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Final Report of VPI's Study of

IMPROVED MODELING AND SOLUTION PROCEDURES FOR NONLINEAR ANALYSIS

NASA Grant No. NSG 1546

Period: July 1, 1978 - June 30, 1979

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Prepared by

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I. INTRODUCTION

The work performed by the author while at the Langley Research Center (LRC) during the period June 1, 1977 through May 31, 1978 on an IPA assigment agreement provided the necessary continuity between the work carried out under the grant NSG 47-004-114 and the present grant NSG 1546. Several ideas evolved or were stimulated as a result of author's collaboration with the technical monitors Robert Hayduk and Robert Thomson of LRC.

A multitude of problems common to the development of a successful nonlinear analysis code prevented ACTION from being able to solve even the simplest of the nonlinear problems during the early stages. Most of these problems having been overcome ACTION's performance looked cautiously optimistic. Hence, the need to validate the ACTION computer code on an aircraft-like structure followed by its finalization for public release were recognized.

II. TECHNICAL PROGRESS

The technical efforts during the past year were concentrated in several different areas. These are discussed below. Each group of paragraphs provides a background, a brief statement of the problem, approach to solution, the progress made and is some cases recommendations for future work.

1. Structuring of the ACTION Code to Handle Relatively Large Scale Problems

In order to make ACTION suitable to handle large scale problems involving finite element models with stringer, frame and membrane elements, it was necessary to undertake the following tasks.

a. Implementation of Analytic Gradients for the Membrane Element

The calculation of gradients of the strain energy of deformation of the membrane element was hitherto being performed by the use of central differences in ACTION. This is nearly twice as more expensive as when the gradients are calculated analytically. This is especially crucial when a model contains a large number of membrane elements. It was thus imperative that gradients of strain energy of deformation of the membrane elements be calculated analytically in ACTION. Accordingly, on lines similar to those used for the frame element explicit expressions were developed for the calculation of the gradients of the strain energy both in the elastic and the inelastic range. In the inelastic range the computations were simplified by the use of the total deformation theory [1]. If incremental flow theory of plasticity is to be used then the definition of the potential function must be altered to allow the use of certain minimum principles in plasticity [2]. This novel formulation using the direct minimization approach appears like a topic suitable for Master's thesis and hence Mr. Lin T. Duong who has been working as a graduate research assistant on the project indicated his interest in pursuing this line of research towards his Master of Science degree. Whether such a formulation will be computationally more efficient or will result in improved response prediction remains to be seen.

b. Quasi-Newton Minimization Algorithms that Exploit Sparsity

Of the two minimization algorithms available in ACTION, Powell's conjugate algorithm can handle problems involving thousands of variables but its convergence rate is at best linear. Hence, it is unlikely that it can be competitive with second order algorithms like the Fletcher's algorithms (BFGS) [3] whose convergence rate is at least superlinear. Fletcher's method is especially suited for solution of transient problems in steps since it has a reasonably good approximation to the variables and the inverse Hessian of the potential surface. The number of minimizations required for convergence to a solution after the first time step is thus a very small fraction of the total number of degrees of freedom. The drawback of Fletcher's algorithm is however it's storage requirements - $N \times (N+1)/2$ for a N degree of freedom problem. This is because Fletcher's algorithm approximates the Hessian inverse rather than the Hessian itself which is usually banded and very sparse. Thus, if an extremely large scale nonlinear transient problem is to be solved using energy minimization technique it is imperative to use a variable metric algorithm which updates the Hessian rather than its inverse and thereby exploits and maintains its sparsity in its march to the minimum. It appeared at one point that for solving problems with as few as 325 degrees of freedom using Fletcher's method the available core on the CDC system may be insufficient.* Accordingly, work was initiated to implement into ACTION a minimization algorithm that exploits sparsity. Such an algorithm based on the work of Schubert [4] has been implemented into one of VPI & SU's version of ACTION. The algorithm, however, remains to be validated. A survey of the state-of-the-art of minimization techniques for extremely large scale problems, which was conducted in this connection, may be found in reference [5].

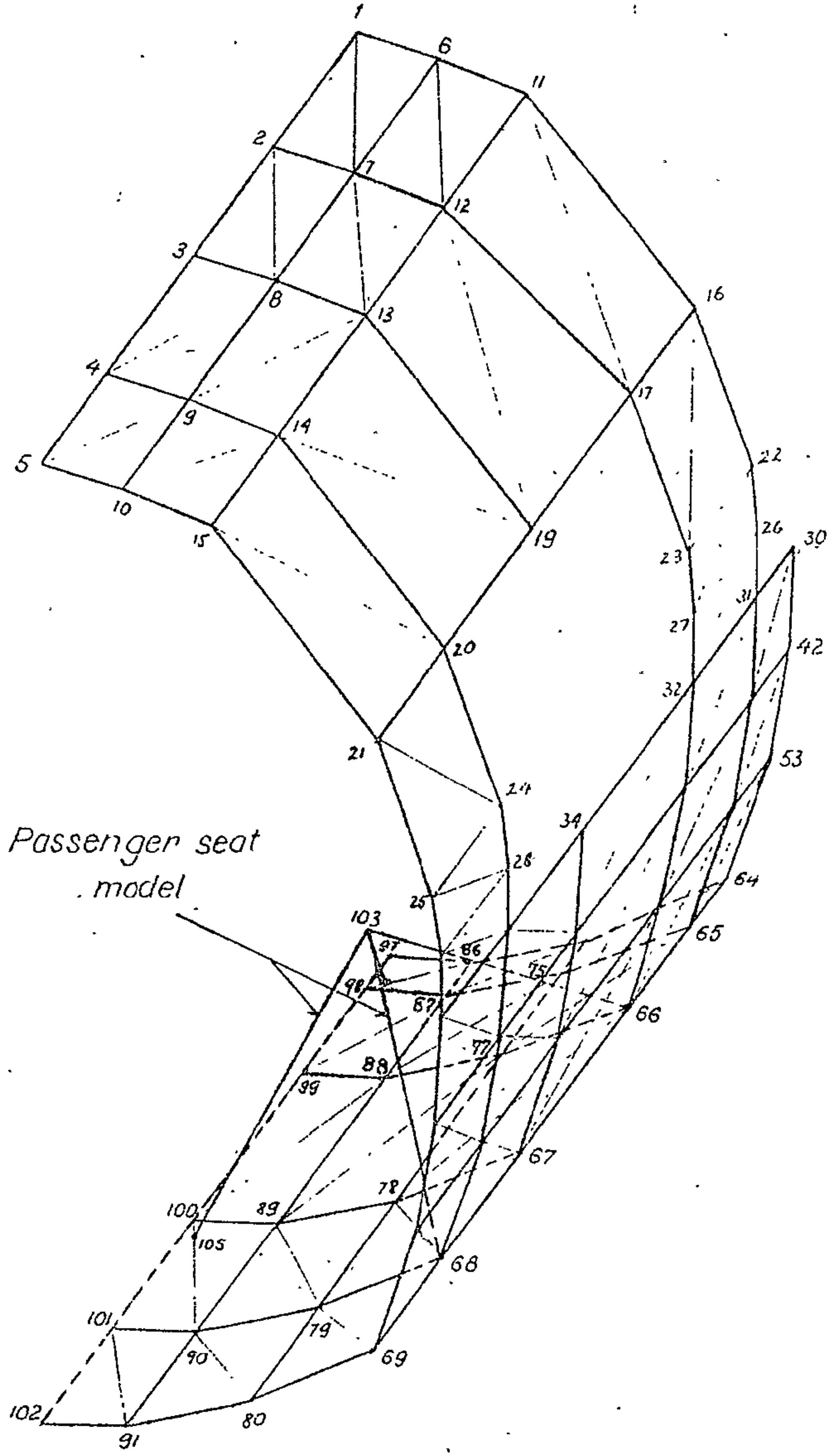
2. Drop Test of the Navajo Fuselage Section:

VPI & SU was entrusted with the task of development of a model for nonlinear crash dynamic analysis of the Navajo fuselage section using the ACTION code. The response characteristics of such a model was to be compared with comparable models developed using DYCAST [6] and KRASH [7]. Such a comparison was subsequently reported in reference [8].

An ACTION model of the Navajo fuselage section was developed using the corresponding DYCAST model (an initial cruder model different from the one reported in [8]) as its basis. This model shown in Fig. 1 has 105 nodes, 209 members (36 stringers, 77 frame elements and 96 membranes), 71 lumped masses and a total of about 336 degrees of freedom. This is the largest model ever attempted using ACTION.

Although the model could be run using the Powell's conjugate gradient minimization algorithm (MIN5) which requires very little storage, it was believed, as explained before, that in going from one step to the next this algorithm does not have the same advantages of the

*An ACTION version which could handle problems with up to about 475 degrees of freedom using the Fletcher's method was put together at VPI & SU. This was possible because of the almost unlimited storage available on the IBM system. Although some preliminary runs were made using this version the task of simulation of large scale problems like the Navajo drop test had to be relegated to the CDC computer at LRC because of extremely tight funding situation at VPI & SU.



g.1 Drop test of eight passenger plane fuselage section

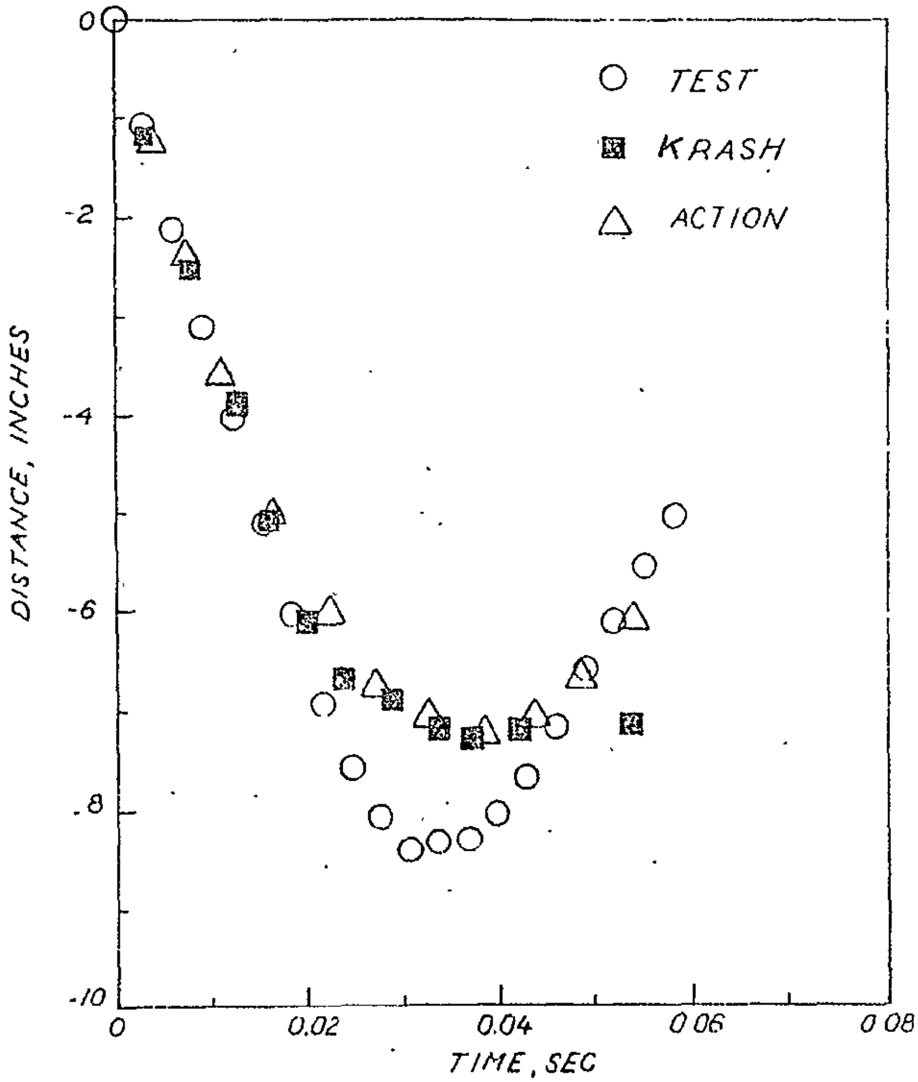
Action model

second order algorithm of Fletcher (MIN7). The latter has a good approximation to the inverse Hessian in addition to a good initial guess for the variables at hand at the end of each time step.

To run this model using the more efficient Fletcher's minimization algorithm in ACTION it was necessary to overlay the ACTION code on NASA's CDC system using the segloader. The assistance of Ms. Barbara Durling of NASA Langley in accomplishing this task is greatly appreciated.

Of simulation interest was the occupant chest motion and its vertical acceleration at the pelvis location. The occupant was modeled by a single lumped mass while the seat was modeled by a set of four nonlinear stringer elements whose stress-strain behavior was based on previous, independent static crash tests on similar seats. This information was provided by LRC. Although, the ground plane capability (terrain model) in ACTION could have been exercised for this problem, in the interest of simplicity, the aircraft substructure was assumed to be in contact with the ground plane at nodes (86) and (91) while a velocity of 330 inches/sec. was imparted to the entire model. Thus one would expect correlation only in the initial phases of the response.

Figures 2 through 5 provide the correlation between the analysis and test results. For additional model and simulation details it is suggested that reference [8] be consulted. This reference also provides a comparison of the performance of the energy minimization technique vis-a-vis the so-called hybrid technique (KRASH) and another technique which utilizes the pseudo force technique (DYCAST).



Occupant chest

Fig. 2 Comparison of Analysis and Test Motions (occupant chest)

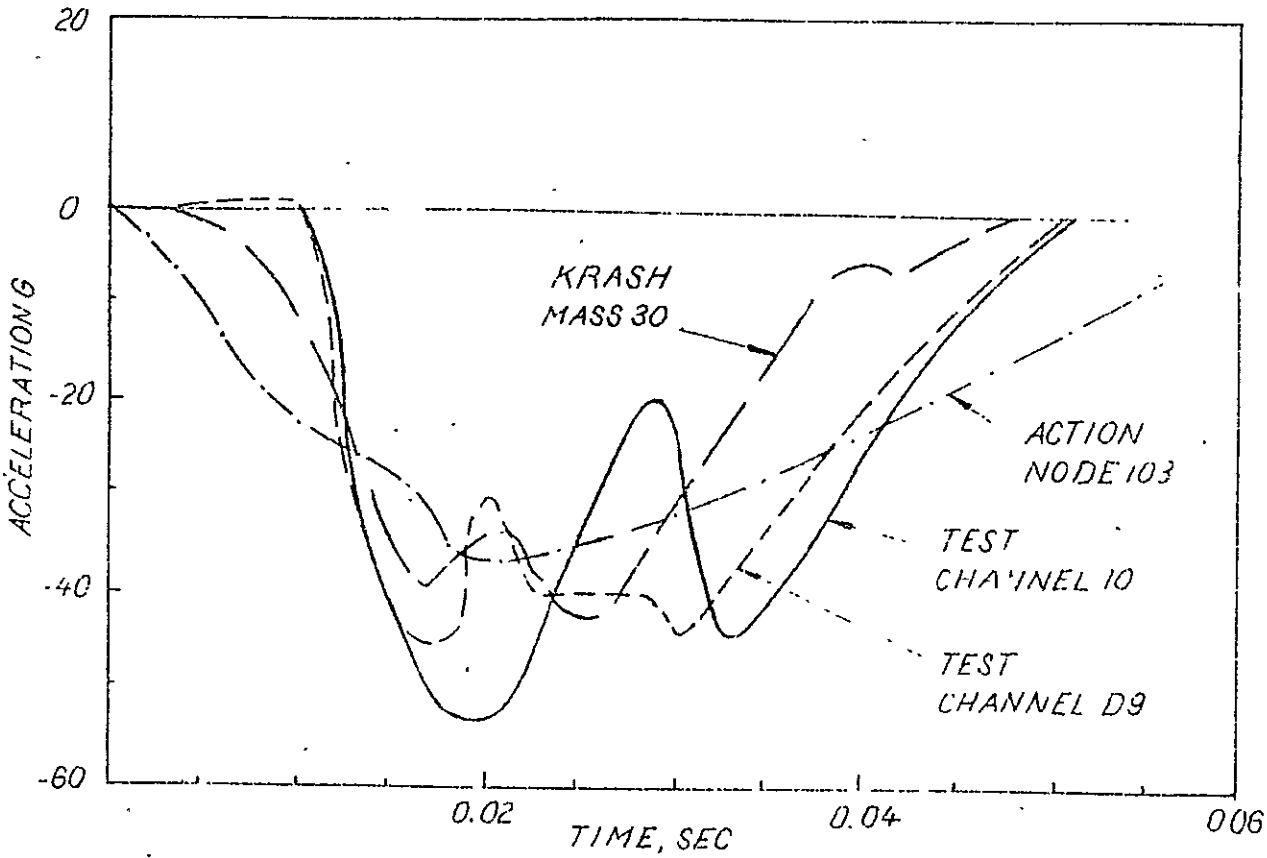


Fig. 3 Comparison of occupant Pelvis vertical Acceleration, Test versus Analysis

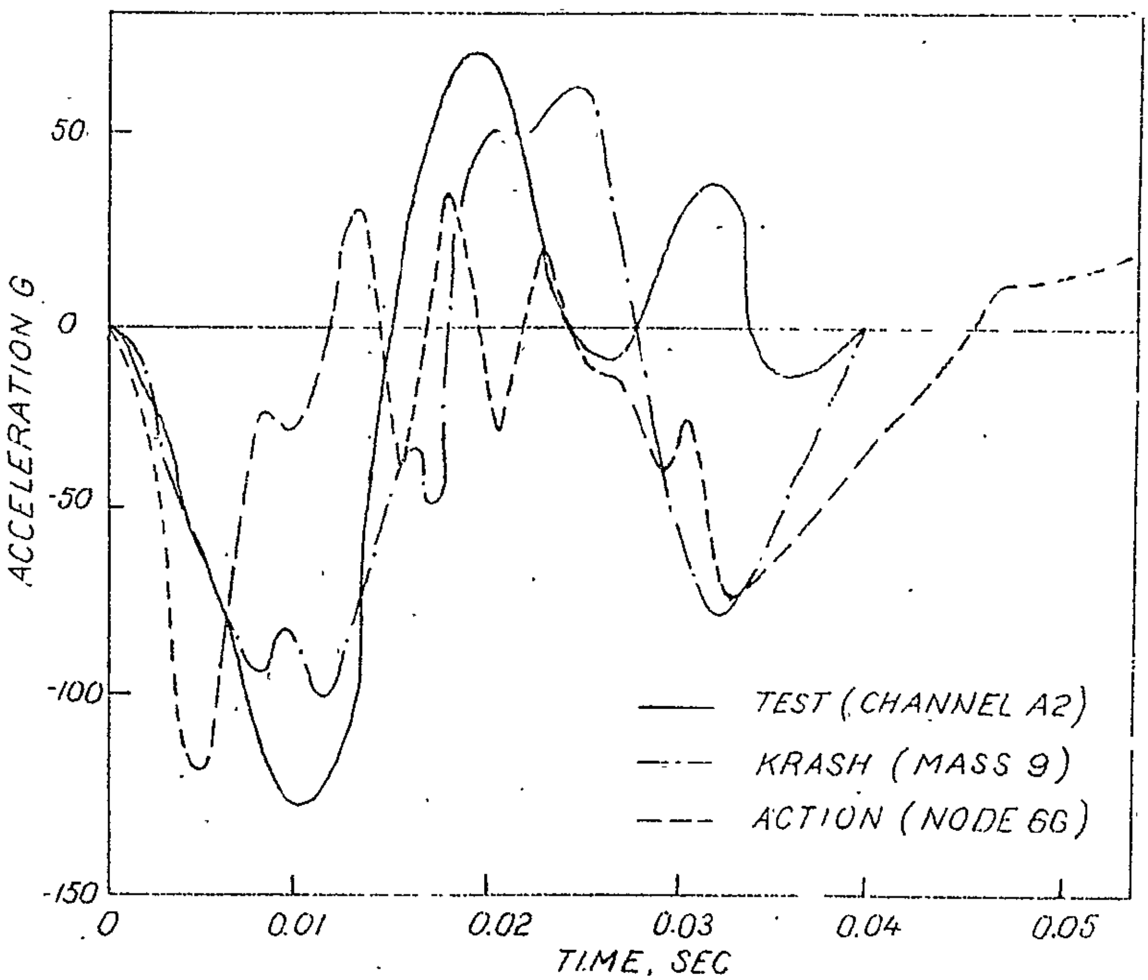


Fig. 4 Comparison of fuselage floor vertical acceleration, Test versus Analysis

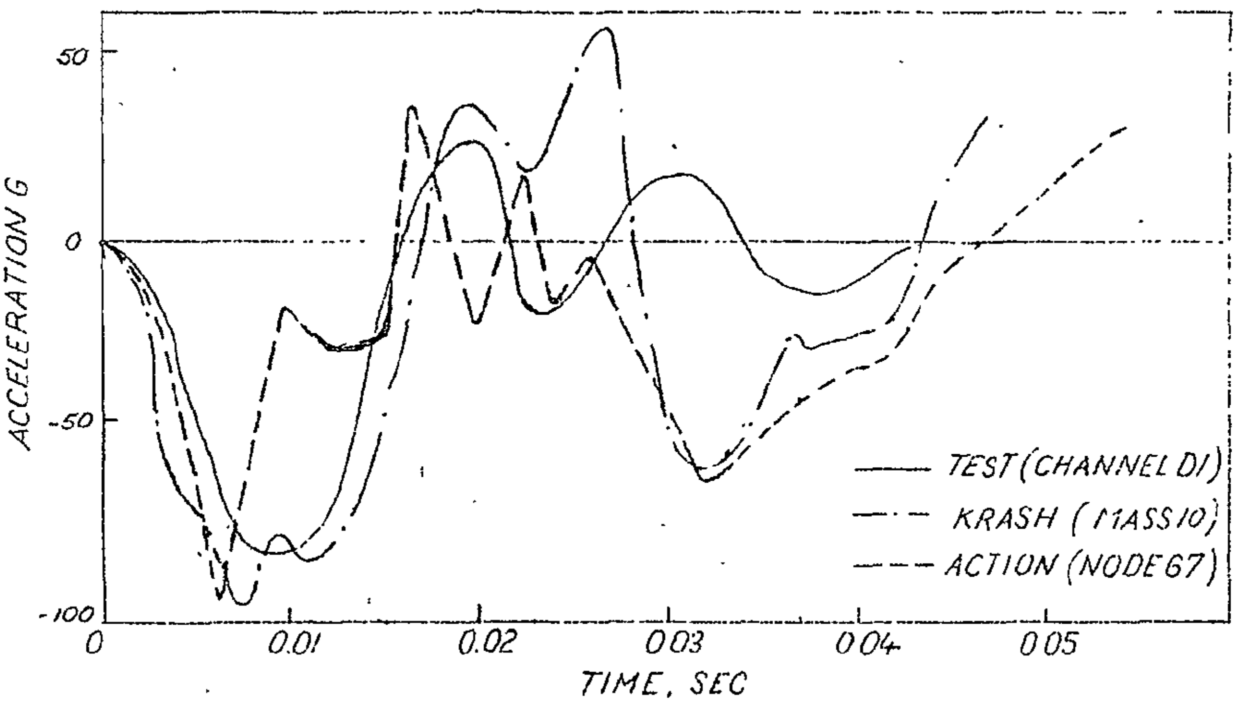


Fig. 5 Comparison of fuselage floor vertical acceleration, Test versus Analysis.

3. Implementation of the Frame Element with Cross-Sectional Flexibility into DYCAST

VPI & SU was entrusted with the task of aiding Greenman Engineers in implementing the frame element with cross-sectional flexibility into their program DYCAST. The frame element with cross-sectional flexibility was developed by VPI & SU in the ACTION code environment [9] which uses the direct minimization approach. Implementation of this element requires the development of a tangent stiffness matrix for such an element. This task was given a very low priority dictated by the conclusions of the study reported in reference [9] and other experiences in simulating the crash response of the Navajo section.

Report [9] concludes that the frame elements in both ACTION and DYCAST which do not currently allow for cross-sectional deformations are adequate in predicting gross response parameters even though the structures in question may undergo severe localized cross-sectional deformations. By gross parameters is meant total energies, load-deflection response, etc. The report goes on to conclude that purely from the point of view of crashworthiness where the trauma measures are determined more by gross parameters it seems highly unlikely that they will be significantly influenced by highly localized cross-sectional deformations. The inclusion of such effects can only make the already expensive nonlinear analysis only more so without any significant pay-off by way of improvement in response. Considering the cost comparisons of KRASH, ACTION & DYCAST in reference [8] vis-a-vis the quality of their respective response predictions this would warrant a careful assessment of the necessity of including such a costly element into either ACTION or DYCAST. In any event if such an implementation is to be carried out into DYCAST at some future date, VPI & SU's cooperation in this matter can always be counted upon. The formulation details outlined in reference [9] will provide the required basis for this purpose.

4. Documentation and Wrap-up of the ACTION Code

a. ACTION Theory Document

The old ACTION theory document was thoroughly revised to reflect the current formulation basis in ACTION (especially the part pertaining to the calculation of analytic gradients and the new minimization algorithms). A copy of the completed rough draft of this document will be mailed to LRC within the next few days. Upon receipt of comments from LRC the document will be revised accordingly and the final version of this document will be completed under NASA's Master Agreement arrangement with VPI & SU (NAS1-15080 Task #10).

b. ACTION User's Guide

The existing ACTION user's manual was revised to be consistent with the final ACTION version to be released through COSMIC. Additional features included analytic gradients, selection of Newmark-Beta parameters, automatic calculation of lumped masses, a new solid rectangular frame cross-section, etc. Several outdated, unvalidated ACTION features have been removed from the present version as called for in the grant proposal. The revised ACTION version, however, remains to be thoroughly tested. This testing will be performed again under the Master Agreement arrangement contract NAS1-15080 Task #10. A copy of the completed rough draft of this document will be mailed to LRC within the next few days for their comments to be implemented in the final version.

c. Publication of VPI&SU Reports and Papers in Referred Journals.

