NASA Technical Memorandum 58221

An Analytical Singularity-Free Solution to The J_2 Perturbation Problem

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July 1979

National Aeronautics and
Space Administration

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Scientific and Technical
Information Office

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SUMMARY

The purpose of this paper is to develop a singularity-free analytical solution to the J_2 perturbation problem in the Earth satellite theory. Elements similar to the classical Poincaré elements, but which are in the extended phase space, are used. The solution, which has the long-period terms eliminated, is developed into a computer algorithm which is then compared for several typical examples with numerically integrated results. The error of the analytical solution is of the order of 500 to 1000 meters after 100 periods of revolution.

INTRODUCTION

The objective of this paper is to develop a singularity-free (for vanishing eccentricity and inclination) solution of the J_2 problem in the satellite theory. The procedure resembles the one described by Lyddane (ref. 1), which rederives Brouwer's satellite theory (ref. 2), by using canonical Poincaré elements instead of the Delaunay elements. A comparable procedure is performed in this report for the newly developed satellite theory, which uses elements similar to the Delaunay elements (DSΦ) presented by Scheifele in reference 3 and the Poincaré elements (PSΦ) presented by Scheifele and Graf in reference 4. The canonical von Zeipel approach is used to arrive at a perturbation solution, which will include only the short-period effects due to J_2.

1. The fast variable (here canonical true anomaly) is eliminated by the use of a canonical transformation, leading to a set of "mean" elements.

2. The remaining Hamiltonian system, which has a Hamiltonian that is free of angle variables, is integrated by quadrature.

This procedure, when used with the DSΦ and PSΦ elements, necessitates changes in both (1) and (2) as follows:

1. The generating function that leads to the "mean" elements and is known from the DSΦ perturbation theory is reformulated in terms of PSΦ variables. Because of the nature of these variables, there will be no more singularities when taking the partial derivatives of this new generating function with respect to the PSΦ variables.

2. The solution of the first-order Hamiltonian system in the DSΦ variables is rearranged in such a way that it can be expressed entirely in PSΦ variables.

The following observations can be made:

1. The use of many convenient regular and carefully selected abbreviations will considerably reduce computation time and core requirements.

2. Typical accuracy is 200 to 500 meters after 50 revolutions when compared with numerically integrated solutions.

3. Although the theory is not restricted to short-period J_2 effects only, for practical purposes it seems adequate at present to take only the short-period J_2 solution into account because the higher zonal harmonics short-period terms are of the order J_2^2 only. Also, the long-period effects due to J_2 are the same order as the higher zonal harmonics.

The author gratefully acknowledges the assistance of Gerhard Scheifele of ACM, Inc. for suggesting that this problem be solved and for patiently providing invaluable aid in the complicated disciplines of Hamilton mechanics and von Zeipel formalism. Dr. Scheifele is the inventor of the Delaunay-Similar and Poincaré-Similar elements. He holds credit for first solving the J_2 satellite problem in the Delaunay-Similar elements, which is absolutely essential for further refinement in the Poincaré-Similar elements as presented here.

The author also wishes to thank Stephen Starke of ACM, Inc. for checking the formulas and for generating the data in table I.

This report was first issued as a Johnson Space Center Internal Note 77-FM-52 JSC-13128 in November 1977.

THE J_2 PROBLEM IN DSϕ VARIABLES

This section briefly outlines the approximate solution (in the sense that only the short-period terms are eliminated) by the von Zeipel technique of the J_2 problem in the DSϕ variables. This outline is taken from reference 4 where a more complete treatment of the J_2 problem in the DSϕ variables may be found.

The canonical DSϕ elements for the unperturbed case may be briefly described with the angle elements

$$\phi = \text{true anomaly} = \tau + \text{constant}$$

$$g = \text{argument of periapsis (classical } \omega)$$

h = argument of the ascending node (classical Ω)

$$\ell = \text{the time element} = \frac{\mu}{(2L)^{3/2}} \tau + \text{constant}$$

and the action elements

Φ = associated with two-body energy

G = total angular momentum

H = axial component of the angular momentum

L = the total energy

The independent variable is τ.

