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TROPOSPHERIC LIMITATIONS ON THE ACCURACY OF PHASE MEASUREMENT OF COORDINATES IN ASTRONOMY

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16. Abstract

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TROPOSPHERIC LIMITATIONS ON THE ACCURACY OF PHASE MEASUREMENT OF COORDINATES IN ASTRONOMY

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This work considers the effect of tropospheric fluctuation effects on the accuracy of phase measurements of coordinates. The nature of the averaging of the trophospheric effects, if N coordinate measurements of duration T with period μ are made, is investigated. Various averaging modes depending on the relation of the various time parameters are investigated. Equations taking into account the correlations between individual observations are presented. It is shown that the correlation interval between the individual observations is always greater than the fluctuation period of tropospheric inhomogeneities typical for a given baseline.

In measuring the coordinates of celestial sources with the help /108* of ground-based optical and radio telescopes, the presence of tropospheric inhomogeneities of the refractive index leads to the generation of random errors in the measured positions of the sources. If a double-element interferometer is considered as the telescope model, then the effect of the troposphere appears random advances in phase of the signal ψ in the troposphere along the routes to the first and second antennas, whose magnitude is determined by the fluctuation

*Numbers in the margin indicate pagination in the original foreign text.

properties of the troposphere. As is well known, it is convenient to consider as a sufficiently general characteristic of these properties the structure function D_l of the electric thickness of the troposphere [1], which is related to the structure function of the phase in the geometric optics approximation in an elementary manner:

$$D_\varphi = \langle [\varphi(t+\Delta t) - \varphi(t)]^2 \rangle = \left(\frac{2\pi}{\lambda}\right)^2 D_l,$$

where λ is the wavelength.

The time structure function of the real troposphere is well approximated by the Karman function:

$$D_\varphi(\Delta t) = 2a^2 \left\{1 - \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\Delta t}{t_0}\right)^\nu K_\nu\left(\frac{\Delta t}{t_0}\right)\right\} \left(\frac{2\pi}{\lambda}\right)^2 \quad (1)$$

with the parameters $a = 10$ cm, $t_0 = 5$ hr and $\nu = 5/6$ [2]. Here $\Gamma(\nu)$ is the gamma function and K_ν is the MacDonald function. The hypothesis of "frozen" turbulence, which is confirmed experimentally for the free atmosphere from the measurements of meteorological parameters up to scales the order of a kilometer [3], permits converting from the time to the spatial structure function $D_\varphi(\rho) = D_\varphi(v\Delta t)$, where ρ is the interferometer baseline and v is the average drift velocity of the tropospheric inhomogeneities, the average wind speed. The existing estimates for v are very indfinite [4]. However, if one assume a compromise value of v = 10 m/sec, which coincides with the typical wind speed for synoptic processes [5], then the overall scale of turbulence in the assumed model will be $\rho_0 = vt_0 = 180$ km. We note that inhomogeneities close to these scales have actually been observed [6]. /109

In the region of scales small as compared to the overall scale of turbulence $\rho \ll \rho_0$ (inertial interval), the Karman approximation (1) gives the well-known "5/3 law" following directly from the Kolmogorov-Obukhov turbulence theory [1]. For large values $\rho > \rho_0$ (energy interval), the Karman function tends asymptotically to the constant value $2a^2(2\pi/\lambda)^2$, determining the maximum advance of the phase difference in the troposphere. It should be noted that, although the Karman approximation may not correspond to actual physical processes in the intermediate range of values of ρ [7], it remains a convenient approximation, which is completely suitable for obtaining estimates in analytic form.

The structure function $D_\varphi(\rho)$ permits evaluating the tropospheric error $\Delta\theta$ arising with an instantaneous measurement of the coordinate of a source: $\Delta\theta \approx \sqrt{D_\varphi(\rho)}$. This error can obviously be reduced if one uses time accumulation. There is the widely held opinion that the nature of the averaging is determined only by the fluctuation period of tropospheric inhomogeneities $t_\rho = \rho/v$ typical for the baseline ρ, which coincides with the time correlation interval of the fluctuations of the refractive index of the tropospheric inhomogeneities. It then seems that one can make $N = T/t_\rho$ independent readings during the time $T > t_\rho$ and, consequently, reduce the tropospheric error by $\sqrt{T/t_\rho}$ times: $\Delta\theta_T \propto \Delta\theta \Big/ \sqrt{\frac{T}{t_\rho}}$. However, if one supports this point of view, then the following paradoxical conclusion is inevitable: the tropospheric angular error $\Delta\theta_T$ in the inertial interval is smaller, the smaller the baseline. Actually, for $\rho < \rho_0$, $\Delta\theta \propto \rho^{\nu-1}$ ($\Delta\theta \to \infty$ as $\rho \to 0$), and thus $\Delta\theta_T \propto \rho^{\nu-1}\sqrt{\rho}/\sqrt{Tv} \propto \rho^{\nu-1/2}/\sqrt{Tv}$, since $\Delta\theta_T \to 0$ as $\rho \to 0$.