Progress in this area was extremely satisfactory. A report [9] entitled "An Investigation into the Effects of Cross-Sectional Flexibility on Response" VPI-E-78-30 was completed and four copies of the same were mailed to LRC (see Appendix A). This study summarizes the findings of the importance of cross-sectional flexibility on gross response and attempts to identify reasons for the discrepancy between analytical and experimental predictions. A technical note entitled "On the Importance of Cross-Sectional Flexibility on Gross Response" has been accepted for publication by the Journal of Computers and Structures (see Appendix B).

A synoptic summarizing the ACTION formulation and validation entitled "Nonlinear Transient Analysis via Energy Minimization" has been accepted for publication by the Journal of American Institute of Aeronautics and Astronautics. The full length paper by the same title has been accepted as a backup NTIS document number N79-22819 for the synoptic. The same full length paper was previously released as a VPI & SU report number VPI-E-79-10 entitled "Nonlinear Transient Analysis of Aircraft-like Structures - Theory and Validation" (see Appendix C). The necessary number of copies were mailed to LRC.

A paper entitled "Energy Minimization versus Pseudo Force Technique for Nonlinear Structural Analysis" has been accepted for publication by the Journal of Computers and Structures (see Appendix D). This paper compares the efficiency of the minimization technique (ACTION) vis-a-vis the pseudo force technique (DYCAST) for the solution of nonlinear problems. It also explores the feasibility of extending the minimization technique for the solution of extremely large scale nonlinear problems competitively.

5. Simplified Modeling Techniques

This is the only task which received the least attention because of lack of time and also because of lack of suitable personnel who could be assigned to tackle this problem. Some nominal progress was, however, made. A literature survey on the topic revealed that ideas similar to those proposed in the proposal have been explored in much greater detail by Kawai and his associates [10]. Their work suggests that ideas like the ones proposed in the proposal should be worthwhile to pursue especially if codes like DYCAST or ACTION can ever hope to be cost effective with codes like KRASH [8].

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[9] M. P. Kamat, "An Investigation into the Effect of Beam Cross-Sectional Flexibility on Response", VPI & SU Report No. VPI-E-78-30, December 1978.

[10] T. Kawai and Y. Toi, "New Element Models in Discrete Structural Analysis",

III. OPERATIONAL SET-UP

The work reported in this report was carried out by the principal investigator, Dr. M. P. Kamat, with the aid of Mr. Linh T. Duong, a graduate research assistant in the department of Engineering Science and Mechanics of VPI & SU. Because of lack of students with the appropriate background and the time required to familiarize them with the ACTION code no attempt was made to hire other graduate research assistants. Instead the principal investigator undertook a larger share of the responsibility inorder that he may meet the grant obligations more fully.

IV. CONCLUSIONS

The work performed under this grant achieved several objectives. It successfully demonstrated ACTION's capability to model and analyze aircraft for nonlinear crash response at a cost which is comparable with that of other similar analytical tools like DYCAST. It was established that computer programs like ACTION & DYCAST in their present status are quite adequate in predicting gross response parameters inspite of the fact that structures in question may undergo severe localized cross-sectional deformations. The theoretical and user's documentation for the ACTION code were nearly finalized for public release. In addition to a few technical reports several papers were published in referred Journals. State-of-the-art survey conducted indicates minimization techniques to be quite suitable for the solution of large scale nonlinear problems. A similar survey also indicates that simplified modeling techniques are not only viable but should be pursued if programs like ACTION or DYCAST are to be cost effective with programs like KRASH.

APPENDIX A

An Investigation Into the Effect

of Beam Cross-Sectional Flexibility on Response

by

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Abstract

A simplified frame element (3D-beam) formulation which accounts for the beam cross-sectional flexibility is presented. The effectiveness of such a formulation as opposed to modeling the frame element using membranes and plate bending elements is investigated on the post-buckling response of a beam column with a thin-walled channel cross-section. The fidelity of the simulation with and without such effects is determined on the basis of a comparison with available experimental results and other independent simulations. The study concludes that in view of the excellent correlation between experimental and mathematical predictions using simulators like ACTION and O-PLANE-MG which do not account for cross-sectional flexibility, a beam element with cross-sectional flexibility will have little pay-off, especially so for the response parameters of importance in crashworthiness.

1. Introduction

Static and dynamic load tests on tubular and angular frame structures conducted by the Dynamic Loads Branch of the NASA Langley Research Center reveal that significant cross-sectional warping and distortions occur near the joints [1]. It is believed that the lack of correlation between experimental and theoretical predictions may be attributed, at least in part, to cross-sectional flexibility during deformation hitherto not accounted for by most mathematical simulators of such nonlinear phenomena. The present investigation was thus initiated with the objective of implementing such a capability into existing programs for nonlinear analysis of structures.

In the context of the cross-sectional flexibility the word "beam" is perhaps a misnomer since it is implicit in its usage that its response is predominantly in the direction of its longest dimension which is supposedly an order of magnitude larger than its other cross-sectional dimensions. Hence when cross-sectional flexibility effects are significant it would be more appropriate to refer to the structure as either a three dimensional solid or in the case of thin-walled open or closed sections as an assemblage of flat or curved plates. In other words, cross-sectional flexibility in the case of thin-walled structures would be most naturally accounted for by idealizing the member or structure with membranes and plate bending finite elements. However, the degrees of freedom of the resulting model would in most cases, be prohibitive for a general nonlinear analysis. Hence, some other cheaper formulation which perhaps sacrifices some of the rigor of the formulation would be desirable.

A thin-walled open section frame (3D-beam) element formulation which abandons the assumptions of the elementary beam theory and accounts for the flexibility of its cross-section in an approximate manner was undertaken. The objective was to develop a model which is simple and in terms of its size well below that which would result from building up the frame element using membrane and plate bending elements. Consequently, certain simplifying assumptions were incorporated in its development. The development was initially carried out in the context of the ACTION simulator. This simulator uses the minimization of the total potential energy for nonlinear analysis [2].

2. Formulation

a. Assumptions: Restricted to beams with thin-walled cross-sections, the formulation focusses on distortions in the plane of the cross-sections only since warping, at least of the unrestrained type is usually accounted for in most conventional frame element development. This is true of the ACTION simulator as well.

The development consists in abandoning the assumption of the elementary beam theory about cross-sections remaining rigid during deformation. Co-ordinates of a select few reference points on the cross-section are treated as additional degrees of freedom with a prescribed co-ordinate variation in between reference points. In essence then this is tantamount to treating the co-ordinates of the integration or quadrature points used for the evaluation of the strain energy of deformation of the frame element as additional degrees of freedom of the model. Since the problem now becomes extremely nonlinear it is reasonable to adopt a linearization procedure of the following type. At the beginning of each load increment elementary beam theory is used to predict overall deformations of the frame element assuming a rigid cross-section. At the end of the increment the load is maintained constant and the additional deformations of the elements of the cross-section are determined by treating the elements as prestressed plate elements with a known initial eccentricity. The shape of the cross-section is determined by minimizing the corresponding potential energy of additional deformations which consists of the strain energy of additional deformations and the potential of the prestress. Using this new cross-section the process is then repeated for the next load increment. Iteration at constant load could be employed to obtain improved results.

b. Formulation Details: Consider the IE cross-section shown in Figure 1. This cross-section has five flange pieces and two web pieces. This element as implemented in the ACTION Simulator has the option of specifying zero thicknesses for all but the two web pieces. This enables the generation of cross-sections of many well-known shapes. Each of the seven pieces is treated like a flat plate element extending between the two end nodes of the beam or frame element. A total of seven reference points (1) through (7) as shown in Figure 1 are chosen on the cross-section at each end of the beam element and half-way between the two nodes. Incidently, these locations of reference points are coincident with the Lobatto integration points (at the ends of the intervals of integrations) used for each of the seven plate elements. The stress-strain history at all of these points is thus available without additional calculation effort or storage. The cross-section is allowed to deform relative to the fixed point (8) in the cross-section. At each of the reference points (1), (3), (4), (5) and (7) only one independent displacement transverse to the respective plate elements [1], [2], [4], [5] and [7] is allowed and at each of the reference points (2) and (6) only one independent displacement transverse to the respective plate elements [2] and [5] is allowed. The other position co-ordinate of each of the reference points is determined on the basis of the additional deformations of the plate elements being inextensional.

For the IE section of Figure 1 the potential energy of additional deformations, π, is thus given by [3]

$$\pi = \pi_1 + \pi_2 \qquad (1\text{-a})$$

where

$$\pi_1 = \sum_{\substack{i=1 \\ i \neq 3\&5}}^{7} \iint_{R_i} \left\{ \tfrac{1}{2} N^{\circ}_{xxi} \left[\frac{\partial}{\partial x}(w_o + w_i) \right]^2 + N^{\circ}_{xyi} \left[\frac{\partial}{\partial x}(w_o + w_i) \right] \left[\frac{\partial}{\partial y}(w_o + w_i) \right] \right\} dxdy$$

$$+ \sum_{\substack{i=3 \\ i \neq 4}}^{5} \iint_{R_i} \left\{ \tfrac{1}{2} N^{\circ}_{xxi} \left[\frac{\partial}{\partial x}(v_o + v_i) \right]^2 + N^{\circ}_{xzi} \left[\frac{\partial}{\partial x}(v_o + v_i) \right] \left[\frac{\partial}{\partial z}(v_o + v_i) \right] \right\} dxdz \qquad (1\text{-b})$$

and

$$\pi_2 = \sum_{\substack{i=1 \\ i \neq 3\&5}}^{7} \iint_{R_i} D_i \left[\left(\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2} \right)^2 - 2(1-\nu) \left\{ \frac{\partial^2 w_i}{\partial x^2} \frac{\partial^2 w_i}{\partial y^2} - \left(\frac{\partial^2 w_i}{\partial x \partial y} \right)^2 \right\} \right] dxdy$$

$$+ \sum_{\substack{i=3 \\ i \neq 4}}^{5} \iint_{R_i} D_i \left[\left(\frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial z^2} \right)^2 - 2(1-\nu) \left\{ \frac{\partial^2 v_i}{\partial x^2} \frac{\partial^2 v_i}{\partial z^2} - \left(\frac{\partial^2 v_i}{\partial x \partial z} \right)^2 \right\} \right] dxdz \qquad (1\text{-c})$$

The deformation shapes w_i and v_i are chosen as in the finite element formulation namely as polynomials with the values of the independent co-ordinates at the six reference points as coefficients of the interpolants. Thus for a typical element shown in Figure 2-a

$$w_i = \sum_{j=1}^{6} w_{ij} N_j(\xi, \eta) \quad i = 1, 2, 4, 6, 7 \qquad (2\text{-a})$$

and $$v_i = \sum_{j=1}^{6} v_{ij} N_j(\xi, \rho) \quad i = 3 \text{ and } 5 \qquad (2\text{-b})$$

where N_j's are the interpolating polynomials given below

$$\left.\begin{aligned}
N_1 &= \tfrac{1}{4}(1-\gamma)(\xi^2-\xi)\\
N_2 &= \tfrac{1}{4}(1+\gamma)(\xi^2-\xi)\\
N_3 &= \tfrac{1}{2}(1-\gamma)(1-\xi^2)\\
N_4 &= \tfrac{1}{2}(1+\gamma)(1-\xi^2)\\
N_5 &= \tfrac{1}{4}(1-\gamma)(\xi^2+\xi)\\
N_6 &= \tfrac{1}{4}(1+\gamma)(\xi^2+\xi)
\end{aligned}\right\} \qquad (2\text{-c})$$

$\xi = \frac{2x}{\ell}$ and γ is equal to either $\frac{2y}{d_i}$ or $\frac{2z}{d_i}$ depending upon whether $i = 1, 2, 4, 6$ and 7 or $i = 3$ and 5. In evaluating π_2 in Equation (1-c) it is assumed that

$$D_i = \frac{E_{Tav} h_i^3}{24(1-\nu^2)} \qquad (3)$$

with $\nu = 0.5$. E_{Tav} is the average tangent modulus for the ith plate element. This is taken to be the average of the tangent moduli at all the nine Lobatto integration points for the element. This approximation enables an explicit evaluation of the integrals in the expression for π_2. Thus typically for the element shown in Figure 2-a

$$\begin{aligned}
(\pi_2)_i = \frac{8}{9}(E_{Tav}) \Bigg[\frac{d_i}{\ell^3} \Big\{ &(w_{i1}-2w_{i3}+w_{i5})^2 + (w_{i2}-2w_{i4}+w_{i6})^2 \\
&+ (w_{i1}-2w_{i3}+w_{i5})(w_{i2}-2w_{i4}+w_{i6}) \Big\} \\
&+ \frac{1}{2\ell d_i} \Big\{ \left(\frac{7}{24}\right) (w_{i2}-w_{i1})^2 + (w_{i6}-w_{i5})^2 \\
&+ \left(\frac{2}{3}\right) \Big\langle (w_{i4}-w_{i3})^2 - (w_{i2}-w_{i1})(w_{i4}-w_{i3}) - (w_{i4}-w_{i3})(w_{i6}-w_{i5}) \Big\rangle \\
&+ \left(\frac{1}{12}\right) \Big\langle (w_{i2}-w_{i1})(w_{i6}-w_{i5}) \Big\rangle \Big\} \Bigg] \qquad i = 1, 2, 4, 6 \;\&\; 7
\end{aligned} \qquad (4)$$

For i=3 or 5, w in Equation (4) is replaced by v. No such explicit expression is however possible for π_1 because of the occurrence of stress resultant terms N°_{xxi}, N°_{xyi} and N°_{xzi} within the integrands. π_1 is therefore evaluated numerically using Lobatto quadratures. For three points in each direction Lobatto quadratures reduces to the well-known Simpson's rule.

c. Simplification of the model: The model described previously may be simplified by reducing the number of reference points per plate element from six to four. For a typical element shown in Figure 2-b equations (2) are then replaced by

$$w_i = \sum_{j=1}^{4} w_{ij} N_j (\xi,\eta) \qquad i = 1, 2, 4, 6, 7 \tag{2-d}$$

and

$$v_i = \sum_{j=1}^{4} v_{ij} N_j (\xi,\rho) \qquad i = 3 \text{ and } 5 \tag{2-e}$$

where N_j's are the interpolating polynomials given below

$$N_1 = \tfrac{1}{4} (1-\xi)(1-\eta)$$

$$N_2 = \tfrac{1}{4} (1-\xi)(1+\eta)$$

$$N_3 = \tfrac{1}{4} (1+\xi)(1-\eta)$$

$$N_4 = \tfrac{1}{4} (1+\xi)(1+\eta)$$

d. Limitations and Deficiencies of the Model: Firstly, like in most other simulators, the frame element material model in ACTION is strictly uniaxial, i.e. interaction between shear and normal stresses is ignored. Shear stresses are thus evaluated by using the shear flow theory of thin-walled structures assuming linear elastic behavior.

Secondly, with changes in cross-section the shift of the position of the shear center should be accounted for. The transverse forces V_Y and V_Z which are assumed to pass through the shear center of the original rigid cross-section will then give rise to additional twisting moments. Because the calculation of the shear center for a thin-walled open cross-section of arbitrary shape is highly complicated in the interest of simplicity and efficiency this is avoided.

Thirdly, for the assumed deformation patterns of the flanges and webs it is not possible to maintain the right angle bends between the flanges and webs at every point along the length. The deformation patterns imply that the plate elements are effectively hinged at some points along intersections. To obtain any sort of slope continuity it will be necessary to assume higher order interpolation functions in the η direction. This defeats the original objective of this model. Hence again in the interest of simplicity such a deficiency is tolerated.

3. Results and Conclusions

a. Discussion of Results:

For validating the proposed model the tests conducted by McIvor et al on thin-walled open section beam columns and reported in reference [4] seemed appropriate. One such test involved the post-buckling response of a beam-column with a thin-walled channel cross-section as shown in Figure 3. To prevent failure by direct compression the column was tested at an inclination of 5° from the vertical with both ends of the column being clamped. A column under these conditions is extremely imperfection sensitive. Hence McIvor suggests assuming an additional 1° offset for the mathematical model to simulate the inherent imperfections of the actual column tested. The post-buckling response of this column as observed in the experiment and as reported in [4] shows extremely severe cross-sectional deformations and is presumably a good problem for validation of the new beam element proposed herein.

Before validating the proposed new beam element it was necessary to assess the importance of the magnitude of the degradation of response without allowing for such effects. This effort was initially undertaken. The results of this study are shown in Figures 3 and 4. Surprisingly enough the responses predicted by both ACTION and O-PLANE-MG [5] correlate extremely well with the experimental prediction. It would appear then that gross responses like load-deflection are not likely to be affected by localized cross-sectional deformations, irrespective of their severity, especially when the beam material models in both ACTION and O-PLANE-MG permit simulation of plastic hinges. This leads very naturally to the investigation of the original problem, namely the NASA angular frame wherein large discrepancies between the mathematical and experimental predictions were responsible for initiating the study of the effects of beam cross-sectional flexibility.

Before we get into a discussion of this study however, it may be appropriate to discuss the results of the attempts of using the new beam model in predicting the post-buckling response of the beam-column of Figure 3. These attempts were not very successful. The simplified beam model with j=4 (see Eqs. 2-d & 2-e) failed to produce any significant cross-sectional deformations and had to be abandoned in place of the more refined model with j=6 (see Eqs. 2-a & 2-b). The new beam elements were employed only between nodes ①-②, ⑥-⑦, ⑦-⑧, ⑧-⑨ and ⑪ - ⑫. This model although instrumental in predicting rather severe distortions of the cross-section troubles in converging to an accurate solution for the prescribed load steps, presumably because of an increased degree of nonlinearity, prevented completion of this study. Drastic reductions in load steps may overcome this problem but only so at a very high computational cost. Limited results of this study are tabulated in Table 1. To say the least, it is disturbing to see that initially the column has a tendency to stiffen leading to a higher axial load carrying capability. This does not seem impossible however, because it is quite likely that the plastic section modulus of the slightly deformed beam cross-section may be higher than that of the undeformed cross-section. With increasing cross-sectional deformations however, the trend is likely to be reversed and this does appear to be characteristic of the results of Table 1. The cross-sectional shapes at different locations along the length of the beam are illustrated qualitatively in Figure 5. While in some places these shapes appear to be intuitively reasonable in other places they do not. This may be the result of the simplicity of the model along with its underlying assumptions. Undoubtedly these results warrant further improvement and an investigation into the accuracy of the results generated using the new beam element.

Next, as regards NASA's angular frame the lack of correlation between mathematical simulation and experimental results, if any, at least in the linear range must be attributed to either (i) lack of inappropriate boundary conditions that is to say boundary conditions of the mathematical model different from those in the experiment; (ii) assumptions regarding rigidity of the bulkheads too conservative; (iii) insufficiently refined model or inappropriate discretization; (iv) lack of inclusion of some important response features like shear deformation.

The results of a study entailing the sensitivity of response (due to a 10 lb total load) to variations in modeling parameters are outlined in Table 2. The maximum deflection of the model of class 4 in Table 2 agrees closely with that of the experimental model. Shear deformation effects which are likely to be important for this frame were not included in this analysis, however. With the inclusion of such effects the mathematical model will undoubtedly be even more flexible than the experimental model.

It would appear that the correlation between the mathematical and experimental models can be substantially improved by using the class 4 model of Table 2 or refinements thereof and allowing for shear flexibility. The frame in question is very much like a vierendel truss wherein the shear forces give rise to secondary bending.

The results of Table 2 suggest that the discrepancy between experimental and mathematical simulations as in Figure 6 are most likely the result of inappropriate modeling of the angular frame and not the result of some highly localized cross-sectional deformations as was anticipated.

b. <u>Conclusions</u>:

It is clear from this study that computer programs like ACTION and O-PLANE-MG in their present status are quite adequate in predicting gross response parameters in spite of the fact that the structures in question may undergo severe localized cross-sectional deformations. By gross parameters we mean total energies, load-deflection responses etc. which are the quantities directly solved for in the displacement formulation. They are not derived quantities like strains, stresses etc. which are derived through spatial differentiation. If derived quantities are of importance we are talking of pointwise correlation rather than a global or gross correlation. In such an event the displacement formulations of programs like ACTION or O-PLANE-MG even with the inclusion of the effects of the cross-sectional flexibility (in the manner outlined in this report) would be inadequate. One will have to then resort to some sort of hybrid or mixed formulations for better response predictions.

Purely from the point of view of crashworthiness where the trauma measures are determined more by gross parameters it seems highly unlikely that they will be significantly influenced by highly localized cross-sectional deformations. The inclusion of such effects can only make the already expensive nonlinear analysis only more so without any significant pay-off by way of improvement in response.

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5. Acknowledgement

The author wishes to express his appreciation to the technical monitor Dr. R. J. Hayduk for consultations and assistance in the use of the simulator 0-PLANE-MG and his model of the NASA frame. This work was funded by NASA grant NGR-47-004-114 and by the NASA-Langley IPA Assignment Agreement.

Figure 1. **GEOMETRY OF IE CROSS-SECTION**

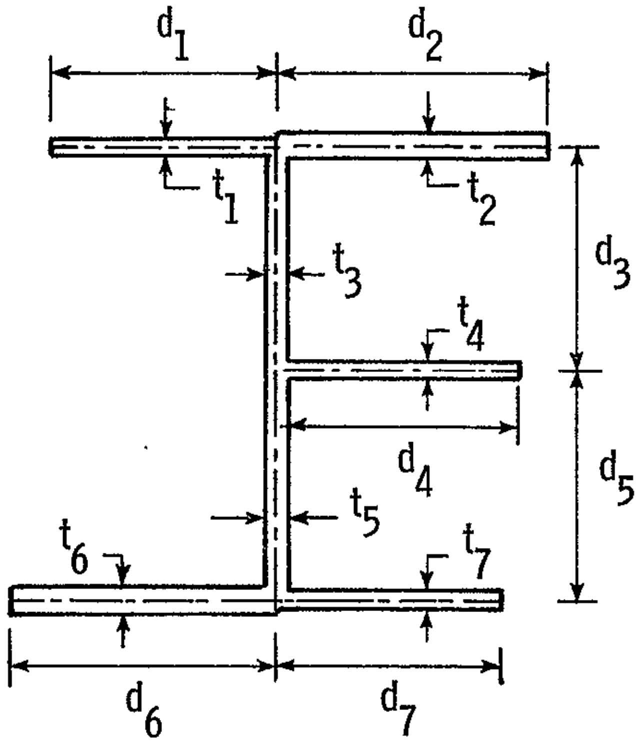


Figure 2. **A TYPICAL PLATE ELEMENT (1 D.O.F. / NODE)**

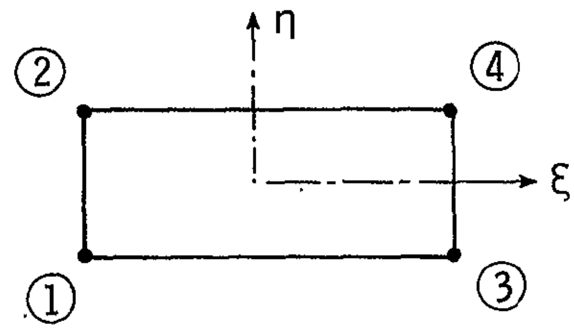


Figure 2-a.

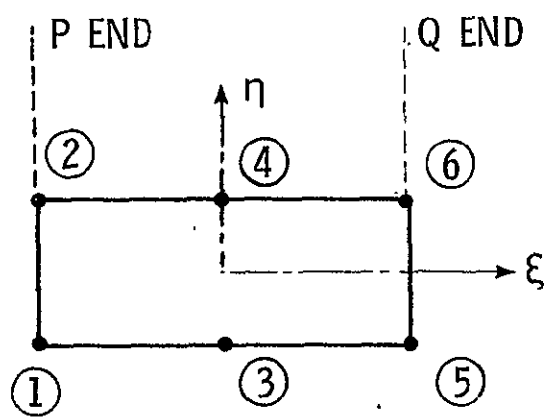


Figure 2-b.