The Hamiltonian for the oblateness perturbation J_2 is

$$F = F_o + \varepsilon F_1 \tag{1}$$

$$F_o = \Phi - \mu/\sqrt{2L} \tag{2}$$

F_o is the unperturbed or two-body Hamiltonian in DSϕ elements and

$$F_1 = \frac{1}{pq}\left\{\left(\frac{b}{2} - \frac{1}{3}\right)(1 + e\cos\phi) - \frac{b}{2}\left[\frac{e}{2}\cos(\phi + 2g) + \cos(2\phi + 2g) + \frac{e}{2}\cos(3\phi + 2g)\right]\right\} \tag{3}$$

is the perturbing Hamiltonian taken from reference 3. The following are abbreviations for some functions of the DSϕ elements:

$$p = \frac{1}{\mu}\left(G - \Phi + \mu/\sqrt{2L}\right)^2$$

$$q = G - \frac{1}{2}\Phi + \mu/2\sqrt{2L}$$

$$b = 1 - H^2/G^2 = \sin^2 I \text{ (I is the inclination)}$$

$$f = 1/_{pq}$$

$$e = \sqrt{1 - \frac{2L}{\mu} \, p} \tag{4}$$

The Hamiltonian equations of motion

$$\frac{d\phi}{d\tau} = \frac{\partial F}{\partial \Phi} \qquad \frac{d\Phi}{d\tau} = - \frac{\partial F}{\partial \phi}$$

$$\frac{dg}{d\tau} = \frac{\partial F}{\partial G} \qquad \frac{dG}{d\tau} = - \frac{\partial F}{\partial g}$$

$$\frac{dh}{d\tau} = \frac{\partial F}{\partial H} \qquad \frac{dH}{d\tau} = - \frac{\partial F}{\partial h}$$

$$\frac{d\ell}{d\tau} = \frac{\partial F}{\partial L} \qquad \frac{dL}{d\tau} = - \frac{\partial F}{\partial \ell} \tag{5}$$

are nonintegrable in this form.

A transformation will now be found

$$\left.\begin{matrix} \phi,g,h,\ell \\ \Phi,G,H,L \end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \phi',g',h',\ell' \\ \Phi',G',H',L' \end{matrix}\right. \tag{6}$$

by means of the von Zeipel method. This method may be systematically employed to eliminate the short-period and long-period terms allowing the integration of the differential equations in the new primed variables to be obtained by simple quadrature. As mentioned above, for the present, only the short-period terms will be considered.

The generating function S to be used to eliminating the short-period effects is

$$S = S_o + \varepsilon S_1 \tag{7}$$

where

$$S_o = \Phi'\phi + G'g + H'h + L'\ell \tag{8}$$

is the identity transformation because

$$\Phi = \frac{\partial S_o}{\partial \phi} = \Phi' \qquad \phi' = \frac{\partial S_o}{\partial \Phi'} = \phi$$

$$G = \frac{\partial S_o}{\partial g} = g' \qquad g' = \frac{\partial S_o}{\partial G'} = g$$

$$H = \frac{\partial S_o}{\partial h} = H' \qquad h' = \frac{\partial S_o}{\partial H'} = h$$

$$L = \frac{\partial S_o}{\partial \ell} = L' \qquad \ell' = \frac{\partial S_o}{\partial L'} = \ell \tag{9}$$

S_1 is also a function of the old (unprimed) angle elements and the new (primed) action elements.

