{N}\frac{\tilde{H}_{2\nu}}{\gamma}\frac{\theta^{2\nu}}{(2\nu)!}=H_0^0+\sum_{\nu=1}^{N}\frac{\tilde{H}_{2\nu}}{\gamma}\frac{\theta^{2\nu}}{(2\nu)!} \tag{51}$$

We note the asymptotic behavior of the pressure in the vicinity of the center was used as a boundary condition in works [1, 3], in numerical solution of the two dimensional problem, where the pressure and, consequently, the enthalpy, in distinction from (51), was erroneously assumed to equal zero ($H(r\leq\varepsilon, \theta)=0$.

Further, in order to find the asymptotic behavior of function $\mu_{k(0)}$, we use initial equation (24) for the energy. We note first that the solution of equation (34) and the approximate solution corresponding to it $E(r, \theta)$, approximated by finite Taylor series (49), exactly satisfies equation (24) at point $\theta=0$ or $\theta'=\pi-\theta=0$ and approximately, at other values of angle θ. In connection with this, we set the difference problem in the vicinity of the center, /18
namely, we break down interval $[0, \theta_S]$ into a series of nodes $\theta_S/(N-1)$, where N is the order of approximation, $\theta_S\leq\pi$, and we demand that, at each value of angle $\theta_j=j\theta_S/(N-1)$, approximate solution $E(r, \theta)$ exactly satisfy equation (24). We then obtain

$$\sum_{\nu=1}^{N}\frac{\theta_j^{2(\nu-1)}}{(2\nu-1)!}\left(\frac{\theta_j}{2\nu}+\varphi^{\pm}(\theta_j)\right)\varepsilon_{2\nu}^{\pm}=0$$

or, with condition of nontrivial solvability (50) taken into account,

$$\left.\begin{aligned} &\sum_{\nu=1}^{N} \frac{\theta^{2(\nu-1)}}{(2\nu-1)!}\left(\frac{\theta_j}{2\nu} + \varphi^{\pm}(\theta_j)\right) P_{2\nu}^{\pm}(\mu_k^{\pm}(0)) = 0, \\ &\Phi_1^{\pm}(\mu_k^{\pm}(0)) = -\frac{1}{2}, \quad j = 1, 2, \dots N-1, \end{aligned}\right\} \tag{52}$$

where

$$\varphi^{-}(\theta_j) = \operatorname{tg}\left(\theta_j + \sum_{k=1}^{N} \mu_k^{-}(0) \sin k\theta_j\right),$$
$$\varphi^{+}(\theta_j') = \operatorname{tg}\left(\theta_j' + \sum_{k=1}^{N} (-1)^k \mu_k^{+}(0) \sin k\theta_j'\right)$$

In this manner, in the vicinity of the center, the asymptotic values of coefficients $\mu_k^{\pm}(0)$, $k=1, 2, \dots, N$ are found from solution of system of equations (52). It is easy to see that $\mu_k^{+}(0) \neq \mu_k^{-}(0)$. Actually, let N be sufficiently large. We assume that $\theta_j = \theta_j'$ and $\mu_k^{+}(0) = \mu_k^{-}(0)$. But, since, by definition, $\phi^{+}(\theta_j, \mu_k) \neq \phi^{-}(\theta_j, \mu_k)$ and $P_{2v}^{+}(\mu_u) \neq P_{2v}^{-}(\mu_k)$, it follows from (52) that $\mu_k^{+}(0) \neq \mu_k^{-}(0)$. Consequently, as above, the asymptotic solution cannot be continued without break from region L^- to region L^+. Below, in the description of the shock wave front search algorithm, it will be determined that, in the vicinity of the center, there is a unique value of angle θ_S, at which Hugoniot condition (7) is satisfied.

We show now that each region of smoothness of solution L^- and L^+ is described by its family of characteristics, the set of points of intersection of which lies on the associated (conical) shock wave. For this purpose, we analyze characteristics (41)

$$\left.\begin{aligned} \frac{dr^-}{r\,d\theta} &= -\operatorname{ctg}\left(\theta + \sum_{k=1}^{N} \mu_k^{-} \sin k\theta\right), \\ \frac{dr^+}{r^+ d\theta} &= -\operatorname{ctg}\left(\theta + \sum_{k=1}^{N} \mu_k^{+} \sin k\theta\right) \end{aligned}\right\} \tag{53}$$