Figure 3. Post-Buckling Response of a Thin-Walled Channel Section Beam-Column

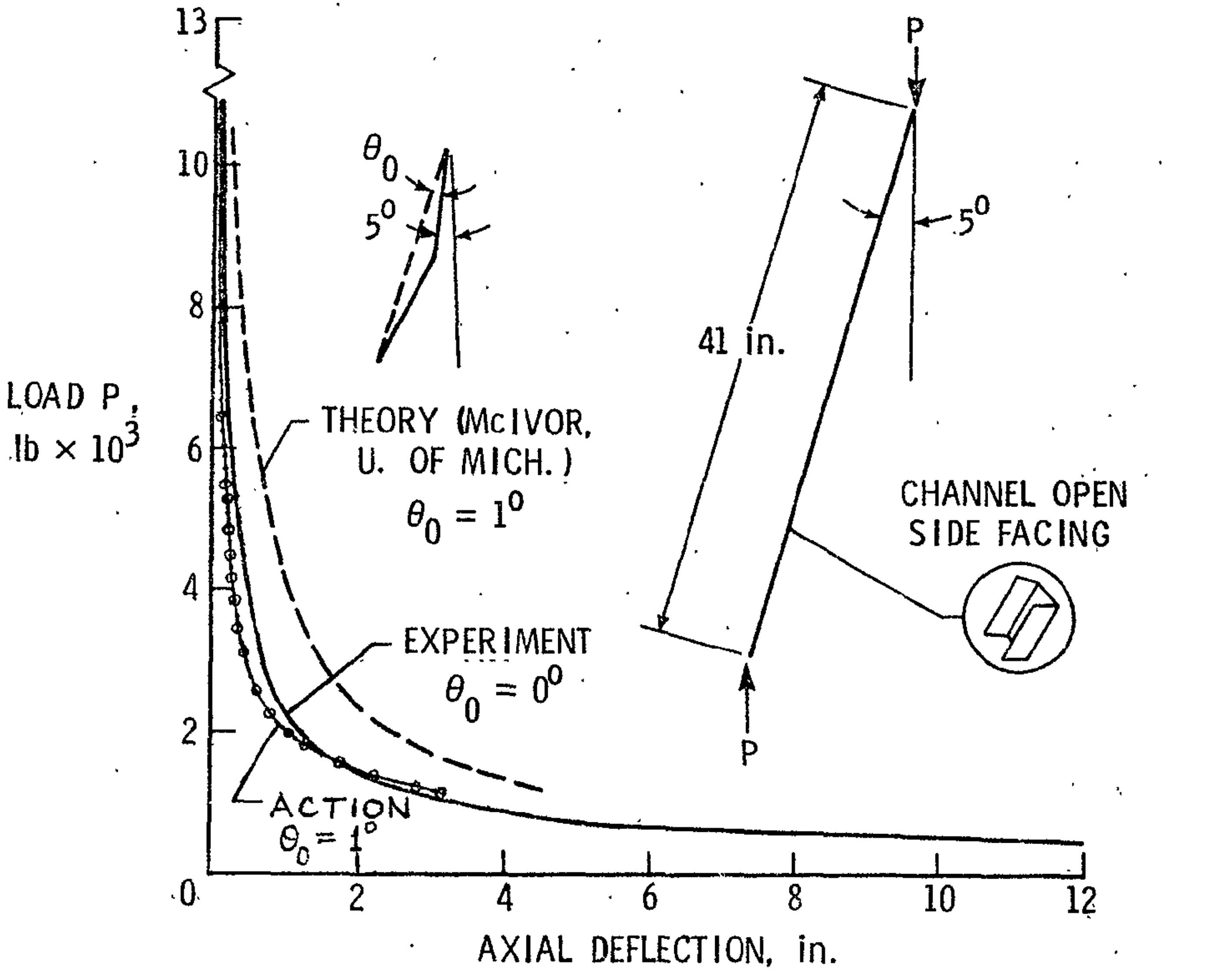


Figure 4. **RESPONSES USING ACTION & PLANS (IGNORING BEAM CROSS-SECTIONAL FLEXIBILITY)**

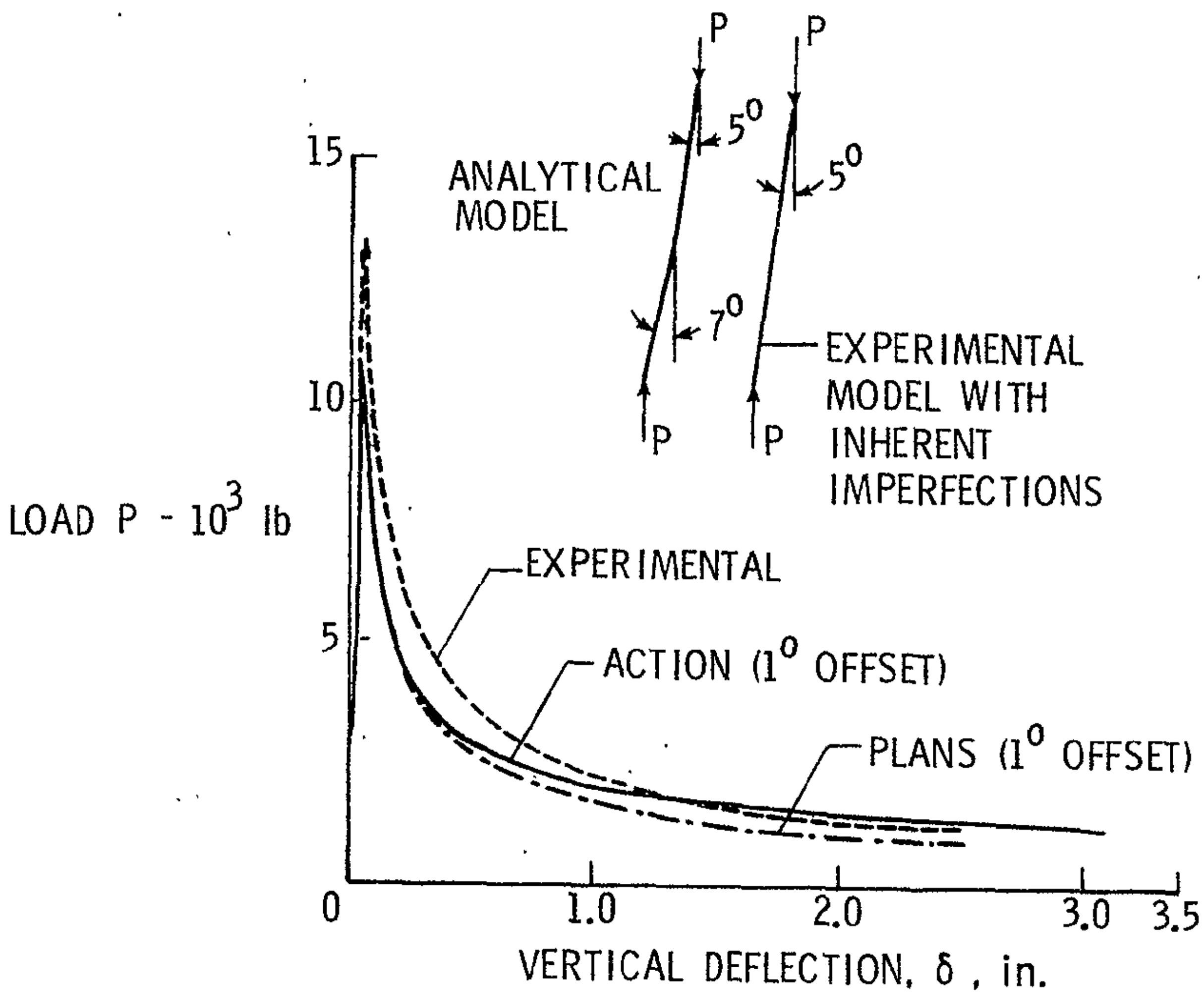


Figure 5. **BEAM CROSS-SECTIONAL FLEXIBILITY**

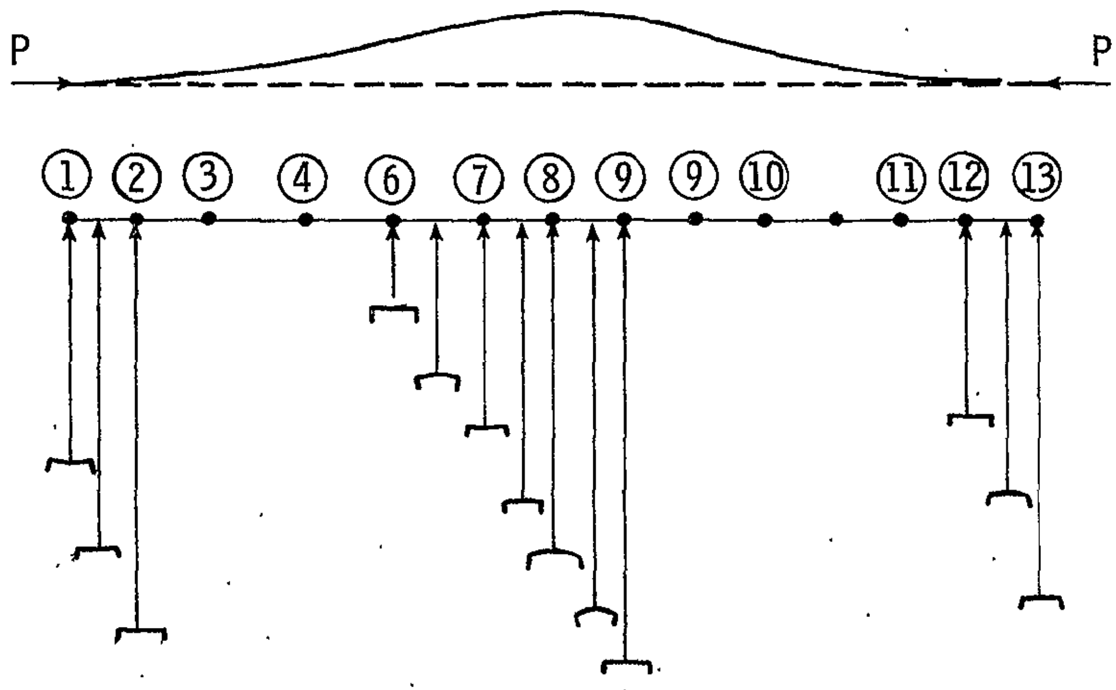
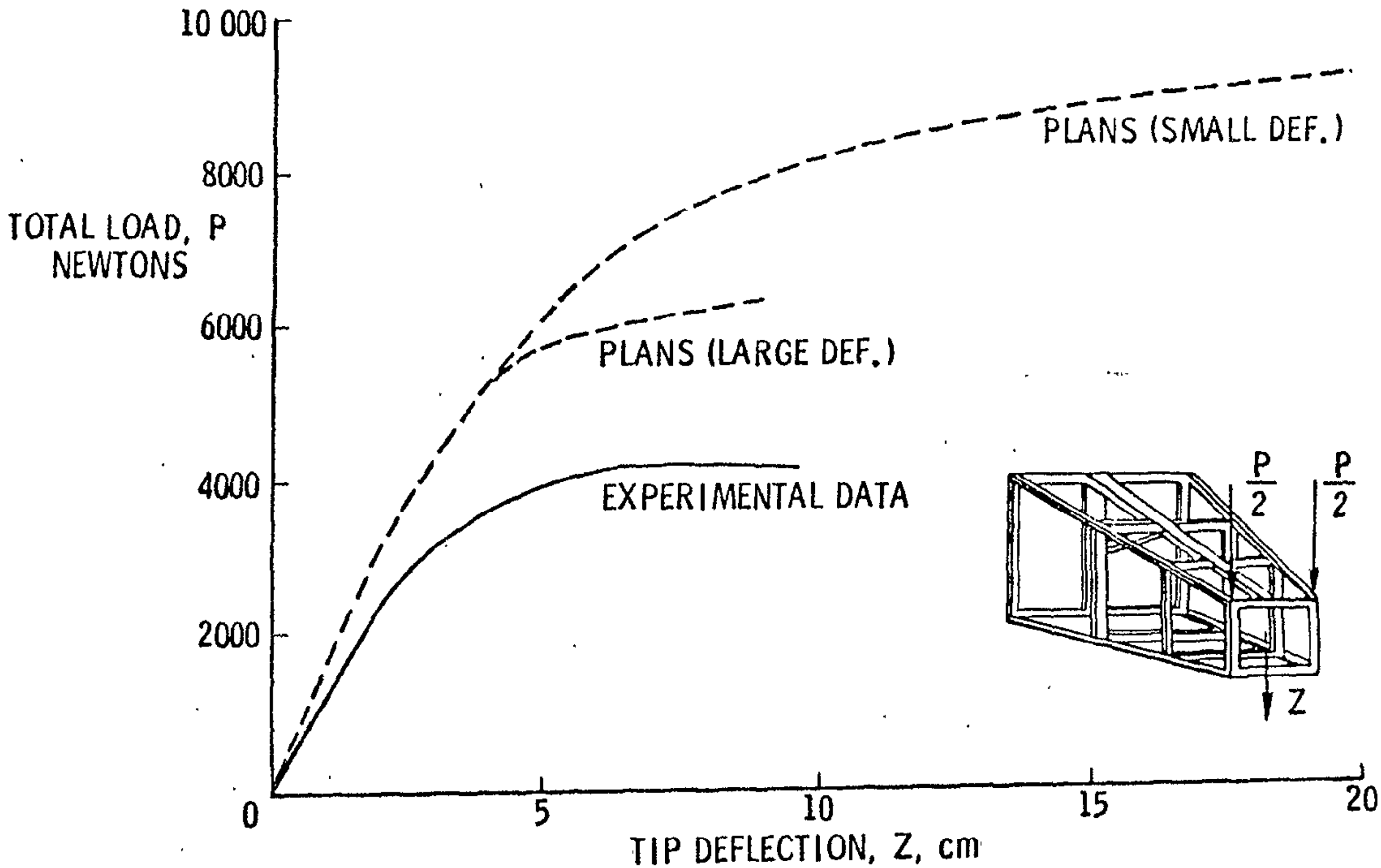


Figure 6. NASA'S ANGULAR FRAME

EXPERIMENTAL DATA AND OLD ANALYTICAL PREDICTIONS



NASA L-4192-21 HAYDUK 2/22-26/75

TABLE 1.

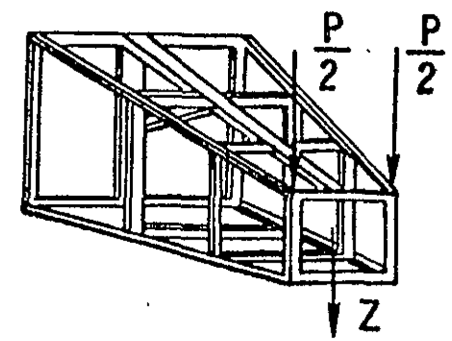
EFFECTS OF CROSS-SECTIONAL FLEXIBILITY (INITIAL STAGES):

VERTICAL DEFL. IN	LOAD P (KIPS)		
	(P_1) WITHOUT CSF	(P_2) WITH CSF	$\Delta P = P_1 - P_2$
0.03	10.228	10.217	0.011
0.08	8.6639	8.7798	-0.1159
0.130	6.0146	6.2641	-0.2495
0.150	5.5161	5.9241	-0.408
0.200	4.6866	4.7734	-0.0868
0.225	4.3972	4.6097	-0.2125

TABLE 2.

SENSITIVITY OF RESPONSE TO MODELLING OF NASA'S ANGULAR FRAME

	TYPE OF MODEL	MAX DEFL. x 10^3 IN. FOR P=10 LB.
		ACTION
1.	A. THREE D.O.F. PER NODE B. BLKHDS. NOT RIGID C. END FULLY FIXED	7.5451
2.	A. SIX D.O.F. PER NODE B. BLKHDS. NOT RIGID C. END FULLY FIXED	8.3433
3.	A. SIX D.O.F. PER NODE B. BLKHDS. RIGID C. END FULLY FIXED	5.6992
4.	A. SIX D.O.F. PER NODE B. BLKHDS. NOT RIGID C. END REPLACED BY A FLEXIBLE BLKHD. WITH PROPER SUPPORT	16.5658



APPENDIX B

On the Importance of Cross-sectional Flexibility on Gross Response*

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Blacksburg, Virginia 24061

ABSTRACT

Most nonlinear analyzers like ACTION and PLANS do not account for cross-sectional flexibility of thin-walled frame members. They do however, permit simulation of plastic hinges. Judging by the excellent correlation between theory and experiment, such analyzers appear quite adequate in predicting reasonably accurately gross response parameters of thin-walled frame structures even though they may undergo severe cross-sectional deformations.

INTRODUCTION

The crush response of structural components of an automobile or aircraft which are built-up from thin-walled open section frame elements entails severe deformations of the cross-section. Static and dynamic load tests on tubular and angular frame structures conducted by the Dynamic Loads Branch of the NASA Langley Research Center revealed that significant cross-sectional warping and distortions occur near the joints[1]. In fact, it was conjectured that the

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* This work was supported by NASA Langley under grant NGR 47-004-114 under the congnizance of the technical monitor, Robert J. Hayduk.

large discrepancies between experimental and theoretical predictions may, at least in part, be attributable to such phenomena which are normally not accounted for in a conventional nonlinear finite element analysis.

Recently, attempts have been made to model the phenomenon of cross-sectional deformations analytically [2],[3]. The purpose of this paper however, is not to evaluate these attempts but rather to assess the degradation of the fidelity of mathematical simulation of the response using nonlinear analyzers like ACTION [4] and PLANS [5] which currently do not permit such modeling.

RESULTS AND CONCLUSIONS

Anderson, McIvor and Kimball have conducted several tests on thin-walled open section columns and these have been reported in reference [6]. These tests seem appropriate for the present investigation. One such test involved the post-buckling elastic-plastic response of a beam-column with a thin-walled channel cross-section as shown in Figure 1. To prevent failure by direct compression, the column was tested at an inclination of 5° from the vertical with both ends of the column being clamped. A column under these conditions is extremely imperfection sensitive. Hence, Anderson et al [6] suggest assuming an additional 1° offset for the mathematical model to simulate the inherent imperfections of the actual column tested. The post-buckling response of this column as observed in the experiment and as reported in reference [6] involves extremely gross cross-sectional deformations and is presumably a good problem for the purposes of this paper.

The column of Figure 1 exhibits a bifurcation phenomenon with a highly unstable branch. For reasons very well known, most nonlinear finite element analyzers like ACTION and PLANS have to resort to a displacement rather than a load incrementation for predicting response along an unstable branch. The mathematical predictions of ACTION and PLANS have been illustrated in Figure 2 along with the experimental and that predicted by UMVCS-1 [7].

Surprisingly enough, the responses predicted by ACTION and PLANS correlate extremely well with the experimental prediction. This excellent correlation between theory and experiment may be the result of several compensating assumptions, especially when one considers the fact that both ACTION and PLANS permit only large rotations with only small strains. The slight differences between the responses of ACTION and PLANS may be due to one or all of the following causes: (i) ACTION uses an energy minimization approach whereas PLANS uses a one-step Newton-Raphson psuedo force technique; (ii) the deformation models of ACTION and PLANS differ in the inelastic range; (iii) the number of quadrature points for integration of energy densities and stresses are not identical. It must be remarked, however, that with an elastic-perfectly plastic stress strain curve, both ACTION and PLANS do indeed possess the capability of simulating a plastic hinge. UMVCS-1 is a program based on the assumption that plastic deformations occur at idealized hinges located at nodal points. A generalized hinge theory which accounts for biaxial bending, torsion and axial loading is employed. The user has to test the different types of joints and cross-sections experimentally to gather data required for the constitutive laws employed in the analytical formulation. In this sense, it is more of a hybrid program.

It is interesting to note that while the UMVCS-1 model is consistently stiffer than the actual that of PLANS is consistently more flexible than the actual and that of ACTION is more like an "average" model. If one were to account for cross-sectional flexibility in ACTION or PLANS, it would appear that this would lead to a deterioration of the excellent correlation.

It appears that the excellent agreement between theory and experiment in Figure 2, in spite of the absence of cross-sectional flexibility, is attributable to the capability of both ACTION and PLANS to permit simulation of plastic hinges. Gross responses like strain energy, load versus deflection are not likely to be affected by highly localized cross-sectional deformations, irrespective of their severity. It is the view of this author that nonlinear analyzers like ACTION and PLANS in their present status seem quite adequate in predicting reasonably accurately gross parameters like total strain energies, load-deflection responses, etc. of structures that undergo severe localized cross-sectional deformations. Gross parameters are parameters which are directly solved for in the displacement formulations of ACTION and PLANS. They are not derived quantities like strains, stresses, etc. which are derived through spatial differentiation. If derived quantities are of importance, one is talking of pointwise correlation rather than a global or gross correlation. In such an event, the displacement formulations of programs like ACTION or PLANS, even with the inclusion of the effects of cross-sectional felxibility, would be inadequate. One will have to then resort to some sort of a hybrid or a mixed formulation with the inclusion of large strain effects for better response prediction.

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ACKNOWLEDGEMENTS

The author very gratefully acknowledges Professor W. J. Anderson and the late Professor Ivor McIvor of the college of Engineering, the University of Michigan for providing the experimental data.

Figure 1.

Figure 1.

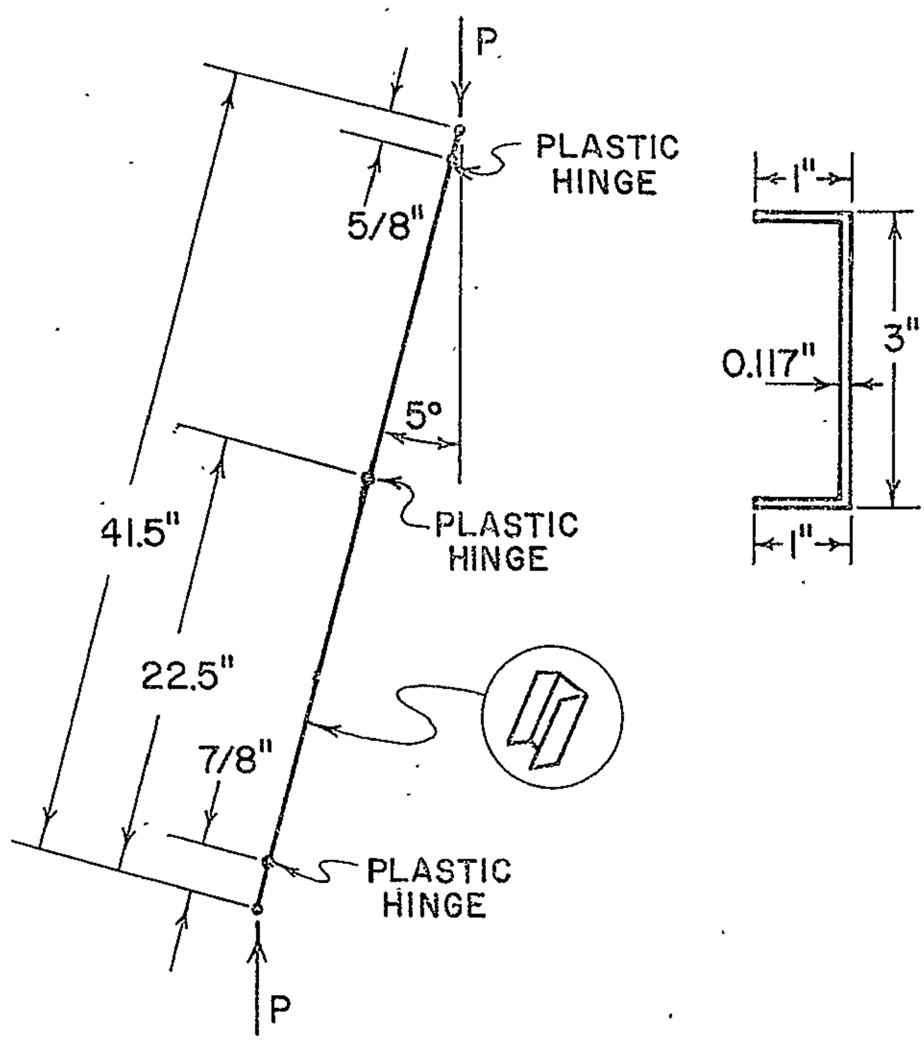

Figure 1.

Figure 1.

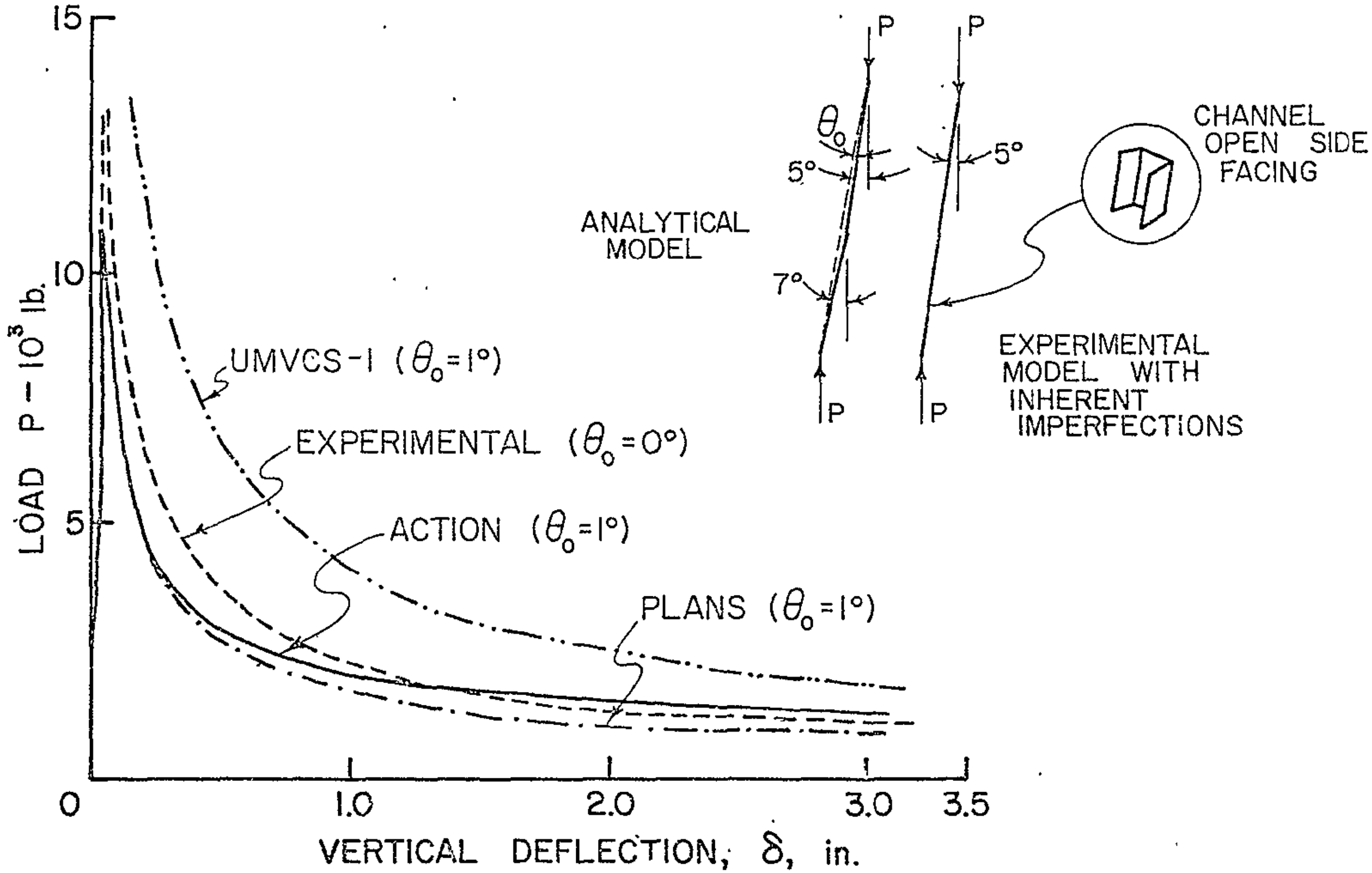

Figure 2

Figure 2.

