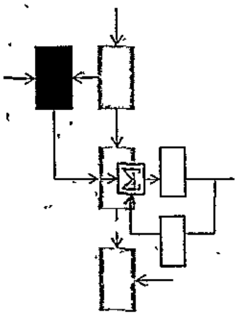
June, 1979

LIDS-TH-916

Research Supported By:

NASA Langley Research Center
Grant NASA/NSG-1553



LINEAR STATE FEEDBACK, QUADRATIC WEIGHTS, AND CLOSED LOOP EIGENSTRUCTURES

(NASA-CR-158755) LINEAR STATE FEEDBACK, QUADRATIC WEIGHTS, AND CLOSED LOOP EIGENSTRUCTURES M.S. Thesis (Massachusetts Inst. of Tech.) 101 p HC A06/MF A01 N79-26844

CSCL 12A G3/64 Unclas 27902

Peter Murray Thompson

Laboratory for Information and Decision Systems

Formerly

Electronic Systems Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

June, 1979 LIDS-TH-916

LINEAR STATE FEEDBACK, QUADRATIC WEIGHTS, AND CLOSED LOOP EIGENSTRUCTURES

by

Peter Murray Thompson

This report is based on the unaltered thesis of Peter Murray Thompson, submitted in partial fulfillment of the requirements for the degree of Master of Science at the Massachusetts Institute of Technology, June 1979. The research was conducted at the M.I.T. Laboratory for Information and Decision Systems, with support provided by the NASA Langley Research Center under grant NASA/NSG-1553.

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

LINEAR STATE FEEDBACK, QUADRATIC WEIGHTS,

AND CLOSED LOOP EIGENSTRUCTURES

by

Peter Murray Thompson

B.S. University of Virginia
(1977)

SUBMITTED IN PARTIAL FULLFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1979

Signature of Author ....Peter M Thompson....................
Department of Electrical Engineering and
Computer Science, May 11, 1979

Certified by ....
Thesis Supervisor

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee on Graduate Students

LINEAR STATE FEEDBACK, QUADRATIC WEIGHTS, AND CLOSED LOOP EIGENSTRUCTURES

by

Peter Murray Thompson

Submitted to the Department of Electrical Engineering and Computer Science on May 11, 1979 in partial fulfillment of the requirements for the Degree of Master of Science.

ABSTRACT

New results are given on the relationships between closed loop eigenstructures, state feedback gain matrices of the linear state feedback problem, and quadratic weights of the linear quadratic regulator. Previous results are presented and gaps in current knowledge are pointed out. Equations are derived for the angles of general multivariable root loci and linear quadratic optimal root loci, including angles of departure and approach. The generalized eigenvalue problem is used for the first time to compute angles of approach. Equations are also derived to find the sensitivity of closed loop eigenvalues and the directional derivatives of closed loop eigenvectors (with respect to a scalar multiplying the feedback gain matrix or the quadratic control weight).

An equivalence class of quadratic weights that produce the same asymptotic eigenstructure is defined, sufficient conditions to be in it are given, a canonical element is defined, and an algorithm to find it is given. The behavior of the optimal root locus in the nonasymptotic region is shown to be different for quadratic weights with the same asymptotic properties.

An algorithm is presented that can be used to select a feedback gain matrix for the linear state feedback problem which produces a specified asymptotic eigenstructure. Another algorithm is given to compute the asymptotic eigenstructure properties inherent in a given set of quadratic weights. This is inherently a structurally unstable problem, unless the system is "generic". Finally, it is shown that optimal root loci for nongeneric problems can be approximated by generic ones in the nonasymptotic region.

Thesis Supervisor: Gunter Stein
Title: Adjunct Professor Electrical Engineering

Thesis Supervisor: Dr. Alan J. Laub
Title: Lecturer

ACKNOWLEDGEMENTS

I am most grateful to my thesis supervisors Gunter Stein and Alan Laub for their help and encouragement. Their technical advice was excellent, but more important was the interest they showed in me as a person. I feel lucky to have had my path cross theirs.

I would like to thank Mike Athans for arranging my financial support and for the interest he has showed in my work.

I would like to thank Arthur Giordani for his excellent technical drawings.

I would like to thank my officemate James Lewis for acting as a sounding board for my ideas as they developed and for proofreading the final copy of this thesis.

Finally, I would like to thank my wife, Michelle, for her excellent job of typing this thesis, for her support, and for making sure I finished on time.

This work was conducted at the Laboratory for Information and Decision Systems and was supported by the NASA Langley Research Center under Grant NASA/NSG-1553.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

Introduction

1.1 Motivation and Summary of Results

The linear state feedback problem [1] is an important tool used for control system design. While any practical design must take many factors into consideration, a very common design objective is to achieve a specified closed loop eigenstructure. By "eigenstructure" we mean both the eigenvalues and the eigenvectors of the closed loop system. Hence, an important relationship in design is between the state feedback gain matrix and the resulting closed loop eigenstructure.

A version of the linear state feedback problem that has recently emerged as an important design is the linear quadratic regulator [2]. It was first studied by theoreticians because of its optimal properties, but it is primarily due to several other properties that design engineers have begun to use it. The linear quadratic regulator is simple to implement provided a full state or at least a reconstructed state is available, it has inherent multivariable capability, and the design algorithms are fully computerized. Desirable closed loop properties exist such as guaranteed stability, guaranteed gain and phase margins [3], and reasonable eigenstructures. Here we are primarily concerned with the relationship between the quadratic weights and the closed loop eigenstructure.

In Chapter II the linear control problems of interest are defined; and then the relationships between the feedback gain matrix, the quadratic weights, and the closed loop eigenstructure are discussed in terms of maps between parameter spaces. Previous results are presented in a tutorial style, references are given, and it is noted where the new results fit in. Some previous results on transmission zeroes are presented, also in tutorial style, in the Appendix.

The new results are now summarized, first for the linear state feedback problem and then for the linear quadratic regulator. In the first problem the feedback gain matrix and the closed loop eigenstructure are points in parameter spaces, and under certain conditions the map between them is one-to-one. In the forward direction this map is an analysis problem and in the reverse direction it is a synthesis problem (selecting a feedback gain matrix to achieve a specified closed loop eigenstructure).

If the state feedback matrix is multiplied by a scalar and this scalar is varied, a curve is traced out in each of the two paramenter spaces. Also, the closed loop eigenvalues trace out a multivariable root locus on the complex s plane. Chapter III derives equations for the angles of the root locus, for the sensitivity of the closed loop eigenvalues (how much they change with respect to a change in the parameter), and for the directional derivatives of the closed loop eigenvectors.

As the feedback gain matrix becomes very large the closed loop eigenstructure approaches certain asymptotic properties. These properties can be parameterized and a map defined between the parameters and the feedback gain matrix. Using this map to find a feedback gain matrix is a synthesis problem that is solved in Chapter V. It is one of many ways to select a feedback gain matrix, but has not to our knowledge appeared in the literature.

A similar procedure is used for the linear quadratic regulator.

The quadratic weights and the closed loop eigenstructure are points in parameter spaces but the map between them is not one-to-one. Many different quadratic weights produce the same closed loop eigenstructure. As an analysis problem this map has received a lot of attention, but as a synthesis problem it has not been used extensively (selecting quadratic weights to achieve a specified closed loop eigenstructure and then computing the feedback gain matrix).

If the weights on the control are multiplied by a scalar and this scalar is varied, a curve is traced out in the parameter spaces of quadratic weights and the closed loop eigenstructures. Also, the closed loop eigenvalues trace out a multivariable optimal root locus on the complex s plane. In Chapter III the behavior of the closed loop eigenstructure is analyzed and equations are derived for angles, sensitivities, and directional derivatives. The same is done when the quadratic weights are dependent in a more general way on a single parameter, and the particular case of analyzing the inverse square method of selecting quadratic weights is treated. Also, in Chapter V it is shown that optimal root loci for so-called "non-generic" problems can be approximated with loci of "generic" ones.

As the control weights become very small the closed loop eigenstructure approaches certain asymptotic properties. Some of the eigenvalues remain finite and others approach infinity. An algorithm is presented in Chapter V that determines how many of each there are and in what manner they approach their limit. The associated eigenvectors are also described.

The asymptotic properties can be parameterized and a map defined between the parameters and the quadratic weights. Using this map to select quadratic weights in a synthesis problem that was first studied by Harvey and Stein [4] and later generalized by Stein [5]. It turns out that many different quadratic weights produce the same asymptotic properties, and in Chapter IV an equivalence class of these quadratic weights is defined. Sufficient conditions are given to be in a particular equivalence class, a canonical element is defined, and an algorithm is presented which computes the canonical element. Finally, the behavior of the optimal root locus for different members of this equivalence class is discussed.

1.2 Notation

Matrices are indicated by capital letters. Scalars and vectors are indicated by small letters. No underlines are used and whether the variable is a scalar or vector is clear from the context. Subspaces are indicated by script letters, with the exception of the R^n, the n^{th} order real vector space. "Im A" and "ker A" are the image and kernel of A. The symbols E^0 and E^∞ indicated equivalence classes as defined in Chapter IV. A^T is the transpose of A, and x_i^H is the Hermitian transpose of the vector x_i. A^{-T} indicates $(A^{-1})^T$ or equivalently $(A^T)^{-1}$.

Equations, examples, lemmas, and theorems are numbered starting from one at the beginning of each chapter. When referenced from within the same chapter only the number is used, otherwise (4.1) means the first occurence in Chapter IV.

A permutation matrix P is a zero matrix with a one in each row and column. PA rearranges the rows of A and AP rearranges the columns of A.

All of the root locus diagrams are in the complex S plane. The x's are the open loop poles and the 0's are the transmission zeroes.

CHAPTER II

Background

2.1 Introduction

The linear control problems studied in this thesis are defined; and then the relationships between the state feedback gains, the quadratic weights, and the closed loop eigenstructure are discussed in terms of maps between parameter spaces. Emphasis is placed on both the general case of the linear state feedback problem and the special case of the linear quadratic regulator.

2.2 Linear Control Problems

2.2.1 Linear State Feedback

Consider the following linear, time invariant system with full state feedback:

$$\dot{x} = Ax + Bu \tag{1}$$

$$u = -\frac{1}{k} Fx \tag{2}$$

$$\text{where } x \in R^n$$
$$u \in R^m.$$

We will always assume that B is full rank. The matrices (A,B) will be either controllable, stabilizable, or neither. The closed loop system matrix is

$$A_{cl} = A - \frac{1}{k} BF. \tag{3}$$

As k is varied from infinity down to zero the closed loop eigenvalues trace out a root locus.

2.2.2 Linear Output Feedback

The outputs and not the full state are used for feedback. Only the case of the same number of inputs and outputs is considered. The equations are

$$\dot{x} = Ax + Bu \tag{4}$$

$$y = Cx$$

$$u = \frac{-1}{k} Ky \tag{6}$$

where $y \in R^m$.

We make the same assumptions about (A,B) as for the linear state feedback problem. C will always be full rank and (C,A) will be either observable, detectable, or neither. The closed loop system matrix is

$$A_{cl} = A - \frac{1}{k} BKC, \tag{7}$$

and k again sweeps out a root locus.

Important quantities associated with the system S(A,B,C) are the transmission zeroes. Here we use the definition of transmission zeroes due to Rosenbrock [6], which is equivalent to the following definition when the number of inputs and outputs are equal [7]. The transmission zeroes are those values of s, not including uncontrollable or unobservable modes, which reduce the rank of

$$\begin{bmatrix} A-sI & B \\ -C & 0 \end{bmatrix}.$$

See Appendix A for further discussion.

2.2.3 Linear Quadratic Regulator

This is an important class of linear state feedback controllers.

The gain matrix F is chosen so that x and u minimize the cost function

$$J = \int_0^\infty (x^T Q x + \rho u^T R u)\,dt \tag{8}$$

where $Q = Q^T \geq 0$

$R = R^T > 0$

and $0 < \rho < \infty$.

The state weighting matrix can be factored into

$$Q = M^T M$$

where Rank(Q) = Rank(M) = p and M is p x m. The matrix is arbitrary to within a premultiplication by a p x p unitary matrix (W such that $W^T W = I$). When Rank(Q) = m then we will use

$$Q = H^T H,$$

where H is m x n. Assume that (A,B) is stabilizable and (M,A) is detectable. The assumptions on Q will sometimes be downgraded to symmetric and not necessarily non-negative definite.

The optimal gain matrix is found by first solving the algebraic Riccati equation [2]

$$0 = Q + A^T P + PA - \frac{1}{\rho} PBR^{-1}B^T P \tag{9}$$

to obtain

$$F = \frac{1}{\rho} R^{-1}B^T P \tag{10}$$

$$u = -Fx \ .$$

The closed loop system matrix is

$$A_{cl}(\rho) = A - BF. \tag{12}$$

The parameter ρ is included to emphasize the dependence of A_{cl} on it. As ρ varies from infinity down to zero the closed loop eigenvalues trace out an optimal root locus.

2.2.4 Hamiltonian System

The Hamiltonian system is defined to be

$$\dot{z} = Zz \tag{13}$$

$$\text{where } Z = \begin{bmatrix} A & -\frac{1}{\rho} BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{14}$$

$$\text{and} \quad z = \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

It is of interest because it describes solutions of the linear quadratic regulator problem [8]. The eigenvalues of Z are symmetric about the imaginary axis. Therefore if s_i is an eigenvalue so is $-s_i$. Those in the left half plane (LHP) are the same as the closed loop eigenvalues of the linear quadratic regulator. If $(x_i^T, \xi_i^T)^T$ is an eigenvector of Z associated with a LHP eigenvalue then the portion x_i is an eigenvector of the linear quadratic regulator. Furthermore, $\xi_i = Px_i$.

The following trick is a useful way of applying root locus methods derived for linear output feedback systems to the optimal root locus. We define a linear output feedback system that has a closed loop system matrix equal to the Hamiltonian system matrix. Let

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -Q & -A^T \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & B^T \end{bmatrix} \quad K = R^{-1},$$

then

$$Z = \tilde{A} - \frac{1}{\rho} \tilde{B} \tilde{K} \tilde{C} .$$

As ρ is varied from infinity down to zero the eigenvalues of Z in the LHP tract out an optimal root locus. This trace is similar to one used by Shaked [9].

2.3 Maps

It is convenient to discuss the linear control problems in terms of maps between parameter spaces. This helps to organize the presentation of previous results and to show where the new results fit in. For quick reference the maps are listed in Table 2.1. In the following sections each map is defined and discussed.

2.3.1 LSF Map

This is a map associated with the linear state feedback problem as described in section 2.2.1. Here we think of the control problem as a map between the space of m x n matrices and the space of closed loop eigenvalues and eigenvectors. The notation used is

$$\text{LSF:} \quad F \rightarrow s_i, \ x_i .$$

Given an F matrix it is always possible to compute A_{cl} and then to compute the eigenstructure of A_{cl}. For the examples in this thesis EISPACK [10] subroutines were used to do this. We note here that non-trivial numerical problems arise when the eigenvalues get too close together or too far apart.

Table 2.1

Maps

	Linear State Feedback	Linear Quadratic Regulator
Forward	LSF: $F \to s_i, x_i$	LQR: $Q, R \to s_i, x_i$
Inverse	ILSF: $s_i, x_i \to F$	ILQR: $s_i, x_i \to Q, R$
Forward, dependent on a parameter	LSF(k): $\frac{1}{k} F \to s_i(k), x_i(k)$ $0 \leq k \leq \infty$	LQR( ): $Q, \rho R \to s_i(\rho), x_i(\rho)$ $0 \leq \rho \leq \infty$
Inverse Asymptotic	IALSF: $\lim_{k \to 0} s_i(k), \lim_{k \to 0} x_i(k) \to \frac{1}{k} F$	IALQR: $\lim_{\rho \to 0} s_i(\rho)$, $\lim_{\rho \to 0} x_i(\rho) \to Q, \rho R$

2.3.2 ILSF Map

This is the inverse of the linear state feedback problem and the notation used is

$$\text{ILSF:} \quad s_i, x_i \rightarrow F.$$

The ILSF map represents a synthesis problem: given a desired eigenstructure find F such that the closed loop system has this desired eigenstructure.

System controllability is an important issue in the ILSF map. Here we think of controllability as the ability to move eigenvalues with state feedback. (For a different type of definition see Willems and Mitter [11]). One way to test for controllability is to pick an F matrix by picking at random each element of F from a dense subset of the real number line, and then using this F to compute the closed loop eigenvalues. Those eigenvalues that do not move are almost surely uncontrollable. If all move then the system is controllable.

A simplified version of the ILSF map is the modal control problem. An F matrix is sought which will result in a desired set of eigenvalues, called modes. The eigenvectors are not specified. A good treatment of this problem is given by Wonham [12]. The main results are given in the form of a lemma. The multiple input results are due to Wonham.

> Lemma 1 (A,B) is controllable if and only if there exist F matrices (many in general) which place the eigenvalues in arbitrary locations. If the system is not controllable then only the controllable modes can be moved. In the single input case F is unique.

Many papers have been written about what to do with the extra design freedom available when F is not unique. One significant paper is by Moore [13]. He shows that the extra freedom can be used to select eigenvectors. His main result is:

> Lemma 2 For the case of distinct closed loop eigenvalues a unique F exists which places the eigenvalues and eigenvectors at specified locations if and only if the system is controllable and for each x_i there exists a ν_i such that
>
> $$[A-s_iI \quad B]\begin{bmatrix} x_i \\ \nu_i \end{bmatrix} = 0$$

So we see that the closed loop eigenvectors must lie in certain m dimensional subspaces determined by the s_i's. If some of the modes are uncontrollable then they must be included in the specified set of eigenvalues, but some freedom still exists in selecting the associated eigenvectors.

Moore's proof is constructive and he gives the following algorithm for finding F. Select the desired s_i (distinct) and the desired eigenvectors x_{id}. These eigenvectors may or may not lie in the permissible subspace, so compute x_i and ν_i by projecting x_{id} onto the permissible subspace (using, for instance, singular value decomposition). Then form the matrices

$$X = [x_1,\ldots,x_n]$$

$$N = [\nu_1,\ldots,\nu_m]$$

If x_i, x_{i+1} and ν_i, ν_{i+1} are complex conjugates replace them by $\mathrm{Re}(x_i)$, $\mathrm{Im}(x_i)$, and $\mathrm{Re}(\nu_i)$, $\mathrm{Im}(\nu_i)$. The gain matrix is then given by

$$F = -NX^{-1}.$$

In the single input case no extra freedom exists to select the x_i's because they are each constrained to a one dimensional subspace.