Actually, as rigorous analysis shows, t_ρ in the general case does not determine the correlation interval and is always significantly greater than t_ρ for $\rho < \rho_0$. The dependence of $\Delta\theta_T$ on the accumulation time T is determined by the mutual relation of the time t_0 characterizing the overall scale of the turbulence and the times t_ρ and T characterizing the spatial and time fluctuations of the tropospheric inhomogeneities. In particular, in the inertial interval of greatest interest for optics, the time accumulation, as will be shown below, effectively weakens the effect of the troposphere only for sufficiently large accumulation times $T > t_0$. Then $\Delta\theta_T$ does not depend on the baseline.

This work considers the nature of the time averaging of the tropospheric angular error and analytic estimates are obtained for the following model of observation procedure — N sequences of observations of duration T with period μ ($\mu/T \geqslant 1$) are carried out. The obtained results develop and generalize the results of [8] in which

some modes of averaging one observation of duration T were studied. In spite of the fact that a double-element interferometer is considered as the model of the measuring device for analysis, the obtained results are also applicable to instruments with filled apertures. Actually, in the case of a continuous aperture with a uniform ample distribution, the fluctuations of the position of the principal lobe determined by the fluctuations of the center of gravity of the radiation pattern differ from the value obtained on the basis of fluctuations of the phase difference at the edges of the aperture only by a coefficient the order of unity (0.97 for a circular aperture [1], 1 - 1.5 for a linear aperture). The latter is related to the fact that small-scale fluctuations (the order of the dimensions of the aperture and smaller) are filtered by the continuous aperture and do not shift the center of gravity of the image in practice, /110
but only lead to its blurring

Spectrum of the phase difference fluctuations and general equations for the dispersion of the phase difference. For further analysis, it is convenient to transform from the structure function (1) to the phase fluctuation spectrum related by the Fourier transformation:

$$W_{\psi}(\omega)=\left(\frac{2\pi a}{\lambda}\right)^{2}\frac{t_{0}}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)}\frac{1}{\left(1+(\omega t_{0})^{2}\right)^{\nu+\frac{1}{2}}}, \quad (2)$$

which is related to the spectral density of the phase difference fluctuations $\delta\psi=\psi_{2}-\psi_{1}$ at two points located at a distance ρ by the relation:

$$W_{\delta\psi}(\omega)=4\sin^{2}\frac{\rho\omega}{2v}W_{\psi}(\omega), \quad (3)$$

where ω is the cyclic frequency. The periodic factor in (3) characterizes the spatial filtration of the tropospheric inhomogeneities with dimensions greater than the instrumental baseline ρ.

Time averaging over the interval T leads to filtration of the fluctuations with frequencies $\omega\geqslant 1/T$:

$$W_{\delta\psi}^{T}(\omega)=\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^{2}W_{\delta\psi}(\omega). \quad (4)$$

The last expression for the spectral density determines the dispersion of the phase difference caused by motion of the tropospheric inhomogeneities for an instrument with a baseline ρ for a single sequence of observation of duration T:

$$\sigma_1^2 \equiv \sigma_{\Delta\varphi}^2(\rho, T) = 2\int_0^\infty W_{\Delta\varphi}^T(\omega)\, d\omega = A\int_0^\infty \frac{\sin^2 \alpha\omega \sin^2 \beta\omega}{\omega^2 \left(1 + (\omega t_0)^2\right)^{\nu+1/2}}\, d\omega, \tag{5}$$

where $\alpha = \rho/2v$; $\beta = T/2$, and

$$A = \frac{8}{\sqrt{\pi}} \frac{t_0}{\beta^4} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu)} \left(\frac{2\pi a}{\lambda}\right)^2. \tag{5a}$$

Let there be N sequences of observations repeated with period μ with a duration of the observations in each sequence $T\,(\mu \geqslant T)$. The time shift of the signal by the amount μ is equivalent to multiplying its spectrum by the quantity $e^{-i\omega\mu}$. Consequently, the total spectral density for N sequences will have the form:

$$W_{\Delta\varphi}^{T,\mu,N}(\omega) = \frac{1}{N^2}\left|1 + e^{-i\omega\mu} + \ldots + e^{-i(N-1)\omega\mu}\right|^2 W_{\Delta\varphi}^T(\omega). \tag{6}$$

Since the expression within the absolute value sign is a sum of terms of a geometric progression with the ratio $e^{-i\omega\mu}$, then, as can be shown:

$$\left|1 + e^{-i\omega\mu} + \ldots + e^{-i(N-1)\omega\mu}\right|^2 = \left|\frac{1 - e^{-iN\omega\mu}}{1 - e^{-i\omega\mu}}\right|^2 = N + 2\sum_{k=0}^{N-1} k\cos(N-k)\,\omega\mu. \tag{7}$$

Thus, the spectral density in the case of N measurements will be:

$$W_{\Delta\varphi}^{T,\mu,N}(\omega) = \frac{A}{N^2} \frac{\sin^2 \alpha\omega \sin^2 \beta\omega}{\omega^2 \left(1 + (\omega t_0)^2\right)^{\nu+1/2}} \left(N + 2\sum_{k=0}^{N-1} k\cos(N-k)\,\mu\omega\right). \tag{8}$$