FIGURE LEGEND

Title

APPENDIX C

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NONLINEAR TRANSIENT ANALYSIS OF
AIRCRAFT-LIKE STRUCTURES-
THEORY AND VALIDATION

by

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I. Introduction

Nonlinear transient analysis of structures has been of increasing interest to engineers by virtue of their interest in minimizing human and property damage resulting from the catastrophic failure of such structures under crash or seismic conditions. Complexities of the structural configuration and its equally complex transient response in the presence of material inelasticity make finite element modeling of such structures a very natural and plausible recourse. Portions of the structures may remain elastic and undergo infinitesimally small deformations while other portions may experience finite deformations and motions and respond inelastically under time-varying loads that may lead to a complete failure of the structure. If finite strains are to be permitted in the model, distinction must be made between undeformed and deformed configurations and the concepts of pseudo stresses and conjugate strain measures which have intricate physical interpretations must be introduced [1]. Furthermore, strictly speaking most elastic-plastic theories which hypothesize an additive decomposition of the total strain into an elastic and a plastic component lose their validity in the large strain domain [2]. Because of this, most developers of nonlinear analysis codes restrict themselves to a small strain formulation but permit finite displacements and rotations thereby allowing buckling and collapse of the structure to occur. There are some indications that this may be adequate for most practical purposes.

With this hypothesis as its basis, the present discussion focuses on the simulation of response of a structure modeled as an assemblage of membrane, frame (3-D beam), stringer elements and rigid links (see Figure 1). The mathematical model is a finite element displacement

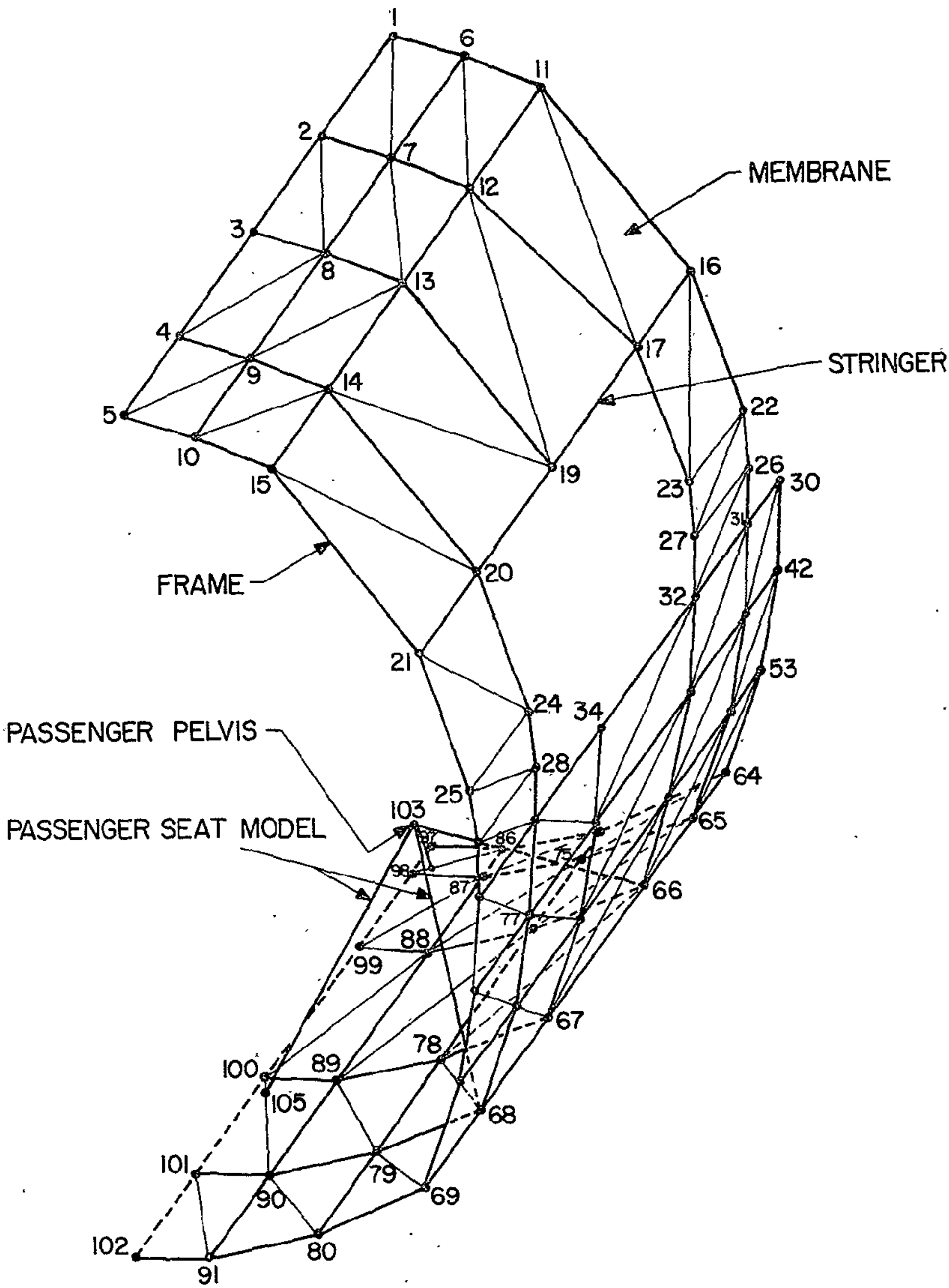

Figure 1. Finite Element Model of an Aircraft Fuselage Substructure

model which consists of discretizing the actual structure by an assemblage of finite elements and approximating the response of each element by a finite number of deformation states expressed as linear functions of the generalized nodal displacements.

Two distinct solution approaches exist: (i) the vector approach and (ii) the scalar approach. In the former, the mathematical model is derived on the basis of the principle of virtual work and reduces to a system of nonlinear second-order differential equations in time. In the latter approach, a scalar or potential function associated with the energy of the model is introduced, minimization of which yields the desired equilibrium configuration. In both approaches a temporal finite difference scheme is utilized to effectively eliminate time as a variable. As a result, in the vector approach the equations of motion are reduced to a system of nonlinear algebraic equations in the unknown nodal parameters of the finite element model[3]-[6]. In the scalar approach, which is of relevance to this report, the problem is reduced to a well known problem in mathematical programming namely the unconstrained minimization of a nonlinear function of several variables.

The scalar approach has been used for nonlinear analysis by previous investigators [7]-[9]. However, with the possible exception of reference [9] most of the previous work using the energy minimization technique has been restricted to static analysis of structures. The algorithm of reference [9] had difficulties in converging to correct solutions because of inherent element formulation deficiencies and use of extremely inefficient and expensive finite difference operations for gradients besides being restricted to stringer and frame element models only. As a result, no meaningful results using the energy minimization approach were obtained. The present formulation overcomes such limitations using analytically derived gradients, consistent element formulations and the best current variable metric update formula [10] for use in unconstrained minimization [11].

II. Minimization Technique for Nonlinear Analysis

a. Formulation Basis

In this case the problem of response prediction is posed as the minimization of a potential function of the unknown nodal parameters of the finite element model. For all structural problems with geometric and material nonlinearities of the type considered herein such a potential function always exists. Although this technique has been hitherto used for mainly positive or negative definite systems, other systems which fail to be positive or negative definite can be handled by using the least squares method or the modified conjugate gradient method with preconditioning [12]. In some cases for such systems displacement incrementation rather than load incrementation in conjunction with conventional unconstrained minimization techniques can also be equally effective [13].

The minimization scheme as applied to the solution of transient nonlinear structural analysis problems consists of minimizing a potential function associated with the system for an assumed relationship between displacements and time. The displacement-time relation for each generalized nodal displacement of a finite element model may be assumed of the form [14]

$$q_{ei} = \beta(\Delta t)^2\ddot{q}_{ei} + (\frac{1}{2} - \beta)(\Delta t)^2\ddot{q}_{0i} + (\Delta t)\dot{q}_{0i} + q_{0i} \quad \text{(1-a)}$$

$$\dot{q}_{ei} = \gamma(\Delta t)\ddot{q}_{ei} + (1 - \gamma)(\Delta t)\ddot{q}_{0i} + \dot{q}_{0i}. \quad \text{(1-b)}$$

where q_{ei} is the i-th generalized nodal displacement at the end of the time step and β and γ are constants. The quantities $\dot{q}_{ei}$ and $\ddot{q}_{ei}$ can now be expressed in terms of the i-th generalized nodal displacement, q_{0i}, velocity, $\dot{q}_{0i}$ and acceleration, $\ddot{q}_{0i}$ at the beginning of the time step and the generalized nodal displacement, q_{ei}, at the end of the time step. It can be easily verified that the equation of equilibrium for an N degree-of-freedom system with lumped masses

$$M_i \ddot{q}_{ei} - F_i + \frac{\partial U}{\partial q_{ei}} = 0 \quad ; \quad i = 1,2...N \tag{2}$$

correspond to the necessary conditions for the functional

$$S = \sum_{i=1}^{N} \{[\frac{1}{2\beta(\Delta t)^2} q_{ei}^2 - (\frac{1}{\beta(\Delta t)^2} q_{0i} + \frac{1}{\beta(\Delta t)} \dot{q}_{0i} + (\frac{1}{2\beta} - 1)\ddot{q}_{0i})q_{ei}]M_i$$

$$- F_i(t+\Delta t)q_{ei}\} + U + C \tag{3}$$

to be stationary. In Equation (3), U is the strain energy and C is an arbitrary constant. Thus, knowing q_{0i}, $\dot{q}_{0i}$ and $\ddot{q}_{0i}$ at time t for any given load F_i at time $(t+\Delta t)$, the functional S may be minimized with respect to the generalized nodal displacements, q_{ei} (i=1,...N), in order to determine the corresponding stable equilibrium configuration. Thus, this scheme satisfies equilibrium at the end of the time step, thereby providing an implicit temporal integration scheme. The size of the time step is automatically controlled so that the error at half time based on interpolated configuration data is less than a prescribed change in total energy. In general, the strain energy U will be a nonlinear function (at the very least a quadratic) of the generalized nodal displacements q_{ei}. Details on the explicit evaluation of U as a function of q_{ei} will be touched upon later.

Of all the available techniques for unconstrained minimization only the quasi-Newton or the variable metric methods have been more frequently used for structural analysis, because of their higher effectiveness [15]. Again, unless one accounts for the sparsity of the Hessian matrix and an update scheme which maintains it, one has to almost invariably resort to some form of a conjugate gradient technique for problems wherein N is an extremely large number. The extension of the minimization techniques to extremely large scale nonlinear structural analysis problems is a subject of separate research [16] in itself and is beyond the scope of this paper.

Most algorithms for unconstrained minimization seek a direction of travel and the amount of travel in that direction. The manner in which these are sought depends upon the sophistication of the particular algorithm invoked. Most often the directions of travel are sought in a manner which guarantees not only a decrease in the value of the function to be minimized at each iteration but also a convergence to the minimum in a finite number of iterations (usually N+1 for an N dimensional space) in the case of quadratic functionals. It is important to note that all functionals are very nearly quadratic in the neighborhood of the minimum. The present formulation uses the well-known Broyden-Fletcher-Goldfarb-Shanno (BFGS) variable metric algorithm which dispenses with the exact line searches while using an Hessian update formula which, in the case of a quadratic functional, guarantees a monotonic convergence of the eigenvalues of the approximating matrix to the inverse Hessian. The iterative scheme is begun with an initial guess which is usually the null vector in the absence of other better estimates. For the variable metric or the conjugate gradient methods the required gradient of S is evaluated either analytically or by a finite difference operation on S. The use of an analytic gradient results in a substantial saving in computational effort. This saving is the result of not only a cheaper gradient evaluation but most often a faster convergence of the solution because of higher accuracy of all computed quantities [15]. The i-th component of the gradient of S can be written as

$$\frac{\partial S}{\partial q_{ei}} = M_i \ddot{q}_{ei} - F_i + \frac{\partial U}{\partial q_{ei}} \quad (4)$$

The term in Eq. (4) requiring significant computational effort is $\frac{\partial U}{\partial q_{ei}}$ as it embraces the geometric and material nonlinearities. Using half-station central differences this is given by

$$\frac{\partial U}{\partial q_{ei}} = \frac{\sum_{k=1}^{m} U_k(q_{e1}, q_{e2}, \ldots q_{ei} + \frac{1}{2}\Delta q_{ei}, q_{ei+1} \cdots q_{eN})}{\Delta q_{ei}} - \sum_{k=1}^{m} \frac{U_k(q_{e1}, q_{e2} \ldots q_{ei} - \frac{1}{2}\Delta q_{ei}, q_{ei+1} \cdots q_{eN})}{\Delta q_{ei}} \tag{5}$$

where Δq_{ei} is a small change in the i-th component and m is the number of members or elements which has the i-th degree of freedom in common. In evaluating the gradient vector analytically [17], each of its component involves the evaluation of only a single function similar to the function for member energy evaluation. Thus,

$$\frac{\partial U}{\partial q_{ei}} = \sum_{k=1}^{m} \int_{v_k} \frac{\partial W}{\partial q_{ei}} dv_k = \sum_{k=1}^{m} \int_{v_k} \left(\frac{dW}{d\bar{\varepsilon}}\right)_k \left(\frac{\partial \bar{\varepsilon}}{\partial q_{ei}}\right)_k dv_k \tag{6-a}$$

where

W = strain energy density

$$\bar{\varepsilon} = \text{effective strain} \begin{cases} = \frac{2}{\sqrt{3}} (\varepsilon_{xx}^2 + \varepsilon_{yy}^2 - \varepsilon_{xx}\varepsilon_{yy} + \frac{3}{4}\gamma_{xy}^2)^{1/2} \text{ for a two dimensional stress state} \\ = \varepsilon_{xx} \text{ for a uniaxial stress state} \end{cases} \tag{6-b}$$

v_k = volume of the k-th element

and for one step incremental loading or unloading

$$\left(\frac{dW}{d\bar{\varepsilon}}\right) = \bar{\sigma} = \text{effective stress} \begin{cases} = (\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\Gamma_{xy}^2)^{1/2} \\ \text{for a two dimensional stress state} \\ = \sigma_{xx} \text{ for a uniaxial stress state} \end{cases} \tag{6-c}$$

Equations (6-a) through (6-c) imply the use of Hencky's total strain theory along with its assumption that in the strain hardening range the inelastic component of the total strain is predominant [18]. This in a way is consistent with the assumption that the total strain can be decomposed into an elastic and a plastic part especially in cases where the strains are large [2]. According to reference [2] it is only when plastic strains are predominant that such a decomposition is justified.

The problem at hand could have equally well been formulated using the incremental flow theories of plasticity in the strain hardening range. The potential function instead of being a function of the total quantities need then be expressed in terms of incremental quanties and the minimization technique can still be used [19]. As a matter of fact, it may be conjectured that the performance of the solution algorithm will perhaps be significantly improved using such a formulation even though the material model may then be slightly more complex. In any event, it is immediately obvious that a significant reduction in computational time will be realized if analytic gradients are used in preference to central difference gradients.

The complexity of the strain energy evaluation for any element is determined by its deformation model. This is discussed next.

b. Deformation model:

The deformation model of the entire structure is synthesized from deformation states of each element of the structure. These states are expressed in terms of generalized displacements of the nodes of the structure at which the elements interface. The displacement field within each element is chosen as a continuously differentiable function of the local spatial coordinates and the generalized nodal displacements. The field maintains interelement continuity of its essential derivatives thereby providing a Galerkin model of the system. The local generalized nodal displacements of each element are then related to the global displacements of the assemblage. These relations, which can be interpreted as transformations of the local coordinate system to the global coordinate system, may be linear or nonlinear depending upon whether the motions and deformations of the elements are infinitesimal or finite. For large rigid body rotations, these transformations are accomplished using Euler angles which are linearly independent by virtue of the fact that the rotations are performed in a prescribed order.

There are three kinematic descriptions most commonly used for characterizing large displacements of finite element models of structures. These are: (i) the total Lagrangian formulation wherein the initial undeformed configuration is the reference configuration, (ii) the updated Lagrangian formulation which uses a total Lagrangian formulation within each load or time step but updates the reference configuration at the end of each step and (iii) the co-rotational or rigid convected coordinate formulation which utilizes a coordinate system rigidly attached to an element and moving with the element. For development of a large rotation formulation vital for crashworthiness studies the use of the total Lagrangian formulation is unsuitable since most structural theories permit only moderately small rotations [20]. The co-rotational formulation decomposes the total displacements into a rigid body motion component and a strain producing component. Thus, with the restriction of small relative rotations within the element, this formulation leads to a simplification of the strain-displacement relationship on the element level while still permitting arbitrarily large rotations of the element. The present deformation model uses the co-rotational or rigid-convected formulation for its kinematic description.

Through appropriate kinematic constraints modeling of massless degrees of freedom or of deformation-free rigid links or even the simulation of contact with an impenetrable, rough plane are easily achieved.

Rigid links can be used to simulate either joint eccentrices or rigid parts of a structure. In the interest of a truly unconstrained minimization Lagrange multipliers or penalty functions are avoided. Rather kinematic constraints are formulated as prescribed displacements under reactive forces provided by the gradients of the strain energy with respect to the corresponding degrees of freedom.

c. Material model:

Although closed form analytic expressions for U can be developed when the material is elastic the same is not true when elements yield. Then the response depends upon the current values of stress components and the past history. Von Mises' yield criterion together with Henckey's total strain theory provides a simple means of calculating strain energy density distributions throughout an element that has yielded. Because total stresses and total strains are no longer linearly related recourse must be made to numerical integration (Gaussian or Lobatto) of the strain energy density over the volume of the element.

It is clear then that when an element yields, the complexity of the strain energy evaluation increases several times in relation to its purely elastic behavior. A number of quadrature points have to be assigned over the volume of the element and using the material model stresses and strain energy densities have to be evaluated at each of these points for known values of strains (Figure 2). The average strain energy density which is simply the weighted sum of these strain energy densities then enables the calculation of the total strain energy. It must be noted, however, that the stress-strain history at each of these quadrature points, which corresponds to a unique location on an idealized effective stress-effective strain curve for the material of the element (Figure 3), must be made available at all times. This places highly increased demands on computer storage as inelastic deformations progress with time.

Thus, a frame element which was strictly a uniaxial member in the elastic range, typified by its cross-sectional area and moments of inertia, requires a full three dimensional characterization in the inelastic range. In other words, frame elements for inelastic analysis require a classification based on the different cross-sections. This development is restricted to frame elements with thin-walled sections of the closed and open (Box, Tube, Elip and E) variety - a characteristic of general aviation aircraft frames. However, in the elastic range the development does permit frame elements with arbitrary cross-sections characterized by their gross section properties. In the interest of simplicity, classical shear flow theory for thin-walled sections is used and certain simplifying assumptions regarding torsion, warping and shear deformations in the inelastic range are introduced. This is characteristi of most nonlinear analyzers mainly because the development of a truly three-dimensional frame element for nonlinear inelastic response is a formidable task perhaps even more challenging than that of the development of a plate bending or a shell element for the same purpose. In fact, in the inelastic range it may be easier to model a thin-walled beam of arbitrary cross-section by an assemblage of a large number of plate and shell elements thereby permitting a faithful representation of very complex effects like restrained warping, torsion, cross-sectional distortions, etc.

The material is assumed to unload elastically. For modeling plasticity under cyclic loading kinematic hardening with an idealized

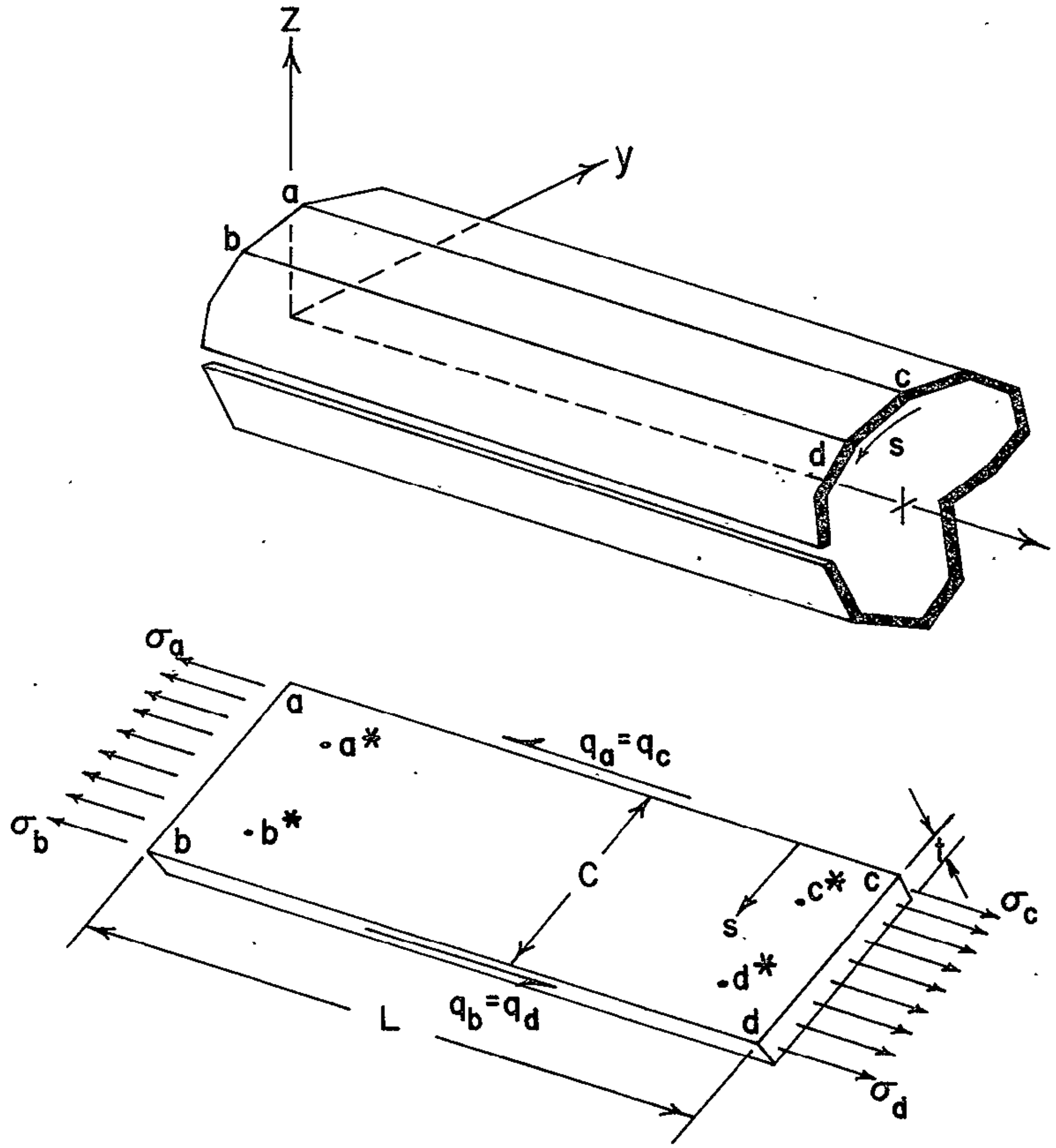

Figure 2. Model for Strain Energy Integration

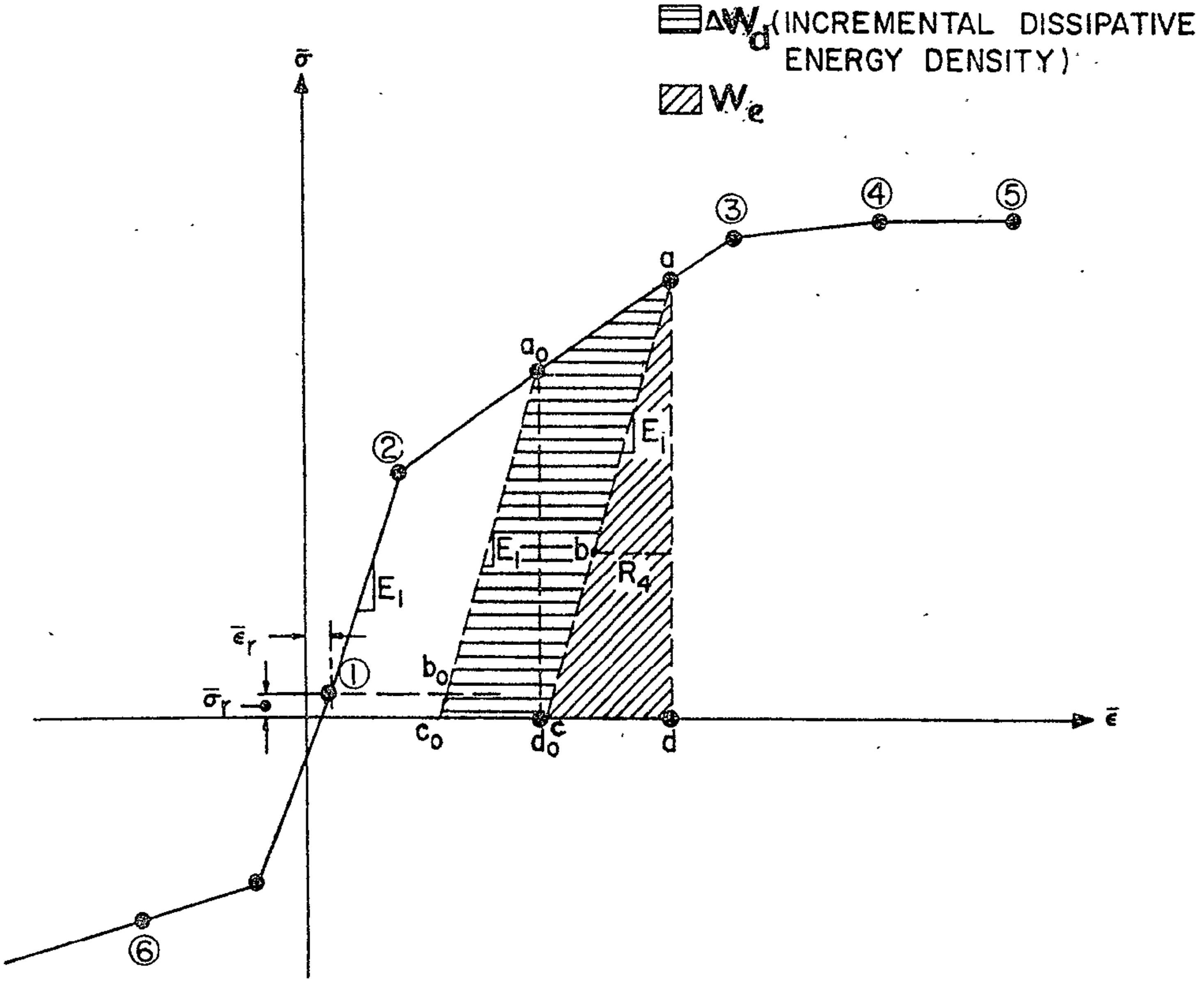

Figure 3. Effective Stress versus Effective Strain

Bauschinger effect is assumed. Specialized elements like the gap elements and stays can be easily modeled simply through an appropriate modification of the material model of the conventional elements.