2.3.3 LSF(k) Map

When k is varied from infinity down to zero a family of linear state feedback problems is produced, which will be denoted by

$$\text{LSF}(k): \ \frac{1}{k} F \to s_i(k),\ x_i(k) \ \text{for } 0 < k \leq \infty \ .$$

As k approaches zero the closed loop eigenstructure approaches certain asymptotic properties. As k varies over its range the s_i's trace out a root locus, which in the single input case is the classical root locus. The multiple input case is more involved and is still an area of current research. As k varies over its range the x_i's rotate in R^n (if x_i and x_{i+1} are complex conjugates then use instead $\text{Re}(x_i)$ and $\text{Im}(x_i)$). Historically very little attention has been given to the behavior of the x_i's.

In Chapter III some new results are derived about the behavior of the closed loop eigenstructure as a function of k. Angles on the root locus are defined, and by using the generalized eigenvalue problem equations are derived to compute the angles. See Appendix A for a brief explanation of the generalized eigenvalue problem. The advantage of using it is that angles can be computed when k=0, even though 1/k is not defined. Also in Chapter III, equations are derived to compute directional derivatives of the x_i's.

2.3.4 Root Locus

In the single input case the root locus methods of classical control can be used to describe the behavior of the closed loop eigenvalues. These methods were first developed by Evans in 1948 [14]. They are described in Eveleigh [15], Melsa and Schwartz [16], and most other classical control textbooks. Descriptions of computer aided plotting routines have appeared in literature as recently as 1978 [17].

For the purpose of analyzing the root locus it is always possible to rephrase the state feedback problem (1-3) as an output feedback problem (4-7). Simply let KC = F.

The starting point for the classical root locus is the output feedback problem with a single input and output. The time domain description (4-6) is Laplace Transformed in order to get the following transfer function, which is the ratio of two polynomials:

$$\frac{y(s)}{u(s)} = c(sI - A)^{-1}b = \alpha \frac{N(s)}{D(s)}$$

$$\text{where } N(s) = \prod_{i=1}^{p} (s-z_i)$$

$$\text{and} \quad D(s) = \prod_{i=1}^{n} (s-p_i).$$

The z_i's are transmission zeroes and the p_i's are open loop poles. The feedback loop is closed by letting

$$u(s) = -\frac{1}{k} y(s) + \overline{u}(s),$$

where $\overline{u}(s)$ is an external input. The closed loop transfer function is

$$\frac{y(s)}{u(s)} = \frac{\alpha N(s)}{D(s) + \frac{\alpha}{k} N(\alpha)} . \tag{15}$$

The closed loop eigenvalues are those values of s which make the denominator equal to zero, and may be plotted as a function of k.

Generalizing the classical root locus methods to the multiple input case has proven to be a difficult task. Kouraritakas, Shaked, and Postlethwaite [18-22] have addressed various aspects of the problem. Progress was slow at first due to a debate over the definition of multivariable transmission zeroes. That debate is beginning to subside and attention is shifting to the behavior of modes that asymptotically approach infinity.

As is often the case in control theory, root locus problems can be attacked with either time or frequency domain techniques. In the frequency domain, the closed loop transfer function must be replaced by a transfer function matrix. Then the closed loop poles are no longer characterized by the denominator in (15) but by an algebraic function (a polynomial in s with coefficients that are polynomials in k). In order to discuss solutions of this algebraic equation concepts such as Riemann surfaces must be introduced. Asymptotic results can be found by using a Newton chart. These complications are sidestepped here by staying in the time domain whenever possible.

Of particular interest is the case when Rank(CB) = m. Why this case is of interest is motivated by Wonham [12], and also [18-22]. This is called the generic case, and it is the only one that will be reviewed in depth. The word "generic" will be used to describe a property which holds everywhere on a set of points except those belonging to a mathematical variety. A "variety" is a locus of points which satisfy a finite number of polynomials [12]. In this case the property is that Rank(CB) = m. The only points for which this property does not hold is when the polynomial det(CB) = 0. In the following example Rank(CB) < m:

$$CB = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .$$

This is the nongeneric case. However, if the 1,1 element of C is perturbed an arbitrarily small amount ε then Rank(CB) = m, which is the generic case.

The following facts are known about the multivariable root locus for the generic case. We assume that (A,B) is controllable and that KCB has no Jordan blocks of size greater than 1 x 1. Then

1) The root locus has n branches, and these are symmetric about the real axis.

2) For large k the branches originate at the open loop poles.

3) For m=1 and $\alpha>0$ (<0), where α is defined in (15), the real axis to the left (right) of an odd number of singularities lies on the locus. This rule does not apply when m>1.

4) As $k \to 0$, (n-m) of the branches stay finite. These are characterized by

$$S^o = \mathrm{diag}(s_1^o, \ldots, s_{n-m}^o)$$

$$X^o = [x_1^o, \ldots, x_{n-m}^o],$$

where each pair s_i^o, x_i^o is a solution of the generalized eigenvalue problem

$$\begin{bmatrix} A-s_i^oI & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_i^o \\ \nu_i^o \end{bmatrix} = 0 \ .$$

The (n-m) finite branches approach the transmission zeroes s_i^o, and the eigenvectors approach the zero directions x_i^o.

5) As $k \to 0$, m branches tend to infinity. These are characterized by

$$S^\infty = \mathrm{diag}(s_1^\infty, \ldots, s_n^\infty)$$

$$N^\infty = [\nu_1^\infty, \ldots, \nu_m^\infty],$$

where each pair s_i^∞, x_i^∞ is a solution of the eigenvalue problem

$$(s_i^\infty I - KCB)\nu_i^\infty = 0.$$

> The m branches approach infinity along asymptotes that have angles with respect to the positive real axis given by $\arg(-s_i^{\infty})$, where "arg(x)" means the argument of the complex number x. The asymptotic radius is $|s_i^{\infty}|/k$. The associated eigenvectors approach $\beta\nu_i^{\infty}$. If the s_i^{∞} are not distinct then the eigenvectors are arbitrary to within a subspace.

Some results are available for the root locus in the nongeneric case [18,22]. The final answers are not in yet and the state of the art is best described as messy. As $k \to 0$, fewer than (n-m) modes remain finite, and these are characterized by the same generalized eigenvalue problem. The rest of the modes group into m patterns that approach infinity. The asymptotes of each pattern meet a pivot point.

2.3.5 Definition of Angles

There are n values of s_i on the root locus for each value of k. If k is perturbed an amount Δk then each s_i will be perturbed by a (possibly very large) amount Δs_i. As $\Delta k \to 0$ then $\Delta s_i/\Delta k$ will approach the constant ds_i/dk, and the angle on the root locus is defined to be

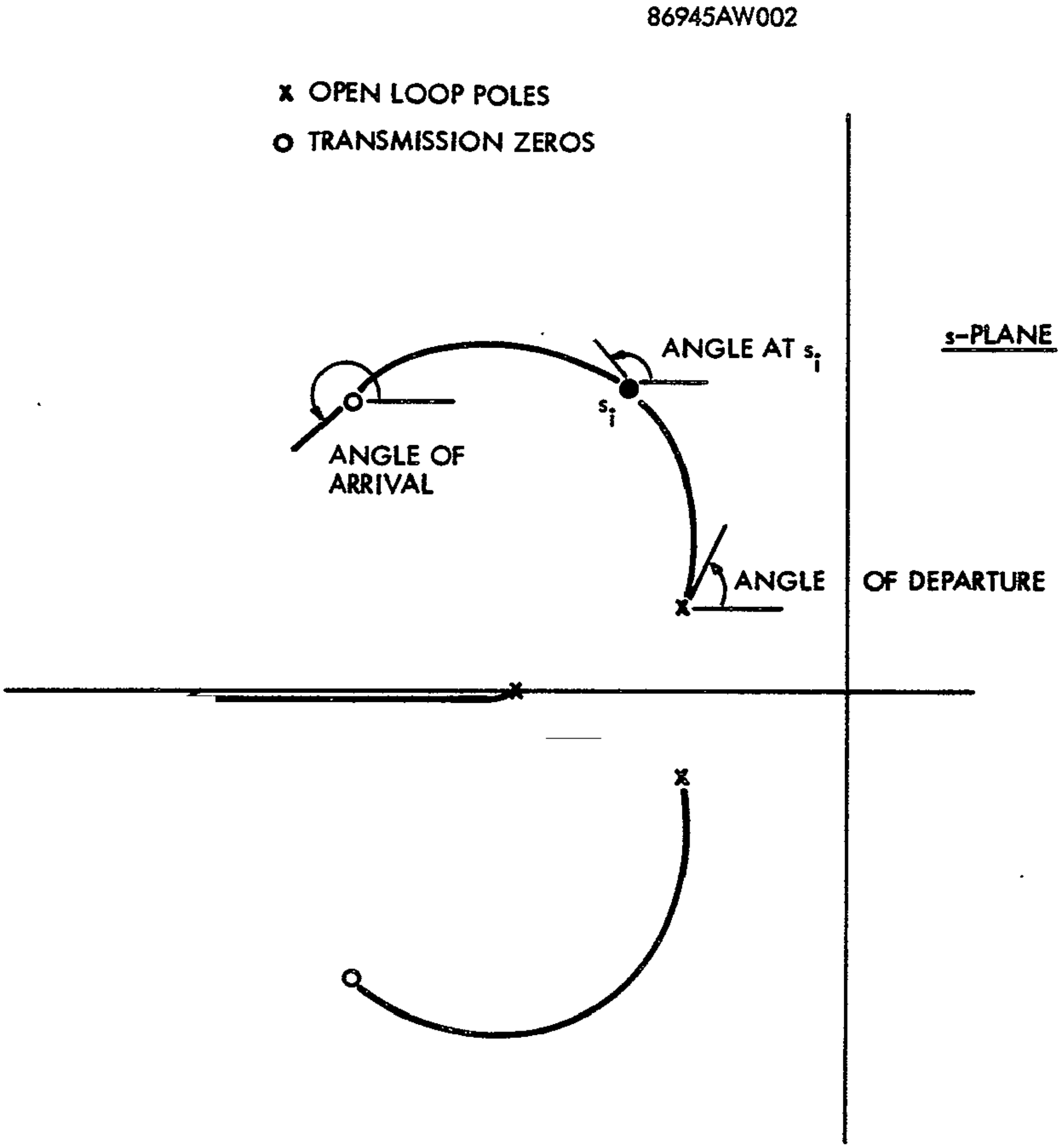
$$\arg(\overline{ds_i}).$$

The angles at the open loop poles ($k \to \infty$) will be called "angles of departure", and the angles at the transmission zeroes ($k \to 0$) will be called "angles of arrival". An example of angles on the root locus is shown in Figure 2.1.

For the single input case standard root locus formulas are available to find the angles of departure and approach. Postlethwaite [21] extends these results to the multiple input case using frequency domain methods. Shaked [19] does so using time domain methods. Here we use new time domain methods (the generalized eigenvalue problem) to extend Shaked's results.

Figure 2.1

Definition of Angles



2.3.6 IALSF Map

This is called the inverse asymptotic linear state feedback map. It is the inverse of the LSF(k) map as $k \to 0$ and is denoted by

$$\text{IALSF:} \quad \lim_{k\to 0} s_i(k),\ \lim_{k\to 0} x_i(k) \to \frac{1}{k} F.$$

The IALSF map represents a synthesis problem: given the desired asymptotic properties of the LSF(k) map characterized by

$$S^o,\ X^o,\ S^\infty,\ \text{and}\ N^\infty;$$

find F such that the closed loop system has these desired asymptotic properties. This problem is solved for the first time in Chapter V. It is similar to synthesis problems solved by Moore [13], Harvey and Stein [4], and Stein [5].

2.3.7 LQR Map

This map is associated with the linear quadratic regulator as described in section 2.2.3. Here we think of the regulator as a map between the matrices Q and R and the closed loop eigenstructure. The notation used is

$$\text{LQR:} \quad Q,R \to s_i,x_i.$$

The LQR map is one of the cornerstones of modern control theory. Major credit is due to Kalman [23].

The steps used to compute the closed loop eigenstructure are symbolically shown by

$$Q,R \to P \to F \to A_{cl} \to s_i,x_i.$$

The most important step is first finding the Riccati solution. Wonham

[24] gives the necessary and sufficient conditions on A, B, Q, R for the existence of a stable closed loop solution.

An active area of research is developing algorithms to compute Riccati solutions. One standard way is to use the eigenvectors of the Hamiltonian system matrix [8]. This method is usually referred to in the control literature as Potter's method but it predates him, (see the references of [25]). The method is symbolically shown by

$$Q,R \rightarrow Z \rightarrow z_i \rightarrow P.$$

A variation of this method with improved numerical properties is by Laub [25] and uses the Schur vectors of Z.

For the example calculations used here we were not interested in P or F, so the following shortcut was used to find the closed loop eigenstructure:

$$Q,R \rightarrow Z \rightarrow s_i, x_i.$$

A simple example shows that the Q and R matrices that produce an optimal gain matrix (and hence a closed loop eigenstructure) are not unique. Multiply Q and R by the same positive constant α. Then the Riccati solution changes from P to αP but F stays the same. Therefore we can define a mathematical equivalence class [26] of Q and R matrices in terms of the property that they produce the same optimal gain matrix.

Perhaps not so well known is that any Q with $Q \geq 0$ and $\text{Rank}(Q) > m$ is equivalent (in the sense used above) to a $\tilde{Q}$ with $\text{Rank}(\tilde{Q}) = m$. This result is used and discussed by Molinari [27] and Harvey and Stein [4], and is due to Popov [28]. As a consequence it is always possible to define a response vector

$$r = Hx$$

$$\text{where } r \in R^m$$

$$\tilde{Q} = H^T H,$$

and such that $\tilde{Q}$ is equivalent to any specified Q. The system S(A,B,H) is important to this thesis.

Necessary and sufficient conditions for (Q,R) and $(\tilde{Q},\tilde{R})$ to be in the same equivalence class have been developed by Molinari [27] and Jameson and Kriendler [29]. Molinari constrains R so that R=I and presents the following result (which is also valid for any $R = R^T > 0$):

Lemma 3 Assume (A,B) is controllable and F is optimal for some (Q,R). Then $(\tilde{Q},R)$ is in the same equivalence class if and only if there exists a real symmetric Y satisfying

$$\text{(i)} \quad \tilde{Q} = A - A^T Y - YA$$

$$\text{(ii)} \quad YB = 0.$$

The Riccati solution changes from P to P+Y, but $F = -R^{-1}B^T(P+Y)$ remains constant becuase $B^T Y = 0$. There is no guarantee that $P+Y \geq 0$ or $\tilde{Q} \geq 0$. When R is not constrained to be constant then a similar result can be extracted (with some difficulty) from the paper by Jameson and Kriendler.

2.3.8 ILQR Map

This map describes the inverse of the linear quadratic regulator and is denoted by

$$\text{ILQR:} \quad s_i, x_i \rightarrow Q, R.$$

The ILQR map represents a synthesis problem and can, in principle, be used to select quadratic weights. This turns out not to be very practical, as we will show in the following discussion. The asymptotic version of this map, presented later, is much more convenient.

The first step is to find the feedback gain matrix which produces the desired eigenstructure. To do this use the ILSF map discussed in section 2.3.2 (specifically the algorithm due to Moore [13]). If (A,B) is controllable then the s_i can be located anywhere, and the x_i must lie in certain m dimensional subspaces.

If the feedback gain matrix exists then a check can be made that it is optimal for some (Q,R). Kalman [30] solved this problem for the single input case. Anderson and Moore [13] generalized this result to the multiple input case. The version of this result presented here is due to Molinari [27].

Lemma 4 Assume (A,B) is controllable and R=I. Then F is optimal for some $Q = Q^T \geq 0$ if and only if $s_i < 0$ and $T^H(j\omega)T(j\omega) \geq I$, where $T(s) = I + F(Is-A)^{-1} B$.

Unfortunately this result is difficult to implement because it must be true for all ω. If Q is only required to be symmetric then we have the following result, also due to Molinari.

Lemma 5 Assume (A,B) is controllable and R=I. Then F is optimal for some $Q = Q^T$ if and only if $s_i < 0$ and FB is symmetric.

Jameson and Kriendler [29] extend this result to a more general class of A, B, and R matrices, again only requiring that Q be symmetric.

If the check on F succeeds then F is optimal for some (Q,R). Jameson and Kriendler give an algorithm that can reach every (Q,R) in the equivalence class of matrices that produce the same F (there is no guarantee that $Q \geq 0$). However, if the check on F fails then no hint is given as to how F, s_i, or x_i should be changed. This is the main problem with using the IALQR map as a way to select quadratic weights.

We note here that T(s) defined in lemma 4 is the return difference equation for the system S(A,B,F), and as such it plays a fundamental role in control system design. It is used to measure the disturbance rejection properties of the system, the ability of the system to follow commands, and the robustness of the system.

2.3.9 LQR(ρ) Map

When ρ is varied from infinity down to zero a family of linear quadratic regulators is produced, which will be denoted by

$$\mathrm{LQR}(\rho): \quad Q,\ \rho R \to s_i(\rho),\ x_i(\rho) \quad \text{for } 0 < \rho < \infty.$$

Only the case where R is linearly dependent on ρ is considered. Similar results, especially for eigenvectors, do not necessarily hold when Q and R are dependent on ρ in an arbitrary way. The closed loop eigenvalues trace out an optimal root locus. Its behavior is studied, as described in section 2.2.4, by using the Hamiltonian system. Of special interest is the asymptotic behavior of the $s_i(\rho)$ and $x_i(\rho)$ as $\rho \to 0$.

Several new results are derived that concern the LQR(ρ) map. In Chapter III equations are derived that compute angles on the optimal root locus and directional derivatives of the $x_i(\rho)$. In Chapter V an algorithm is presented that computes the asymptotically infinite behavior of the optimal root locus, and a result is presented that shows how an optimal root locus can be approximated by another.

2.3.10 Optimal Root Locus

In the single input case the optimal root locus can be described using classical root locus techniques. The trick is to recognize that the optimal closed loop poles are the left half plane eigenvalues of $\det(sI-Z) = 0$. Assume that $Q = h^T h$, where h is 1 x n, and that $R = r > 0$. Then by using determinant identities it can be shown that [8]

$$\det(sI-Z) = (-1)^n[\phi(s)\phi(-s) + \frac{1}{r\rho}\psi(s)\psi(-s)] \tag{16}$$

$$\text{where } \phi(s) = \det(sI-A) = \prod_{i=1}^{n} (s-p_i)$$

$$\text{and} \quad \bar{\phi}(s) = \phi(s)h^T(sI-A)^{-1}b = \alpha\prod_{i=1}^{p} (s-z_i).$$

The p_i's are the open loop poles and the z_i's are the transmission zeroes of the system $S(A,b,h)$. Equation (16) is analogous to the denominator of (15).