By integrating (8) by parts, we arrive at the following expression for the dispersion of the phase difference:

$$\sigma_N^2 = \sigma_{l\varphi}^2(\rho,\ T,\ \mu,\ N) = 2\int_0^\infty W_{l\varphi}^{T,\mu,N}(\omega)\,d\omega =$$

$$= AN^{-1}\left\{\int_0^\infty \frac{\alpha \sin 2\alpha\omega \sin^2\beta\omega + \beta \sin 2\beta\omega \sin^2\alpha\omega}{\omega(1+(\omega t_0)^2)^{\nu+1/2}}\,d\omega - \right.$$

$$\left. -(2\nu+1)\,t_0^2\int_0^\infty \frac{\sin^2\alpha\omega \sin^2\beta\omega}{(1+(\omega t_0)^2)^{\nu+3/2}}\,d\omega\right\} +$$

$$+ 2AN^{-2}\sum_{k=0}^{N-1} k\left\{\int_0^\infty \frac{\alpha \sin 2\alpha\omega \sin^2\beta\omega + \beta \sin 2\beta\omega \sin^2\alpha\omega}{\omega(1+(\omega t_0)^2)^{\nu+1/2}}\times\right.$$

$$\times \cos(N-k)\mu\omega\,d\omega - (2\nu+1)\,t_0^2\int_0^\infty \frac{\sin^2\alpha\omega \sin^2\beta\omega}{(1+(\omega t_0)^2)^{\nu+3/2}}\cos(N-k)\mu\omega\,d\omega -$$

$$\left. -(N-k)\mu\int_0^\infty \frac{\sin^2\alpha\omega \sin^2\beta\omega \sin(N-k)\mu\omega}{\omega(1+(\omega t_0)^2)^{\nu+1/2}}\,d\omega\right\} \equiv (I_1+I_2)+(J_1+J_2+J_3). \qquad (9)$$

It is not difficult to show that the sum of the integrals $\sigma_1^2 = I_1 + I_2 = \sigma_1^2/N$. If $J_1 + J_2 + J_3 = 0$, then the individual sequences were averaged as statistically independent. Consequently, the group of terms related to the integrals J_1, J_2, J_3 characterize the effect of the correlations between the individual sequences.

By using simple trigonometric relations, the integrals I_1, J_1 and J_3 can be reduced to the sum of the integrals of the form:

$$\int_0^\infty \frac{\sin\gamma\omega\,d\omega}{\omega(1+(\omega t_0)^2)^{\nu+1/2}} = \frac{\gamma}{2t^\nu}\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left\{\Gamma(\nu)\,{}_1F_2\left(\frac{1}{2};\ 1-\nu,\ \frac{3}{2};\ \left(\frac{\gamma}{2t_0}\right)^2\right) - \right.$$

$$\left. -\frac{\Gamma(1-\nu)}{\nu(2\nu+1)}\left(\frac{\gamma}{2t_0}\right)^{2\nu}{}_1F_2\left(\nu+\frac{1}{2};\ \nu+\frac{3}{2};\ \nu+1;\ \left(\frac{\gamma}{2t_0}\right)^2\right)\right\} \equiv \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\frac{2t_0}{\gamma}Y_\nu\left(\frac{\gamma}{2t_0}\right). \qquad (10a)$$

where ${}_1F_2(a,\ b,\ c,\ z)\ |z>0|$ is the generalized hypergeometric series, while the integrals I_2 and J_2 can be reduced to the sum of the integrals:

$$\int_0^\infty \frac{\cos\gamma\omega\,d\omega}{(1+(\omega t_0)^2)^{\nu+3/2}} = \frac{\sqrt{\pi}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}\times\begin{cases} 2^\nu\Gamma(\nu+1), & \gamma=0 \\ \left(\frac{\gamma}{t_0}\right)^\nu K_\nu\left(\frac{\gamma}{t_0}\right), & \gamma\neq 0\end{cases} \equiv \frac{\sqrt{\pi}}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)t_0}X_{\nu+1}\left(\frac{\gamma}{t_0}\right). \qquad (10b)$$

which is the Basset integral representation for the MacDonald function [9].

Then, by keeping in mind the integral expressions (10), we have:

$$I_1 = \frac{\sqrt{\pi}}{2\Gamma\left(\nu+\frac{1}{2}\right)} A N^{-1} t_0 \left\{ Y_\nu\left(\frac{\alpha}{t_0}\right) + Y_\nu\left(\frac{\beta}{t_0}\right) - \frac{1}{2} Y_\nu \left|\frac{\alpha+\beta}{t_0}\right| \right\};$$

$$J_1 + J_2 = \frac{\sqrt{\pi}}{2\Gamma\left(\nu+\frac{1}{2}\right)} A N^{-2} t_0 \sum_{k=0}^{N-1} k \left\{ Y_\nu \left|\frac{2\alpha \pm (N-k)\mu}{2t_0}\right| + Y_\nu \left|\frac{2\beta \pm (N-k)\mu}{2t_0}\right| - \right.$$
$$\left. - \frac{1}{2} Y_\nu \left|\frac{2\alpha + 2\beta + (N-k)\mu}{2t_0}\right| - \frac{1}{2} Y_\nu \left|\frac{2\alpha \pm 2\beta - (N-k)\mu}{2t_0}\right| - 2Y_\nu \left|\frac{(N-k)\mu}{2t_0}\right| \right\};$$