The details of the total strain energy calculations for the different element types considered and the transformations relating element behavior to global variables is relegated to the appendix.

III. Results and Discussion

The effectiveness of the minimization technique in solving nonlinear problems is very much a function of not only the size of the load or time step but also the extent and type of the nonlinearity-geometric or material and even the type of the temporal discretization scheme used which is to say the assumed values of β and γ in Eq. (I). With this in mind, the process of selection of problems for validation was geared towards providing an evaluation of the techniques under different types of nonlinearities. Problems belonging to three distinct classes namely: (i) quasi-static, elastic with geometric nonlinearities, (ii) quasi-static, elastic-plastic with geometric nonlinearities and (iii) transient, elastic-plastic with geometric nonlinearities were selected. Independent solutions or experimental results for these problems were available for comparison purposes.

Figure 4 shows the case of a rod-spring problem wherein the stiffness of the spring is just enough to prevent a snap-through and provide a single-valued load deflection response. Most researchers regard this problem as geometrically highly nonlinear. Using stringer elements with load steps as high as 1 lb., the energy minimization solution is indistinguishable from the easily obtainable exact solution to this problem. Higher load steps could have been chosen but caution must be exercised with extremely large load steps since the performance (the number of minimizations required for convergence) of the minimization algorithm may be adversely affected. In other words, the computational effort within a load step may increase substantially enough to offset the savings accured from fewer load steps.

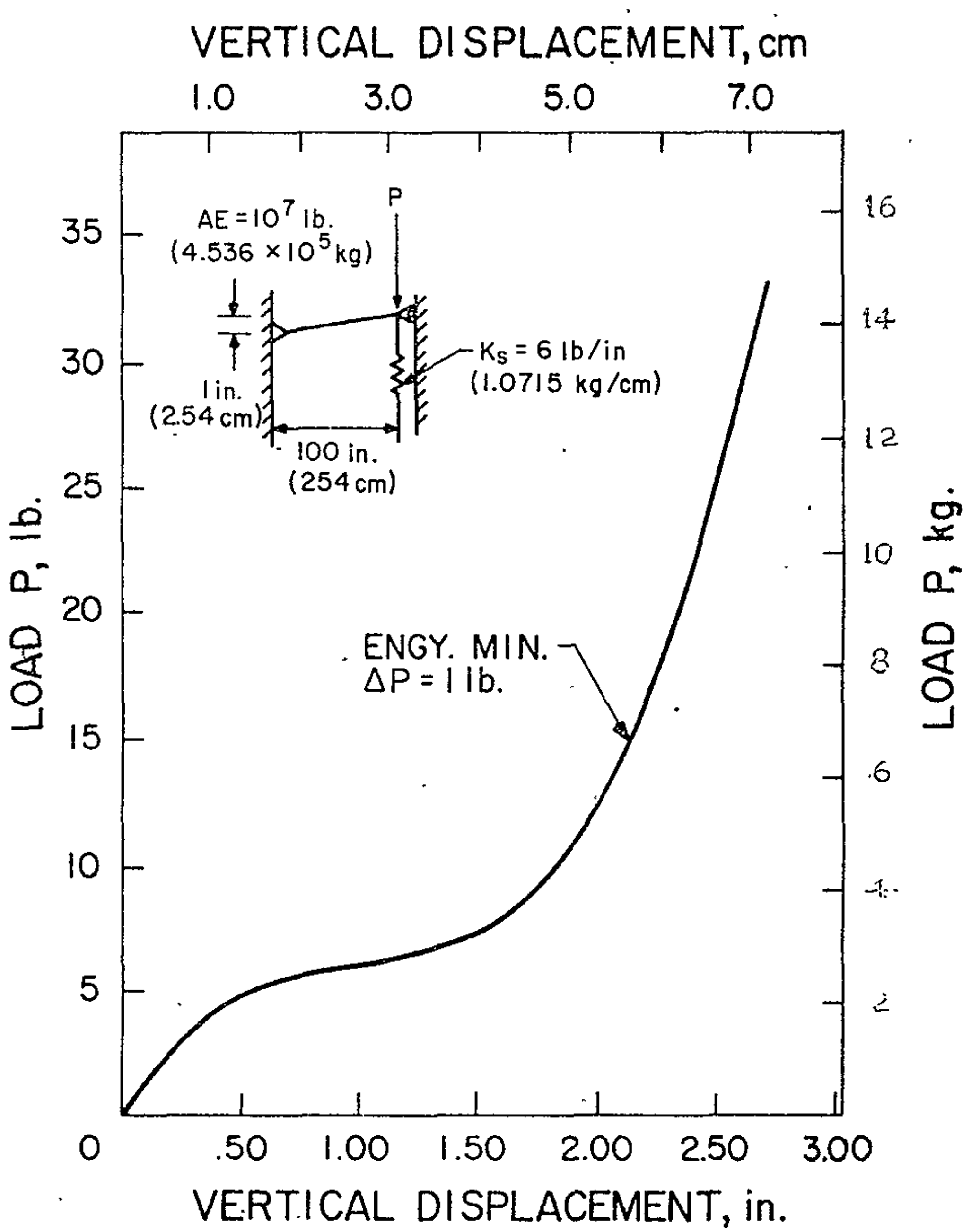

Figure 4. The Rod-Spring Problem

Figure 5 provides yet another example of such a nonlinearity except that in this case the load response curve is no longer single valued but is a composite of stable and unstable branches. Using straight-forward load incrementation, it is possible to locate only the stable equilibrium configurations as indicated in Figure 5. Using displacement incrementation, however, the entire load response curve can be easily obtained. The response predicted by energy minimization agrees very closely with that predicted by the nonlinear analyzer developed by Stricklin and Haisler [21] for an identical model of the shallow arch using frame elements.

While both of the previous problems involved only geometric nonlinearities, Figure 6 presents the case wherein both material and geometric nonlinearities interact. The experimental prediction of the post-buckling, elastic-plastic response of this beam-column with a thin-walled channel cross-section was the result of a test carried out by Anderson et al [22]. To prevent failure by direct compression, the column was tested at an inclination of 5° from the vertical with both ends of the column being clamped. A column under these conditions is highly imperfection sensitive and hence Anderson et al assume an additional 1° offset, as shown in the figure, for the mathematical model hoping to simulate the inherent imperfections of the actual column tested. Because the response involves a highly unstable branch, displacement incrementation had to be used in place of load incrementation. The response predicted by energy minimization agrees extremely well with the experimental prediction and even more so by comparison with that predicted using UMVCS-1[23].

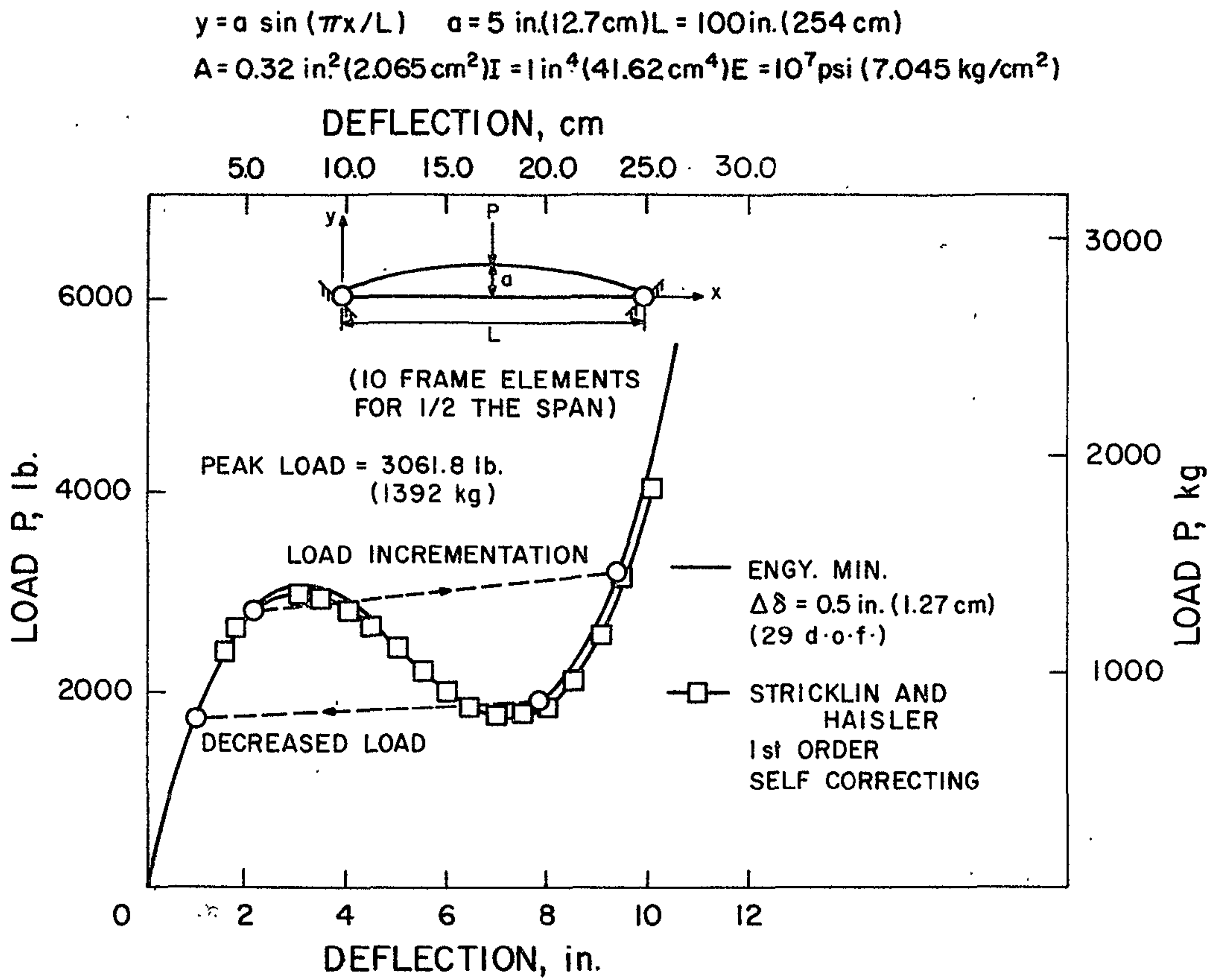

Figure 5. Snap-Through of a Shallow Arch

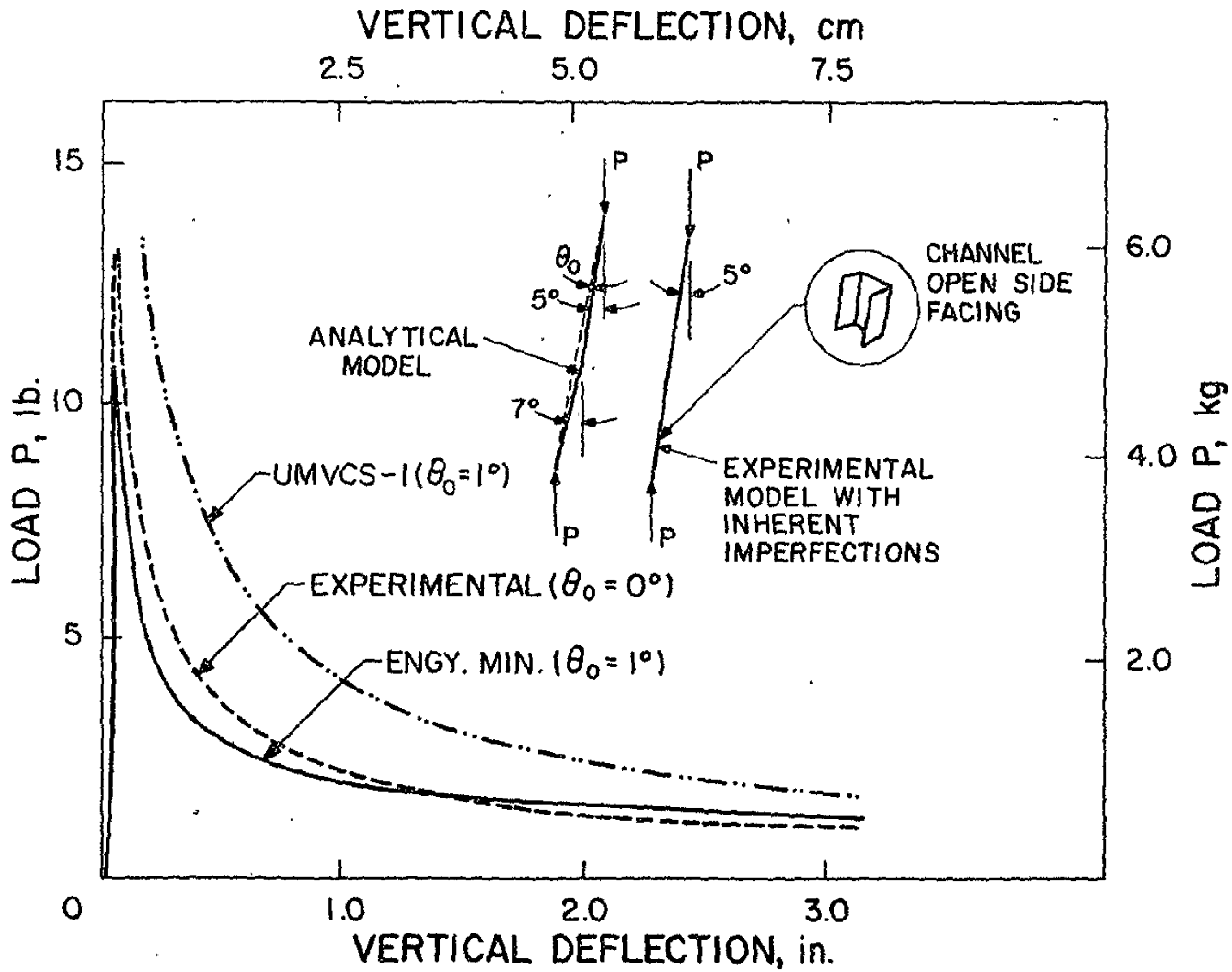

Figure 6. Post-Buckling Elastic-Plastic Response of a Thin-Walled Column

Figure 7 illustrates the case of the transient response in the presence of both geometric and material nonlinearities. The impulse is large enough to cause the entire beam to respond inelastically while experiencing moderately large relative rotations. The experimental response for this beam was obtained by Krieg et al [24]. It is immediately obvious that the quality of the response prediction is very much a function of the values of β and γ. This is not to say that optimum values of β and γ exist which guarantee optimum fidelity of the response prediction. As a matter of fact, optimum values of β and γ appear to be very much problem dependent. Use of $\beta = 0.276$ and $\gamma = 0.55$ has been recommended by Goudreau and Taylor [25] for smoothing out the high frequency oscillations. Again, the response may be significantly affected by the use of a consistent mass matrix and with rotatory inertia and shear deformation effects included.

With the possible exception of the problem of Figure 8, all the previous problems involved only a relatively few degrees of freedom. Furthermore, the state of stress in all of these problems was essentially uniaxial for all practical purposes. Using constant strain membrane elements the maximum strain in the vicinity of a notch in the direction of loading is determined and compared with the experimental results. The agreement between the two predictions is good but could perhaps be improved upon by the use of nonlinear strain displacement relationships in the co-rotational coordinate system.

Next, by way of a reasonably large scale problem consider the drop test of a twin engine, low wing Navajo substructure conducted by NASA Langley's Impact Dynamics Research Facility under the auspices of the joint NASA-FAA general aviation crash test program. The substructure

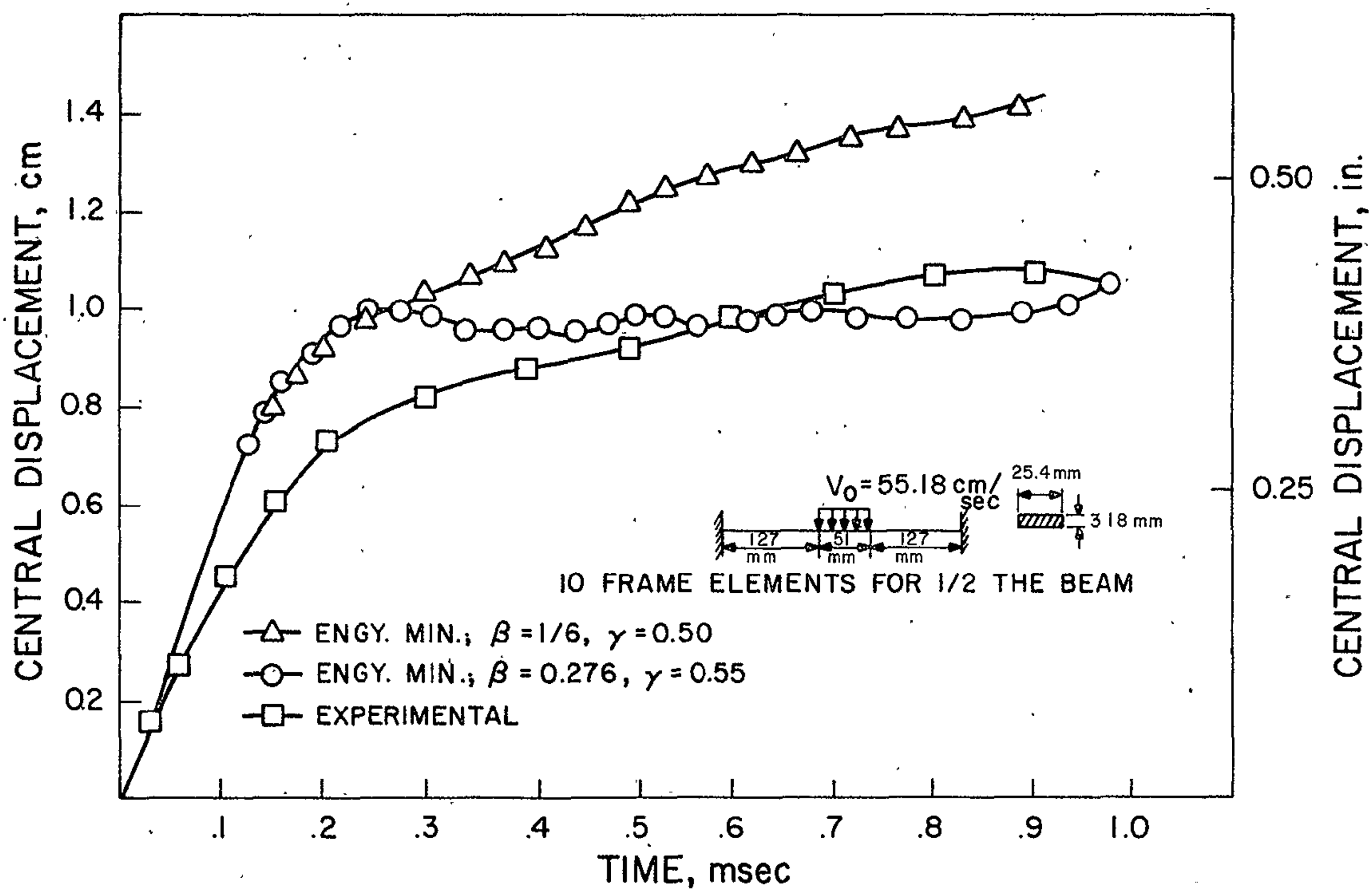

Figure 7. Impulsively Loaded Clamped Beam

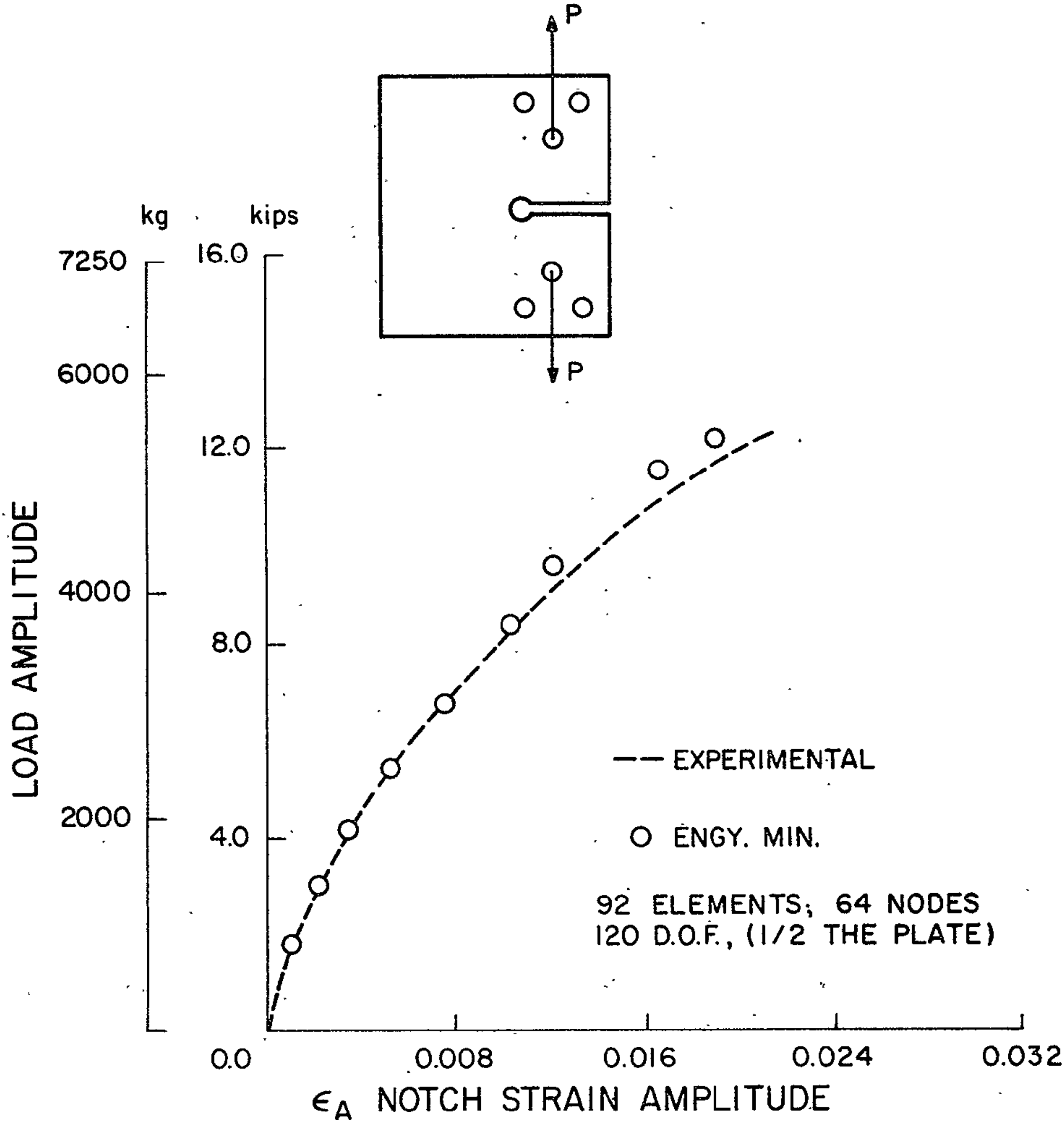

Figure 8. The Notch Problem

including the disposition of the occupants and their seats being symmetric about B.L.0.0, a finite element model of only half the structure as indicated in Figure 1, suffices. The depth of the substructure floor is implicit in the frame elements used for this portion of the substructure. Of simulation interest was the occupant chest motion and its vertical acceleration at the pelvis location. The occupant was modeled by a single lumped mass while the seat was modeled by a set of four nonlinear stringer elements whose stress-strain behavior was based on previous, independent static crash tests on similar seats. Although, the ground plane capability could be exercised for this problem, in the interest of simplicity, the aircraft substructure was assumed to be in contact with the ground plane at nodes (86) through (91) while a velocity of 330 inches/sec was imparted to the entire model. Thus one would expect correlation in only in the initial phases of the response.

Figures 9 and 10 provide the correlation between the analysis and test results. The chest motion history was obtained from high-speed film analysis, performed by NASA Langley. The reduced data for the acceleration trace at the pelvis location was obtained by NASA by least-square-fit filtering technique. The differences in the two traces on the left and right side of the substructure indicate slightly unsymmetrical test conditions. The time of occurrence of the initial peak and its magnitude obtained by analysis agrees reasonably well with the corresponding test values. For additional model and simulation details the interested reader should consult reference [26]. This reference also provides a comparison of the performance of the energy minimization technique vis-a-vis the so-called hybrid technique and another technique which utilizes the vector approach.