Since point $r(\theta)=0$ is a common point of family of characteristics (53), here, for region L^+, it is sufficient to determine the common slope of characteristics r^+, which go out from point $r(\theta)=0$. For this, in the vicinity $r \to 0$, we consider angles $\theta = \pi - \theta'$, $\theta' \leq 1$,

$$\frac{r\,d\theta'}{dr} = \operatorname{tg}\left(\pi - \theta' + \sum_{k=1}^{N} \mu_k^{+} \sin k(\pi - \theta')\right) = -\operatorname{tg}\left(\theta' + \sum_{k=1}^{N} \mu_k^{+} (-1)^k \sin k\theta'\right) =$$
$$= -\operatorname{tg}\left(\theta'\left(1 + \sum_{k=1}^{N} \mu_k^{+}(0)(-1)^k k\right)\right) = -\operatorname{tg}\left(\theta' \Phi_1^{+}(\mu^{+})\right).$$

Since the flow is not potential in region L^+, by virtue of nontrivial solvability conditions (50), $\Phi_1^+(\mu_k^+(0))=-1/2$, it follows from this that $\frac{d\theta}{dr}>0$, i.e., $\theta(r)$ decreases with increase in r. /19

This corresponds to the slope of the curves presented in region I in Fig. 2.

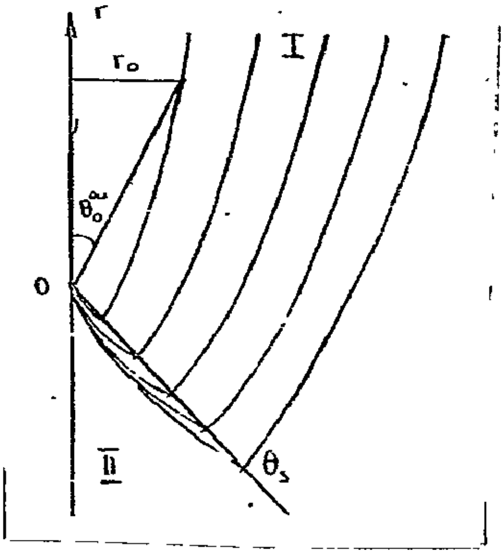

Fig. 2.

We now make a qualitative analysis of the behavior of the first family of curves for large values of $r(\theta)$. Because of the awkwardness of the expression, we limit ourselves here to one term of the representation of $\mu_1(r)$, on the assumption, by virtue of conditions (3), that it is small at large r and has an asymptotic form at infinity

$$\mu_1(r)=\frac{\mu}{r}, \quad \mu_1 \ll 1.$$

Then,

$$\frac{dr^-}{d\theta}=-\frac{r^-\cos\left(\theta-\frac{\mu}{r^-}\sin\theta\right)}{\cos\left(\theta-\frac{\mu}{r}\sin\theta\right)} \cong \frac{r^-\left(\cos\theta+\frac{\mu}{r^-}\sin^2\theta\right)}{\sin\theta\left(1-\frac{\mu}{r^-}\cos\theta\right)},$$

from which, disregarding $\frac{\mu}{r^-}\cos\theta$, we find

$$r^-(r_0,\theta_0^\infty,\theta)=r_0(\theta_0^\infty)\sin\theta_0^\infty\sin\theta\Big/\left\{\sin\theta\sin\theta_0^\infty+\frac{r_0}{\mu}\left[\sin 2\theta_0^\infty-\sin 2\theta+\frac{1}{2}\left(\arcsin\cos\theta_0^\infty-\arcsin\cos\theta\right)\right]\right\},$$

where $r_0(\theta_0^\infty)$ is the sighting distance for angle θ_0^∞. /20

It follows from the resulting expression that $r^-(r_0, \theta_0^\infty, \theta) \xrightarrow[\theta\to\pi]{} 0$. Since, by virtue of estimate (45), $\mu(r)$ has a finite limit in the vicinity $r\to 0$, here, we use asymptote $\mu_1(r)=\mu_0 e^{-r}$, and we evaluate the behavior of the curves near the center. On the assumption that $\theta=\frac{\pi}{2}+\theta'$, we obtain

$$\frac{dr^-}{d\theta'}=-\frac{r^-\cos\left(\frac{\pi}{2}+\theta'-\mu_0 e^{-r}\cos\theta'\right)}{\sin\left(\frac{\pi}{2}+\theta'-\mu_0 e^{-r}\cos\theta'\right)}=-\frac{r^-\sin\left(\mu_0 e^{-r}\cos\theta'-\theta'\right)}{\cos\left(\mu_0 e^{-r}\cos\theta'-\theta'\right)}$$

Let $\mu_0>1$. It follows from this that a value of θ' is found, such that

$$\mu_0 e^{-r}\cos\theta'-\theta'\geq 0, \quad \frac{d\theta'}{dr}=-\xi^2\frac{d\theta'}{d\xi}, \quad \xi=\frac{1}{r}, \quad \text{i.e., } \frac{d\theta'}{d\xi}>0.$$

which proves the increase of θ' and, together with it, $\theta(r)=\frac{\pi}{2}+\theta'$ as $r\to 0(\mu\to\infty)$.