$$I_2 = \frac{\sqrt{\pi}}{2^{\nu+2}\Gamma\left(\nu+\frac{1}{2}\right)} A N^{-1} t_0 \left\{ 2^\nu \Gamma(\nu+1) - X_{\nu+1}\left(\frac{2\alpha}{t_0}\right) - X_{\nu+1}\left(\frac{2\beta}{t_0}\right) + \frac{1}{2} X_{\nu+1} \left|\frac{2\alpha \pm 2\beta}{t_0}\right| \right\};$$

$$J_3 = -\frac{\sqrt{\pi}}{2^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)} A N^{-2} t_0 \sum_{k=0}^{N-1} k \left\{ X_{\nu+1}\left(\frac{(N-k)\mu}{t_0}\right) - \frac{1}{2} X_{\nu+1} \left|\frac{2\alpha + (N-k)\mu}{t_0}\right| - \right.$$
$$- \frac{1}{2} X_{\nu+1} \left|\frac{2\beta + (N-k)\mu}{t_0}\right| + \frac{1}{4} X_{\nu+1} \left|\frac{2\alpha + 2\beta \pm (N-k)\mu}{t_0}\right| +$$
$$\left. + \frac{1}{4} X_{\nu+1} \left|\frac{2\alpha - 2\beta + (N-k)\mu}{t_0}\right| \right\}. \qquad (11)$$

Here and below, ($\pm$) denotes summation over the corresponding two terms: $f(x \pm y) = f(x+y) + f(x-y)$. We draw attention to the obvious fact that as $\mu \to \infty$, $J_1 + J_2 + J_3 \to 0$; there is no correlation between the individual sequences and, as a consequence, $\sigma_N = \sigma_1 / \sqrt{N}$. /112

The extremely cumbersome expression for the dispersion of the phase difference fluctuation (11) caused by the motion of tropospheric inhomogeneities becomes accessible for simple analysis in a majority of cases of greatest importance for observational astronomy.

<u>Asymptotic expressions for the dispersion of the phase difference</u>. The nature of the averaging of σ_N depends on the relation of five time parameters: t_0, $t_\rho = \rho/v = 2\alpha$, $T = 2\beta$ and the total observation time $T_{ef} = (N-1)\mu + T$.

A. Small t_ρ, T_{ef}. Let all the characteristic time parameters t_ρ, T_{ef}, and consequently T, μ, be small as compared to the time scale t_0 determined by the overall scale of the turbulence. In other words, we will consider the case of small baselines ($\rho < 100$ km) and small accumulation times ($T < 1$ hr).

Since the functions $Y_\nu(z)$ and $X_{\nu+1}(z) (\nu > 0)$ have the following asymptotic expressions for small $z \ll 1$:

$$Y_\nu(z) \approx z^\nu \left\{ \Gamma(\nu) - \frac{\Gamma(1-\nu)}{\nu(2\nu+1)} z^2 \right\};$$

$$X_{\nu+1}(z) \approx \frac{\pi}{2\sin\pi\nu} \left\{ \frac{z^{2\nu+2}}{2^{\nu+1}\Gamma(\nu+2)} - \frac{2^{\nu+1}}{\Gamma(-\nu)} - \frac{z^2}{2^{1-\nu}\Gamma(1-\nu)} \right\}, \qquad (12)$$

we obtain for (11) after some transformation:

$$I_1 = -\frac{\nu+1}{\nu+\frac{1}{2}} I_2 = \frac{4}{2\nu+1}\left(\frac{2\pi a}{\lambda}\right)^2 N^{-2} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{T}{2t_0}\right)^{2\nu} \left\{ \frac{1}{2}\left|1 \pm \frac{\rho}{vT}\right|^{2\nu+2} - \left(\frac{\rho}{vT}\right)^{2\nu+2} - 1 \right\};$$

$$I_1 + I_3 = -\frac{\nu-1}{\nu+\frac{1}{2}} I_2 = \frac{4}{2\nu+1}\left(\frac{2\pi a}{\lambda}\right)^2 N^{-2} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{2t_0}{T}\right)^2 \times$$

$$\times \sum_{k=0}^{N-1} k \left(\frac{(N-k)\mu}{2t_0}\right)^{2\nu+2} \left\{ 2 + \frac{1}{2}\left|1 \pm \frac{\rho + T_e}{(N-k)\mu v}\right|^{2\nu+2} + \frac{1}{2}\left|1 \pm \frac{\rho - T_e}{(N-k)\mu v}\right|^{2\nu+2} - \left|1 \pm \frac{\rho}{(N-k)\mu v}\right|^{2\nu+2} - \left|1 \pm \frac{T_e}{(N-k)\mu v}\right|^{2\nu+2} \right\}. \qquad (13)$$

We note that the expression $\sigma^2 - \frac{1}{2(\nu+1)}(I_1 + I_2 + I_3)$ determined by (13) coincides as $\mu \to \infty$ and N = 1 with the analogous expression obtained by another method in [4]. We also draw attention to the fact that $(I_1 + I_3)|_{\mu=T} + I_1(T) = NI_1(NT)$, as must be, since for $\mu = T$, $T_{ef} = NT$.