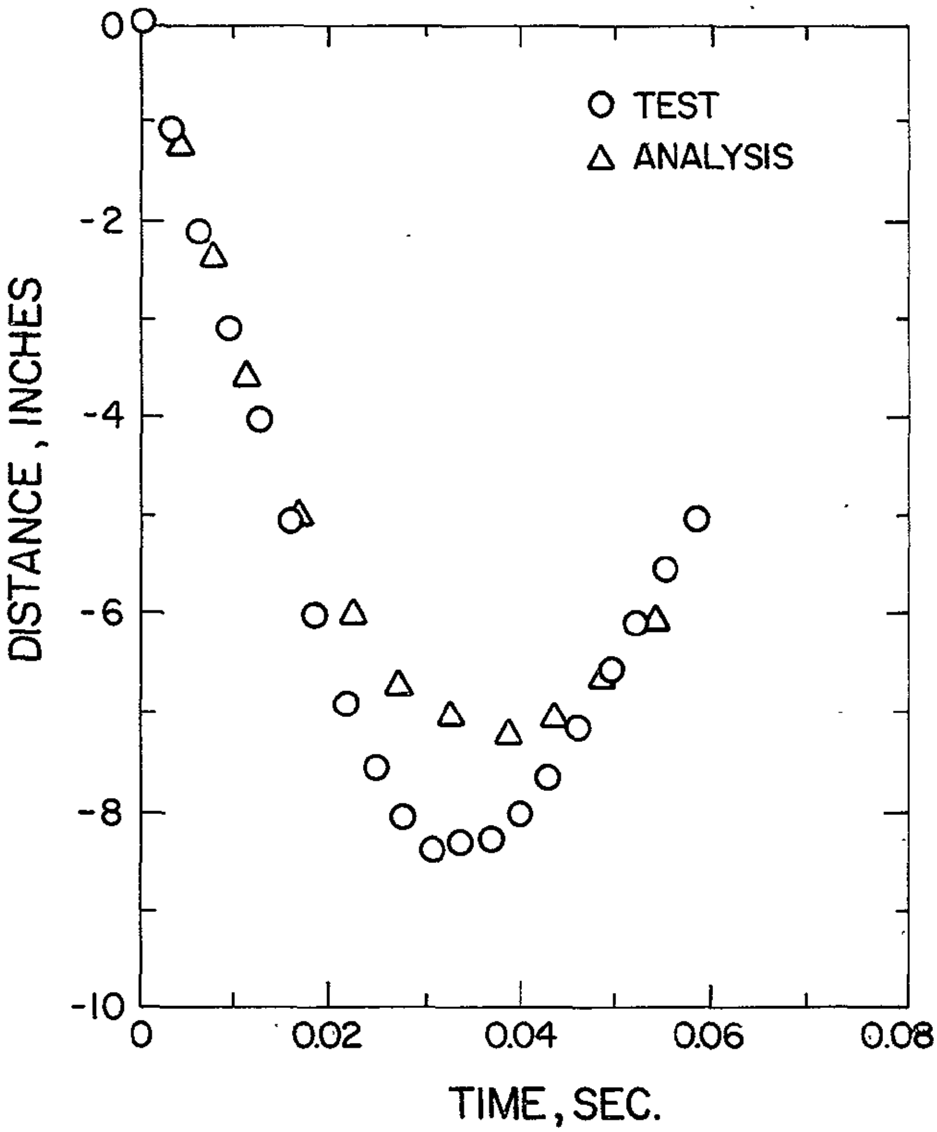

Figure 9. Occupant Chest Motion - Test versus Analysis

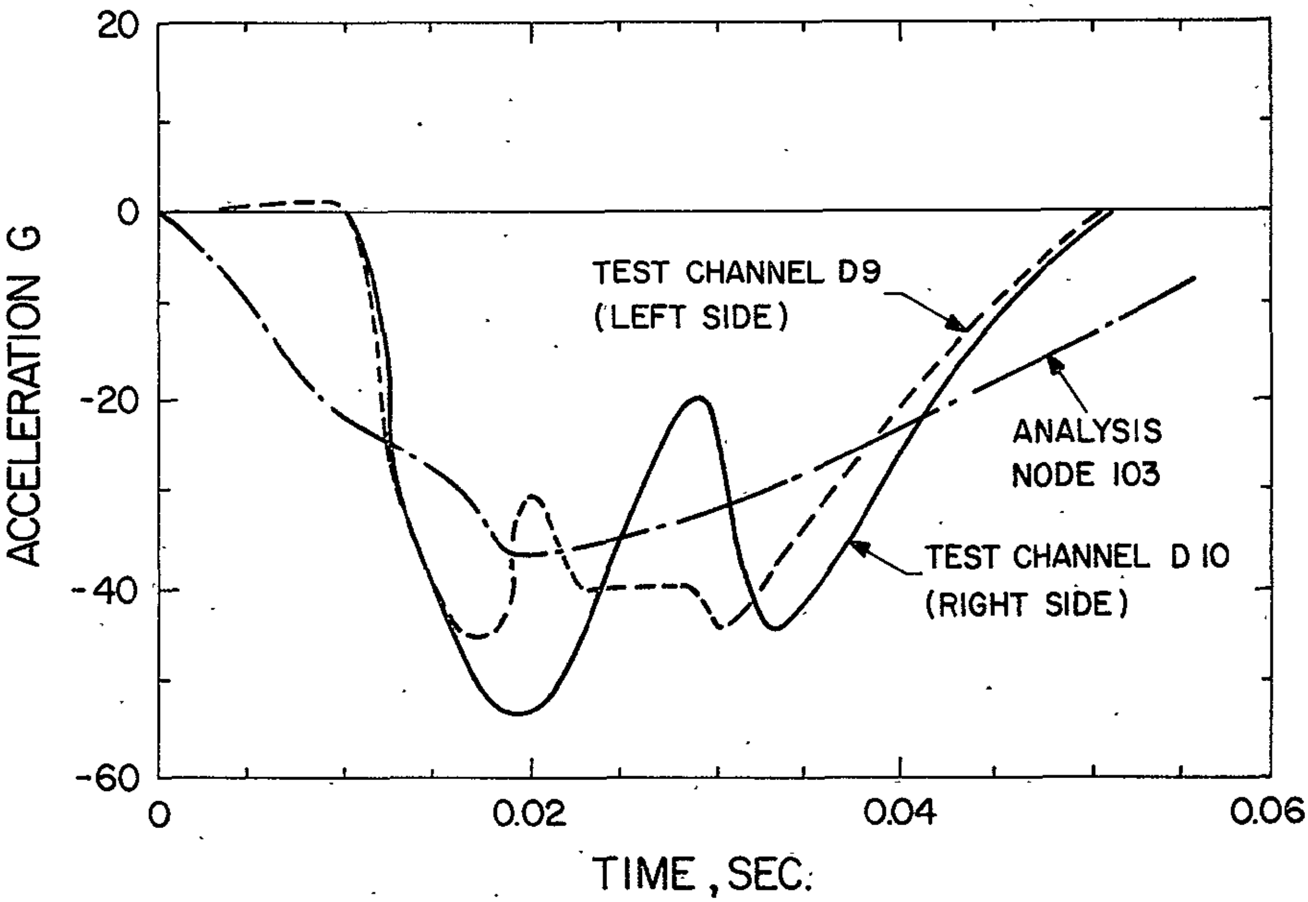

Figure 10. Occupant Pelvis Vertical Acceleration - Test versus Analysis

Indeed, one may say that this demonstration of the effectiveness of the minimization technique as a tool for nonlinear analysis has been, restricted to problems with a relatively few degrees of freedom. For such small scale problems the energy minimization technique has been shown to be at least comparable to, if not better than, the pseudo force technique [16]. Extensions to large scale problems like a full aircraft may involve several thousands of degrees of freedom. The state-of-the-art in nonlinear transient analysis in general, does not appear to be at a point where such large scale problems can be solved efficiently and with any high degree of confidence in the simulation fidelity. Likewise the effectiveness of the present technique for response prediction of such large scale structures remains to be demonstrated. Using preconditioned conjugate gradient technique or variable metric methods which exploit sparsity, it is believed that this is no longer an insurmountable task.

IV. Acknowledgements

The development of the computer code used for the solution of problems of Figures 4 through 10 was the work of several people including the author. Although, many of them have no longer been associated with its development for quite some time, their contributions, especially those of D. E. Killian and N. F. Knight, Jr., are nonetheless gratefully acknowledged. This work was supported by the NASA Langley Research Center under grants NGR 47-004-114, author's IPA assignment agreement and NSG 1576. Dr. Robert J. Hayduk was the responsible technical monitor. Appreciation is extended to both Dr. R. J. Hayduk and Dr. R. G. Thomson for supporting this research and for their faith in the energy minimization technique.

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VI. Appendix

Evaluation of the Total Strain Energy

The scalar approach for the solution of problems of structural analysis requires that the strain energy of the system be expressed, explicitly or implicitly, as a function of the global generalized nodal displacements of the finite element model.

From a known vector of the generalized nodal variables in the global co-ordinate system, consistent with the prescribed boundary conditions, a vector of local generalized variables in the co-rotational co-ordinate system of each element is established through transformations which are functions of its geometry and its rigid body rotations. The assumption of deformation patterns of the element as functions of these local generalized nodal variables (interpolating polynomials) yields element strains. Recourse to the element material model then yields the corresponding stresses and strain energy densities at various predetermined points (quadrature points) over the extent of the element. Barring purely elastic response, a simple weighted summation of these quantities over the element volume yields stress resultants and strain energies respectively. For purely elastic response these are provided by well-known closed form expressions. For the elastic-plastic response the strain energy density may be decomposed into an elastic part and an incremental-dissipative part thereby providing an estimate of the total energy of the system that has been dissipated through inelastic deformations. Thus, as shown in Figure 3 for a system with M elements

$$U = \sum_{i=1}^{M} U^i = \sum_{i=1}^{M} U_e^i + \Delta U_d^i = \sum_{i=1}^{M} \left(\int_{V_i} W_e{}^i dv + \int_{V_i} \Delta W_d{}^i dv \right) \qquad \text{(A-1)}$$

where the dissipative energy ΔU_d^i is the incremental dissipative energy computed from the previous stress state typied by the point a_0 on the effective stress-effective strain curve of Figure 3.

In the following sections expressions for the strain-displacement relations are developed for the different elements.

(i) Stringer Element

A structural component of uniform cross section which is initially straight and which is capable of resisting only axial loads is known as a stringer element.

From Figure (A-1) it can be seen that for assumed nodal displacements (U_p, V_p, W_p) and (U_q, V_q, W_q) of nodes p and q, the change in length, DL, of the element is given by

$$DL = [(X_q + U_q - X_p - U_p)^2 + (Y_q + V_q - Y_p - V_p)^2 + (Z_q + W_q - Z_p - W_p)^2]^{1/2} - [(X_q - X_p)^2 + (Y_q - Y_p)^2 + (Z_q - Z_p)^2]^{1/2} \qquad \text{(A-2)}$$

which can be simplified to

$$DL = L[1 + \frac{2(\Delta X\Delta U+\Delta Y\Delta V+\Delta Z\Delta W)}{L^2} + \frac{\Delta U^2+\Delta V^2+\Delta W^2}{L^2}]^{1/2} - L \qquad \text{(A-3)}$$

Δ being the difference operator for q and p end values. Assumption of the usual linear interpolation function in the corotational coordinate system then yields

$$\varepsilon = (\frac{DL}{L}) \qquad \text{(A-4)}$$

(ii) Frame Element (3D Beam)

A frame element (Figure A-2) is a structural component which is initially straight and which undergoes axial, bending and torsional deformations resulting from finite displacements and rotations of its ends.

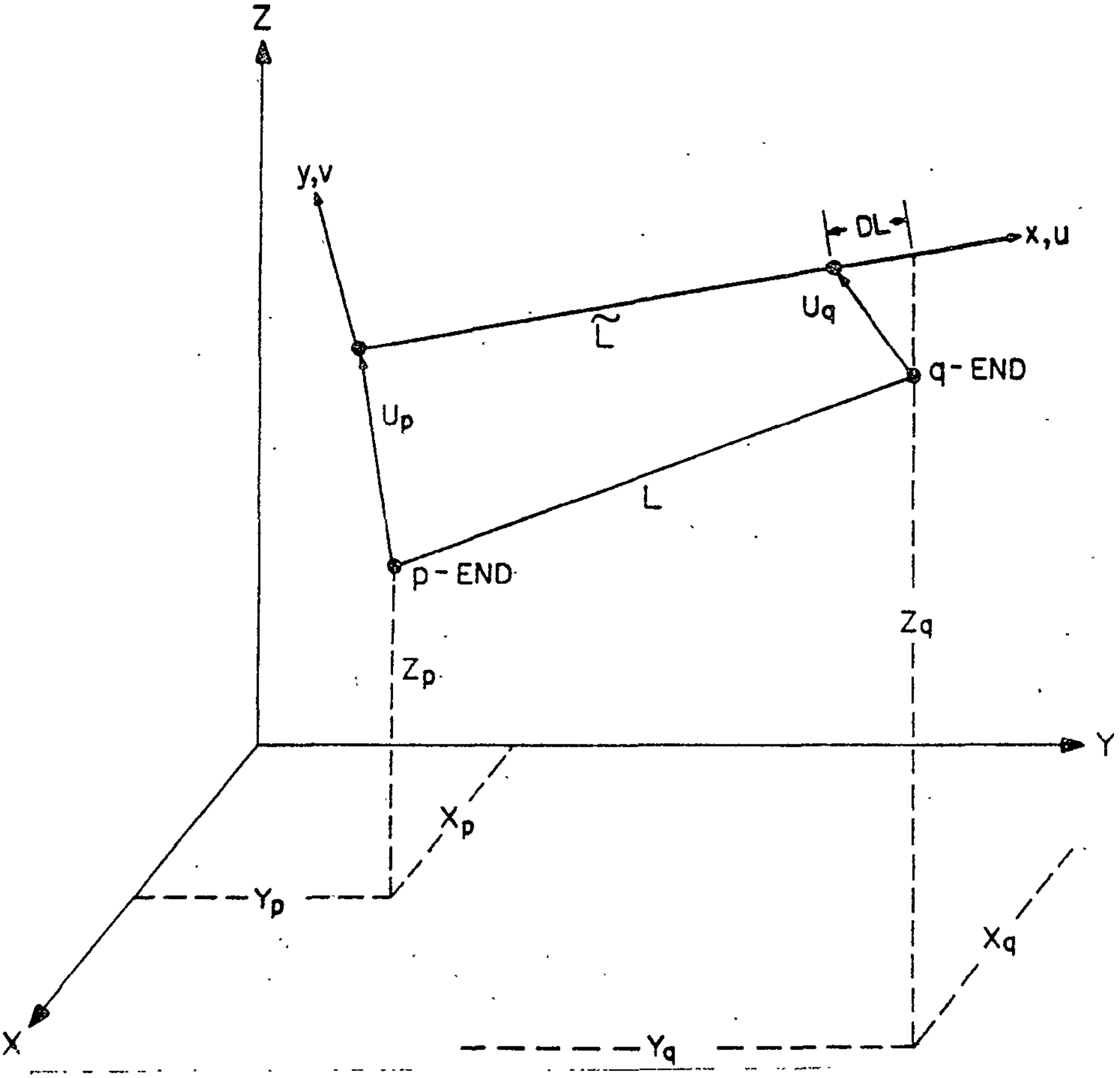

Figure A-1. Deformation of a Truss Element

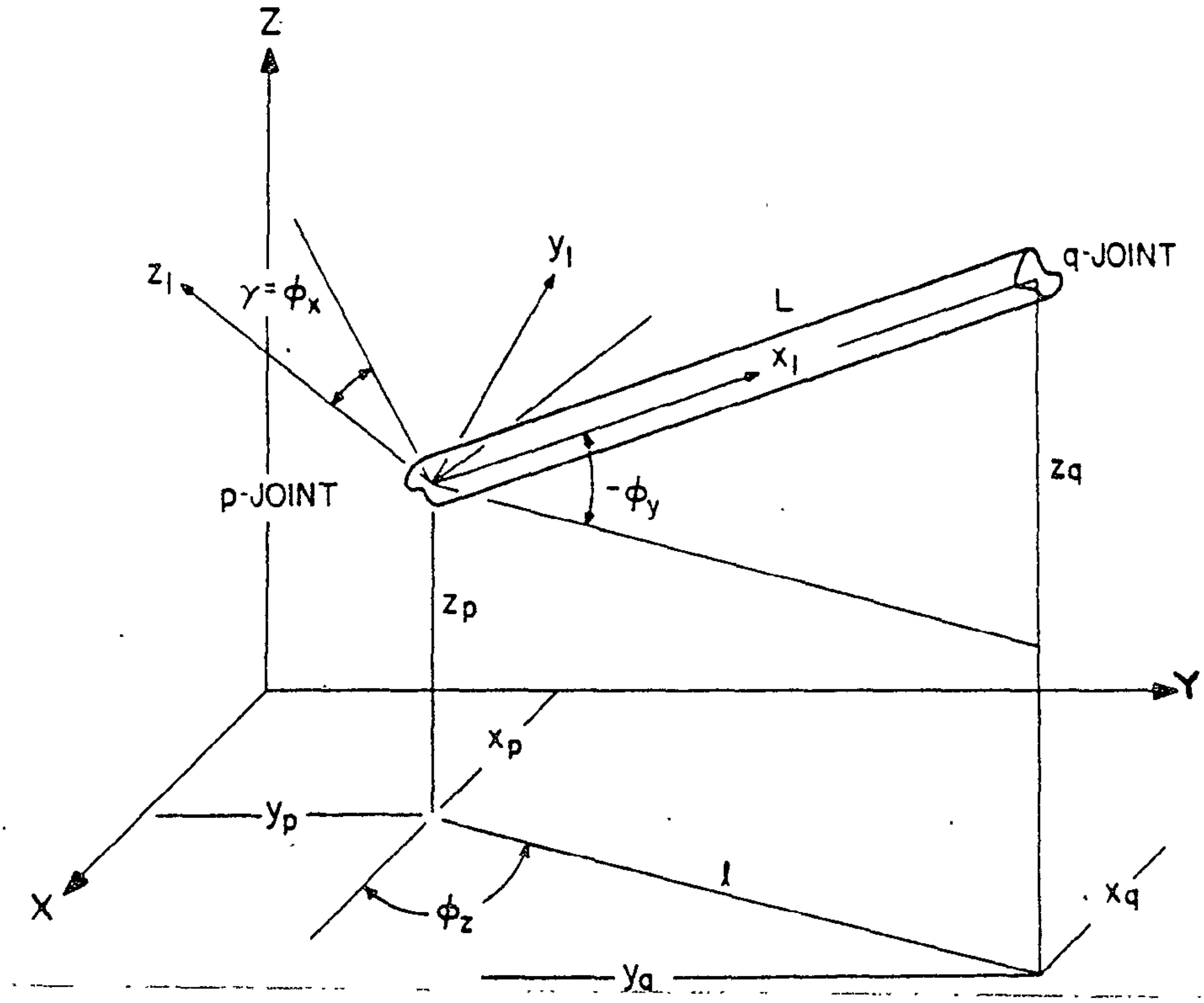

Figure A-2. Frame Element Orientation

From Figure (A-3), the displacements of the end q relative to the end p can be seen to be

$$\delta\underline{u} = (\underline{R}_q - \underline{R}_p) - \underline{L} + (\underline{U}_q - \underline{U}_p) \tag{A-5}$$

or in terms of the three components as

$$\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix} = [T]_p \begin{Bmatrix} X_q - X_p \\ Y_q - Y_p \\ Z_q - Z_p \end{Bmatrix} - \begin{Bmatrix} L \\ 0 \\ 0 \end{Bmatrix} + [T]_p \begin{Bmatrix} U_q - U_p \\ V_q - V_p \\ W_q - W_p \end{Bmatrix} \tag{A-6}$$

where again U_i, V_i and W_i (i=p or q) denote the global displacements of the nodes. The matrix $[T]_p$ can be shown to be [9]

$$[T]_p = [T_1(\phi_x, \phi_y, \phi_z)][T_1(\theta_{xp}, \theta_{yp}, \theta_{zp})] \tag{A-7}$$

with

$$[T_1(\alpha_x,\alpha_y,\alpha_z)] = \begin{bmatrix} c_y c_z & c_y s_z & -s_y \\ -c_x s_z + s_x s_y c_z & c_x c_z + s_x s_y s_z & s_x c_y \\ s_x s_z + c_x s_y c_z & -s_x c_z + c_x s_y s_z & c_x c_y \end{bmatrix} \tag{A-8}$$

$c_i = \cos\alpha_i$ and $s_i = \sin\alpha_i$ for i=x,y and z. Angles ϕ_x, ϕ_y and ϕ_z are the initial orientation angles described in Figure (A-2) and angles θ_{xp}, θ_{yp} and θ_{zp} are the rigid body rotations of the end p. In deriving Eq. (A-7) Euler angle transformations are implied with the order of the rotations being α_z, α_y and α_x.

Similarly, with the restriction of small relative rotations within the element, the rotations ψ_x, ψ_y and ψ_z of the end q relative to the end p are

$$\begin{Bmatrix} \psi_x \\ \psi_y \\ \psi_z \end{Bmatrix} = [T]_p \begin{Bmatrix} \theta_{xq} - \theta_{xp} \\ \theta_{yq} - \theta_{yp} \\ \theta_{zq} - \theta_{zp} \end{Bmatrix} \tag{A-9}$$

With the relative generalized displacements $\{\delta u, \delta v, \delta w\}$ and $\{\psi_x, \psi_y, \psi_z\}$ known the usual deformation patterns of the reference axis of the beam element in the co-rotational co-ordinate system are assumed

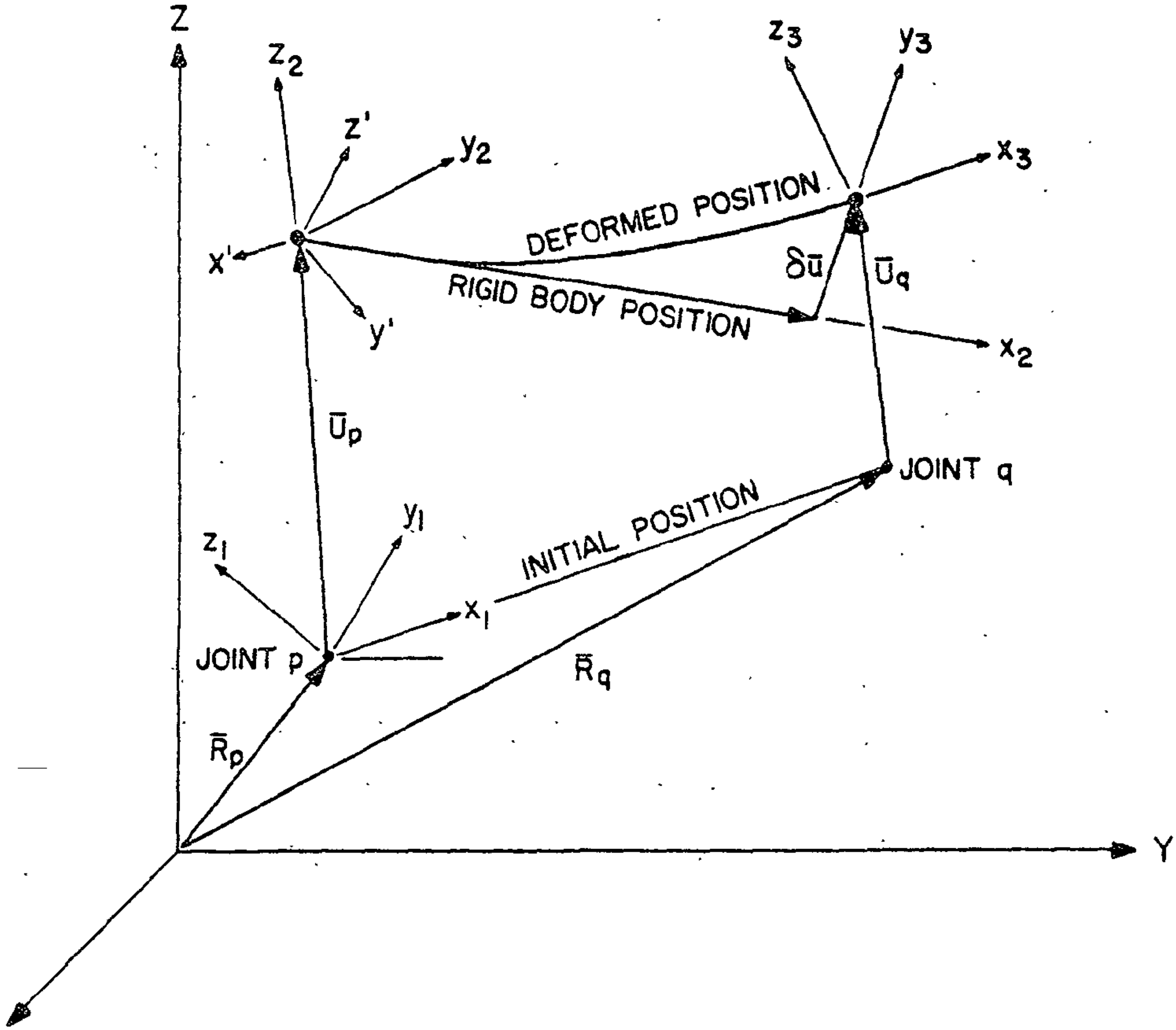

Figure A-3. Deformation of a Frame Element to be

$$u(\xi) = \xi \frac{\delta u}{L}$$

$$v(\xi) = \frac{1}{L} (3\xi^2 - 2\xi^3)(\delta v - z_s\psi_x) + (\xi^3 - \xi^2)\psi_z$$

$$w(\xi) = \frac{1}{L} (3\xi^2 - 2\xi^3)(\delta w + y_s\psi_x) - (\xi^3 - \xi^2)\psi_y \qquad \text{(A-10)}$$

$$\beta = \xi\psi_x$$

where $\xi = x/L$ and y_s and z_s are the co-ordinates of the shear center of the cross-section of the beam. The strain of the reference axis can then be shown to be

$$\varepsilon = \frac{\delta u}{L} - \eta[\frac{6}{L} (1-2\xi)(\delta v - z_s\psi_x) + 2(3\xi-1)\psi_z] - \zeta[\frac{6}{L} (1-2\xi)(\delta w + y_s\psi_x) - 2(3\xi-1)\psi_y] \qquad \text{(A-11)}$$

with $\eta = y/L$ and $\zeta = z/L$. In the above equations it is implicitly assumed that the lateral displacements and twists are referenced to a longitudinal axis through the shear center while the axial displacements and rotations are referenced to the centroidal axis. As shown in reference [27] this assumption necessitates the introduction of an additional degree of freedom in the axial direction in the interest of equilibrium satisfaction in the inelastic range.

(iii) Membrane Element

The membrane element of Figure (A-4) is a plane triangular thin plate element under constant strain. The element can undergo large rigid body motions but its deformation is restricted to only in-plane stretching resulting from finite displacements of its vertices.