Before the heyday of Kalman, state space techniques, and Riccati equations it was recognized by Chang [32] that (16) can be used to plot the single input optimal root locus. He suggested using the root square locus, which is done by rewriting (16) as a polynomial in $s' = s^2$, and then plotting the classical root locus on the s' plane. Kalman [30] and Kwakernaak and Sivan [8] give rules for plotting the single input optimal root locus on the s plane. These rules involve plotting (16) on the s plane using classical root locus techniques and then only keeping the left half side.

Now we move on to the multiple input case of the optimal root locus. Rynaski [33] developed a multivariable version of the root square locus, but it is very cumbersome even for the case of two inputs. The procedure used here is to use the Hamiltonian system $S(\tilde{A},\tilde{B},\tilde{K}\tilde{C})$ and apply the multivariable root locus results [34]. The generic case is when MB is full rank (where $Q = M^TM$). Both this and the nongeneric case will be reviewed. The symmetry about the imaginary axis of the eigenvalues of the Hamiltonian system makes the analysis easier. The notation used here is the same as used by Stein [5].

1) The optimal root locus has n branches, which stay entirely in the LHP, and are symmetric about the real axis.

2) For ρ large the branches originate at $-|\mathrm{Re}\ p_i| + j\ \mathrm{Im}\ p_i$. (This is equal to p_i if p_i is in the LHP, otherwise to the mirror image of p_i about the imaginary axis).

3) For m=1 and (n-p) even (odd) the negative real axis to the left of an odd (even) number of singularities lies on the locus. This rule does not apply when $m > 1$.

4) As $\rho \to 0$, p of the branches stay finite. These are characterized by

$$S^o = \mathrm{diag}(s_i^o, \ldots, s_p^o)$$

$$X^o = [x_1^o, \ldots, x_p^o],$$

where each pair s_i^o, x_i^o is a solution of the generalized eigenvalue problem

$$\begin{bmatrix} \tilde{A} - s_i^o I & \tilde{B} \\ -\tilde{C} & 0 \end{bmatrix} \begin{bmatrix} z_i^o \\ \nu_i^o \end{bmatrix} = 0,$$

$$\text{where } z_i^o = \begin{bmatrix} x_i^o \\ \xi_i^o \end{bmatrix}.$$

If H is available then a lower dimensional generalized eigenvalue problem can be solved using the system S(A,B,H). If s_i and x_i are the finite solutions then $s_i^o = -|\mathrm{Re}\ s_i| + j\ \mathrm{Im}\ s_i$, but $x_i^o = x_i$ only if the corresponding s_i is in the LHP. The p branches that stay finite approach the s_i^o, and the associated eigenvectors approach the x_i^o. In the generic case $p = n - m$, in the non-generic case $0 \leq p < n - m$

5) In the generic case the asymptotically infinite behavior is characterized by

$$S^\infty = \mathrm{diag}(s_1^\infty, \ldots, s_m^\infty)$$

$$N^\infty = [\nu_i^\infty, \ldots, \nu_m^\infty],$$

where each pair s_i^∞, ν_i^∞ is a solution of the eigenvalue problem

$$[(s_i^{\infty})^2 \, I - R^{-1} \, B^T Q B] \nu_i^{\infty} = 0.$$

The m branches which approach infinity stay on the negative real axis with an asymptotic radius of $s_i^{\infty}/\rho^{\frac{1}{2}}$. (These are first order Butterworth patterns, contrary to Wonham [12] p. 318). The associated eigenvectors approach $B\nu_i^{\infty}$.

6) In the nongeneric case the infinite behavior is characterized by the same S^{∞} and N^{∞} and also by a multi-index γ. The (n-p) branches that approach infinity group into m Butterworth patterns. The ith pattern is n_ith order, with asymptotic radii equal to

$$\omega_i = \left(\frac{s_i^{\infty}}{\rho^{\frac{1}{2}}} \right)^{1/n_i}$$

Properties of Butterworth patterns are summarized in Table 2.2. There are n_i vectors which form a basis for the subspace associated with each pattern. These are

$$B\nu_i^{\infty}, \; AB\nu_i^{\infty}, \ldots, A^{n_i-1} B\nu_i^{\infty}.$$

The multi-index γ lists the n_i's in the following way:

$$\gamma = (01, \; 11, \; 21, \ldots, [n_1-1]1, \; 02, \ldots, [n_2-1]2, \ldots, [n_m-1]m).$$

If we define the controllability matrix

$$U = [BN^{\infty}, \ldots, ABN^{\infty}, \ldots, A^{n-1} \, BN^{\infty}],$$

which is an n x (nm) matrix, then each term (i,j) of the multi-index defines the column $A^i B\nu_j^{\infty}$ of U. The collection of all columns defined by γ we call U^{γ}, therefore

$$U^{\gamma} = [B\nu_1^{\infty}, \ldots, A^{n_1-1} B\nu_1^{\infty}, \ldots, A^{n_m-1} B\nu_m^{\infty}].$$

The columns of U^{γ} form a basis for the subspaces spanned by all m Butterworth patterns. In the generic case $U^{\gamma} = BN^{\infty}$.

Computing S^{∞}, N^{∞}, and γ in the nongeneric case is numerically an ill-posed problem because arbitrarily small changes in B or H cause the problem to become generic with only first order patterns. The s_i^{∞}'s and ν_i^{∞}'s associated with first order patterns are the nonzero solutions of the eigenvalue problem given in step 5.

Table 2.2

Butterworth Patterns

Order	Angles that the asymptotes make with the negative real axis
1	$0°$
2	$\pm 45°$
3	$0°, \pm 60°$
n (odd)	$\pm \frac{\ell}{n} 180° \quad \ell = 0, 1, \ldots, \frac{n-1}{2}$
n (even)	$\pm \frac{1}{n} (\ell + \frac{1}{2}) 180°, \ \ell = 0, 1, \ldots, \frac{n}{2} - 1$

The results for the multiple input case have been established over the last several years. The attraction of the finite branches to the transmission zeroes of S(A,B,H) is stated in Kwakernaak and Sivan [8]. Both Kouvaritakas [34] and Shaked [19] use the Hamiltonian system to find the transmission zeroes. Neither use the generalized eigenvalue problem, as done here, which can easily be used to compute the transmission zeroes and angles of arrival. Kwakernaak and Sivan [8] conjecture that the infinite modes group into Butterworth patterns, and Kwakernaak [35] later used some algebraic function theory to prove this. Wonham [12] gives a theorem which describes the behavior of the optimal root locus in the generic case. The asymptotic behavior of the eigenvectors was established by Harvey and Stein [4] in the generic case and by Stein [5] in the nongeneric case. Algorithms to find S^∞ and γ are given by Shaked [19] and Postlethwaite [36], but neither addresses the inherent numerical instability of their solutions. An algorithm to find S^∞, N^∞, and γ is given here in Chapter V.

2.3.11 The IALQR Map

The last of the maps to be defined is the inverse of the LQR(ρ) map as $\rho \to 0$, and it is denoted by

$$\text{IALQR:} \quad \lim_{\rho\to 0} s_i(\rho),\ \lim_{\rho\to 0} x_i(\rho) \to Q, \rho R.$$

This again represents a synthesis problem: given the desired asymptotic properties of the LQR(ρ) map characterized by

$$S^o,\ X^o,\ S^\infty,\ N^\infty,\ \text{and}\ \gamma;$$

find Q and R such that the optimal regulator has these desired properties. This synthesis problem is posed and solved by Harvey and Stein [4] in the generic case and by Stein [5] in the nongeneric case.

The algorithm presented by Stein starts with the following assumptions: The S^o matrix has p stable diagonal elements. The p columns of X^o are linearly independent and for each there exists a $\nu_i{}^o$ such that $(A-s_i{}^o)x_i{}^o + B\nu_i{}^o = 0$. Furthermore, the $x_i{}^o$ do not lie in the image of U^γ. (To avoid complex arithmetic S^o can be made block diagonal and the complex conjugate columns x_i and x_{i+1} can be replaced by Re x_i and Im x_i). The S^∞ matrix has m nonzero and positive diagonal elements. The N^∞ matrix in invertible, and the multi-index γ is such that $n_1 + \ldots + n_m = n - p$.

Let P be a permutation matrix that switches around the columns of U^γ in an arbitrary manner except that the last m columns of $U^\gamma p$ are $A^{n_1-1} B\nu_1{}^\infty, \ldots, A^{n_m-1} B\nu_m{}^\infty$. Then we have

<u>Lemma 6</u> The quadratic weighting matrices (not unique) that produce the desired asymptotic properties are given by

$$H = [0,I]\,[X^o,\ U^\gamma P]^{-1}$$

$$R = (N^\infty)^{-T}(S^\infty)^{-2}(N^\infty)^{-1}$$

$$Q = H^T H.$$

The last step in the synthesis problem is to solve the linear quadratic regulator problem for several values of ρ and "trade off" control energy with eigenvalue and eigenvector placement.

2.3.12 <u>Selecting Quadratic Weights</u>

The following quote from Athans [2] in 1971 is still true today.

> The selection of weighting matrices in the quadratic criterion is not a simple matter. Usually they are selected by the designer on the basis of engineering experience coupled with alternate simulation runs for different trial values. There is no universal agreement on precisely how these are to be selected for any given application.

At the heart of the problem is that many different and sometimes contradictory specifications have to be lumped into a single cost function. Over the years several ad hoc methods have been developed, the best known being the inverse square method due to Bryson. A few remarks are made in Chapter III about this method. See the references in Harvey and Stein [4] for other methods. None can be considered uniquely satisfactory.

Some encouraging progress has been made on the problem of selecting quadratic weights to produce a desired asymptotic eigenstructure. The algorithm due to Stein [5] presented in the previous section is an example of this. There remain, however, other types of specifications which cannot adequately be described using a closed loop eigenstructure. For example, constraints may exist on feedback gain levels, or adequate stability margins may have to be assured. The relationship between quadratic weights and these types of specifications needs further research.

CHAPTER III

Angles

3.1 Introduction

Equations are derived for the angles on the root locus of the output feedback problem. The angles can be computed at any point including the points of departure and approach. By using the generalized eigenvalue problem the angles of approach can easily be computed. These results are then applied to the optimal root locus of the linear quadratic regulator by using the Hamiltonian system. Equations are also derived to find the sensitivity of the closed loop eigenvalues and the directional derivatives of the closed loop eigenvectors. Finally, a more general dependence of Q and R on ρ is considered and equations for angles and sensitivities are used to analyze the inverse square method of selecting quadratic weights.

Only the case of distinct closed loop eigenvalues is considered. The equations derived in this chapter are not valid at the points where there are multiple closed loop eigenvalues, and the equations cannot easily be extended to handle these cases.

3.2 The Ouput Feedback Problem

3.2.1 Finding the Closed Loop Eigenstructure Using the Generalized Eigenvalue Problem

The closed loop system matrix for the output feedback problem is

$$A_{cl} = A - \frac{1}{k} BKC. \tag{1}$$

The closed loop eigenvalues, eigenvectors, and left eigenvectors are defined in the usual way by

$$(A_{cl} - s_i I)x_i = 0 \tag{2}$$

$$y_i^H (A_{cl} - s_i I) = 0 \tag{3}$$

Now we state a lemma which we shall use in the remainder of this chapter.

Lemma 1 The s_i, x_i, and y_i^H are solutions of the generalized eigenvalue problems.

$$\begin{bmatrix} A - s_i I & B \\ -C & -kK^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ \nu_i \end{bmatrix} = 0 \tag{4}$$

$$\begin{bmatrix} y_i^H & \eta_i^H \end{bmatrix} \cdot \begin{bmatrix} A - s_i I & B \\ -C & -kK^{-1} \end{bmatrix} = 0. \tag{5}$$

Remark k=0 is now allowed, in which case (4) and (5) can be used to find the transmission zeroes, zero directions, and left zero directions.

Remark When $k > 0$ there are exactly n finite solutions to (4) and (5), and when k=0 there are anywhere from 0 to n-m finite solutions.

To prove lemma 1, from (4) we see that

$$(A-s_i I)x_i + B\nu_i = 0$$

$$\nu_i = -\frac{1}{k} K C x_i$$

$$(A-s_i I)x_i - \frac{1}{k} B K C x_i = 0, \tag{6}$$

and then (2) follows immediately from (6). Therefore the s_i and x_i that are solutions of the generalized eigenvalue problem are the closed loop eigenvalues and eigenvectors. In a similar way (5) can be reduced to (3), and the proof is complete.

Lemma 1 is not a new result, but it is not well known. Its proof is simple and direct, and it provides valuable insight into the feedback control problem. For example, Laub used this result to show the relationship of two methods of computing transmission zeroes (see Appendix A). It turns out that the ν_i's and η_i's can be used to compute angles of arrival to the transmission zeroes.

3.2.2 Finding ds_i/dk Using the Eigenvalue and Generalized Eigenvalue Problems

In the eigenvalue problems (2) and (3) the first derivative of A_{cl} with respect to k exists everywhere in the open interval $(0,\infty)$. We use this fact to find ds_i/dk, the derivative of the closed loop eigenvalue with respect to k.

<u>Lemma 2</u> For any k in the interval $(0,\infty)$, and for any distinct s_i,

$$\frac{ds_i}{dk} = \frac{1}{k^2} \frac{y_i^H \, B \, K \, C \, x_i}{y_i^H x_i} . \tag{7}$$

To prove lemma 2 differentiate (2) to get

$$\frac{d}{dk} (A_{cl}-s_iI) \, x_i + (A_{cl}-s_iI) \frac{d}{dk} x_i = 0. \tag{8}$$

Multiply on the left by the left eigenvector y_i^H, which cancels the second term and leaves

$$y_i^H \frac{d}{dk} (A_{cl}-s_iI) \, x_i = 0.$$

Rearrange to get

$$\frac{ds_i}{dk} = \frac{y_i^H\left(\frac{dA_{cl}}{dk}\right)x_i}{y_i^H x_i} .$$

Differentiate A_{cl} to get (7). It is always possible to normalize the eigenvectors so that $y_i^H x_i = 1$, in which case the denominator in (7) can be removed. This completes the proof.

Lemma 2 is a standard result that can be extracted from the higher mathematics of Kato [37]. Wilkinson [38] gives a more readable derivation of a similar result - the sensitivity of s_i to changes in A_{cl}. Shaked [19] derives and uses (7).

In the generalized eigenvalue problems (4) and (5) the first derivative of the large matrices with respect to k exist everywhere in the semi-open interval $[0,\infty)$. We use this fact to again find ds_i/dk.

<u>Lemma 3</u> For any k in the interval $[0,\infty)$ and for any distinct s_i,

$$\frac{ds_i}{dk} = \frac{-\eta_i^H K^{-1}\eta_i}{y_i^H x_i} . \tag{9}$$

To prove lemma 3 use obvious substitution of (4) to get

$$[L(k) - s_i M]v_i = 0.$$

Differentiate to get

$$\frac{d}{dk}(L - s_i M)\ v_i + (L - s_i M)\frac{dv_i}{dk} = 0.$$

Multiply on the left by the left eigenvector, call it u_i^H, which will cancel the second term and leave

$$u_i^H \frac{d}{dk} (L - s_i M) v_i = 0.$$

Rearrange to get

$$\frac{ds_i}{dk} = \frac{u_i^H \left(\frac{dL}{dk}\right) v_i}{u_i^H M v_i} .$$

Differentiate L and substitute back the original terms to get (9).

This completes the proof. Stewart [39] has worked on a problem similar to this - the sensitivity of s_i to changes in L and M.

3.2.3 Angles on the Root Locus

Theorem 1 The angles on the root locus, for any value of k in the interval $[0,\infty]$ and for any distinct s_i, are found by

$$\arg (ds_i) = \arg \left(\frac{-y_i^H \, B \, K \, C \, x_i}{y_i^H \, x_i} \right) \quad 0 < k \leq \infty \tag{10}$$

$$\arg (ds_i) = \arg \left(\frac{\eta_i^H \, K^{-1} \, \nu_i}{y_i^H \, x_i} \right) \quad 0 \leq k < \infty . \tag{11}$$

Remark The angles of departure are found using (10) with $k = \infty$ (to be more precise, let $\ell = 1/k$ and use $\ell = 0$). The angles of approach are found using (11) with $k = 0$.

Remark (10) is due to Shaked [19], (11) is new.

We prove theorem 1 by showing (10) and then (11) are correct. To derive (10) start with (7) of lemma 2, the formula for the derivative of an eigenvalue. We have that

$$\arg \left(\frac{ds_i}{dk} \right) = \arg \left(\frac{1}{k^2} \; \frac{y_i^H \, B \, K \, C \, x_i}{y_i^H x_i} \right),$$

and then the angle is

$$\arg(ds_i) = \arg(dk) + \arg\left(\frac{1}{k^2}\right) + \arg(-1)$$
$$+ \arg\left(\frac{-y_i^H B K C x_i}{y_i^H x_i}\right).$$

Now $\arg(-1) = 180°$, and because k is positive and varies negatively from infinity to zero it follows that $\arg(1/k^2) = 0°$ and $\arg(dk) = 180°$. Therefore (10) is true.

We cannot use k=0 in (10) becuase A_{cl} is not defined for k=0. It is very awkward to use a limiting argument as $k \to 0$ (as does Shaked [19]) because $Cx_i \to 0$ and $y_i^H B \to 0$, as seen from (4) and (5).

To derive (11) start with (9) of lemma 3, the formula for the derivative of a generalized eigenvalue. We note that (9) is well defined for k=0. The formula for the angle is

$$\arg(ds_i) = \arg(dk) + \arg(-1) + \arg\left(\frac{\eta_i^H K^{-1} \nu_i}{y_i^H x_i}\right).$$

Since $\arg(dk) = 180°$ and $\arg(-1) = 180°$, (11) follows. This completes the proof.

When k is in the interval $(0,\infty)$ either (10) or (11) can be used to find angles. To show this we note from (4) and (5) of lemma 1 that

$$Cx_i = -k\, K^{-1} \nu_i$$

$$y_i^H B = k\, \eta_i^H K^{-1}.$$

Substitute these into (10), and the result by inspection is (11). Except for k very close to or equal to zero, (10) is better to use for computing angles because it involves solving an eigenvalue problem of lower dimension than the corresponding generalized eigenvalue problem. When k is very close to zero, however, (11) is a more reliable equation. The tradeoffs when k is very close to zero are the same as those for solving transmission zeroes using (2), an eigenvalue problem with high gain feedback; or (4), a generalized eigenvalue problem. Laub [40] gives a good discussion of these tradeoffs and concludes that the generalized eigenvalue problem is better (for computing transmission zeroes).