In the approximation under consideration, the following rigorous inequality is satisfied for the terms characterizing the correlations between the individual sequences $\sigma^2 - I_1 + I_2 + I_3$.

$$4(2\nu-1)\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \sum_{k=0}^{N-1} \frac{k}{N^2}\left(\frac{(N-k)\mu}{2t_0}\right)^{2\nu}\left(\frac{\rho}{(N-k)\mu v}\right)^2 \leqslant \sigma_0^2 \Big/ \left(\frac{2\pi a}{\lambda}\right)^2 \leqslant$$
$$\leqslant 2\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(1 - \frac{1}{N}\right)\left(\frac{\rho}{2vt_0}\right)^{2\nu}. \qquad (14)$$

The right side of this inequality corresponds to the condition
$T_{ef} \ll t_0$ when the effect of the correlation terms are minimum for given ρ, T_{ef}, and μ. The left side of the inequality is satisfied for $\mu \gg T, t_0$, when the effect of the correlations between the individual sequences are minimum for given ρ and T, and can serve as a good approximation of the exact expressions for σ_0^2 even for $\mu > 5T, 5t_0$. We also note that there is the estimate convenient for further calculations:

$$4(2\nu-1)\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \sum_{k=0}^{N-1} \frac{k}{N^2}\left(\frac{(N-k)\mu}{2t_0}\right)^{2\nu}\left(\frac{\rho}{(N-k)\mu v}\right)^2 \leqslant$$
$$\leqslant 2\nu(2\nu-1)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(1 - \frac{1}{N}\right)\left(\frac{\mu}{2t_0}\right)^{2\nu}\left(\frac{\rho}{\mu T}\right)^2. \qquad (14a)$$

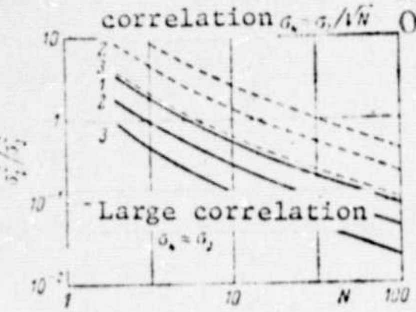

Figure 1. Dependence of σ_N/σ_1 on the number of sequences N for T = 0.1 sec

Dotted line — $\mu, T = 10$;
solid lines — $\mu, T = 1000$, 1 — $\rho = 1$; 2 — 10;
3 — 100 m

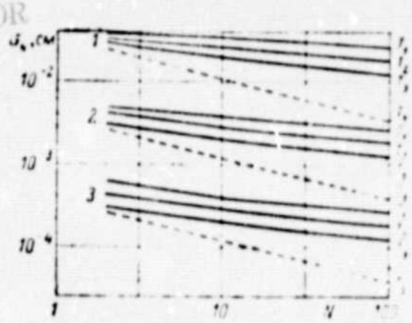

Figure 2. Dependence of σ_N on number of observations N for $\mu, T = 10$.
1 — $\rho = 100$; 2 — 10; 3 — 1 m; $T_1 = 0.1$; $T_2 = 1$; $T_3 = 10$
sec; dotted lines — $\sigma_1/\sqrt{N}$.

According to (14), the effect of the correlation terms are significant even for significant ratios μ/T and μ/t_ρ (Figure 1). Actually, σ_N^2 depends weakly on N and the simple accumulation of the number of sequences N for specified T_{ef} and ρ does not lead to a significant decrease in the tropospheric error σ_N^2 (Figure 2). The latter actually indicates that when all the characteristic time parameters are less than t_0, the effect of time accumulation is scarcely perceptible.

B. Small μ and t_ρ, large T_{ef}. This mode differs from that considered above in that the total observational time T_{ef} can become larger than the characteristic time t_0 for a sufficiently large number of sequences N. In this case, as follows from (11), the summation in expression (13) for I_1, J_2 and J_3 must be extended only to $k = N^* - 1$, where N* is the integer part of the ratio t_0/μ. Consequently, in this case, if $\mu > T, t_\rho$,

$$\sigma_N^2 \approx \left(\frac{2\pi a}{\lambda}\right)^2 2\nu(2\nu-1)\frac{(N^*-1)N^*}{N^2}\left(\frac{\mu}{2t_0}\right)^{2\nu}\left(\frac{\rho}{ut}\right)^2.$$

An increase in the number of sequences in this mode can obviously lead to a significant decrease in σ_N^2.

C. Small T and t_ρ, large T_{ef}.* Let the baseline ρ be small as compared to the overall scale of the turbulence ρ_0 and the accumulation time T in an individual sequence be less than the time t_0 typical for the troposphere. If the interval between individual sequences $\mu > t_0$, then $\sigma_3^2 = 0$, as is easily seen from (11), so that the sequences are statistically independent where, according to (12) and (13): /114

$$\sigma_N^2 = \frac{\sigma_1^2}{N} = \frac{2}{N}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\left(\frac{2\pi a}{\lambda}\right)^2 \begin{cases} \left(\frac{\rho}{2vt_0}\right)^{2\nu}, & T < t_\rho; \\ \left(\frac{\rho}{vT}\right)^2\left(\frac{T}{2t_0}\right)^{2\nu}, & T > t_\rho. \end{cases} \quad (15)$$

It is not difficult to see that within an individual sequence the time averaging occurs very slowly $(T > t_\rho)$ or not at all $(T < t_\rho)$. This result is quite natural from the point of view of case A. Actually, the individual sequence of duration T can be considered as the sum of a large number of observations of duration T/N between which, according to (14), there are strong correlations.