The function S_1 is a periodic function of the fast variable ϕ and is found from the von Zeipel equations,

$$F_o'(\Phi',L') = F_o(\Phi',L') \tag{10}$$

$$\frac{\partial F_o}{\partial \Phi'} \frac{\partial S_1}{\partial \phi} = -F_{1P}(\Phi',G',H',L',\phi,g) \tag{11}$$

Equation (10) shows that the unperturbed part of the Hamiltonian is left unchanged by the transformation. Equation (11) is used to determine the generating function S_1. F_{1P} is the periodic portion of F_1 and has a mean value of zero. The new perturbing Hamiltonian is found from

$$F_1' \, (\Phi',G',H',L',\_,g = F_1(\Phi',G',H',L',\phi,g) - F_{1P}(\Phi',G',H',L',\phi,g) \tag{12}$$

The periodic portion of F_1 is found from equation (3) by inspection

$$F_{1P} = \frac{1}{pq} \left\{ \left(\frac{b}{2} - \frac{1}{3}\right) e \cos \phi - \frac{b}{2} \left[\frac{e}{2} \cos (\phi + 2g) + \cos (2\phi + 2g) + \frac{e}{2} \cos (3\phi + 2g)\right] \right\} \tag{13}$$

Solving the partial differential equation (11) for S_1,

$$\begin{aligned} S_1 = &-\left(\frac{1}{2} f'e'b' + \frac{1}{3} f'e' \sin \phi\right) \\ &+ \frac{1}{4} f'e'b' \sin (\phi + 2g) \\ &+ \frac{1}{4} f'b' \sin (2\phi + 2g) \\ &+ \frac{1}{12} f'e'b' \sin (3\phi + 2g) \end{aligned} \tag{14}$$

The abbreviations f', e', b', p', q' are the same as those in equations (4) except they depend on the primed elements.

The new (primed) Hamiltonian is

$$F' = F_o' + \epsilon F_1' \tag{15}$$

where from equations (3), (12), (13)

$$F_1' = \frac{1}{2p'q'} \left(\frac{1}{3} - \frac{H'^2}{G'^2}\right) \tag{16}$$

Because F' depends only upon the elements Φ', G', H', L', the problem is now solved.

The Hamilton equations of motion, equation (5) (but with primes on all the canonical variables), yield the first-order solution. This operation will be performed in a following section.

The generating function (eq. (7)) where S_o and S_1 are given by equations (8) and (14) defines the transformation between the primed and unprimed variables in either direction through first order. During this transformation, there are numerical problems for certain types of orbits. For both $I = 0$ and $e = 0$, singularities occur because $\sin I$ and e occur in the denominators of certain terms. In the next section, the transformation to the PSϕ canonical elements will be given. The PSϕ elements do not have singularities for vanishing inclination and eccentricity.

THE J_2 PROBLEM IN THE PSϕ ELEMENTS THROUGH FIRST ORDER

The transformation among the DSϕ and PSϕ variables may be given as

$$
\begin{aligned}
\rho_1 &= \Phi & \sigma_1 &= \phi + g + h \\
\rho_2 &= \sqrt{2(\Phi - G)}\ \cos(g + h) & \sigma_2 &= -\sqrt{2(\Phi - G)}\ \sin(g + h) \\
\rho_3 &= \sqrt{2(G - H)}\ \cos h & \sigma_3 &= -\sqrt{2(G - H)}\ \sin h \\
\rho_4 &= L & \sigma_4 &= \ell
\end{aligned}
\tag{17}
$$

A transformation between primed and unprimed PSϕ elements is sought. The approach taken will be to transform the generating function S_1 given by equation (14) from DSϕ elements to PSϕ elements by direct substitution. That is,

$$
S_1(\Phi',G',H',L',\phi,g) \xrightarrow{(17)} \overline{S}_1\left(\rho_1',\rho_2',\rho_3',\rho_4',\sigma_1,\sigma_2,\sigma_3,\ -\right) \tag{18}
$$

Here it is assumed that equations (17) remain valid whether in the primed or unprimed system.