3.2.4 Sensitivity of the Closed Loop Eigenvalues

For each eigenvalue on the root locus we can write the approximation

$$\frac{ds_i}{dk} \approx \frac{\Delta s_i}{\Delta k} .$$

Therefore

$$|\Delta s_i| \approx \left|\frac{ds_i}{dk}\right| \cdot |\Delta k| .$$

The term $|ds_i/dk|$ is defined as the sensitivity of s_i. One use of the sensitivity is that to a first order approximation a change $|\Delta k|$ will move the closed loop eigenvalue s_i a distance $|\Delta s_i|$ in the direction $\arg(ds_i)$. It is immediate from lemmas 2 and 3 that

<u>Lemma 4</u> For any k in the interval $[0,\infty]$ and for any distinct s_i,

$$\left|\frac{ds_i}{dk}\right| = \frac{1}{k^2}\left|\frac{y_i^H \, B \, K \, C \, x_i}{y_i^H x_i}\right| \qquad 0 < k \leq \infty \tag{12}$$

$$\left|\frac{ds_i}{dk}\right| = \left|\frac{\eta_i^H \, K^{-1} \, \nu_i}{y_i^H \, x_i}\right| \qquad 0 \leq k < \infty. \tag{13}$$

Wilkinson [38] discusses in detail the sensitivity of eigenvalues and eigenvectors, including the case of multiple eigenvalues. It turns out to be much easier to derive bounds on $|ds_i/dk|$ for the multiple eigenvalue case than to derive an expression for ds_i/dk. For our purposes we will continue to assume that the eigenvalues are distinct.

<u>Example</u>

Several root loci are plotted for an output feedback problem, and the angles are computed in order to verify the preceding results. The same system S(A,B,KC) is used as by Shaked [19].

$$A = \begin{bmatrix} -4 & 7 & -1 & 13 \\ 0 & 3 & 0 & 2 \\ 4 & 7 & -4 & 8 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -5 & 2 & -2 \\ 8 & -14 & 0 & 2 \end{bmatrix}$$

Three different output feedback matrices are used, Shaked used K = I.

<u>Case # 1</u>

$$K = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Case # 2</u>

$$K = I$$

<u>Case # 3</u>

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 50 \end{bmatrix}$$

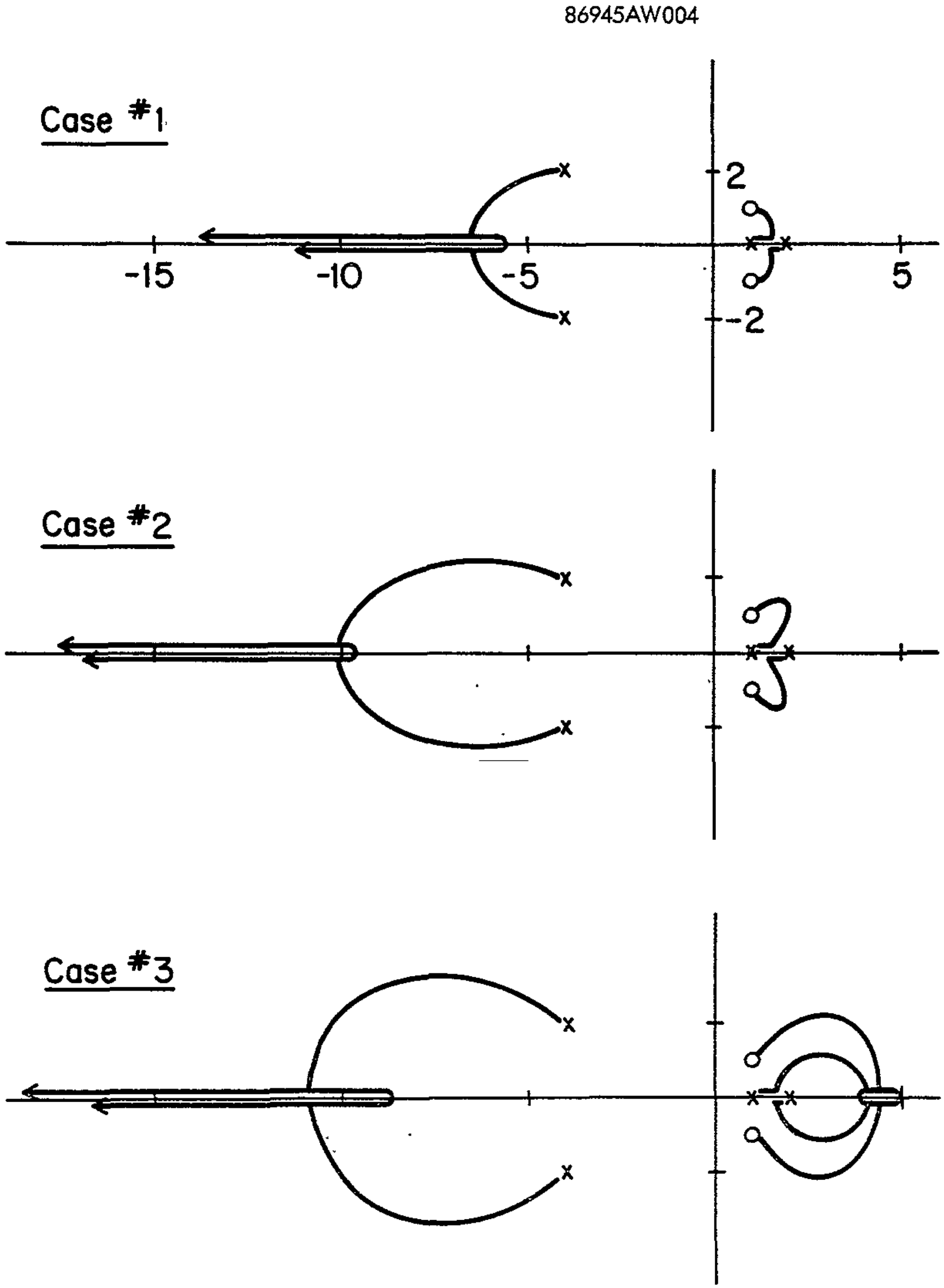


Figure 3.1

Root Loci of a Linear System with Output Feedback

Table 3.1

Angles of Departure and Approach for Example 3.1

Case	Angles of Departure			Angles of Approach
	$-4 \pm 2i$	1	2	$1 \pm i$
1	$\pm$ 173°	0°	180°	$\mp$ 170°
2	$\pm$ 149	0	180	$\mp$ 121
3	$\pm$ 135	0	180	$\mp$ 114

Table 3.2

Points and Angles on the Root Loci of Example 3.1

Angles are in parentheses

Case	k	Left half plane				Right half plane			
1	100	- 4.45±2.02i	(∓ 177°)			1.14	(0°)	1.94	(180°)
	10	-11.4	(180)	-5.16	(0)	1.65±0.42i	(± 81)		
	1	-84.1	(180)	-6.69	(180)	1.53±0.85i	(± 137)		
	10^{-6}	$-8x10^{7}$	(180)	$-3x10^{6}$	(180)	1.00±1.00i	(∓ 170)		
2	100	- 4.10±2.06i	(± 151)			1.15±	(0)	1.93	(180)
	10	- 4.86±2.40i	(± 161)			1.81±0.65i	(± 57)		
	1	-10.34±0.71i	(∓ 113)			2.34±1.33i	(± 85)		
	10^{-6}	$-8x10^{6}$	(180)	$-3x10^{6}$	(180)	1.00±1.00i	(∓ 121)		
3	100	- 5.85±3.18i	(± 161)			2.57±1.16i	(± 15)		
	10	-20.5	(180)	-8.7	(0)	4.20±0.48i	(∓ 62)		
	5	-37	(180)	-8.6	(180)	4.01	(0)	5.00	(0)
	1	-159	(180)	-12.0	(180)	3.84±1.75	(± 140)		
	10^{-6}	$-1.5x10^{8}$	(180)	$-8x10^{6}$	(180)	1.00±1.00i	(∓ 114)		

Numerical results are given in Tables 3.1 and 3.2, and the root loci are plotted in Figure 3.1.

As expected in all three cases the branches of the locus depart from the open loop poles. Two of the branches stay entirely in the LHP with different angles of departure in all three cases. The branches meet on the negative real axis and then continue in opposite directions as might be expected in the single input case. However, the branch that goes to the right eventually turns around, and then both branches approach infinity on the negative real axis with different asymptotic radii in each case. The other two branches stay entirely in the RHP, so the system is always unstable. These branches eventually arrive at the transmission zeroes at $1 \pm i$ with different angles of arrival in each case. The path taken in the third case is unusual, to say the least.

Shaked's paper from which this example is taken contains some errors which will be pointed out here. His calculation for the angles of departure contains numerical errors. More importantly, his formula for the angles of arrival (3.16b) is incorrect due to an error in the derivation after (3.15). This leads to the incorrect conclusion that angles of arrival are independent of the output feedback matrix K.

3.3 The Linear Quadratic Regulator

Theorem 2 The angles on the optimal root locus, for any value of ρ in the interval $[0,\infty]$, and for any distinct s_i, are found by

$$\arg(ds_i) = \arg\left\{ \frac{-1}{w_H z_i} w_H^H \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} z_i \right\} \qquad 0 < \rho \leq \infty \qquad (14)$$

$$\arg(ds_i) = \arg\left(\frac{\eta_i^H R \nu_i}{w_i^H z_i}\right) \qquad 0 \leq \rho < \infty \qquad (15)$$

Remark The angles of departure are found using (14) with $\rho = \infty$, and the angles of arrival by using (15) with $\rho = 0$.

Theorem 2 is proved by applying (10) and (11) of theorem 1 to the Hamiltonian system $S(\tilde{A},\tilde{B},\tilde{K}\tilde{C})$. The vectors z_i and w_i^H are the right and left eigenvectors of Z, the Hamiltonian system matrix; and the vectors ν_i and η_i^H are found by solving generalized eigenvalue problems analogous to (4) and (5). Note that

$$\tilde{B}\tilde{K}\tilde{C} = \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} .$$

This completes the proof. An example of an optimal root locus is given in Chapter IV after a discussion of asymptotic equivalence classes.

The sensitivity of the optimal closed loop eigenvalues is found by applying (12) and (13) of lemma 4 to the Hamiltonian system.

The number of computations needed to compute (14) and (15) can be reduced by using the following identities. First, using (4) and (5) it can be shown that

$$\nu_i = \eta_i .$$

Now let $z_i = (x_i^H, \xi_i^H)^H$ be the eigenvector associated with s_i in the LHP, let s_i be the mirror of s_i in the RHP, and let $\bar{s}_i = (\bar{x}_i^H, \bar{\xi}_i^H)^H$ be the associated eigenvector. Then

$$w_i^H = [-\bar{\xi}_i^H, \bar{x}_i^H] \quad .$$

3.4 Directional Derivatives of the Closed Loop Eigenvectors

We found ds_i/dk for the eigenvalue problem in section 3.2.2. Now we continue that discussion and find dx_i/dk. The references are again Kato [37] and Wilkinson [38]. The results are

Lemma 5 For any value of k in the interval $(0,\infty)$, and for any distinct s_i,

$$\frac{ds_i}{dk} = \beta_{ii} \tag{16}$$

$$\frac{dx_i}{dk} = v_i + b_i x_i \tag{17}$$

$$\text{where } \beta_{ji} = \frac{y_j^H \left(\frac{dA_{cl}}{dk}\right) x_i}{y_j^H \; x_i}$$

$$v_i = \sum_{\substack{j=1 \\ j\neq 1}}^{n} \frac{-\beta_{ji}}{s_i - s_j} \; x_j$$

$$b_i = \frac{-x_i^H v_i}{x_i^H x_i} . \tag{18}$$

Equation (16) is derived in section 3.2.2 and is included here for convenience. To derive (17) we will need the following:

$$x_i^H \frac{dx_i}{dk} = 0. \tag{19}$$

This can be explained as follows: $x_i(k)$ and $x_i(k+\Delta k)$ can be normalized to lie on the same hypersphere, as $\Delta k \to 0$ then $[x_i(k+\Delta k) - x_i(k)]/\Delta k \to dx_i/dk$, which is tangent to the hypersphere, and therefore is orthoganol to x_i.

Now, since the eigenvectors x_i form a basis for R^n we can write

$$-\left[\frac{d}{dk}(A_{cl} - s_i I)\right] x_i = Xc \tag{20}$$

$$\frac{dx_i}{dk} = Xb, \tag{21}$$

where b and c are vectors and b_i and c_i are components of these vectors. Substitute (20) and (21) into (8) to get

$$(A_{cl} - s_i I)Xb = Xc. \tag{22}$$

Multiply (20) on the left by y_j^H, note that $y_j^H x_i = 0$ for $i \neq j$, and then after some algebra

$$c_j = \begin{cases} \beta_{ji} & j \neq i \\ 0 & j = i \end{cases}.$$

By manipulating (22) we see that $(s_j - s_i)b_j = c_j$ for $j=1,\ldots,n$. Solve for b_j to get

$$b_j = \begin{cases} \dfrac{-\beta_{ji}}{s_j - s_i} & j \neq i \\ \text{undetermined} & j = i \end{cases}. \tag{23}$$

Substitute (23) into (21) to get (17). The only thing not determined is b_i, which we can find by multiplying (17) on the left by x_i^H to get

$$x_i^H \frac{dx_i}{dk} = x_i^H v_i + b_i x_i^H x_i .$$

By (19) the left hand side is zero, and therefore (18) follows. This completes the proof of lemma 5.

The directional derivatives for the optimal eigenvectors can be found in a similar way. Use (17) with the Hamiltonian system to get

$$\frac{dz_i}{dk} = \begin{bmatrix} \dfrac{dx_i}{dk} \\ \\ \dfrac{d\xi_i}{dk} \end{bmatrix}$$

and then extract dx_i/dk.

3.5 Inverse Square Quadratic Weights

One common ad hoc method for selecting quadratic weights is the inverse square method. Here we give a brief explanation of the method and show one way to analyze it if the first guess of quadratic weights is not satisfactory.

Start with the quadratic cost function

$$J = \int_0^\infty (\bar{x}^T Q x + u^T R u)\,dt.$$

Require that Q and R be diagonal, and then the cost function can be rewritten

$$J = \int_0^\infty \left(\sum_{i=1}^{n} q_{ii}\, x_i^{\,2} + \sum_{i=1}^{m} r_{ii}\, u_i^{\,2} \right) dt.$$

Decide on a maximum allowable deviation for each state and control and call these x_{imax} and u_{imax}. Then select the weights so that each term has equal contribution to the cost function at maximum deviation, i.e.

$$q_{ii} = \frac{1}{x_{imax}^2}\ , \qquad\qquad r_{ii} = \frac{1}{u_{imax}^2}\ .$$

With Q and R chosen in this manner, compute the optimal gain matrix. If the system behaves well with this controller then do not make any changes in Q and R.

If the system does not behave well with this controller then Q and R must be changed. Suppose the state $x_i(t)$ is sluggish and needs to be speeded up. It may suffice to increase q_{ii} or to decrease the weighting on the control that predominantly controls x_i. This scheme, however, may not work. Suppose the predominant mode of x_i is s_i, in other words $x_i(t) \approx c\, e^{-s_i t}$, where c is a constant. If s_i is near a transmission zero then decreasing the control weight will cause s_i to move closer to the transmission zero, and this may even "slow down" s_i.

Now we give a method to determine how the modes change with respect to a change in one or more of the diagonal elements of Q or R. Simply use the following lemma which follows easily from the previous results of this chapter.

Lemma 6 Let $Q(\rho)$ and $R(\rho)$ be dependent on ρ, and let

$$Z = \begin{bmatrix} A & -BR(\rho)^{-1}B^T \\ -Q(\rho) & -A^T \end{bmatrix}.$$

Then for any ρ such that $dZ/d\rho$ is well defined, and for any distinct s_i,

$$\arg(ds_i) = \arg(d\rho) + \arg\left[\frac{w_i^H\left(\frac{dZ}{d\rho}\right)z_i}{w_i^H\, z_i}\right] \tag{24}$$

$$\left|\frac{ds_i}{d\rho}\right| = \left|\frac{w_i^H \frac{dZ}{d\rho}\, z_i}{w_i^H\, z_i}\right| . \tag{25}$$

Suppose R is constant and

$$Q = \mathrm{diag}(q_{11},\ldots,\rho q_{ii},\ldots,q_{nn}).$$

Then $dZ/d\rho$ will have all zeroes except for q_{ii}. Each of the modes s_i will move an approximate distance

$$|\Delta s_i| \approx \left|\frac{ds_i}{d\rho}\right| \, |\Delta\rho|$$

in the direction $\arg(ds_i)$.

CHAPTER IV

Equivalence Classes of Quadratic Weights

4.1 Introduction

As noted in Chapter II, quadratic weighting matrices are not unique. Rather, many choices produce the same controller gains. This chapter provides a characterization of two classes of equivalent weights, and for one of them defines a unique canonical element. These characterizations may prove useful in design.

Both E^o and E^∞ equivalence classes are defined. Then sufficient conditions are given for different quadratic weights to be in the same E^∞ equivalence class (in other words to have the same asymptotic properties). A canonical element of the E^∞ class is defined, and an algorithm is given to find it. Finally, it is shown by example that the closed loop eigenstructure (and hence the optimal root locus) in the nonasymptotic region can be different for members of the same E^∞ equivalence class.

4.2 Definitions of the E^o and E^∞ Equivalence Classes

(Q,R) and $(\tilde{Q},\tilde{R})$ are members of an equivalence class if they produce the same optimal gain matrix. We use the following notation:

$$(Q,R)\ E^o\ (\tilde{Q},\tilde{R}),$$

where E^o has the meaning "produces the same optimal gain matrix as."

This is not the only type of equivalence class that can be defined. $(Q,\rho R)$ and $(\tilde{Q},\rho\tilde{R})$ are members of an equivalence class if they produce the same asymptotic properties (which are S^o, X^o, S^∞, N^∞, and γ). We use the following notation:

$$(Q,\rho R)\ E^{\infty}\ (\tilde{Q},\rho\tilde{R}),$$

where E^{∞} has the meaning "produces the same asymptotic properties as." The ρ is included in the parenthesis to emphasize the dependence on the control weight.

4.3 Sufficient Conditions to be in the E^{∞} Equivalence Class

In the form of two lemmas we present sufficient conditions for $(Q,\rho R)$ and $(\tilde{Q},\rho\tilde{R})$ to be in the same E^{∞} equivalence class.