D. Small t_ρ, large T. Let the accumulation time T in an individual sequence be greater than t_0, and the baseline be small as compared to the overall scale of the turbulence. If $\Delta T = \mu - T$ is the time interval between the individual sequences, then we have beyond the dependence of the relation between ΔT, T and t_ρ, according to (11):

$$I_1 = \frac{16N^{-1}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{t_0}{T}\right)^2 Y_\nu\left(\frac{\rho}{2vt_0}\right);$$

$$I_2 = \frac{8N^{-1}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{t_0}{T}\right)^2\left\{X_{\nu+1}\left(\frac{\rho}{vt_0}\right) - 2^\nu\Gamma(\nu+1)\right\}; \quad (16a)$$

$$I_1 + I_3 = \frac{16}{\Gamma(\nu)}\frac{N-1}{N^2}\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{t_0}{T}\right)^2\left\{Y_\nu\left(\frac{\Delta T}{2t_0}\right) - \frac{1}{2}Y_\nu\left|\frac{\Delta T \pm \rho/v}{2t_0}\right|\right\}; \quad (16b)$$

$$I_2 = \frac{2^{3-\nu}}{\Gamma(\nu)}\frac{N-1}{N^2}\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{t_0}{T}\right)^2\left\{X_{\nu+1}\left(\frac{\Delta T}{t_0}\right) - \frac{1}{2}X_{\nu+1}\left|\frac{\Delta T + \rho/v}{t_0}\right|\right\}.$$

When the time interval between individual sequences $\Delta T > t_0$, $\sigma_3^2 = 0$ and the individual sequences become statistically independent, which is quite natural since the characteristic time t_0 determines the

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*Case C is the transition from case A to case B, since for $\mu > t_0$, $N^* = 1$ and $\sigma_3^2 = 0$, and for $T_{ef} < t_0$, $N^* \approx N$.

natural scale of the correlation of the tropospheric inhomogeneities. Since $t_\rho < t_0$, keeping in mind (12), we have according to (16a):

$$\sigma_N^2 = \frac{\sigma_1^2}{N} = \frac{2}{N}\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{\rho}{vT}\right)^2. \qquad (17)$$

If the time interval sequences $\Delta T < t_0$, then from (12) and (16):

$$\sigma_N^2 = 2\left(\frac{2\pi a}{\lambda}\right)^2\left(\frac{\rho}{vTN}\right)^2\left(1+\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(N-1)\left(\frac{\Delta T}{2t_0}\right)^{2\nu}\right). \qquad (18)$$

Thus, for $T > t_0$, the effect of the time averaging of the tropospheric advance of the phase difference becomes very perceptible, while the dependence of σ_N on the accumulation T is general and does not depend on the value of the index ν.

E. Large t_ρ and small T. Let the dimension of the baseline ρ be larger than the overall scale of the turbulence ρ_0, which occurs for observations on radio interferometers with extra long baselines (ELBR). The accumulation time for ELBR is limited by the instability of the time and frequency standards, and is always less than t_0.

If $T_{ef} < t_0$ in the approximation under consideration, then /115
according to (11) and (12):

$$I_1 = -2I_2, \quad \frac{I_1 + I_3}{N-1} = \frac{2I_2}{N-1} = 4N^{-1}\left(\frac{2\pi a}{\lambda}\right)^2,$$

and the individual sequences thus have maximum correlation since:

$$\sigma_N^2 = \sigma_1^2 = 2\left(\frac{2\pi a}{\lambda}\right)^2. \qquad (19)$$

It is obvious that (19) is a direct consequence of (1) with $\rho > \rho_0$, and is the maximum possible advance of the phase difference in the troposphere.

If for a small accumulation time $T < t_0$, the period between sequences μ exceeds the characteristic scale t_0, then beyond the dependence on the relation between t_ρ and μ, the individual sequences become statistically independent, $\sigma_3^2 = 0$ and

$$\sigma_N^2 = \frac{\sigma_1^2}{N} = \frac{2}{N}\left(\frac{2\pi a}{\lambda}\right)^2.$$

In both cases, as should be expected, the tropospheric advance of the base difference depends neither on the duration of the sequence nor on the baseline.

<u>Tropospheric limitations with coordinate measurements.</u> The tropospheric advance of the phase difference σ_N, the expressions for which were obtained above, leads to a mean-square error in the measurement of the angular position of the source $\Delta\theta_N = \sigma_N / \frac{2\pi}{\lambda} \rho$ (radian) $= 2 \cdot 10^5 \sigma_N / \frac{2\pi}{\lambda} \rho$ (arcsec).