Thus, it has been established that there are two families of curves $r^-(r_0, \theta_0^\infty, \theta)$ and $r^+(0, \theta_0, \theta)$, which move towards each other. Consequently, there must exist such angles θ_S, that $r^-|_{\theta_S} = r^+|_{\theta_S}$, which completes the proof of the theorem.

Note

Since the structure of equations (33)-(37) permits discontinuous solution with respect to variable r, the existence of stationary conical shock waves, with the apex in the center, does not exclude a frontal shock wave, and, thus, the pattern of gas flow from the gravitating center can be the superposition of two shock fronts.

5. Conical Shock Wave Front Search Algorithm

As was shown above, solution of the stationary gravitational gas dynamics equations in regions L, by means of the nonlinear Fourier algorithm, is reduced to the solution of approximating systems of common differential equations in subregions L^- and L^+, separated by shock wave boundary $\mathcal{L}_S(r, \theta_S)$, the location of which was previously unknown. Let $\vec{n}$ and $\vec{\tau}$ designate the unit normal and tangent vectors at each point of $\mathcal{L}_S(r, \theta_S)$. Then, condition (7) should be satisfied in $\mathcal{L}_S(r, \theta_S)$. Let, in a small vicinity of the center, the shock wave front be described by the generatrix of a cone, which goes out from the center at angle of inclination θ_S. Then, relationships (7), after simple transformations, are written in the form

$$\frac{m_+(r,\theta_s)}{m(1,\theta_s)} = \frac{\cos^2\Phi(r,\theta_s)}{\cos^2\Phi_+(r,\theta_s)}, \exp\left(\frac{\lambda^+}{E^+}-\frac{\lambda}{E}\right) = \frac{H_+(r,\theta_s)}{H(r,\theta_s)} \cdot \frac{tg^{\gamma-1}\Phi_+(r,\theta_s)}{tg^{\gamma-1}\Phi(r,\theta_s)}, \tag{54}$$

$$\frac{\frac{\gamma}{\gamma-1} H_+(r,\theta_s) + m_+ \sin^2\Phi_+}{\frac{\gamma}{\gamma-1} H(r,\theta_s) + m \sin^2\Phi} = \frac{\operatorname{tg}\Phi_+(r,\theta_s)}{\operatorname{tg}\Phi(r,\theta_s)}, \quad \frac{m_+}{2} + H_+ = \frac{m}{2} + H, \tag{54}$$

where notations (14) and (25) are used /21

$$\left.\begin{aligned} & m_-(r,\theta_s) = \sum_{\nu=0}^{N} m_{2\nu}^-(r)\frac{\theta_s^{2\nu}}{(2\nu)!}, \quad m_+(r,\theta_s) = \sum_{\nu=0}^{N} m_{2\nu}^+(r)\frac{(\pi-\theta_s)^{2\nu}}{(2\nu)!}, \\ & E_-(r,\theta_s) = \sum_{\nu=0}^{N} E_{2\nu}^-(r)\frac{\theta_s^{2\nu}}{(2\nu)!}, \quad E_+(r,\theta_s) = \sum_{\nu=0}^{N} E_{2\nu}^+(r)\frac{(\pi-\theta_s)^{2\nu}}{(2\nu)!}, \\ & \lambda_-(r,\theta_s) = \sum_{\nu=0}^{N} \lambda_{2\nu}^-(r)\frac{\theta_s^{2\nu}}{(2\nu)!}, \quad \lambda_+(r,\theta_s) = \sum_{\nu=0}^{N} \lambda_{2\nu}^+(r)\frac{(\pi-\theta_s)^{2\nu}}{(2\nu)!}, \\ & \lambda = \lambda_-, \quad m = m_-, \quad E = E_-, \quad H = H_-, \\ & \Phi^{\pm}(r,\theta_s) = \theta_s + \sum_{k=1}^{N} \mu_k^{\pm}(r)\sin k\theta_s \end{aligned}\right\} \tag{55}$$