Lemma 1 Assume (A,B) is controllable and F is optimal for some (Q,R). Then

$$(Q,\rho R)\ E^{\infty}\ (\tilde{Q},\rho R)$$

if there exists an n x n real symmetric Y such that

$$\text{(i)}\quad \tilde{Q} = Q - A^TY - YA$$

$$\text{(ii)}\quad YB = 0.$$

To prove this we first show that for any $\rho > 0$ the closed loop eigenvalues of the optimal regulator are the same when either Q or $\tilde{Q}$ is used (the R is the same in both cases). The closed loop eigenvalues are the eigenvalues of Z, the Hamiltonian matrix, and the eigenvalues are invariant under the following similarity transformation:

$$UZU^{-1} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} A & -\frac{1}{\rho}BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix}$$

$$= \begin{bmatrix} A & -\frac{1}{\rho}BR^{-1}B^T \\ -Q+A^TY+YA & -A^T \end{bmatrix}.$$

Using the same transformation the eigenvectors of Z change from $z_i = (x_i^T, x_i^TP)^T$ to $z_i = (x_i^T, x_i^T(P+Y))^T$. The x_i portions are the closed loop eigenvectors of the optimal regulator and these do not change. Since for any $\rho > 0$ the closed loop eigenstructure is invariant, the asymptotic properties as $\rho \to 0$ are invariant. This completes the proof.

To show that the closed loop eigenstructure is the same for any $\rho > 0$ we could alternatively have used a result due to Molinari [27] that was reviewed in section 2.3.8. We note that it is not necessarily true that $\tilde{Q} \geq 0$ or $P+Y \geq 0$. The conditions given here are only sufficient. We were not able to prove (and not able to find a counterexample) for the converse.

For the next lemma let D and E be diagonal matrices with positive diagonal elements d_i and e_i such that

$$DE = S^{\infty} .$$

All of the other terms are defined in sections 2.3.10 and 2.3.11.

> Lemma 2 Assume that (A,B) is controllable; that F is optimal for some (Q,R); that the asymptotic properties are S^{0}, X^{0}, S^{∞}, N^{∞}, and γ; and that
>
> $$\nu_i^{\infty T} \nu_i^{\infty} = 1 \qquad \text{for } i = 1,\ldots,m$$
>
> Then
>
> $$(Q,\rho R) \; E^{\infty} \; (\tilde{Q},\rho\tilde{R})$$
>
> if there exists a D and Y such that
>
> $$H = [0,\ D]\,[X^{0},\ U^{\gamma}\ P]^{-1} \tag{1}$$
>
> $$\tilde{R} = (N^{\infty})^{-T}\ E^{-2}\ (N^{\infty})^{-1} \tag{2}$$
>
> $$\tilde{Q} = H^{T}H - A^{T}Y - YA \tag{3}$$

Before proving this we note that equations (1-3) are similar to those used by Stein [5] in his algorithm for selecting quadratic weights, as reviewed in section 2.3.11. He sets D=I, $E=S^{\infty}$, and Y=0; and he places no restriction on the magnitudes of the ν_i^{∞}'s.

The proof of lemma 2 is in two parts. The first part is to show that subtracting $A^TY + YA$ in (3) does not change the asymptotic properties, but this was established in lemma 1. The second part is to show that S^{∞} can be split into D and E without changing the asymptotic properties. The proof that Stein uses to prove his algorithm is only trivially changed when S^{∞} is split into D and E, so we will not repeat it here. This completes the proof.

Only sufficient conditions are given in lemma 2 for quadratic weighting matrices to be in the same E^{∞} equivalence class. We do not yet know if these conditions are necessary, in other words if by changing D and Y every member of the E^{∞} equivalence class can be reached.

The assumptions about the magnitudes of the ν_i^{∞}'s are made without loss of generality. These vectors are used to specify directions, and their magnitudes do not change the asymptotic properties. If the ν_i^{∞}'s are not of unit magnitude then it is always possible to find a diagonal matrix G such that the columns of $\bar{N}^{\infty} = N^{\infty}G$ are of unit magnitude. Then in (1) and (2) we can replace N^{∞} by $\bar{N}^{\infty}$, E by EG^{-1}, and D by DG without changing H and $\tilde{R}$.

Several other changes can be made in (1-3) without affecting $\tilde{R}$ and $\tilde{Q}$. The H in (1) can be premultiplied by an m x m unitary matrix (W such that $WW^T = I$). Then in (3) when Q is formed the influence of W is lost. In (1) the magnitudes and the order of the columns of X^0 and all but the last m columns of $U^{\gamma}\rho$ can be changed without changing H. In (1) the order of the last m columns of $U^{\gamma}\rho$, and in (2) the order of the ν_i^{∞}'s can be changed without changing either H or $\tilde{R}$, as long as the corresponding d_i's and e_i's are changed.

4.4 A Canonical Element of the E^{∞} Equivalence Class

We define the canonical element of the E^{∞} equivalence class to be the $(\tilde{Q},\rho\tilde{R})$ reached by (1-3) when $D=I$, $E=S^{\infty}$, and $Y=0$. By "canonical" we mean unique, but by changing D and Y many other canonical elements could be defined. The choice used here has an uncomplicated structure and agrees with Stein's algorithm [5] when the ν_i^{∞}'s are unit magnitude.

Given any $(Q,\rho R)$ it is always possible to find the canonical element. Use a generalized eigenvalue problem to find S^{O} and X^{O}, as described in section 2.3.10. Use the algorithm in section 5.2 to find S^{∞}, N^{∞}, and γ. Then use (1-3) with $D=I$, $E=S^{\infty}$, and $Y=0$ (and with $\nu_i^{\infty T}\nu_i^{\infty} = 1$) to find the canonical element $(\tilde{Q},\rho\tilde{R})$.

4.5 Behavior of the Optimal Root Locus in the Nonasymptotic Region

Members of the E^{∞} equivalence class have the same asymptotic properties. They may or may not, however, have the same closed loop eigenstructure for $\rho > 0$. As a consequence the optimal root loci may look very different in the nonasymptotic region. This is important because when selecting a Q and R using (1-3) the final choice of ρR uses a $\rho>0$.

Suppose a $(Q,\rho R)$ is computed using equations (1-3). If D and E are kept the same and Y is changed to get a different Q then the closed loop eigenstructure will be the same for any $\rho>0$. This we know is true by the proof of lemma 1 (and also due to Molinari [27]). For any scalar $\alpha > 0$ if D is changed to αD and E to $(1/\alpha)E$ then $(Q,\rho R)$ will change to $(\alpha^2 Q,\rho\alpha^2 R)$, and it is easy to see that the closed loop eigenstructure is the same for any $\rho > 0$. However, if D and E are changed in a more complicated way (change d_i to $\alpha_i d_i$ and e_i to $(1/\alpha_i)e_i$, where the α_i's are all positive and not all equal) then the closed loop eigenstructure will not be the same for any $\rho > 0$. Example 4.1 shows how the optimal root locus can change in the nonasymptotic region.

Example 4.1

We consider a linear system with the following A and B matrices.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -4 & 0.1 & 1 \\ 0.1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The asymptotic properties of an optimal linear quadratic regulator are specified, and equations (1-3) are used to compute the quadratic weighting matrices. Six different values of D and E are chosen. The purpose of this example is to show that the choice of D and E does not change the asymptotic properties but dramatically changes the behavior of the optimal root loci in the finite region.

The linear system has four states and two inputs. The first input drives a first order subsystem with a pole at -1.0. The second input drives a third order subsystem which can be broken down into a damped oscillator with poles at $-2 \pm i$ and a first order "actuator" with a pole at -5. The 0.1 terms in the A matrix couple the two subsystems.

The asymptotically finite properties are specified by

$$s^o_{1,2} = 0.5 \pm 3.0i \qquad x^o_{1,2_d} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ v \end{bmatrix} \pm \begin{bmatrix} 1 \\ 1 \\ 0 \\ v \end{bmatrix} i$$

The "d" means that the eigenvectors are desired and not necessarily obtainable. The "v" means that the element is not specified. The result of projecting the complex eigenvectors onto the obtainable subspaces is

$$x^{o}_{1,2} = \begin{bmatrix} .341 \\ .561 \\ 0 \\ .232 \end{bmatrix} \pm \begin{bmatrix} -.244 \\ 1.146 \\ 0 \\ 4.476 \end{bmatrix} i.$$

The asymptotically infinite properties are specified by

$$S^{\infty} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \; N^{\infty} = I, \quad \gamma = (01, 02).$$

In each of the six cases there are two first order Butterworth patterns.

The D, E, R, and H matrices are shown in Table 3.1. The optimal root loci are in Figure 3.1. We see that the behavior of the loci in the finite region is dependent on the choice of D and E. The angles of departure and approach are different in each of the six cases and are listed in Table 3.2. Points on the loci for different values of ρ are listed in Table 3.3.

In the first case the 2,2 element of R is bigger than the 1,1 element by a factor of 3200, and for ρ "not too small" this causes the subsystems to decouple. Since the second input is heavily weighted the pole at -1 does not move much. On the other hand the branches of the locus associated with the third order subsystem (driven by the first input) start to behave like the optimal root locus of a single input system. Two of the branches form a second order Butterworth pattern and the third branch approaches a transmission zero somewhere to the left of -5. Finally, as ρ gets "very small" the subsystems couple together and the specified asymptotic properties are achieved. The branches that start at $-2 \pm i$ eventually meet on the real axis and form two first order Butterworth patterns.

In the second case the decoupling of the two subsystems is less apparent. The branches of the locus that start at $-2 \pm i$ meet on the negative real axis at about -4.8. One of the branches goes to the left and forms a first order pattern. The other goes to the right, joins the branch that starts at -1, and then they approach the transmission zeroes.

In cases 3 through 6 the branches that leave the open loop poles at $-2 \pm i$ eventually make it to the transmission zeroes at $-.5 \pm 3i$. Given just the locations of the open loop poles and the transmission, the locus in case 4 is probably the most "desirable" pattern. This case allows a meaningful tradeoff between control weight and asymptotic properties. It happens to correspond to the canonical member of the E^{∞} equivalence class, as defined in the previous section. For a wide range of values of D and E the locus does in fact look like the one in case 4. Not until the 2,2 element of R is smaller than the 1,1 element by a factor of one million does the behavior of the locus in the finite region change significantly. The behavior in cases 5 and 6 cannot easily be explained. The subsystems do not appear to decouple as in cases 1 and 2.

From the above example it is not obvious how to choose the D and E matrices. Since only the ratios of the diagonal elements of the D and E matrices are important there are m-1 degrees of freedom available to the designer. (In the single input case there are no extra degrees of freedom). It may be true that the extra degrees of freedom can be used up by specifying angles of approach and departure, but we do not know of any algorithms that allow you to do this, nor do we know of a check to determine which angles are valid.

Figure 4.1

Optimal Root Loci

86945AW005

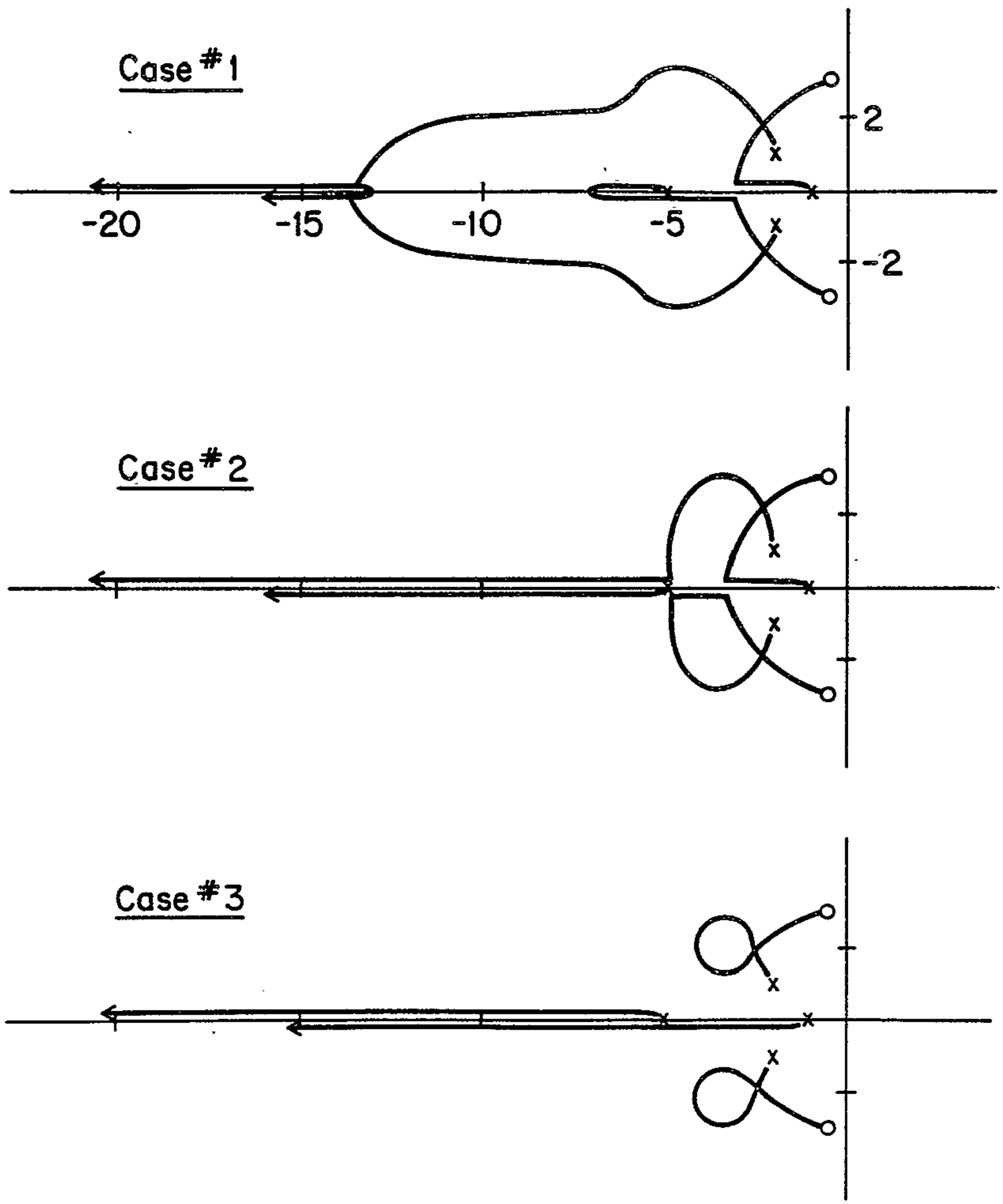
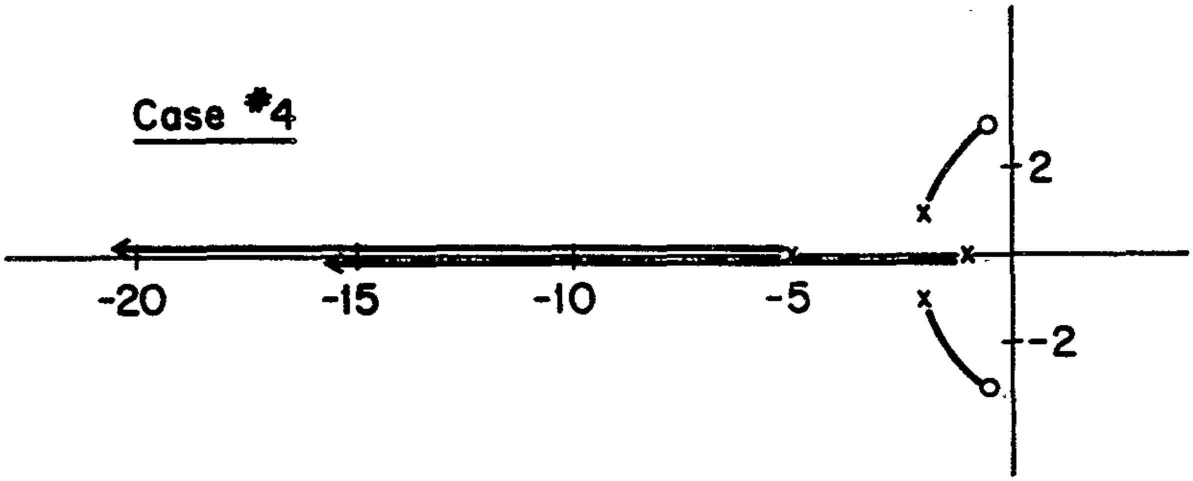
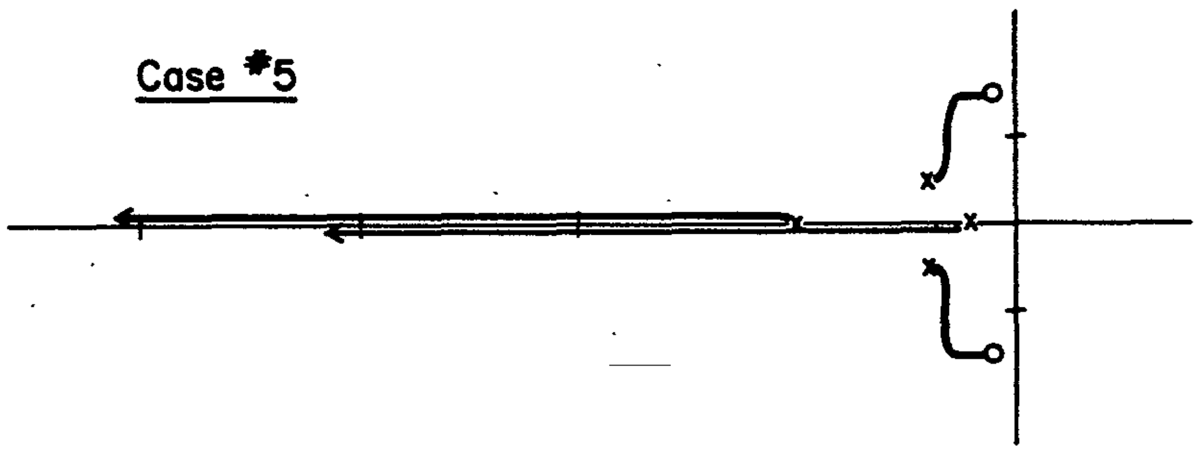