The nature of the dependence of $\Delta\theta_N$ on the baseline and the averaging time T_{ef} is determined by the combined effect of two effects — spatial (the parameter t_ρ) and time (the parameters T, μ and T_{ef}) filtrations. It is seen directly from the expression for the spectral density (8) that the effectiveness of both effects depends mainly on the relation of t_ρ, T, μ and the time parameter t_0 typical for the troposphere. The spatial and time averaging of $\Delta\theta_N$ becomes rather significant only when the corresponding time parameter is greater than t_0. This means that the main contribution to the tropospheric error for measuring the angular position of a source in introduced by large-scale (low-frequency) tropospheric inhomogeneities. Their effect can be reduced only on baselines large as compared to the overall scale of the turbulence or for averaging times T large as compared to t_0.

1. Baselines small as compared to the overall scale of the turbulence $\rho < \rho_0 (t_\rho < t_0)$. In this case, in accordance with the comments made above, a decrease in the tropospheric error can be achieved only due to an increase in the accumulation time in the individual sequence or period between sequences μ, since the effect of the spatial filtration is scarcely perceptable in this mode.

a) Accumulation time in the sequence $T < t_0$. If the period between sequences $\mu > t_0$, then the observations in the individual sequences are statistically independent and $\Delta\theta_N = \Delta\theta_1/\sqrt{N}$, where, according to (15), the asymptotic expression for $\Delta\theta_1$ will be:

$$\Delta\theta_1 = \begin{cases} 3.75/\rho^{1/6}, & \text{if} \quad T \leqslant t_\rho; \\ 1.71/T^{1/6}, & \text{if} \quad T \geqslant t_\rho. \end{cases} \tag{20}$$

where ρ is expressed in centimeters and T in seconds. The precise dependences of $\Delta\theta_1$ on ρ and T are presented in Figure 3 and 4. The dashed lines here indicate the results of calculations under the assumption that t_ρ determines the correlation interval between instantaneous measurements, while $\sqrt{T/t_\rho}$ is the number of independent readings during the time T. It is not difficult to see that this

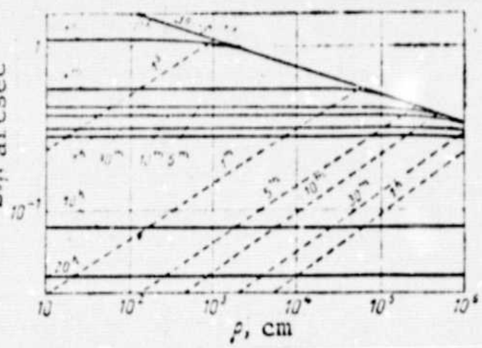

Figure 3. Dependence of the angular error of a single measurement $\Delta\theta_1$ of duration T on the baseline β for $t_\rho < t_0$.

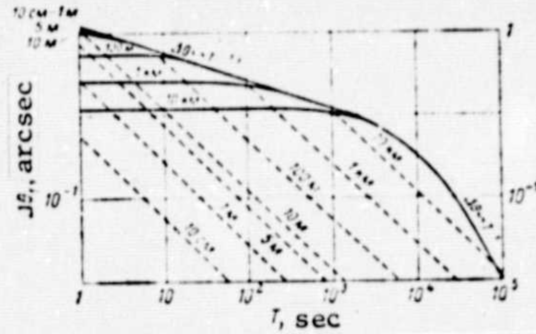

Figure 4. Dependence of the angular error of a single measurement $\Delta\theta_1$ over the baseline ρ on the accumulation time T for $t_\rho < t_0$.

point of view is true only in the trivial case $T \leqslant t_\rho$, while for $T \gg t_\rho$ this leads to estimates differing by an order of magnitude from the actual. /116

The mode $T < t_\rho$, which corresponds to an instantaneous readout for a given baseline, is realized extremely rarely in practice (in radio astronomy — for large antennas and for bright sources) since the accumulation time T is determined not only by the fluctuation properties of the troposphere, but also by the noise properties of the recording apparatus. As a rule, the signal/noise ratio becomes significant only for $T \to t_\rho$, and the tropospheric error (20) in this case does not depend on the baseline and decreases gradually with increasing T [8].

If the period between the sequences $\mu < t_0$, then, as has already been stated, the individual sequences are not statistically independent, in connection with which the averaging of $\Delta\theta_N$ in the general case goes more slowly than the law $1/\sqrt{N}$. According to (14) and (14a), the asymptotic expressions for the angular tropospheric error have the form:

$$\Delta\theta_N = \begin{cases} \frac{1.71}{\sqrt{N}} T^{-1/6} \sqrt{1 + 0.6 \frac{N^*(N^*-1)}{N} \left(\frac{T}{\mu}\right)^{1/3}}, & \text{if} \quad \mu \geqslant T \geqslant t_0; \\ \frac{3.5}{\sqrt{N}} \rho^{-1/6} \sqrt{1 + 0.706 \frac{N^*(N^*-1)}{N} \left(\frac{t_0}{\mu}\right)^{1/3}}, & \text{if} \quad \mu \geqslant t_0 \geqslant T. \end{cases} \tag{21}$$

where ρ is measured in centimeters and μ and T in seconds.