Now, the algorithm of the solution of the complete problem briefly consists of the following. In regions L^- with boundary conditions (3) at infinity, the Cauchey problem (system (33)-(37)) is solved and, then, at each point (r, θ), the approximate solution is regenerated according to relationships (55). We proceed to regions L^+. Here, in the general case, the boundary conditions at infinity are unknown. Therefore, as the boundary conditions for the Cauchey problem in L^+, asymptotic conditions (42)-(46) should be used, with the solution $\tilde{m}_0=2$ in the vicinity of the center. Let L^+ be bounded by angle $\theta_S<\pi$, which must be found. As the first step, we assume $\theta_S=\theta^{(0)}$, where $\theta_S^{(0)}$ is some randomly assigned angle $\theta_S^{(0)}\in(0, \pi)$. We designate $\theta_S'=\pi-\theta_S$, and we break down interval $[0, \theta_S']$ into $(N-1)$ segments, so that $\theta_j'(\theta_S)=j\theta'_S/(N-1)$. Then, as was shown above, the asymptotic values of coefficients $\mu_k^+(0, \theta_S)$ aare found from solution of system (52) an

$$\left.\begin{aligned} & \sum_{k=1}^{N}(-1)^k\mu_k^+(\theta_s)k = -\frac{3}{2}, \\ & \sum_{\nu=1}^{N}\frac{\theta_j'^{\,2(\nu-1)}}{(2\nu-1)!}\left(\frac{\theta_j'}{2\nu} + \operatorname{tg}\left(\theta_j' + \sum_{k=1}^{N}(-1)^k\mu_k^+(\theta_s)\sin k\theta_j'\right)\right)\beta_{2\nu}^+ = 0, \\ & j = 1, 2, \ldots N-1 \end{aligned}\right\} \tag{56}$$

In turn, from relationships (54), for functions $\beta(\theta_S)=\cos\Phi_+(\theta_S)$, we obtain an expression which, with the arithmetic calculations omitted, is written in the form

$$\beta_{1,2}^2 = \frac{(\gamma-1)\xi^2(1\pm q) + (2-\gamma)z m \sin^2\Phi}{2(\gamma-1)(\cos^2\Phi\,\xi^2 + z^2\sin^2\Phi)}\cos^2\Phi, \tag{57}$$

where the following designations are used

$$\xi(\theta_s)=\frac{\gamma}{\gamma-1}H+m\sin^2\Phi,\quad Z=\xi-\frac{2-\gamma}{2(\gamma-1)}m,$$
$$q(\theta_s)=\sqrt{1+2\frac{\sin^2\Phi}{\xi}\left(\frac{2-\gamma}{\gamma-1}\right)m\left(1-\frac{2-\gamma}{2(\gamma-1)}\frac{m}{\xi}(1+\cos^2\Phi)\right)} \tag{58}$$

Since, in passage of the shock wave front, the inequality $m_+<m$ must occur, $\beta(\theta_S)>\cos\Phi$ is equivalent. Therefore, further, we assume $\beta=\beta_1$, if the condition $\beta\lesssim 1$ is satisfied simultaneously here. Now, since the values of $\mu_k^+(\theta_S^{(0)})$ are known from solution of system (56), with given $\theta_S=\theta_S^{(0)}$, from the equality

$$\beta(\theta_S^{(0)})=\cos\left(\theta_S^{(1)}+\sum_{K=1}^{N}(-1)^K\mu_K(\theta_S^{(0)})\sin K\theta_S^{(1)}\right),$$

we find a new value of $\theta_S^{(1)}$. If it turns out that $\theta_S^{(0)}\neq\theta_S^{(1)}$, we assume $\theta_S^{0}=\theta_S^{(1)}$ and, again, we find the solution of system (56) and $\beta(\theta_S^{(1)})$. We will continue this procedure, until the desired value $\theta_S=\lim_{\ell}\theta_S^{(\ell)}$ is found.