The orientation of the element is uniquely determined by the global co-ordinates of its three vertices, p°, q° and r°. The relative displacements δu_q, δu_r and δv_r defined in Figure (A-4) can be seen to be

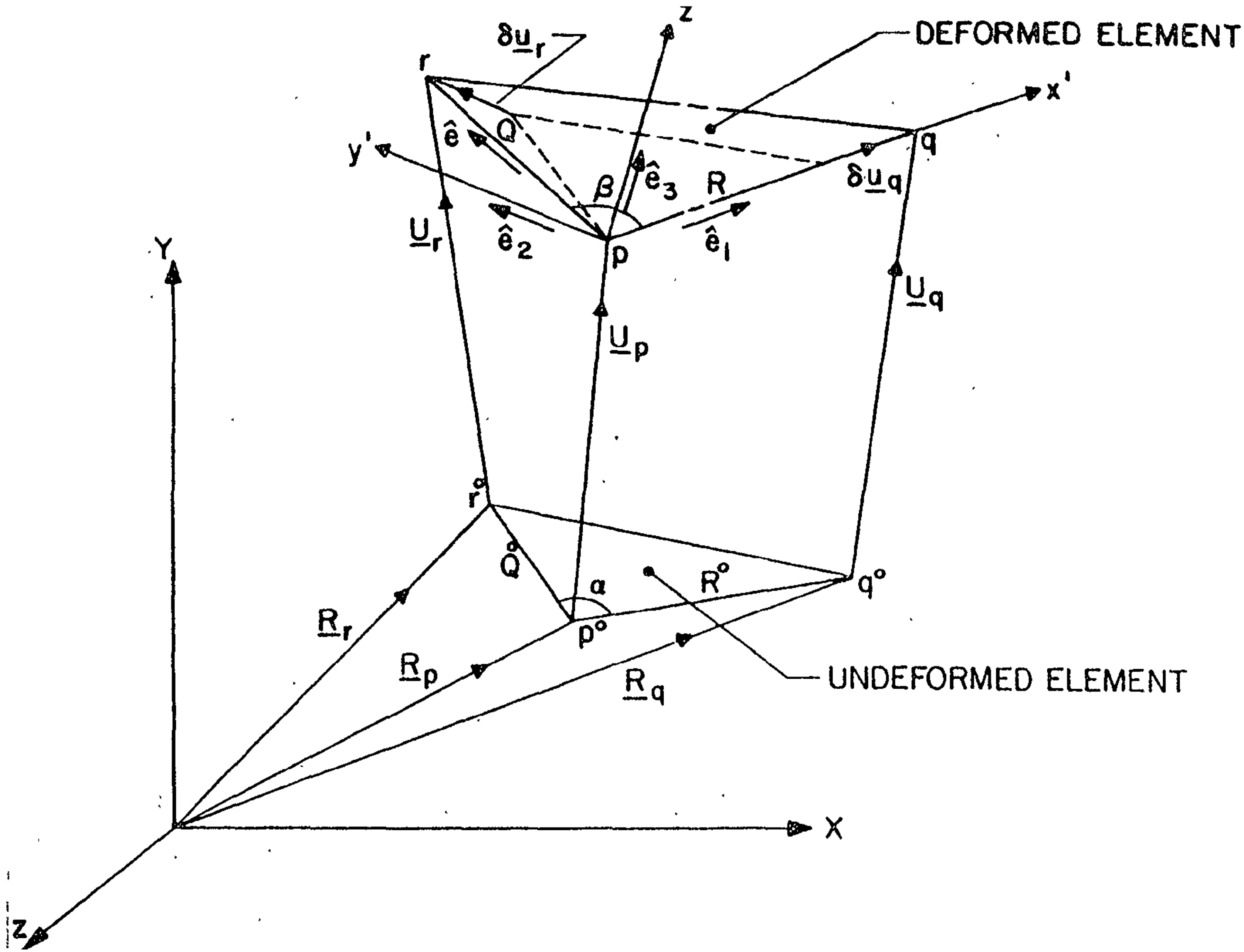

Figure A-4. Deformation of a Membrane Element

$$\delta u_q = R - R^\circ$$
$$\delta u_r = Q \cos \beta - Q^\circ \cos \alpha \tag{A-12}$$
$$\delta v_r = Q \sin \beta - Q^\circ \sin \alpha$$

with

$$R^\circ = (X_{rp}^2 + Y_{rp}^2 + Z_{rp}^2)^{1/2}$$

$$R = [(X_{rp} + U_{rp})^2 + (Y_{rp} + V_{rp})^2 + (Z_{rp} + W_{rp})^2]^{1/2}$$

$$Q^\circ = (X_{qp}^2 + Y_{qp}^2 + Z_{qp}^2)^{1/2}$$

$$Q = [(X_{qp} + U_{qp})^2 + (Y_{qp} + V_{qp})^2 + (Z_{qp} + W_{qp})^2]^{1/2} \tag{A-13}$$

$$\cos \alpha = (X_{qp}X_{rp} + Y_{qp}Y_{rp} + Z_{qp}Z_{rp})/(Q^\circ R^\circ)$$

$$\cos \beta = [(X_{qp} + U_{qp})(X_{rp} + U_{rp}) + (Y_{qp} + V_{qp})(Y_{rp} + V_{rp}) + (Z_{qp} + W_{qp})(Z_{rp} + W_{rp})]/(QR)$$

and typically $\alpha_{ij} = \alpha_i - \alpha_j$. Next, as in the case of the two previous elements, deformation patterns u(x,y) and v(x,y) in the co-rotational co-ordinate system when expressed in terms of the local nodal displacements yield

$$\left.\begin{aligned} u(x,y) &= u_p + \left(\frac{\delta u_q}{R^\circ} \right)x + \left[\frac{\delta u_r}{Q^\circ \sin\alpha} - \frac{\delta u_q}{R^\circ} \cot\alpha \right]y \\ v(x,y) &= v_p + \left(\frac{\delta v_q}{R^\circ} \right)x + \left[\frac{\delta v_r}{Q^\circ \sin\alpha} - \frac{\delta v_q}{R^\circ} \cot\alpha \right]y \end{aligned}\right\} \tag{A-14}$$

with the strains ε_{xx}, ε_{yy} and γ_{xy} determined on the basis of the small deformation theory as

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \left(\frac{\delta u_q}{R^\circ} \right) \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\delta v_r}{Q^\circ \sin\alpha} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\delta u_r}{Q^\circ \sin\alpha} - \frac{\delta u_q}{R^\circ} \cot\alpha \right). \end{aligned}\right\} \tag{A-15}$$

Thus, with the assumption that the total deformation theory of plasticity is applicable, the effective strain and effective stress defined by Eqs. (6-b) and (6-c) yield estimates of the stresses and strain energy densities from the material model. It is obvious that the integrations over the volume of the element are rendered trivial by virtue of the assumption that strains and hence the stresses and strain energy densities within the element are constant.

(iv) Rigid Link

Rigid link is an element which merely translates and rotates without any appreciable deformations. The element is identified by two nodes located with reference to the global co-ordinate system. One of these two nodes is referred to as the master or primary node, p, with a maximum of six independent degrees of freedom. The motions of the slave or secondary node or nodes, q, are determined purely from kinematics by setting the left hand side of equation (A-5) to zero. Knowing the dependent displacements of the secondary nodes,

$$\{U\}_q = \{U\}_p + [T]_p^T\{L\} + \{R\}_p - \{R\}_q \qquad \text{(A-16)}$$

the contribution, to the total potential energy, of loads applied directly at the secondary nodes can thus be determined.

APPENDIX D

Energy Minimization Versus Pseudo Force Technique
for Nonlinear Structural Analysis*

by

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Abstract

The effectiveness of using minimization techniques for the solution of nonlinear structural analysis problems is discussed and demonstrated by comparison with the conventional pseudo force technique. The comparison involves nonlinear problems with a relatively few degrees of freedom. A survey of the state-of-the-art of algorithms for unconstrained minimization reveals that extension of the technique to large scale nonlinear systems is possible.

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*This work was performed while the first author was detailed at NASA Langley on an Inter-Governmental Personnel Act assignment for one year.

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I. Introduction

The prediction of transient linear or nonlinear response of structures is almost invariably accomplished by using a temporal finite-difference scheme to effectively eliminate time as a variable and reduce the system to a set of algebraic equations in the unknown nodal variables of the finite element discretization. Finite differencing in time may be either of the explicit or implicit type.

Most nonlinear analyzers linearize response within a time step and use an explicit scheme [1]-[3] or an implicit scheme [3]-[4], while a select few do not linearize response within the time step and use an implicit scheme [5] or an explicit scheme [6]. At the present time there are no clear cut guidelines or criteria for the selection of these procedures. Use of hybrid explicit-implicit (on-off) schemes have been advocated and is the subject of current and future research [7]-[9].

For schemes which do not linearize the response within a time or load step, several different techniques for the solution of the nonlinear equations may be used. Such techniques have been discussed at great lengths by Bergan [10] and Stricklin et al. [11]. Of particular interest is the technique utilizing the minimization algorithms of mathematical programming. This approach has been used successfully for nonlinear structural analyses [5], [12]-[14]. In this case, the problem of finding the solution of the equilibrium equations can be equivalently posed as the one corresponding to the minimum value of a potential function

In the past there has been considerable skepticism with regard to the effectiveness of the energy minization technique, when compared to other methods, such as the pseudo force technique. It has been also claimed that the extension of the minimization algorithms to large scale problems is virtually impossible. The purpose of this paper is to demonstrate by comparison with the pseudo force technique that minimization techniques have and can overcome some of these objections and that future improvements in minimization algorithms will further improve their efficiency.

II. Summary of Minimization Techniques for Nonlinear Analysis

The solution of the equilibrium equations can be equivalently posed as the minimization of a potential function. For all structural problems with geometric and material nonlinearities of the type considered herein, such a potential function always exists. Although this technique has been previously used for mainly positive or negative definite systems, other systems can be handled by using least squares methods or modified conjugate gradient methods with preconditioning.

The minimization problem as applied to the solution of transient nonlinear structural problems (reduction to the static case is obvious) consists of minimizing a potential function associated with the system for an assumed relationship between displacements and time. The displacement-time relation for each generalized nodal variable of the finite element model may be assumed to be of the form [16].

$$X_{ei} = \beta(\Delta t)^2 \ddot{X}_{ei} + (\frac{1}{2} - \beta)(\Delta t)^2 \ddot{X}_{oi} + (\Delta t)\dot{X}_{oi} + X_{oi} \quad . \tag{1-a}$$

$$\dot{X}_{ei} = \gamma(\Delta t)\ddot{X}_{ei} + (1 - \gamma)(\Delta t)\ddot{X}_{oi} + \dot{X}_{oi} \tag{1-b}$$

where X_{ei} is the i-th generalized nodal displacement at the end of the time step. The β and γ constants are selected by the analyst to define the integration algorithm. It can be easily verified that the equilibrium equation corresponding to the i-th degree of freedom of a lumped mass system

$$M_i \ddot{X}_{ei} - F_i + \frac{\partial U}{\partial X_{ei}} = 0 \tag{2}$$

is the stationary condition of the functional

$$S = \sum_{i=1}^{N} \left[\frac{1}{2\beta(\Delta t)^2} X_{ei}^2 - \left(\frac{1}{\beta(\Delta t)^2} X_{oi} + \frac{1}{\beta(\Delta t)} \dot{X}_{oi} + \left(\frac{1}{2\beta} - 1\right) \ddot{X}_{oi} \right) X_{ei} \right] M_i$$

$$- F_i X_{ei} + U + C \tag{3}$$

where C is an arbitrary constant, U is the strain energy, and M_i and F_i are the lumped mass and external force, respectively, associated with the i-th degree of freedom.

Thus, once the assumption of the displacement-time relation is made, the minimization approach, unlike the incremental stiffness approach, solves the actual nonlinear problem within a given load or time step without linearization. Consequently iteration at constant load to improve equilibrium or force imbalance at the end of a load or time step is not required.

In the realm of mathematical programming, the algorithms used for unconstrained minimization can be broadly classified into three distinct classes stemming from the level of computational sophistication: (i) the zeroth order requiring only function evaluations, (ii) the first order requiring evaluation of the gradient as well as the function and (iii) the second order requiring in addition a variable metric related to the curvatures of the functional S. Only the techniques belonging to the latter two categories have been more frequently used for structural analysis, because of their higher effectiveness in comparison with zeroth order techniques [17].

For first and second order methods the required gradient of S can be evaluated either by a finite difference operation on S or analytically. The use of an analytic gradient results in a substantial saving in computational effort. This saving is the result of not only a cheaper gradient evaluation but in most cases, a faster convergence to the solution because of higher accuracy of all computed quantities [17].

The i-th component of the gradient of S can be written as

$$\frac{\partial S}{\partial X_{ei}} = M_i \ddot{X}_{ei} - F_i + \frac{\partial U}{\partial X_{ei}} \tag{5}$$

The term requiring an analytic expression is $\left(\frac{\partial U}{\partial X_{ei}}\right)$ which can be evaluated as

$$\frac{\partial U}{\partial X_{ei}} = \sum_{k=1}^{m} \int_{V_k} \frac{\partial W}{\partial X_{ei}} dV_k = \sum_{k=1}^{m} \int_{V_k} \left(\frac{dW}{d\bar{\varepsilon}}\right)_k \left(\frac{\partial \bar{\varepsilon}}{\partial X_{ei}}\right)_k dV_k \tag{6}$$

where

W - the strain energy density for the k-th member

m - the number of members or elements which have the i-th degree in common

V_k - volume of element k

$$\bar{\varepsilon} = \text{effective strain} \begin{cases} = \frac{2}{\sqrt{3}} \left(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 - \varepsilon_{xx}\varepsilon_{yy} + \frac{3}{4}\gamma_{xy}^2\right)^{1/2} \\ \quad \text{for two dimensional stress state} \\ = \varepsilon_{xx} \text{ for one dimensional stress state} \end{cases} \tag{7-a}$$

For one step incremental loading or unloading

$$\frac{dW}{d\bar{\varepsilon}} = \bar{\sigma} \text{ effective stress} \begin{cases} = \left(\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2\right)^{1/2} \\ \quad \text{for two-dimensional stress state} \\ = \sigma_{xx} \text{ for one dimensional stress state} \end{cases} \tag{7-b}$$

Thus

$$\frac{\partial U}{\partial X_{ei}} = \sum_{k=1}^{m} \int_{V_k} \bar{\sigma}_k \left(\frac{\partial \bar{\varepsilon}}{\partial X_{ei}}\right)_k dV_k. \tag{7-c}$$

Equations (7-a) through (7-c) imply the use of Henckey's total strain or the deformation theory of plasticity. In addition, unloading takes place elastically with the load path described by the initial elastic portion of the curve and for modeling plasticity under cyclic loading kinematic hardening is prescribed with an idealized Bauschinger effect.

The term $\frac{\partial U}{\partial X_{ei}}$ in Eq. (7-c) involves a volume integral which is very similar to that required for a member energy evaluation. Hence, each component of the analytic gradient vector involves approximately the same calculation effort as one member energy evaluation, two if the node is common to two elements, three if the node is common to three elements and so on. Consequently a significant reduction in the number of member energy evaluations and in CPU time should be realized if analytic gradients are used instead of finite difference gradients. This has been vividly demonstrated for problems involving different types of nonlinearities in reference [17].

III. Summary of Pseudo Force Technique

The pseudo force technique, as applied here, uses the initial strain concept for the treatment of material nonlinear behavior and an incremental updated Lagrangian formulation for the geometric nonlinear behavior [43], [44]. The governing equation at the (n+1)th time step for the transient nonlinear analysis of a discretized structural problem is

$$[M]\{\Delta\ddot{u}_{n+1}\} = \{\Delta P_{n+1}\} + \{\Delta Q_{n+1}\} - [K_n^{(0)} + K_n^{(1)}]\{\Delta u_{n+1}\} + \{R_n\} \tag{8}$$

with the following definitions:

$$\begin{aligned}
\{\ddot{u}_{n+1}\} &= \{\ddot{u}_n\} + \{\Delta\ddot{u}_{n+1}\} \\
\{P_{n+1}\} &= \{P_n\} + \{\Delta P_{n+1}\} \\
\{Q_{n+1}\} &= \{Q_n\} + \{\Delta Q_{n+1}\} \\
\{R_n\} &= \{P_n\} + \{Q_n\} - [M]\{\ddot{u}_n\} - \{F_n\}
\end{aligned} \tag{9}$$

where $[K_n^{(0)}]$ - linear stiffness matrix

$[K_n^{(1)}]$ - nonlinear geometric stiffness matrix

$\{\Delta P_{n+1}\}$ - increments of generalized nodal forces

$\{\Delta Q_{n+1}\}$ - increments of the effective plastic load

$\{R_n\}$ - vector of residual forces due to equilibrium imbalance

$\{\Delta u_{n+1}\}$, $\{\Delta\dot{u}_{n+1}\}$, $\{\Delta\ddot{u}_{n+1}\}$ - increments of displacement, velocity and acceleration

$[M]$ - mass matrix

$\{F_n\}$ - vector of internal forces.

The equations for the static case are obtained by simply neglecting the inertia term in Equation (8). For the transient solution a convenient finite difference approximation is the central difference integration algorithm. The recurrence relations for this temporal operator are

$$\begin{aligned}
\{\Delta u_{n+1}\} &= 2\{\Delta u_n\} - \{\Delta u_{n-1}\} + \Delta t^2 \{\Delta\ddot{u}_n\} \\
\{\Delta\dot{u}_n\} &= \frac{\{\Delta u_{n+1}\} - \{\Delta u_{n-1}\}}{2\Delta t}
\end{aligned} \tag{10}$$

where Δt is the time step. The integration procedure is explicit because $\{\Delta u_{n+1}\}$ is obtained directly from Eq. (10) using previous information, and it is also notable that $\{\Delta \ddot{u}_{n+1}\}$ is obtained from Eq. (8) without factorization of the (effective) stiffness matrix in the step-by-step solution.

Equation (8) is valid for large elastic-plastic deformation provided that the appropriate nonlinear terms are included in the strain-displacement relations and that the total strain increment can be simply decomposed into elastic and plastic components [43]. With an assumption on the plastic strain distribution within an element, the effective plastic load increment can be expressed as $\{\Delta Q_{n+1}\} = [k^*] \{\Delta \varepsilon_{n+1}\}$ where $\{\Delta \varepsilon_{n+1}\}$ is the increment of nodal or element plastic strain and $[k^*]$ is the initial strain stiffness matrix used to represent initial strains and to reflect the assumed distribution of both total and plastic strains within an element. Incremental plasticity relations are used to determine values of stress and plastic strain developed throughout the loading history. This technique has the capability of handling both strain hardening and ideally plastic behavior.

IV. Energy Minimization Versus Pseudo Force Techniques

Structural analysis computer programs differ significantly in their treatment of elastic-plastic response. Most explicit codes appear to tolerate violation of equilibrium in the inelastic range while implicit codes cannot get by with such a deficiency. At least two well-known explicit codes, after conversion to implicit type using the same element deformation and material models, experienced difficulty in converging to a solution and violation of equilibrium. For frame elements, the implicit formulation requires an axial displacement field in the inelastic range which is an order higher than the linear field which is commonly used for a two noded beam-column element in the elastic range [19]. Furthermore, the number of quadrature points used for integration of strain energy densities, stresses, etc. differ from code to code. Thus, even if the two finite element models are identical in the undeformed state they differ signficantly as deformation proceeds. This factor has to be taken into account before undertaking an efficiency assessment of the two procedures.

An obvious follow-on to the work reported in reference [17] was a comparative efficiency evaluation of pseudo force techniques (incremental linearization-explicit) versus energy minimization techniques (nonlinear-implicit). For the purposes of Figures 1 through 5, to be discussed below, analytic rather than finite difference gradients were employed and Fletcher's algorithm [18], more commonly known as the BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm, was used for unconstrained minimization of the functional S.

Figure 1 illustrates the case of a rod-spring problem involving a single degree of freedom but regarded as a highly nonlinear problem. The technique using energy minimization predicted the 'exact' solution using load increments as high as 1 lb. (Larger increments could have been tried but were not attempted. For quasi-static loading conditions the energy minimization code does not provide an automatic selection of load steps guided by error tolerances or the like). The pseudo force technique using 1/10th of the load increments did not do quite as well. Since it is only a single degree of freedom problem a comparison of running times was not considered meaningful although the energy minimization technique was several times faster than the pseudo force technique.

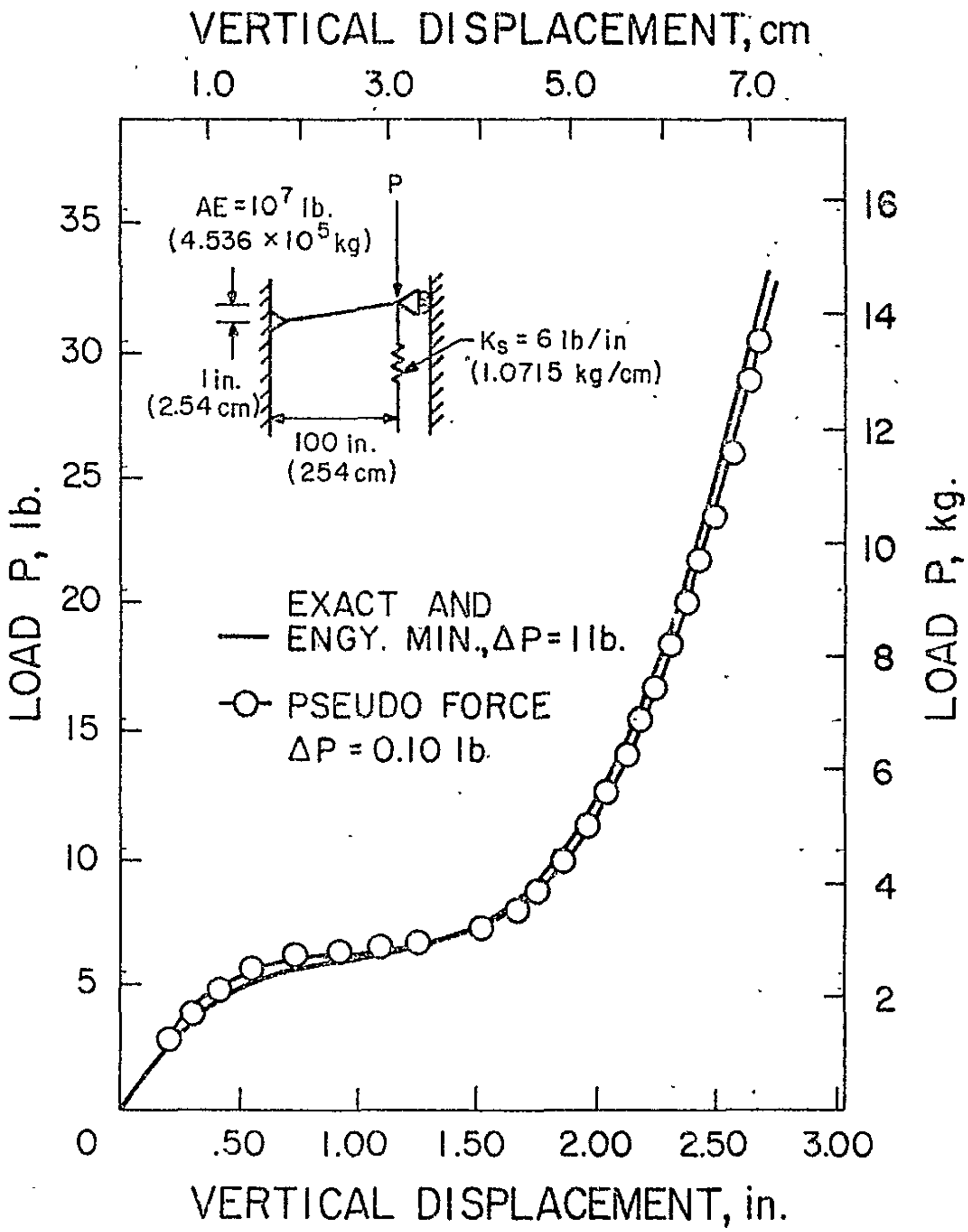

VERTICAL DISPLACEMENT, cm
1.0
3.0
5.0
7.0
35
30
25
20
15
10
5
0
LOAD P, lb.
16
14
12
10
8
6
4
2
LOAD P, kg.
P
AE = 10^7 lb.
(4.536 × 10^5 kg)
1 in.
(2.54 cm)
100 in.
(254 cm)
Ks = 6 lb/in
(1.0715 kg/cm)
EXACT AND
ENGY. MIN., ΔP = 1 lb.
PSEUDO FORCE
ΔP = 0.10 lb.
.50
1.00
1.50
2.00
2.50
3.00
VERTICAL DISPLACEMENT, in.

Figure 1

Next, Figure 2 illustrates the case of the classic snap through of a shallow arch. Since the unstable equilibrium branch presents computational problems using both techniques, displacement incrementation rather than load incrementation was used. The problem using the pseudo force technique was run on a CDC 6600 machine (taking 130.29 CPU secs), which is known to be about 2 1/2 times slower than the CYBER 175. The energy minimization method took 9.618 CPU seconds on the CYBER 175. A comparison of the equivalent running times shows that the pseudo force technique is about five times slower in comparison with the energy minimization technique and the quality of the response prediction does not appear to be as good. Using strictly load incrementation the energy minimization technique using potential energy as the potential function can locate only stable equilibrium configurations as indicated in Figure 2 by the circles and dotted load path. The pseudo force technique with load incrementation becomes singular at the limit load where the tangential stiffness vanishes. As a point of reference the square symbols represent a first order self-correcting solution from [11].

Figure 3 illustrates the post-buckling elastic-plastic response of a thin-walled channel cross-section column. Again, displacement rather than load incrementation is used because of the unstable equilibrium branch. For the range of deformations considered the entire column, built-up from 12 frame elements, responded inelastically. Figure 3 indicates that the energy minimization technique is comparable to the pseudo force technique for this problem.

Figure 4 provides an example of transient response in the presence of both geometric and material nonlinearities. The problem consists of a clamped-clamped beam of rectangular cross-section subjected to an explosive loading over a central region. The experimental data was taken from the literature [41].