Figure 4.1 Continued

86945AW006





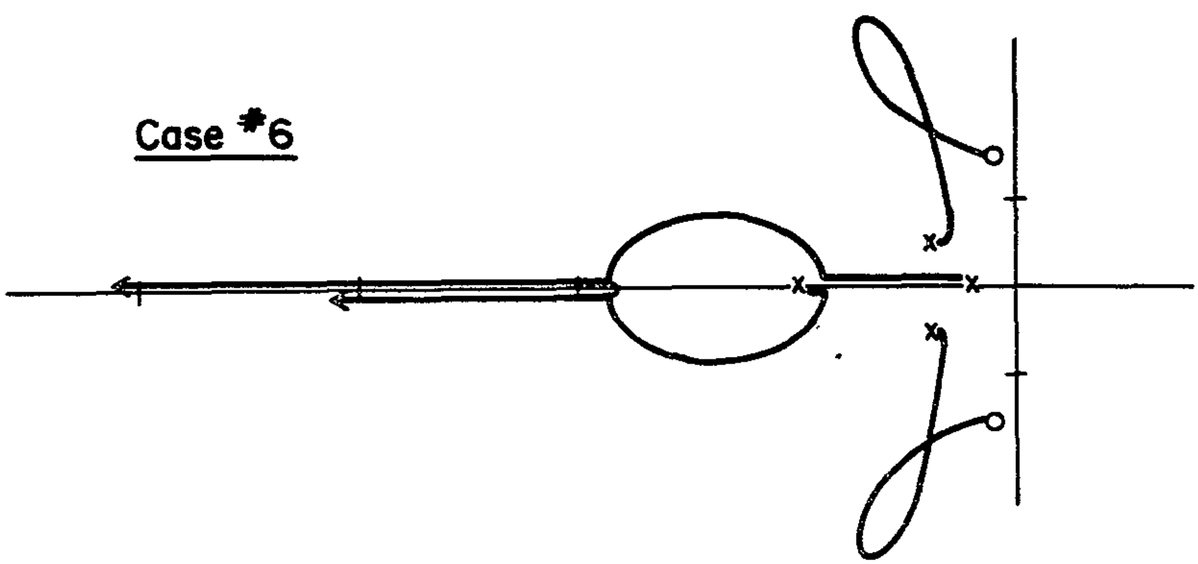


Table 4.1

<u>Matrices Used in Example 4.1</u>

Case	D	E	R	H (where $Q = H^T H$)
1	$\begin{bmatrix} 1 & \\ & 56.6 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & .0353 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 800 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 240 & -170 & 0 & 56.6 \end{bmatrix}$
2	$\begin{bmatrix} 1 & \\ & 46.0 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & .0435 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 528 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 195 & -138 & 0 & 46.0 \end{bmatrix}$
3	$\begin{bmatrix} 1 & \\ & 40 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & .05 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 400 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 170 & -120 & 0 & 40 \end{bmatrix}$
4	$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & .25 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 4.25 & -3 & 0 & 1 \end{bmatrix}$
5	$\begin{bmatrix} 1 & \\ & .002 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 10^3 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 10^{-6} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ .0085 & -.006 & 0 & .002 \end{bmatrix}$
6	$\begin{bmatrix} 1 & \\ & .0002 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 10^4 \end{bmatrix}$	$\begin{bmatrix} 1 & \\ & 10^{-8} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ .00085 & -.0006 & 0 & .0002 \end{bmatrix}$

Table 4.2

Angles of Departure and Approach for Example 4.1

Case	Angles of Departure			Angles of Approach
	$-2 \pm i$	-5	-1	$-.5 \pm 3i$
1	$\pm$ 100.8°	180°	180°	$\pm$ 11.7°
2	$\pm$ 99.3	180	180	$\pm$ 11.9
3	$\pm$ 98.1	180	180	$\pm$ 12.1
4	$\pm$ 70.5	180	180	$\pm$ 21.5
5	$\pm$ 4.9	180	180	$\pm$ 1.4
6	$\mp$ 10.2	0	180	$\mp$ 9.3

Table 4.3

Points on the Optimal Root Loci of Example 4.1

Case	ρ	Closed Loop Eigenvalues			
1	10	$-2.20 \pm 1.53i$		-5.08	-1.28
	1	$-3.03 \pm 2.66i$		-5.55	-1.42
	.1	$-5.16 \pm 3.32i$		-6.93	-1.67
	.01	-18.5	-13.8	$-2.17 \pm 2.01i$	
	10^{-16}	$-2x10^8$	-10^8	$-0.50 \pm 3.00i$	
2	10	$-2.13 \pm 1.41i$		-5.09	-1.24
	1	$-2.77 \pm 2.42i$		-5.60	-1.42
	.1	$-4.30 \pm 2.94i$		-7.68	-1.79
	.01	-19.7	-12.2	$-1.90 \pm 2.28i$	
	10^{-16}	$-2x10^8$	-10^8	$-0.50 \pm 3.00i$	
3	10	$-2.09 \pm 1.34i$		-5.09	-1.21
	1	$-2.61 \pm 2.28i$		-5.63	-1.42
	.1	$-3.84 \pm 2.76i$		-8.01	-1.91
	.01	$-1.73 \pm 2.42i$		-20.1	-11.6
	10^{-16}	$-0.50 \pm 3.00i$		$-2x10^8$	-10^8
4	10	$-2.00 \pm 1.07i$		-5.09	-1.04
	1	$-1.82 \pm 1.47i$		-5.72	-1.41
	.1	$-1.33 \pm 2.33i$		-8.85	-3.32
	.01	$-0.74 \pm 2.88i$		-21.1	-1.01
	10^{-16}	$-0.50 \pm 3.00i$		$-2x10^8$	-10^8
5	10	$-1.73 \pm 1.18i$		-5.08	-1.87
	1	$-1.65 \pm 2.06i$		-5.69	-2.77
	.1	$-1.66 \pm 2.89i$		-8.84	-3.88
	.01	$-1.04 \pm 2.99i$		-21.1	-10.1
	10^{-16}	$-0.50 \pm 3.00i$		$-2x10^8$	-10^8
6	10	$-1.89 \pm 3.11i$		$-4.81 \pm 0.89i$	
	1	$-2.36 \pm 4.52i$		$-5.99 \pm 2.89i$	
	.1	$-3.05 \pm 5.91i$		$-8.31 \pm 1.58i$	
	.01	$-3.72 \pm 5.50i$		-21.1	-10.9
	10^{-16}	$-0.50 \pm 3.00i$		$-2x10^8$	-10^8

CHAPTER V

Related Results

The three sections in this chapter contain new results concerning the relationships between linear state feedback, quadratic weights, and the closed loop eigenstructure.

5.1 An Algorithm for Selecting F to Produce Desired Asymptotic Properties

In this section an algorithm is given to implement the IALSF map. Given the asymptotic properties of the linear state feedback problem characterized by S^0, X^0, S^∞, and N^∞; we find a feedback gain matrix $(1/k)F$ that produces these properties as $k \to 0$. Only the generic case is considered, when Rank(CB) = m; and (A,B) is assumed to be controllable.

The algorithm presented here is analogous to Harvey and Stein's algorithm [4] for selecting quadratic weights (the IALQR map). The IALQR map gives a way to trade off eigenvalue and eigenvector placement with control energy, and indirectly this affects the feedback gains. The IALSF map gives a way to directly trade off eigenvalue and eigenvector placement with feedback gains, and indirectly this affects control energy.

S^0, X^0, S^∞, and N^∞ are assumed to satisfy the following conditions: The S^0 matrix has (n-m) distinct diagonal elements. The columns of X^0, as is always the case for closed loop eigenvectors, must be linearly independent and for each there must a ν_i^0 such that $(A - s_i^0 I)x_i^0 + B\nu_i^0 = 0$. Furthermore, it is assumed that the x_i^0 are not in the image of B. (To stay in real arithmetic, let S^0 be block diagonal and replace complex conjugates x_i^0 and x_{i+1}^0 with $\mathrm{Re}(x_i^0)$ and $\mathrm{Im}(x_i^0)$). The S^∞ matrix is assumed to have m nonzero diagonal elements and N^∞ is assumed to be invertible.

> <u>Theorem 1</u> The unique F matrix which produces the specified asymptotic properties is
>
> $$F = N^\infty S^\infty (N^\infty)^{-1} [0,I] [X^0,B]^{-1}. \tag{1}$$

To prove this we first show that F has the desired asymptotically finite eigenstructure. It suffices to show that the s_i^0 and x_i^0 satisfy the generalized eigenvalue problem

$$\begin{bmatrix} A - s_i^0 I & B \\ -F & 0 \end{bmatrix} \begin{bmatrix} x_i^0 \\ \nu_i^0 \end{bmatrix} = 0 .$$

This is true because by assumption there exists a ν_i^0 such that $(A - s_i^0 I)x_i^0 + B\nu_i^0 = 0$, and by the way F is constructed $Fx_i^0 = 0$. Since FB is full rank there are n-m solutions to this generalized eigenvalue problem - namely the n-m specified values of s_i^0 and x_i^0.

Next we must show that F has the desired asymptotically infinite eigenstructure. It suffices to show that the s_i^∞ and ν_i^∞ satisfy the eigenvalue problem

$$(sI - FB)\nu_i^\infty = 0$$
$$\text{where} \quad s = \frac{-s_i^\infty}{k} .$$

The parameter s is the ith infinite mode as a function of s_i^∞ and k. The s_i^∞ and ν_i^∞ satisfy the eigenvalue problem because by the way that F is constructed

$$FB = N^\infty S^\infty (N^\infty)^{-1}.$$

The last thing to show is that F is unique. We know that FB is uniquely defined by S^∞ and N^∞. (If some of the s_i^∞'s are equal then the ν_i^{∞}'s are arbitrary within a subspace but FB remains unchanged). From the way that F is constructed we see that no extra freedom exists to choose F. (If F is multiplied by a scalar α then each s_i^{∞} changes to αs_i^{∞}, and the asymptotic properties are not the same.) The proof is complete.

Example 5.1 The same A and B matrices are used as in example 4.1, namely that

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -5 & -4 & 0.1 & 1 \\ 0.1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The asymptotic properties of this system are specified for three cases, and for each the F matrix is computed using (1). The root loci are shown in Figure 5.1.

The asymptotically finite behavior for each case is specified by

$$s_{1,2}^{o} = -3 \pm 2i \qquad x_{1,2}^{o} = \begin{bmatrix} 0 \\ 0.571 \\ 0 \\ -1.14 \end{bmatrix} \pm \begin{bmatrix} -.286 \\ .859 \\ 0 \\ .571 \end{bmatrix} i .$$

The infinite behavior is different for each case. The specifications and the resulting F matrices are shown in Table 5.1.

In the first case we have that $s_1^{\infty} = 1$ and $s_2^{\infty} = 2$. There are two infinite modes that stay on the negative real axis. In the second case $s_{1,2}^{\infty} = 1 \pm \sqrt{3}\, i$. Since $\arg(-s_{1,2}^{\infty}) = \pm 120°$, the two infinite modes approach infinity along asymptotes that make angles of $\pm 120°$ with the positive real axis. In the third case $s_{1,2}^{\infty} = -\sqrt{3} \pm i$, and therefore

Figure 5.1

Root Loci of a Linear System with State Feedback

86945AW007

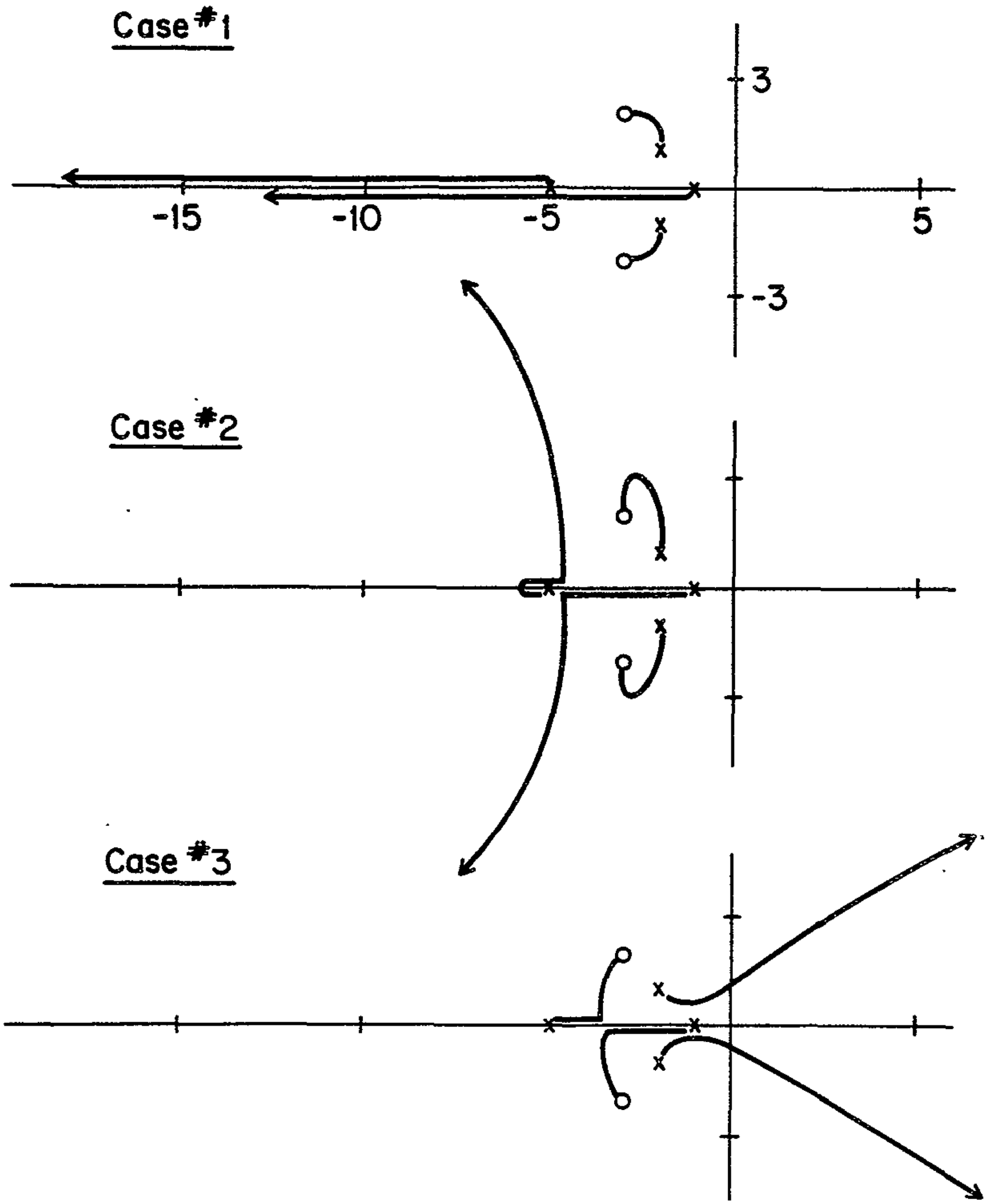


Table 5.1

Matrices Used in Example 5.1

Case	$N^{\infty} S^{\infty} (N^{\infty})^{-1}$	F
1	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 16 & 4 & 0 & 2 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & 0 & 1 \\ 16 & 4 & -4 & 2 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 1 \\ -4 & -2\sqrt{3} \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & 0 & 1 \\ -27.713 & -6.928 & -4 & -3.464 \end{bmatrix}$

Table 5.2

Angles of Departure and Approach for Example 5.1

Case	Angles of Departure			Angles of Approach
	$-2 \pm i$	-5	-1	$-3 \pm 2i$
1	$\pm$ 98.1°	180°	180°	$\mp$ 161.7°
2	$\pm$ 92.8	180	180	$\mp$ 34.3
3	$\mp$ 79.1	0	180	$\pm$ 32.5

the two infinite modes approach infinity along asymptotes that make angles of $\pm 30^{\circ}$ with the positive real axis. For high enough gain this closed loop system goes unstable. The angles of departure and approach are different for each case and are listed in Table 5.2.

5.2 An Algorithm for Finding the Asymptotically Infinite Behavior of the Optimal Root Locus

Given the A, B, Q, and R matrices the objective is to find S^{∞}, N^{∞}, and γ; which are used to characterize the asymptotically infinite behavior of the optimal root locus. We assume that (A,B) is controllable and (M,A) is observable (where $Q = M^T M$, Rank(Q) = Rank(M) = p, and M is p x n).

The algorithm described here is a variation of Shaked's [9], which can be used to find S^{∞} and γ. The changes made to N^{∞} simplify the algorithm, the main reason for that being the use of subspaces of R^m spanned by the vectors ν_i^{∞}. After some definitions the algorithm is presented in the form of a theorem. The theorem is proved, the U_i matrices are discussed, and then an example is given.

Define the matrices

$$G_1 = B^T Q B$$
$$G_2 = (AB)^T Q A B$$
$$\vdots$$
$$G_i = (A^{i-1}B)^T Q A^{i-1} B,$$

and

$$G_i = J_i^T J_i$$
$$J_i = M A^{i-1} B.$$

Define the subspaces of R^m

$$\mathcal{U}_o = R^m$$

$$\mathcal{U}_1 = \mathcal{U}_o \cap \ker J_1$$

$$\mathcal{U}_2 = \mathcal{U}_1 \cap \ker J_2$$

$$\vdots$$

$$\mathcal{U}_i = \mathcal{U}_{i-1} \cap \ker J_i,$$

with the dimensions such that

$$\dim \mathcal{U}_{i-1} - \dim \mathcal{U}_i = p_i,$$

$$\Sigma\, p_i = m.$$

Finally, define the matrices

U_i = matrix whose columns form a basis for $\mathcal{U}_i$, without loss of generality let $U_o = I$.

N_i^∞ = $m \times p_i$ matrix whose columns are the ν_i's corresponding to the p_i ith order Butterworth patterns. If $p_i = 0$ then N_i^∞ is missing and there are no ith order Butterworth patterns.

S_i^∞ = $p_i \times p_i$ diagonal matrix whose elements are the s_j^∞'s corresponding to the p_i ith order Butterworth patterns. If $p_i = 0$ then S_i^∞ is missing.

Using the above definitions we see that

$$N^\infty = [N_1^\infty, \ldots, N_k^\infty]$$

$$S^\infty = \mathrm{diag}(S_1^\infty, \ldots, S_k^\infty),$$

where $k \leq n - m + 1$ is the highest order Butterworth pattern.

In the generic case $k = 1$ and $\mathcal{U}_1 = 0$.

Theorem 2 For $i = 1,\ldots,k$ consider the Jordan canonical forms

$$(U_{i-1}^T RU_{i-1})^{-1} U_{i-1}^T G_i U_{i-1} = [W_1, W_2] \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix} [W_1, W_2]^{-1} . \qquad (2)$$

Λ is a diagonal matrix with real eigenvalues ≥ 0, and

$$N_i^{\infty} = U_{i-1} W_1$$

$$S_i^{\infty} = \Lambda^{\frac{1}{2}} .$$

Remark If there are no ith order Butterworth patterns then W_1 and Λ will be missing.