The next step is the search for asymptotic values of $m_+(r,\theta)$, $E_+(r,\theta)$ and $\lambda_+(r,\theta)$ in the vicinity of the center. For this purpose, it advisable to return to initial differential equations (15)-(20), which, with the asymptotic behavior of the solution taken account, take the form

$$\left.\begin{array}{l}\varphi\dfrac{dE}{d\theta}+E(r,\theta)=E_0,\quad \varphi(\theta)=\varphi^+(\theta),\ E_0=E(r,\theta=\pi),\\ \varphi\dfrac{d\lambda}{d\theta}+\lambda(r,\theta)=\lambda_0,\quad \lambda_0=\lambda(r,\theta=\pi),\ \theta\leq\theta_s,\ r\leq\varepsilon>0\end{array}\right\} \tag{59}$$

$$\varphi\frac{dH}{d\theta}+\frac{H}{\gamma E}\left(\lambda\frac{E_0}{E}-\lambda_0\right)(1+E\varphi^2)=\varphi^2\left(E+\left(\frac{1}{2}-\Phi_1(\theta)\right)m-H_0^0\right), \tag{60}$$

$$\frac{\varphi}{2}\frac{dm}{d\theta}+\frac{m}{m-(\gamma-1)H}\left[E(r,\theta)-H\left(\frac{5-3\gamma}{2}+(\gamma-1)\left(\Phi_1(\theta)+\frac{\varphi}{T}\right)\right)\right]=E_0-H_0^0, \tag{61}$$

where, in reducing requations (18)-(20) to equations (60)-(61), the properties of equations (59) were used, and ε was a previously assigned small number:

$$T=\operatorname{tg}\theta,\quad \lim_{\theta\to\pi}\frac{\varphi}{T}=\Phi_1(\pi).$$

It can be shown that, from symmetry requirements $\left.\frac{dm}{d\theta}\right|_{\theta=\delta}=0$, $H_0^0=0$ follows.

Thus, the problem in the vicinity of the center was determined in the general case, to within random constants E_0 and λ_0, for the selection of which, the involvment of some physical considerations if required. By virtue of the continuity of the energy on the lines of symmetry, we subsequently will assume that $E_0=E_0^-$, $\lambda_0=\lambda_0^-$. As /23 analysis shows, system (59)-(61) is solved numerically more simply in order to obtain the values of the coefficients sought in expressions (55) from the numerical results. Therefore, here, we break down interval $[\theta_S, \pi]$ into N angular segments $(\pi-\theta_S)/N$, so that $\theta_j=\theta_S+j\Delta\theta$. Then, by using the scheme of second order of approximation of equations (59), we obtain the difference scheme

$$\varphi_k(E_{k+1}-E_{k-1})+2\Delta\theta E_k=2\Delta\theta E_0,\ k=1,2,\dots N-1, \tag{62}$$

for the solution of which, in turn, we use the trial run method [7]

$$E_k=a_k E_{k+1}+\eta_k,\ k=N-1,\dots 0, \tag{63}$$

where

$$a_k=\frac{-\varphi_k}{2\Delta\theta-\varphi_k a_{k-1}},\quad \eta_k=\frac{2\Delta\theta E_0+\varphi_k\eta_{k-1}}{2\Delta\theta-\varphi_k a_{k-1}},\quad a_0=0,\ \eta_0=E_-(r,\theta_S)$$

The solvability of (63), as is known, is connected with stability of the trial runs, which always occurs with $\theta_S\in(\pi/2, \pi)$, since, in this case, $\phi_k=\phi(\theta_k)<0$, $a_k\geq 0$, k=1, 2, ..., N. Equation (59) is solved for $\lambda(r\leq\varepsilon,\theta)$, by means of the same algorithm. After the numerical values of $E_k=E(r\leq\varepsilon, \theta_k)$, $\lambda_k=\lambda(r\leq\varepsilon, \theta_k)$ are found, we approximate them by series (55), and we then determinetthe values of E_{2v}, λ_{2v}.

The asymptotic values of $m_+(r, \theta)$ and $H_+(r, \theta)$ are found in a similar manner where relationships (54) are used as the boundary conditions for equations (60)-(61) at $\theta=\theta_S$ and, with $\theta=\pi$, from the asymptote we have $m_+(r\leq\varepsilon, \pi)=\frac{2}{v}$, $H_+(r\leq\varepsilon, \pi)=H_0^0=0$. Here, in distinction from equation (59), together with the requirement for second order of approximation, an iteration scheme for solvability of the difference analog of equations (60)-(61) also should be used.

Thus, the solutions of equations (59)-(61) completely define the problem in region L^+ in the vicinity of the center and are the initial conditions for solution of system of equations (33)-(37). Since the shape of the surface of the shock wave front can change with distance from the center, it is advisible further to determine the position of the shock wave from the intersection of the $r^+(0, \theta_0, \theta)$ and $r^-(r_0, \theta_0^\infty, \theta)$ curves which, in turn, permits determination of the flow in the total region $L(0 \le r < \infty, \theta \in [-\pi, \pi])$.

The algorithm reported above was programmed for a BESM-6 /24
computer. The results of the calculations and discussion of them will be published separately.

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