The operation involved in making the transformation from S_1 to $\overline{S}_1$ is done by temporarily dropping the distinction between the primed and unprimed PSϕ variables. By direct substitution of equations (17), a temporary S_1 (ρ_k, σ_k) is obtained in terms of the unprimed variables. Then, by making the assumption that the primed and unprimed PSϕ variables are related through the order of the perturbing parameter ε, that is $\rho_k = \rho_k' + O(\varepsilon)$ and $\sigma_k = \sigma_k' + O(\varepsilon)$, and eliminating ρ_k in favor of ρ_k', the temporary $\overline{S}_1$ becomes $S_1(\rho_k + O(\varepsilon), \sigma_k)$. Terms of $O(\varepsilon)$ are then neglected to obtain the relation

$$\bar{S}_1 = -\frac{Q'}{2p'q'G'^2}\left\{\left[\frac{1}{3}\left(G'^2 - 3H'^2\right)\rho_2' - \frac{1}{2}\left(\rho_2'c' - \sigma_2 s'\right)\right]\sin\sigma_1\right.$$

$$+\left[\frac{1}{3}\left(G'^2 - 3H'^2\right)\sigma_2 + \frac{1}{2}\left(\rho_2's' - \sigma_2 c'\right)\right]\cos\sigma_1$$

$$-\frac{c'}{2Q'}\sin 2\sigma_1 + \frac{s'}{2Q'}\cos 2\sigma_1$$

$$\left.-\frac{1}{6}\left(\rho_2'c' + \sigma_2 s'\right)\sin 3\sigma_1 + \frac{1}{6}\left(\rho_2's' - \sigma_2 c'\right)\cos 3\sigma_1\right\}$$

PSϕ The abbreviations of equation (4) may also be expressed in terms of the variables by use of equation (17)

$$p' = \frac{1}{\mu}\left[G' - \rho_1' + \mu/\sqrt{2\rho_4'}\right]^2 \qquad q' = G' - \frac{1}{2}\rho_1' + \frac{\mu}{2\sqrt{2\rho_4'}} \tag{20}$$

Also, there are now some new abbreviations:

$$Q' = \frac{\sqrt{\rho_4'}}{\mu}\left[\frac{2\mu}{\sqrt{2\rho_4'}} + G' - \rho_1'\right]^{1/2} \tag{21}$$

$$G' = \rho_1' - \frac{1}{2}\left(\rho_2'^2 + \sigma_2^{\,2}\right)$$

$$H' = G' - \frac{1}{2}\left(\rho_3'^2 + \sigma_3^{\,2}\right)$$

$$s' = (G' + H')\rho_3'\sigma_3$$

$$c' = (G' + H')\left(\frac{\rho_3'^2 - \sigma_3^{\,2}}{2}\right)$$

Numerically, through $O(\varepsilon)$, it makes no difference in equations (19), (20), and (21) whether the primed or unprimed PSϕ variables are used. Algebraically in finding the derivatives $\frac{\partial \bar{S}_1}{\partial \sigma_k}$ and $\frac{\partial \bar{S}_1}{\partial \rho'_k}$, the difference between the primed and unprimed quantities is very important and must be strictly observed. For example, in the equation for G' (middle of equations (21) above), primed values of ρ_1 and ρ_2 are used, but the unprimed value of σ_2 is used.

The unperturbed portion of the generating function is once again an identity transformation.

$$\bar{S}_o = \rho'_1\sigma_1 + \rho'_2\sigma_2 + \rho'_3\sigma_3 + \rho'_4\sigma_4 \tag{22}$$

The generating function through first order is

$$\bar{S}\,(\rho'_k,\sigma_k) = \bar{S}_o + \varepsilon\,\bar{S}_1 \tag{23}$$

The transformation between the primed and unprimed variables is therefore

$$\sigma'_k = \frac{\partial \bar{S}}{\partial \rho'_k} = \sigma_k + \varepsilon\,\frac{\partial \bar{S}_1}{\partial \rho'_k}$$

$$\rho_k = \frac{\partial \bar{S}}{\partial \sigma_k} = \rho'_k + \varepsilon\,\frac{\partial \bar{S}_1}{\partial \sigma_k} \tag{24}$$