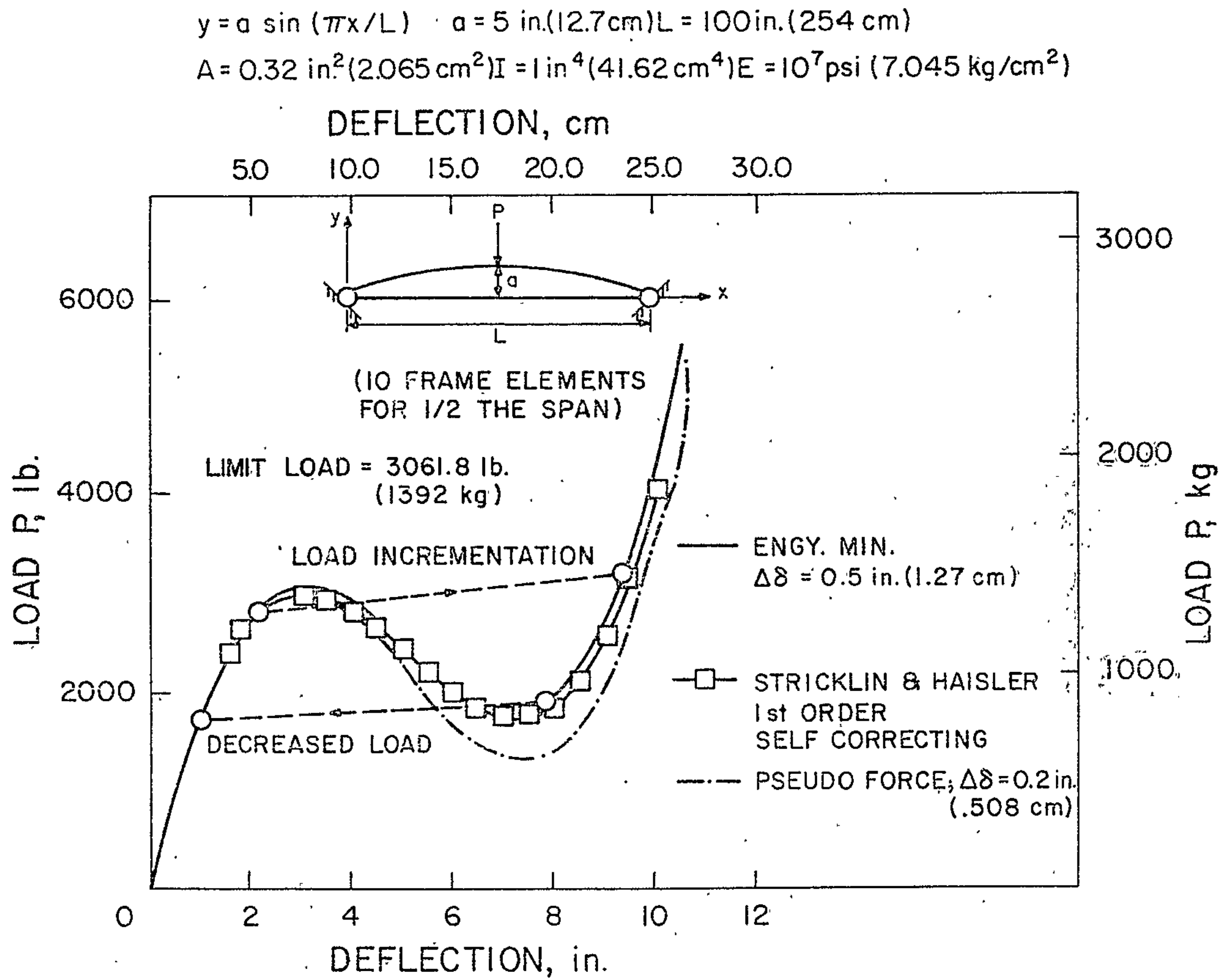
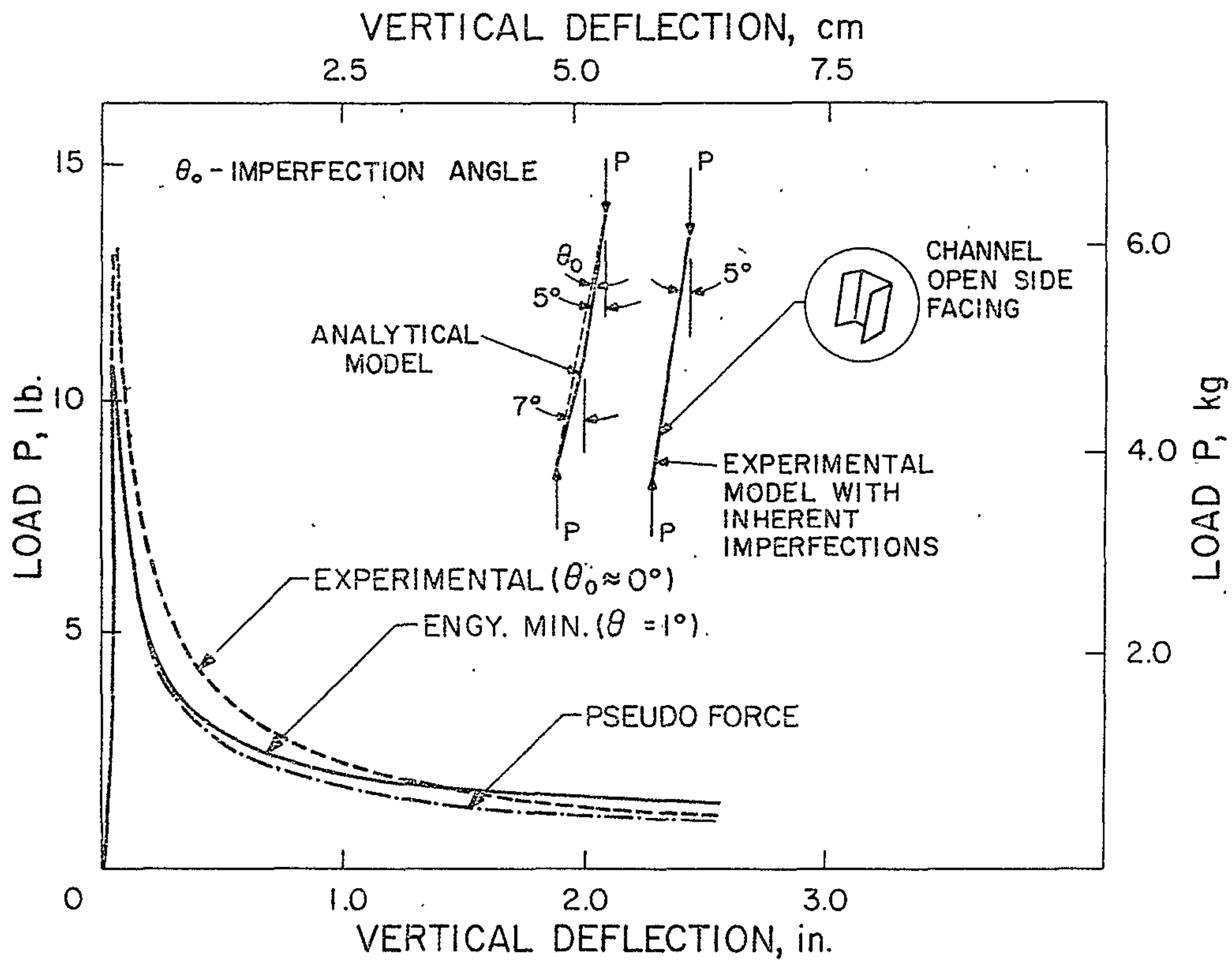
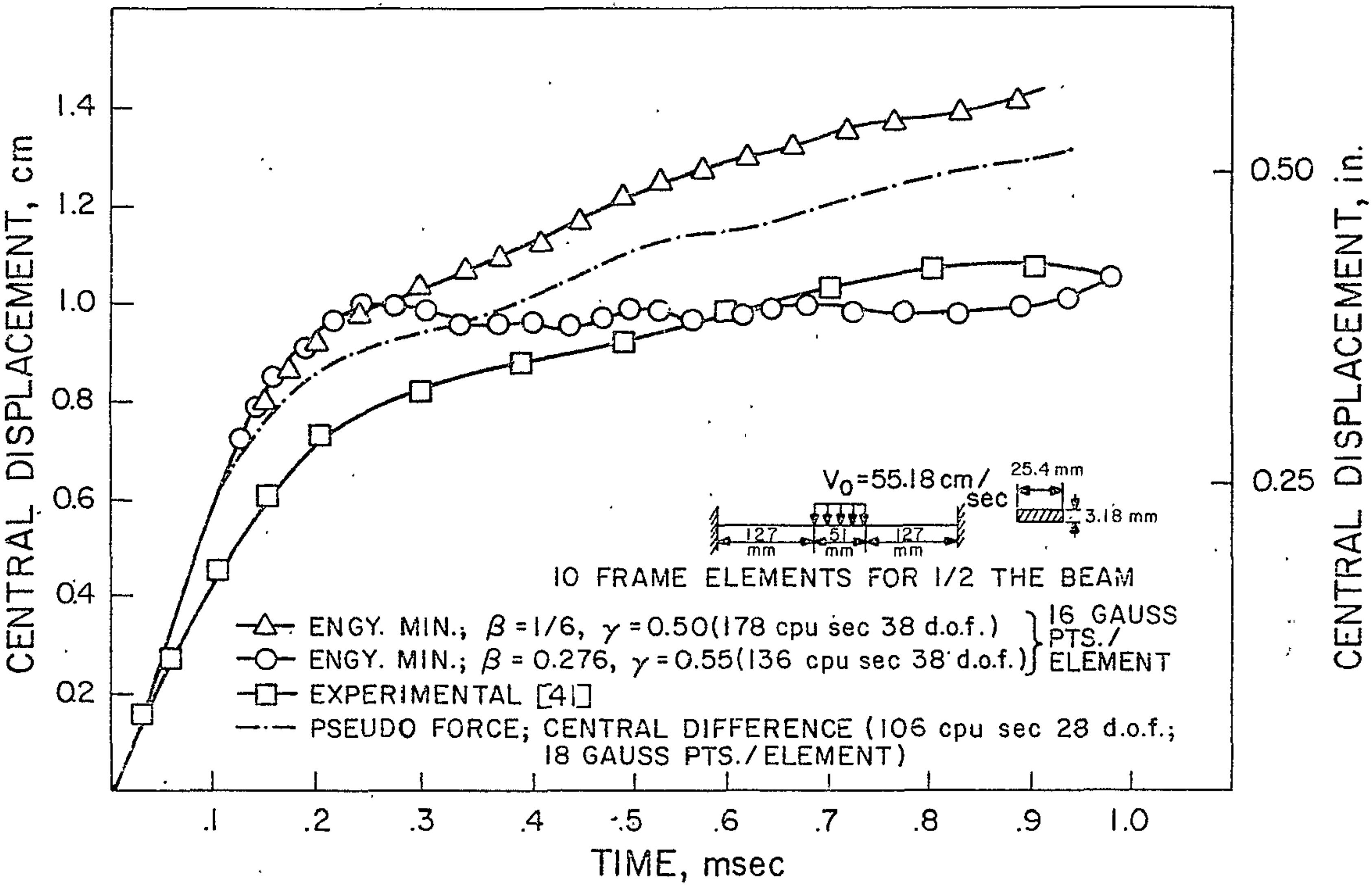

y = a sin (πx/L) a = 5 in.(12.7cm) L = 100 in.(254 cm)
A = 0.32 in.² (2.065 cm²) I = 1 in⁴ (41.62 cm⁴) E = 10⁷ psi (7.045 kg/cm²)
DEFLECTION, cm
5.0
10.0
15.0
20.0
25.0
30.0
y
P
a
x
L
(10 FRAME ELEMENTS
FOR 1/2 THE SPAN)
LIMIT LOAD = 3061.8 lb.
(1392 kg)
LOAD INCREMENTATION
DECREASED LOAD
ENGY. MIN.
Δδ = 0.5 in. (1.27 cm)
STRICKLIN & HAISLER
1st ORDER
SELF CORRECTING
PSEUDO FORCE; Δδ = 0.2 in.
(.508 cm)
LOAD P, lb.
6000
4000
2000
0
2
4
6
8
10
12
DEFLECTION, in.
LOAD P, kg
3000
2000
1000

Figure 2



VERTICAL DEFLECTION, cm
2.5
5.0
7.5
15
10
5
0
LOAD P, lb.
6.0
4.0
2.0
LOAD P, kg
θ₀ - IMPERFECTION ANGLE
P
P
θ₀
5°
5°
7°
ANALYTICAL MODEL
CHANNEL OPEN SIDE FACING
EXPERIMENTAL MODEL WITH INHERENT IMPERFECTIONS
P
P
EXPERIMENTAL (θ₀ ≈ 0°)
ENGY. MIN. (θ = 1°)
PSEUDO FORCE
1.0
2.0
3.0
VERTICAL DEFLECTION, in.

Figure 3



CENTRAL DISPLACEMENT, cm
1.4
1.2
1.0
0.8
0.6
0.4
0.2
CENTRAL DISPLACEMENT, in.
0.50
0.25
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
TIME, msec
V0=55.18 cm/sec
25.4 mm
3.18 mm
127 mm
51 mm
127 mm
10 FRAME ELEMENTS FOR 1/2 THE BEAM
ENGY. MIN.; β=1/6, γ=0.50(178 cpu sec 38 d.o.f.)
ENGY. MIN.; β=0.276, γ=0.55(136 cpu sec 38 d.o.f.)
16 GAUSS PTS./ELEMENT
EXPERIMENTAL [41]
PSEUDO FORCE; CENTRAL DIFFERENCE (106 cpu sec 28 d.o.f.; 18 GAUSS PTS./ELEMENT)

Figure 4

The impulse is large enough to cause the entire beam to behave inelastically. In this case, the amount of nonlinearity within a time step is very much dependent not only on the size of the time step but also on the temporal algorithm used. That is to say it is very much dependent on the values of β and γ. Thus, it is rather difficult to settle upon a common denominator for comparison of the two procedures. In addition, all the previous comments regarding the violation of equilibrium in the elastic range and the complications for the implicit techniques thereof, hold true for this case as well. With this in mind, one would tend to conclude that the efficiencies of the two procedures are comparable, although the pseudo force technique appears to have a slight edge over the energy minimization technique when the entire structure responds inelastically. The particular pseudo force technique code used here for comparison purposes provided the options of using either the central difference or Adams predictor-corrector algorithm. Of these two algorithms the former was found to be the more efficient and hence it is used for comparison with the energy minimization technique which employed variations of the Newmark-Beta algorithm. One should note in passing, however, that the transient response appears to be tremendously influenced by the values of β and γ. This is not to say that optimum values of β and γ exist which can guarantee optimum fidelity of the response prediction. As a matter of fact, optimum values of β and γ appear to be very much problem dependent and it may indeed pay-off to use a hybrid explicit-implicit (on-off) scheme.

Finally, Figure 5 provides the case of a structure built-up from two dimensional membrane elements. The experimental results for the notched tensile specimen are taken from [42]. A comparison of the running times for this problem reveals that the time of execution using the pseudo force technique is

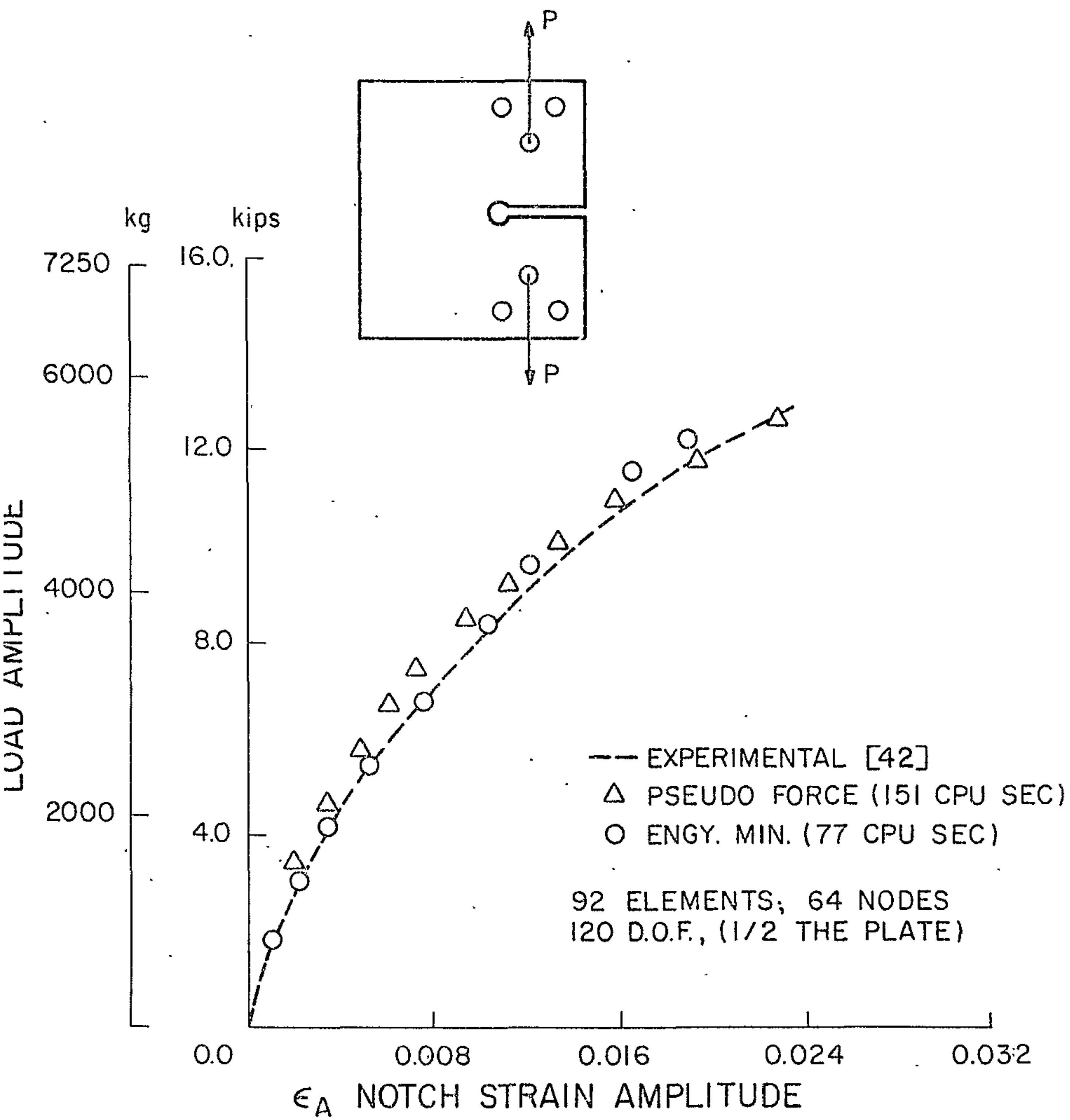

P
P
kg
kips
7250
6000
4000
2000
16.0
12.0
8.0
4.0
0.0
LOAD AMPLITUDE
0.008
0.016
0.024
0.032
ϵ_A NOTCH STRAIN AMPLITUDE
EXPERIMENTAL [42]
PSEUDO FORCE (151 CPU SEC)
ENGY. MIN. (77 CPU SEC)
92 ELEMENTS; 64 NODES
120 D.O.F., (1/2 THE PLATE)

Figure 5

nearly twice that required by energy minimization. This may be in part due to the fact that the pseudo force technique formulates the problem using the incremental flow theory of plasticity whereas the energy minimization technique uses the simpler deformation theory of plasticity which postulates the existence of a strain energy density function of the total stresses and strains as in Eq. (7). A comparison of the responses, however, brings out quite vividly the effectiveness of both the techniques. It may be remarked in passing that the problem at hand could have equally well been formulated using the incremental flow theory when energy minimization is used as the solution procedure. The potential function, in this case, instead of being a function of the total quantities need then be expressed in terms of incremental quantities [20].

With the possible exception of the problem of Figure 5 all the other problems involve only a relatively few degrees of freedom. However, based on the results of Figures 1 through 5 it can be safely concluded that at least for small scale problems the energy minimization technique is better suited than the pseudo force technique for solving highly nonlinear problems. If anything, it is the extent of the inelasticity requiring very costly numerical volume integrations which mars its performance. Ignoring the scale of the problem for the moment, such a conclusion can have profound implications in the selection of a technique for analyzing the highly nonlinear crush response of vehicles wherein inelastic deformations are usually confined to a very small portion of the entire structure with a major portion behaving either elastically or like a rigid body.

Results of Figures 1 through 5 certainly dispel the skepticism with which investigators in the past have regarded the effectiveness of the energy minimization technique. Needless to say, the solution algorithms of the pseudo force techniques can be considerably improved upon resulting in a much higher efficiency. However, by the very repetitive nature of the computations of an implicit technique like the energy minimization technique, savings of as high as 50 percent in the computational effort may be realized simply by an expert restructuring of the computer logic alone.

It has been also claimed that the extension of the minimization algorithms to large scale problems is virtually impossible. One would not have challenged the veracity of such an assertion back in the sixties, but not any longer. Within the past decade mathematicians and computer scientists have extended the scope of the mathematical programming techniques significantly.

IV. Extension of Minimization Techniques to Large Scale Nonlinear Systems

Reference [17] concludes that for general nonlinear structural analysis, Fletcher's variable metric method (BFGS) [18] with analytic gradients is the best minimization algorithm of those considered therein. This algorithm is initially very slow in coverging to a solution presumably because it uses the null vector as an initial guess for the unknown variables and the identity matrix as an approximation to the inverse Hessian matrix [H]. Thus the solution for the first time or load step is a strong function of the total number of degrees of freedom of the problem. The same is not true, however, of the subsequent time or load steps. Having a good approximation to the inverse Hessian [H] and a good initial estimate of the variables the number of minimizations required for convergence in subsequent steps is only a small fraction of the total number of degrees of freedom. Although, Fletcher's variable metric algorithm is a very powerful tool its storage requirements (upper or lower half of the symmetric matrix [H] requiring $n \times (n + 1)/2$ storage locations for a n

degree of freedom system) limits its applicability to small scale problems. To alleviate the problems of storage one could fall back upon conjugate gradient algorithms of Fletcher-Reeves [21] and Powell [22]. But both these algorithms have been found to be not nearlyas efficient as the variable metric (BFGS) or the Davidon-Fletcher-Powell's (DFP) algorithms [23] presumably because of some ill-conditioning. So an extension of the minimization algorithms to large scale problems centers on reducing the storage requirements of the second order quasi-Newton methods (BFGS, DFP, etc.) or improving the efficiency of the first order conjugate gradient techniques. It is essentially these very features which have received the tremendous attention of the mathematicians and the computer scientists in the past few years.

The convergence rate of the conventional conjugate gradient method has been considerably improved by the use of preconditioning. Briefly, preconditioning involves the modification of the residuals (components of the gradient of the potential function) and working with the pseudo residuals rather than the residuals themselves. The modification in designed to reduce the spectral condition number of the Hessian of the potential surface and thereby accelerate convergence to the minimum. Preconditioned conjugate gradient or generalized conjugate gradient methods have been used for both linear and nonlinear problems with a great degree of success [24]-[28]. Recently, Axelsson has extended the use of generalized conjugate gradient techniques to mixed finite element variational formulations [29]. Again, using preconditioning Axelsson [29] and Widlund [30] have demonstrated the effectiveness of the generalized conjugate gradient method in solving problems pertaining to nonpositive definite systems. The details on the application of the generalized conjugate gradient technique to nonlinear elliptic boundary value problems in irregular regions, as for instance the nonlinear problems of structural analysis using a finite element discretization, are found in reference [27]. An additional improvement is an attempt to eliminate convergence problems resulting from round-off errors by a modification of the linear searches [31]-[32].

In order to be able to use second order quasi-Newton methods for large scale problems one has to exploit the fact that the Hessian of the potential surface for most finite element discretizations is symmetric and banded with a very narrow band width. However, the inverse of a banded matrix is not necessarily banded and since most variable metric methods in their original form required approximations to the inverse of the Hessian the fact that the Hessian is banded could not be exploited. Recently however, several investigators [33]-[36] have modified the variable metric methods by utilizing an approximation to the Hessian rather than its inverse in their march to the minimum without destroying the sparsity of the approximation during its updating at every step. It has been shown that by doing so the modified method still retains the convergence properties of the original method of Broyden [33] which did not account for sparsity [37]. This is a major step in the direction of extending the minimization techniques for the solution of large scale nonlinear problems of structural analysis.

Disgressing for a moment, the differential equations of motion describing the nonlinear impulsive response of structures are stiff because of the presence of very high and low frequency components in the initial stages of the response. In fact, a boundary layer phenomenon exists such that within the boundary layer the use of a standard explicit integration technique will lead to stability and round-off problems or in the case of an implicit technique will lead to inaccuracies which may destroy the reliability of the computed transient. Specialized techniques utilizing the concepts from the singular perturbation theory have been used to tackle such stiff differential equations [37]-[39]. When using minimization techniques to solve such problems the boundary layer effect exhibits itself as an ill-conditioned minimization problem. Boggs [40] has extended the singular perturbation techniques of Miranker and others [37]-[39] to modify the conventional variable metric methods so as to accelerate the convergence to the minimum value.

Several other investigators [45]-[47] have advocated the use of self-scaling variable metric algorithms. These algorithms use a two-parameter family of approximations to the inverse Hessian and determine conditions on one of the parameters to improve the condition number of the approximated Hessian inverse. The effectiveness of such scaling in conjunction with the Hessian updates themselves rather than the inverse updates is a matter that needs further investigation.

Thus, two alternatives are available in extending the minimization techniques to large scale nonlinear systems, namely: (i) the preconditioned conjugate gradient technique or (ii) the variable metric methods that exploit sparsity and utilize singular perturbation theory or scaling to eliminate ill-conditioning. The relative performance of these two alternatives is unknown at the moment, but it is clear that they both offer the promise of being able to solve extremely large scale problems efficiently. Their performance vis a vis the pseudo force or other techniques remains to be established.

VI. Concluding Remarks

A comparative efficiency evaluation of the pseudo force technique (incremental linearization - explicit) versus energy minimization techniques (nonlinear - implicit) is presented in this paper for the purpose of dispeling some of the skepticism of the past with regard to the effectiveness of the energy minimization technique. The use of analytic gradients rather than finite difference gradients significantly improves the efficiency of the energy minimization technique in the solution of nonlinear structures problems. For small scale problems the energy minimization technique is better suited than the pseudo force technique for solving highly nonlinear problems.

In the past few years the mathematicians and computer scientists have been attacking the problem areas which inhibit the extension of the minimization algorithms to large scale problems. Two alternatives are presently available, namely: (i) the preconditioned conjugate gradient technique or (ii) the variable metric methods that exploit sparsity and utilize singular perturbation theory or scaling to eliminate ill-conditioning.

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Figure Legend