The proof is by induction and uses the fact that all s_i and ν_i (including those describing the asymptotically infinite behavior) must satisfy

$$[\rho R + \Phi^T(-s)\Phi(s)]\nu = 0 \qquad (3)$$

$$\text{where } \Phi(s) = M(sI - A)^{-1}B$$
$$= M\left(\frac{1}{s} I + \frac{1}{s^2} A + \ldots\right)B.$$

Equation (3) is derived by Harvey and Stein [4]. It can also be found by plugging $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ of the Hamiltonian system (defined in section 2.2.4) into (3.4) and manipulating the result. An expanded version of $\Phi^T(-s)\Phi(s)$ is shown below.

$$\Phi^T(-s)\Phi(s) = -\frac{1}{s^2} G_1$$
$$+ \frac{1}{s^3} (-J_1^T J_2 + J_2^T J_1 \quad)$$
$$+ \frac{1}{s^4} (-J_1^T J_3 + G_2 \quad - J_3^T J_1 \quad)$$

$$+\frac{1}{s^5}\left(-J_1^TJ_4 + J_2^TJ_3 - J_3^TJ_2 + J_4^TJ_1\right)$$

$$+\frac{1}{s^6}\left(-J_1^TJ_5 + J_2^TJ_4 - G_3 + J_4^TJ_2 + J_5^TJ_1\right)$$

$$\vdots$$

$$+\frac{1}{s^i}\sum_{j=1}^{i-1}(-1)^j J_j^T J_{i-j}$$

The first step is to show that the theorem is valid for $i = 1$. Assume without loss of generality that first order patterns exist. Rewrite (3) as

$$\left[\rho R + \frac{1}{s^2}B^TQB + o\left(\frac{1}{s^2}\right)\right]\nu = 0.$$

As $\rho \to 0$ the $\frac{1}{s^2}$ term dominates and this becomes

$$\left[\lambda I - R^{-1}B^TQB\right]\nu = 0$$

where $\lambda = \rho s^2$.

The eigenvalues of $R^{-1}B^TQB$ are real and ≥ 0. (This is because the eigenvalues are the same as those of $R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}}$, which is a matrix of the form X^TX, which is known to have real eigenvalues ≥ 0). From the Jordan canonical form (2) of $R^{-1}B^TQB$ we see immediately that $N_1^\infty = W_1$. For each nonzero eigenvalue we have that $s^2 = -\lambda/\rho$, and the solutions of s are $\pm\lambda^{\frac{1}{2}}/\rho^{\frac{1}{2}}$. As $\rho \to 0$ the branch of s in the LHP is a first order Butterworth pattern with $s_i^\infty = \lambda^{\frac{1}{2}}$. The ν_j^∞ not associated with first order patterns lie in the kernel of $R^{-1}B^TQB$, which is equal to $\mathcal{U}_1$. These ν_j^∞'s are not "trapped" by $R^{-1}B^TQB$.

The next step in the induction is to assume that N_{i-1}^∞ and S_{i-1}^∞

are valid and then show that N_i^∞ and S_i^∞ are valid. The ν_j^∞'s corresponding to ith order patterns must lie in $\mathcal{U}_{i-1}$, therefore $N_i^\infty = U_{i-1}W_1$ for some W_1. The case where there are no ith order patterns is trivial, because the W_1 is missing. Therefore assume that there exists at least one ith order pattern. The next step is crucial. Factor out of (3) the influence of the ν_j^∞'s corresponding to lower order patterns. Do so by multiplying (3) on the left and right as shown below:

$$U_{i-1}^T[\rho R + \Phi^T(-s)\Phi(s)]U_{i-1}\omega = 0.$$

After some work this reduces to

$$\left[\rho U_{i-1}^T R U_{i-1} + (-1)^{i-1}\frac{1}{s^{2i}}U_{i-1}^T G_i U_{i-1} + o\left(\frac{1}{s^{2i}}\right)\right]\omega = 0.$$

As $\rho \to 0$ the first term dominates and this can be written

$$[\lambda I - (U_{i-1}^T R U_{i-1})(U_{i-1}^T G_i U_{i-1})]\omega = 0$$

$$\text{where } \lambda = (-1)^i \rho s^{2i}.$$

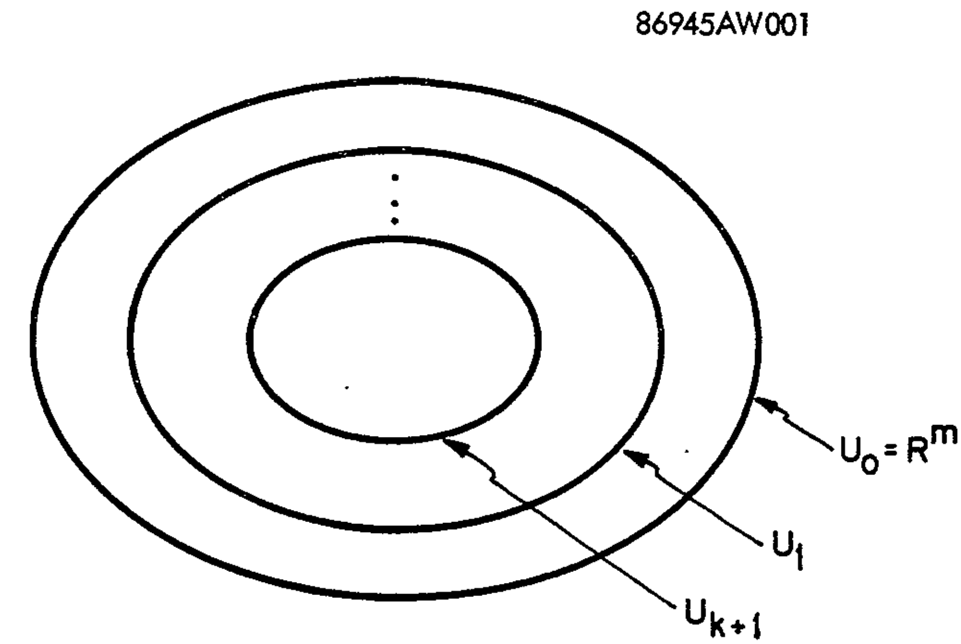
The eigenvalues are real and ≥ 0. From the Jordan canonical form (2) we see that $N_i^\infty = U_{i-1}W_1$. The solutions of s in the left half plane form an ith order Butterworth pattern with $s_j^\infty = \lambda^{\frac{1}{2}}$. The ν_j^∞ not corresponding to 1^{st} through ith order patterns must lie in $\mathcal{U}_i$. This completes the proof.

The $\mathcal{U}_i$ subspaces are shown in Figure 5.2. The following properties of the $\mathcal{U}_i$ are simple consequences of the definitions and the above theorem.

(i) $0 = \mathcal{U}_k \subseteq \mathcal{U}_{k-1} \subseteq \cdots \subseteq \mathcal{U}_o = R^m$

Figure 5.2

The U_i Subspaces



(ii) $\mathcal{U}_i = \text{ImN}_{i+1}^{\infty} + \ldots + \text{Im } N_k^{\infty}$

(iii) $\mathcal{U}_i = \mathcal{U}_{i-1}$ if there are no ith order patterns.

Use the following algorithm to find the U_i matrices. In simple examples this can be done by hand. In more difficult examples use the software described below.

1) Find a basis for $\mathcal{U}_1 = \ker J_1$, call it U_1.

2) If dim $\mathcal{U}_1 = 0$ then let k=1 and stop.

3) Let i=2.

4) Find a basis for ker J_i.

5) Find a basis for $\mathcal{U}_i = \mathcal{U}_{i-1} \cap \ker J_i$, call it U_i.

6) If dim $\mathcal{U}_i = 0$ then let k=i and stop.

7) Go to 4.

Singular value decomposition (SVD) can be used to find an orthonormal basis for the ker J_i. Since ker J_i = ker G_i and G_i is symmetric, another way to find a basis for ker J_i is to use the eigenvectors associated with the zero eigenvalues of G_i. SVD can also be used to find an orthonormal basis for the intersection of two subspaces, see section 3 of Laub's report [41] for more details. An orthonormal basis is not necessary for our purposes. FORTRAN subroutines exist in EISPACK [10] to compute the SVD and the eigenvalue decomposition.

The problem of finding Butterworth patterns of order greater than one is numerically an ill-posed problem because an arbitrarily small change in A, B, or Q can cause a change in the order of the Butterworth patterns. By using proven software we have tried to minimize the numerical problems, but we cannot get rid of them.

<u>Example 2</u> Use the same A and B matrices as in example 1. Given

H and R the problem is to find S^{o}, X^{o}, S^{∞}, N^{∞}, and γ for the optimal root locus. Let

$$H = \begin{bmatrix} -65 & 0 & 0 & 1 \\ 100 & 10 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} .01 & -.01 \\ -.01 & .1211 \end{bmatrix}.$$

Using a computer program that finds the transmission zeroes of S(A,B,H) by using a generalized eigenvalue problem we see that

$$s_i^{o} = -10 \qquad x_1^{o} = \begin{bmatrix} -.089 \\ .891 \\ 0 \\ -5.79 \end{bmatrix}$$

We now implement the algorithm described in this section for finding S^{∞}, N^{∞}, and γ.

$$G_1^{\frac{1}{2}} = HB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{U}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_2^{\frac{1}{2}} = HAB = \begin{bmatrix} 0 & -5 \\ 1 & 10 \end{bmatrix} \qquad \mathcal{U}_2 = \mathcal{U}_1 \cap \ker G_2^{\frac{1}{2}} = 0$$

The number of 1^{st} order patterns is $\dim \mathcal{U}_o - \dim \mathcal{U}_1 = 1$.

The number of 2^{nd} order patterns is $\dim \mathcal{U}_1 - \dim \mathcal{U}_2 = 1$.

The Jordan canonical forms are

$$R^{-1}G_1 = \begin{bmatrix} 0 & 9 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$(U_1^T R U_1)^{-1} \, U_1^T G_2 U_1 = 100$$

Therefore

$$s_1^{\infty} = 3 \qquad N^{\infty} = [\nu_1^{\infty}\nu_2^{\infty}] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$s_2^{\infty} = 10 \qquad \gamma = (01,\ 02,\ 12)$$

5.3 Approximating the Optimal Root Locus

In this section we are concerned with optimal root loci that have Butterworth patterns of order greater than one. What we will show is that any optimal root loci can be approximated by another that has only first order Butterworth patterns, and this approximation can be made arbitrarily precise for an arbitrary distance out along the asymptotes. This result is significant because it allows any optimal root locus problem to be treated as a generic problem, and the generic problem is numerically better conditioned than the nongeneric problem.

The basic idea will first be presented in words. Then a more formal proof will be given using two lemmas and a theorem. Finally, an example will be given.

Start with the well known fact that if Rank(HB) = m then the optimal root locus will have m first order Butterworth patterns. (If Q cannot be factored into H^TH then use M instead of H). In the first lemma it is proved that if Rank(HB) < m then H can always be perturbed an arbitrarily small amount such that the new H times B is full rank. In the second lemma an old result is quoted that essentially says that the eigenvalues of a matrix are continuous with respect to the elements of the matrix. Then in the theorem the Hamiltonian system matrix, for some fixed value of ρ, is perturbed a small amount so that the eigenvalues move less than a prespecified amount and the rank condition on

HB is changed. For ρ smaller than this the infinite modes of the optimal root locus will fall into first order patterns.

Lemma 1 Let H be a real m x n matrix and B a real n x m matrix. If B is full rank then there exists a ΔH such that for all $\alpha \neq 0$ $\text{Rank}(H+\Delta H)B = m$.

Proof Let $\Delta H = W^T$, where $\text{Im}(W) = \text{Im}(B) - \text{Im}(H^T) \cap \text{Im}(B)$.

The next lemma is copied from Wilkinson [38] and is due to Ostrowski. The bound given is not computationally useful due to the $\alpha^{1/n}$ term.

Lemma 2 Let A and B be n x n matrices with elements that satisfy the relationships

$$|a_{ij}| < 1 \qquad |b_{ij}| < 1$$

Then if λ' is an eigenvalue of $A + \alpha B$, there is an eigenvalue λ of A such that

$$|\lambda' - \lambda| < (n+2)(n^2\alpha)^{1/n}.$$

Now we consider two optimal root loci. Let λ be an eigenvalue on the root locus generated by A, B, H, and R. For some $\rho = \rho_o$ specify a ball of radius ε centered around each λ. Let λ' be an eigenvalue generated by A, B, $H + \alpha\Delta H$, and R. We have the following result.

Theorem 3 For every $\rho_o > 0$ and $\varepsilon > 0$ there exists an $\alpha > 0$ and ΔH such that

(i) $\text{Rank}(H + \alpha\Delta H)B = m$

and for every ρ in the interval $\rho_o < \rho < \infty$

(ii) $|\lambda' - \lambda| < \varepsilon$.

The proof for the case Rank(HB) = m is trivial because we can let $\Delta H = 0$. When Rank(HB) < m choose a ΔH such that $\text{Rank}(H + \alpha\Delta H)B = m$ for all $\alpha > 0$, which is always possible from lemma 1. Next, use the fact that the optimal root loci corresponds to the LHP eigenvalues of

$$Z = \begin{bmatrix} A & -\frac{1}{\rho} BR^{-1}B^T \\ -H^TH & -A^T \end{bmatrix}.$$

Let $\rho = \rho_o$. When H is changed to $H + \alpha\Delta H$ then Z changes to

$$Z + \alpha\Delta Z = Z + \alpha\begin{bmatrix} 0 & 0 \\ H^T H\Delta H + \Delta H^T H + \alpha\Delta H^T\Delta H & 0 \end{bmatrix}.$$

The $\alpha\Delta H^T\Delta H$ term can be dropped because for α sufficiently small the largest element of $\alpha\Delta H^T\Delta H$ will be less than the largest of $H^T\Delta H + \Delta H^T H$

Choose β such that the largest element of $(1/\beta)Z$ is less than one in absolute magnitude. Without loss of generality assume also that the largest element of $(1/\beta)\Delta Z$ is less than one in absolute magnitude, which will be true for α sufficiently small. Apply lemma 2 to get

$$|\lambda' - \lambda| < \beta(2n + 2)(4n^2\alpha)^{1/2n}$$

where 2n is used instead of n because Z is 2n x 2n. The β term is included because if λ is an eigenvalue of Z then λ/β is an eigenvalue of $(1/\beta)Z$. Now choose α such that

$$|\lambda' - \lambda| < \beta(2n + 2)(4n^2\alpha)^{1/2n} < \varepsilon.$$

this can always be done, even though α may be very small. The theorem has been proved for $\rho = \rho_o$.

For $\rho > \rho_o$ recompute Z and call it Z'. Let β' be its largest element and note that $\beta' \leq \beta$. Then using the same α in (4) we see that the distance between the eigenvalues of Z' and $Z' + \alpha\Delta Z$ is also less than ε. This completes the proof.

Example 3 Use the same A and B matrices as in examples 1 and 2, and let

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad R = I.$$

The optimal root locus has no finite modes. The four infinite modes group into a first and third order Butterworth pattern with $N^{\infty} = I$, $S^{\infty} = I$, and $\gamma = (01, 02, 12, 22)$. Now suppose H is perturbed in such a way that $\text{Rank}(H + \alpha\Delta H)B = m$. Let

$$\Delta H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (H + \alpha\Delta H)B = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}.$$

The optimal root locus is plotted for $\alpha = 0$, .01, and .001. The results are in Table 5.1 and Figure 5.3. We see that when $\alpha > 0$ the third order pattern shifts into a first order pattern and two new transmission zeroes appear. For the case when $\alpha = .01$ the two optimal root loci are within $\varepsilon = .1$ until ρ is less than .001. For the case when $\alpha = .001$ the approximation is within $\varepsilon = .13$ until ρ is less than 10^{-5}.

Figure 5.3

A Third Order Butterworth Pattern Shifting to First Order

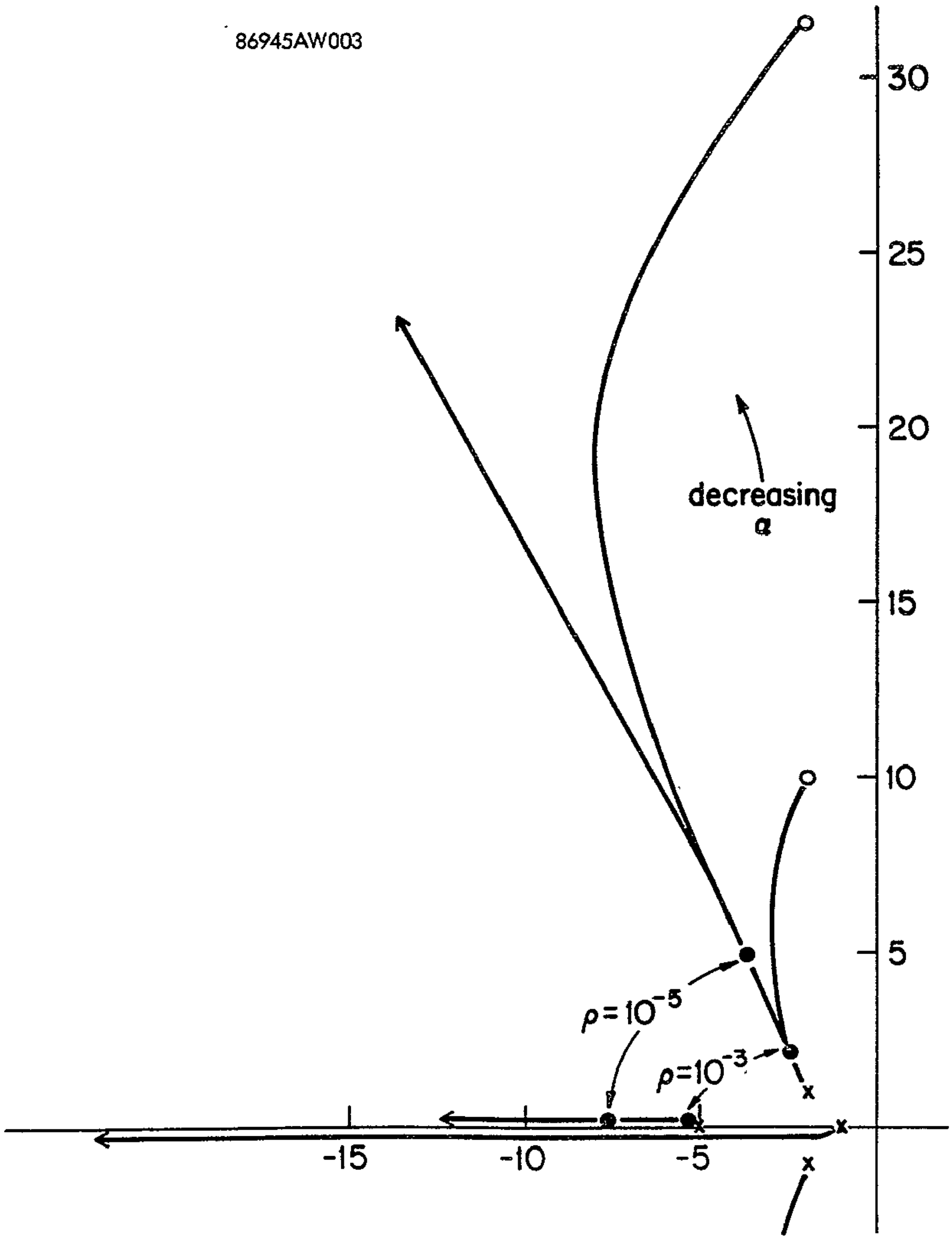


Table 5.3

Approximations of an Optimal Root Locus

ρ	Closed Loop Eigenvalues		
Case #1: $\alpha=0$			
10^{-1}	-3.31	-5.00	$-2.00 \pm i\ 1.01$
10^{-3}	-31.6	-5.17	$-2.23 \pm i\ 1.68$
10^{-5}	-316	-7.61	$-3.75 \pm i\ 5.25$
10^{-16}	-10^{8}	-464	$-232 \pm i\ 402$
Case #2: $\alpha=10^{-3}$			
10^{-1}	-3.31	-5.00	$-2.00 \pm i\ 1.01$
10^{-3}	-31.6	-5.18	$-2.22 \pm i\ 1.69$
10^{-5}	-316	-7.74	$-3.66 \pm i\ 5.27$
10^{-16}	-10^{8}	-10^{5}	$-2.00 \pm i\ 31.6$
Case #3: $\alpha=10^{-2}$			
10^{-1}	-3.31	-5.00	$-2.00 \pm i\ 1.01$
10^{-2}	-31.6	-5.27	$-2.19 \pm i\ 1.76$
10^{-3}	-316	-8.97	$-2.96 \pm i\ 5.33$
10^{-16}	-10^{8}	-10^{6}	$-2.00 \pm i\ 10.0$

CHAPTER VI

Conclusion

The linear state feedback problem was studied in this thesis, and as a special case the linear quadratic regulator was studied. Of primary interest was the relationship between the state feedback gain matrix and the closed loop eigenstructure of the linear state feedback problem, and the relationship between the quadratic weights and the closed loop eigenstructure of the linear quadratic regulator. Several new results were derived which will help both the analysis and the design of multivariable linear control systems.