Figure 5 presents the exact dependences of $\Delta\theta_N$ on the period between sequences for the two modes (21). The estimates for other cases can be obtained by using Figures 1 - 3. /117

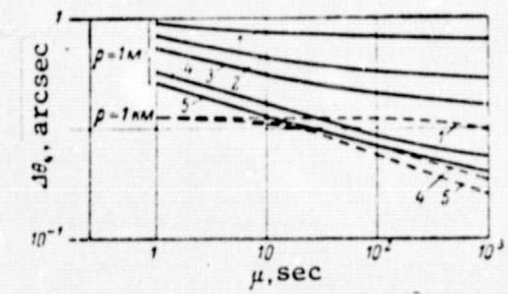

Figure 5. Dependence of the angular error $\Delta\theta_N$ on the period between sequences for T - 1 sec:
1 — N = 2; 2 — 5; 3 — 10; 4 — 50; 3 — 100

It is not difficult to see that the averaging law (21) coincides with the averaging law of statistically independent sequences only for significant ratios μ/T, μ/t_0 or N/N^*. We note that for $T_{ef} < t_0 (N^* < N)$ and specified ρ, T, and μ, one can always find such a value N, beginning with which an increase in the number of sequences does not lead to a notable decrease in the tropospheric error, and its dependence in this case on μ has the general nature: $\Delta\theta_N \approx 0''.86/\mu^{1/6}$.

Finally, the completely obvious consequences of (14) and (15) are the absence of a time averaging effect for $T_{ef} < t_0 - \Delta\theta_N = 3''.5\rho^{1/6}$.

b) The accumulation time in the sequence $T > t_0$. In the mode when the accumulation time T is greater than the characteristic t_0, there occurs time averaging of large-scale tropospheric inhomogeneities (with frequencies $\omega > 1/t_0$) which, as has already been noted, make the main contribution to the error in measuring the angular position of a source. If the time interval between sequences $\Delta T = \mu - T < t_0$, then according to (18) the averaging of the tropospheric error will be very intensive:

$$\Delta\theta_N = \frac{278\cdot 10^{9}}{NT}\sqrt{1+1.5(N-1)10^{-7}(\Delta T)^{2}}, \quad (22)$$

where the more effective, the smaller ΔT (Figure 6). The latter is quite obvious since the case of continuous measurements $\Delta T \to 0$ $(T_{ef} = NT)$ is the most favorable for decreasing the tropospheric error.

If $\Delta T > t_0$, then individual sequences become statistically independent, and according to (19):

$$\Delta\theta_N = 278\cdot 10^{9}/\sqrt{N}\,T. \quad (23)$$

The tropospheric error does not depend on the baseline ρ in both cases under consideration.

We note in conclusion that even for a very large total observation time, according to (22) and (23), it is difficult to obtain accuracy better than 0.05 arcsec (Figures 3, 4, 6).

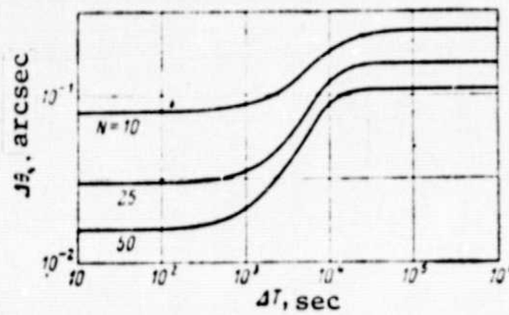

Figure 6. Dependence of the error of the angular measurement $\Delta\vartheta_N$ on the time interval between sequences ΔT for T = 10 sec

2. Baselines large as compared to the overall scale of the turbulence $\rho > \rho_0$ $(t_\rho > t_0)$. In this case, which is typical for measurements of ELBR, $T \ll t_0$, and there is no time averaging within the sequence since:

$$\Delta\theta_N = 278\cdot 10^{9}\rho, \quad (24)$$

where ρ is measured in centimeters. A decrease in the tropospheric error occurs only due to saturation of $\sigma_{\Delta\varphi}$ in the energy interval, and the effect which in some sense "clears" the troposphere. In /118

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spite of the absence of time averaging, the effect of tropospheric effects can be limited with the help of ELBR on baselines the order of several thousand kilometers, as is well known, to thousandths of arcsec per unit measurement. According to (2), ELBR observations can be considered statistically independent only for $\rho \gtrsim l_0$ when $\Delta\theta_N = \Delta\theta_1/\sqrt{N}$.

Conclusion. The presented estimates show that the fluctuation effects of the troposphere strongly limit the accuracy of coordinate measurements even for large accumulation times $(T > t_0)$. This conclusion is true to the greatest degree for measurements carried out on small baselines $(\rho < \rho_0)$, which are typical for optical astronomy. The troposphere limits the accuracy of the coordinate measurements to hundredths of an arcsec even with exposures of several hours.

The use of the methods of differential astronomy or the reference object method can significantly weaken the effect of the troposphere. However, this assertion is true only for very small angular distances ϕ between the observed and reference objects: $\rho_{ef} = \phi h_{ef} \leq \rho$, where $h_{ef} \approx 10$ km is the effective height of the troposphere; ρ is the diameter of the telescope or the baseline of the interferometer. In this case, the phase advance is determined by the effective baseline ρ_{ef}, and the angular error will thus decrease with decreasing ϕ. In connection with this remark, all the estimates obtained in the work are completely applicable both for absolute as well as for differential measurements in optical and radio astronomy.

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