The relationships were discussed in terms of maps between parameter spaces. The names of the maps used here are not standard and not important, but this seems to be a natural way to discuss the relationships. The similarities between the linear state feedback problem and the linear quadratic regulator become clear when using these maps, and these similarities were exploited in this thesis.

In Chapter III equations were derived to compute angles on the root locus and the optimal root locus, including angles of departure and approach. For the first time the generalized eigenvalue problem was used to compute the angles of approach. Then the quadratic weights were defined to be continuously dependent on ρ and equations were given for the direction and rate of change of the closed loop eigenvalues with respect to ρ. These equations were used to analyze the inverse square method of selecting quadratic weights. All of the above results are valid only at the points where eigenvalues are distinct. Extending these results to multiple eigenvalue points appears to be a nontrivial problem.

In Chapter IV two different kinds of equivalence classes of quadratic weighting matrices were defined. Most of the attention was given to the E^{∞} equivalence class - those quadratic weighting matrices that produce the same asymptotic eigenstructure. Sufficient conditions were given for matrices to be in the E^{∞} equivalence class, a canonical element was defined, and an algorithm given to find the canonical element. Then it was shown by example that the closed loop eigenstructures can be different in the nonasymptotic region for members of the same E^{∞} equivalence class.

More research needs to be done concerning the E^{O} and E^{∞} equivalence classes. Necessary conditions need to be derived for matrices to be in the same E^{∞} equivalence class, and these conditions are probably not much more complicated than the sufficient conditions already derived. It would also be beneficial to divide up the E^{∞} equivalence class according to the equivalence relation that the closed loop eigenstructure be the same in the nonasymptotic region. Having the same angles of departure and approach may be a way to do this, having the same D and E may be another. Similar questions apply to the E^{O} equivalence class. We do not yet know if all members of the E^{O} equivalence class (those quadratic weighting matrices that produce the same optimal gain matrix) have the same asymptotic properties. If not then the E^{O} equivalence class can be further split up. Within this (possibly) smaller set of quadratic weighting matrices a canonical element can be defined that has the following properties (and possibly others in order to assure uniqueness): the Q is of rank m and S(A,B,H) (where $Q = H^{T}H$)

has transmission zeroes in the left half plane. To find this canonical element the asymptotic properties of the original (Q,R) can be found, appropriate D, E, and Y matrices can be chosen, and then using equations (4.1-4.3) the canonical (Q,R) can be computed.

Three related results were derived in Chapter V. The first was a synthesis solution for the linear state feedback problem. An algorithm was given that finds a feedback gain matrix $\frac{1}{k}$ F that produces specified asymptotic properties as $k \to 0$. Only the generic case was considered, so the obvious next thing to do is to extend this algorithm to handle nongeneric cases. Also, this technique may yield insight into selecting a feedback gain matrix K for the output feedback problem. In the generic case K can be chosen to arbitrarily select S^∞ and N^∞, but K has no effect on S^O, X^O, or γ.

The next result was an algorithm to compute S^∞, N^∞, and γ implicit in a set of quadratic weights. With this algorithm and previous results for S^O and X^O, all of the implicit asymptotic regulator properties can now be found. With obvious similarity to the output feedback problem, the asymptotic properties S^O, X^O, and γ can be found from A, B, and Q; and S^∞ and N^∞ can be found from A, B, Q, and R. The next thing to do is to modify this algorithm to find S^∞, N^∞, and a multi-index similar to γ for the output feedback problem. This will be more difficlut because the closed loop eigenvalues are no longer guaranteed to be symmetric about the imaginary axis. The elements of S^∞ are found by solving various eigenvalue problems, and because symmetry is not guaranteed the eigenvalue problems may have Jordan blocks of size 2 x 2 or greater.

The final result shows that an optimal root locus with Butterworth patterns of order greater than one can be approximated by an optimal root locus with all first order Butterworth patterns, and this approximation can be made arbitrarily precise for an arbitrary distance out along the asymptotes. This result may help to analyze optimal root loci. Determining the order of Butterworth patterns is numerically a badly conditioned problem, so by restricting our attention to a finite region of the complex s plane we can use the better conditioned problem of analyzing first order patterns.

REFERENCES

1. Chen, C.T., Introduction to Linear System Theory, Holt, Rinehart and Winston, Inc., New York, 1970.

2. Athans, M., "The Role and Use of the Stochastic Linear-Quadratic-Guassian Problem in Control System Design," IEEE Trans. Auto. Control, Vol. AC-16, No. 6, pp. 529-551, Dec. 1971.

3. Safonov, M.G., and M. Athans, "Gain and Phase Margins for Multi-loop LQG Regulators," IEEE Trans. Auto. Control, Vol. AC-22, No. 2, April 1977.

4. Harvey, C.A., and G. Stein, "Quadratic Weights for Asymptotic Regulator Properties," IEEE Trans. Auto. Control, Vol. 23, No. 3, June 1978.

5. Stein, G., "Generalized Quadratic Weights for Asymptotic Regulator Properties," to be published in IEEE Trans. Auto. Control, Aug. 1979.

6. Rosenbrock, H.H., State Space and Multivariable Theory, Thomas Nelson and Sons Ltd., London, 1970.

7. Davison, E.J., and S.H. Wang, "Properties and Calculations of Transmission Zeroes of Linear Multivariable Systems," Automatica, Vol. 10, pp. 643-658, 1974.

8. Kwakernaak, H., and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

9. Shaked, U., "The Asymptotic Behavior of the Root-Loci of Multi-variable Optimal Regulators," IEEE Trans. Auto. Control, Vol. AC-23, No. 3, June 1978.

10. Garbow, B.S., et al., Matrix Eigensystems Routines - EISPACK Guide Extension, Lecture Notes in Computer Science, Vol. 51, Springer-Verlag, Berlin, 1977.

11. Willems, Jan C., and S.K. Mitter, "Controllability, Observability, Pole Allocation, and State Reconstruction," IEEE Trans. Auto. Control, Vol. AC-16, No. 6, Dec. 1971.

12. Wonham, W.M., Linear Multivariable Control: A Geometric Approach, Springer-Verlag, Berlin, 1974.

13. Moore, B.C., "On the Flexibility Offered by State Feedback in Multi-variable Systems Beyond Closed Loop Eigenvalue Assignment," IEEE Trans. Auto. Control, Vol. AC-21, No. 5, pp. 689-691, Oct. 1976.

14. Evans, W.R., Am. Inst. Elect. Electron. Engrs., Vol. 67, p. 547, 1948.

15. Eveleigh, V.W., Introduction to Control Systems Design, McGraw-Hill Book Co., New York, 1972.

16. Melsa, J.L., and D.G. Schultz, Linear Control Systems, McGraw-Hill Book Co., New York, 1969.

17. Pan, C.T., and K.S. Chao, "A Computer-Aided Root-Locus Method," IEEE Trans. Auto. Control, Vol. AC-23, No. 5, Oct. 1978.

18. Kouvaritakas, B., and U. Shaked, "Asymptotic Behavior of Root-Loci of Linear Multivariable Systems," Int. J. Control, Vol. 23, No. 3, pp. 297-340, 1976.

19. Shaked, U., "The Angles of Departure and Approach of the Root-Loci of Linear Multivariable Systems," Int. J. Control, Vol. 23, No. 4, pp. 445-457, 1976.

20. Shaked, U., "The Intersection of the Root-Loci of Multivariable Systems with the Imaginary Axis," Int. J. Control, Vol. 25, No. 4, pp. 603-607, 1977.

21. Postlethwaite, I., "The Asymptotic Behavior, The Angles of Departure, and the Angles of Approach of the Characteristic Frequency Loci," Int. J. Control, Vol. 25, pp. 677-695, 1977.

22. Kouvaritakas, B., "Gain Margins and Root Locus Asymptotic Behavior in Multivariable Design, Parts I and II," Int. J. Control, Vol. 27, No. 5, pp. 705-751, 1978.

23. Kalman, R.E., "Contribution to the Theory of Optimal Control," Bol. Soc. Mat. Mexico, pp. 102-119, 1960.

24. Wonham, W.M., "On a Matrix Riccati Equation of Stochastic Control," SIAM J. Control, Vol. 6, No. 4., pp. 681-697, 1968.

25. Laub, A.J., "A Schur Method for Solving Algebraic Riccati Equations," Lab. for Infor. and Decision Systems, MIT, LIDS-R-859, Oct. 1978.

26. Denham, M.J., "Canonical Forms for the Identification of Multivariable Linear Systems," IEEE Trans. Auto. Control, Vol. AC-19, No. 6, Dec. 1974.

27. Molinari, B.P., "The Stable Regulator and Its Inverse," IEEE Trans. Auto. Control, Vol. AC-18, No. 5, pp. 454-459, Oct. 1973.

28. Popov, V.M., "Hyperstability and Optimality of Automatic Systems with Several Control Functions," Rev. Roam. Sci. Tech. Electrotech. Energ., Vo. 9, pp. 629-690, 1964.

29. Jameson, A., and F. Kreindler, "Inverse Problem of Linear Optimal Control," SIAM J. Control, Vol. 11, pp. 1-19, Feb. 1973.

30. Kalman, R.E., "When is a Linear System Optimal?," Trans. ASME J. Basic Engr., ser. D, Vol. 86, pp. 51-60, Mar. 1964.

31. Anderson, B.D.O., and J.B. Moore, Linear Optimal Control, Prentice-Hall, Englewood Cliffs, N.J., 1971.

32. Chang, S.S.L., Synthesis of Optimal Control Systems, McGraw-Hill, New York, 1961.

33. Rynaski, E.G., "The Multivariable Root Square Locus - An Optimal System Design Aid," Cornell Aeronautical Lab., Inc., about 1965.

34. Kouvaritakas, B., "The Optimal Root Loci of Linear Multivariable Systems," Int. J. Control, Vol. 28, No. 1, pp. 33-62, 1978.

35. Kwakernaak, H., "Asymptotic Root Loci for Multivariable Linear Optimal Regulators," IEEE Trans. Auto. Control, Vol. AC-23, No. 3, June 1978.

36. Postlethwaite, I., "A Note on the Char. Frequency Loci of Multivariable Linear Optimal Regulators," IEEE Trans. Auto. Control, Vol. AC-23, No. 4, August 1978.

37. Kato, T., Perturbation Theory for Linear Operators, 2nd Edition, Springer-Verlag, Berlin, 1976.

38. Wilkinson, J.H., The Algebraic Eigenvalue Problem, Oxford University Press, London, 1965.

39. Stewart, G.W., "On the Sensitivity of the Eigenvalue Problem $Ax = \lambda Bx$," SIAM J. Numer. Anal., Vol. 9, pp. 669-686, 1972.

40. Laub, A.J., and B.C. Moore, "Calculation of Transmission Zeroes Using QZ Techniques," Automatica, Vol. 14, pp. 557-566, 1978.

41. Laub, A.J., "Linear Multivariable Control. Numerical Considerations," Electronic Systems Lab., MIT, ESL-P-833, July 1978.

42. MacFarlane, A.G.J., and N. Karcanias, "Poles and Zeroes of Linear Multivariable Systems: A Survey of the Algebraic, Geometric, and Complex-Variable Theory," Int. J. Control, Vol. 24, No. 1, pp. 33-74, 1976.

43. Davison, E.J., and S.H. Wang, "Remark on Multiple Transmission Zeroes of a System," Automatica, Vol. 12, p. 195, 1976.

44. Davison, E.J., and S.H. Wang, "An Algorithm for the Calculation of Transmission Zeroes of the System (C,A,B,D) Using High Gain, Output Feedback," IEEE Trans. Auto. Control, Vol. AC-23, No. 4, pp. 738-741, Aug. 1978.

APPENDIX A

Transmission Zeroes

Definition

Several definitions of transmission zeroes exist in the literature and all have some validity. Common sense dictates that the definition reduces to the usual one in the SISO case (the roots of the numerator polynomial of the transfer function) and that the transmission zeroes have some physical meaning. One definition that meets these requirements is due to Rosenbrock [6], which is the one used here. Though it is too early to know for sure, the control community appears to be settling down to this definition.

Rosenbrock defines the system matrix

$$P(s) = \begin{bmatrix} A - sI & B \\ -C & 0 \end{bmatrix} \tag{1}$$

and his definition of transmission zeroes is given in terms of the minors of P(s). An equivalent definition uses the Smith McMillan form of P(s) [42]. Since for our purposes we have the same number of inputs and outputs we use the following equivalent definition [7,43]: The transmission zeroes are those values of s, including multiplicities but not including uncontrollable or unobservable modes, that reduce the rank of P(s). We note that this definition allows the degenerate case where the whole complex plane reduces the rank of P(s); this can happen even if B and C are full rank and the system is controllable and observable. We note also in the square case that the determinant of P(s) is equal to

$$\det(A-sI)\det[C(A-sI)^{-1}B], \tag{2}$$

and another equivalent definition of transmission zeroes is the roots of (2). This definition is used by Kwakernaak and Sivan [8].

To demonstrate that Rosenbrock's definition has physical meaning [42] we consider the linear output feedback system of section 2.2.2. Assume that it is controllable and observable and then Laplace transform (2.4, 2.5) to get

$$\begin{bmatrix} A - sI & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}.$$

Transmission through the system is blocked (the output will be zero) for certain values of s, x, and u. Those values of s for which this is true are transmission zeroes, and the corresponding values of x are called zero directions.

The Generalized Eigenvalue Problem

The problem is to find all finite λ and their associated eigenvectors v which satisfy

$$Lv = \lambda Mv, \tag{3}$$

where L and M are real p x p matrices not necessarily full rank. There will be from 0 to p finite solutions. If M is invertible then premultiplication by M^{-1} changes the generalized eigenvalue problem into an eigenvalue problem and there will be exactly p solutions. Stable and reliable FORTRAN subroutines exist in EISPACK [10] to solve the generalized eigenvalue problem.

Computing Transmission Zeroes

Three different methods for computing transmission zeroes are dis-

C-2

cussed. The relationship between the first two methods is an unpublished result due to Laub. Only the special case of an equal number of inputs and outputs and no feedforward term is discussed.

Laub and Moore [40] discuss in detail using the generalized eigenvalue problem as a way to compute transmission zeroes. In place of (3) use

$$\begin{bmatrix} A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \lambda \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} .$$

The finite solutions λ reduce the rank of Rosenbrock's system matrix (1) and therefore are transmission zeroes. There are anywhere from 0 to (n-m) transmission zeroes, and in the generic case when Rank(CB) = m there are exactly (n-m) solutions. The portion x of the associated eigenvector is the zero direction.

Davison and Wang [44] use high gain feedback and an eigenvalue problem to compute transmission zeroes. First they prove that as $k \to 0$ the finite closed loop eigenvalues of

$$A_{cl} = A - \frac{1}{k} BKC \tag{5}$$

approach the transmission zeroes. Their method is to find a "suitably small" value of k, compute the eigenvalues of A_{cl}, and then determine which are finite. The eigenvectors associated with the finite eigenvalues are the zero directions.

These two methods are closely related. The connection is the result derived in section 3.2.1 that the closed loop eigenvalues and eigenvectors are solutions of the generalized eigenvalue problem

$$\begin{bmatrix} A - \lambda I & B \\ -C & -kK^{-1} \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = 0. \tag{6}$$

Equations (5) and (6) give the same answers for $k > 0$ and in the limit as $k \to 0$. The only difference is that in (6) k can be set exactly to zero. Equation (6), incidentally, is an obvious way of proving that as $k \to 0$ the closed loop eigenvalues approach the transmission zeroes.

A third approach is found in various papers by MacFarlane, Karcanias, Kouvaritakas, and Shaked [for instance 42,18]. They define N and M such that the rows of the $(n-m) \times n$ matrix N form a basis for the ker (B^T) and the columns of the $n \times (n-m)$ matrix M form a basis for the ker(C). Then the transmission zeroes are the roots of the polynomial.

$$\det(NAM - \lambda NM) = 0. \tag{7}$$

This result can be derived using similarity transformations on the matrices in (4), see [42,18] for details. In the generic case when Rank(CB) = m then Rank(NM) = (n-m) and (7) is an eigenvalue problem, ie the (n-m) eigenvalues of $(NM)^{-1}NAM$ are the transmission zeroes. In the nongeneric case when Rank(CB) < m then it is not clear how to continue, other than to treat (7) as a generalized eigenvalue problem.

The first method, by Laub and Moore, has the best numerical properties and is the one to use. While all three methods give excellent results in some cases, Laub and Moore give examples where the other two methods break down. In Davison and Wang's method it is not obvious how to choose a suitably small k, and in some cases the accuracy of the answer is critically dependent on this choice. Furthermore, Davison and Wang's method gives no indication when the answers are in error; and therefore their method is unreliable. The method by MacFarlane et al.

unnecessarily requires rank determination, which numerically is a very difficult thing to do, and can introduce errors into the computations.