The computation of the partial derivatives

$$\frac{\partial \bar{S}_1}{\partial \rho'_k} \quad \text{and} \quad \frac{\partial \bar{S}_1}{\partial \rho_k}$$

are straightforward and are facilitated by the introduction of additional abbreviations. Let

$$w = Q'/2p'q'$$

$$y = \sum_{k=1}^{3}\left(\delta_k\eta_k + \gamma_k\zeta_k\right)$$

$$\delta_1 = \frac{1}{3}\left(G'^2 - 3H'^2\right)\rho_2 - \left(\frac{1}{2}\rho_2'c' - \sigma_2 s'\right)$$

$$\delta_2 = -c'/2Q'$$

$$\delta_3 = -\frac{1}{6}\left(\sigma_2 s' + \rho_2'c'\right)$$

$$\gamma_1 = \frac{1}{3}\left(G'^2 - 3H'^2\right)\sigma_2 + \frac{1}{2}\left(\rho_2's' + \sigma_2 c'\right)$$

$$\gamma_2 = s'/2Q'$$

$$\gamma_3 = \frac{1}{6}\left(\rho_2's' - \sigma_2 c'\right)$$

$$\eta_1 = \sin\sigma_1$$

$$\eta_2 = \sin 2\sigma_1$$

$$\eta_3 = \sin 3\sigma_1$$

$$\zeta_1 = \cos\sigma_1$$

$$\zeta_2 = \cos 2\sigma_1$$

$$\zeta_3 = \cos 3\sigma_1 \tag{25}$$

and the generating function (eq. (19)) becomes

$$\bar{S}_1 = -\frac{1}{G^2}\,wy \tag{26}$$

It is at this point that the advantage of the PSϕ elements over the DSϕ elements for the J_2 problem occurs. In the derivatives $\frac{\partial\bar{S}_1}{\partial\rho_k'}$ and $\frac{\partial\bar{S}_1}{\partial\sigma_k}$, no singularities occur for vanishing inclination and eccentricity. At the close of the second section, it was seen that singularities will occur for these two conditions in the DSϕ solution. However, when the Hamiltonian F' in the PSϕ equations is completed, it is found that the Hamiltonian differential equations are nonintegrable. However, the solution can be found by returning to the DSϕ solution and transforming it by direct substitution into the PSϕ variables. This will be done in the following section.

USE OF THE DSϕ SOLUTION

From equations (5) (with primes on the DSϕ variables) and equation (16), the solution in the DSϕ elements is given by

$$\phi' = \phi'_o + \left(\tau - \tau_o\right) + \frac{\varepsilon}{2} f'_1(b' - 2/3)\left(\tau - \tau_o\right)$$

$$g' = g'_o + \frac{\varepsilon}{2}\left[f'_2(b' - 2/3) + f'b'_2\right]\left(\tau - \tau_o\right)$$

$$h' = h'_o + \frac{\varepsilon}{2} f'b'_3\left(\tau - \tau_o\right)$$

$$\ell' = \ell'_o + \frac{\mu}{(2L')^{3/2}}\left(\tau - \tau_o\right) + \frac{\varepsilon}{2} f'_4(b' - 2/3)\left(\tau - \tau_o\right) \tag{27}$$

where ϕ_o, g_o, h_o, ℓ_o are the initial $\tau = \tau_o$ values of the angle elements.

The elements Φ', G', H', L' are all constant and

$$f'_1 = \partial f'/\partial \Phi'$$

$$f'_2 = \partial f'/\partial G'$$

$$f'_4 = \partial f'/\partial L'$$

$$b'_2 = \partial b'/\partial G'$$

$$b'_3 = \partial b'/\partial H' \tag{28}$$

The approach to be taken here is to find expressions in terms of the DSϕ elements that will relate the PSϕ elements at an arbitrary value of the independent variable to the PSϕ elements at initiation.

Add the first three equations of equation (27) to obtain

$$\phi' + g' + h' = \phi'_o + g'_o + h'_o + \left(\tau - \tau_o\right)$$
$$+ \frac{\varepsilon}{2}\left[f'_1(b' - 2/3) + f'_2(b' - 2/3) + f'b'_2 + f'b'_3\right]\left(\tau - \tau_o\right)$$

The left-hand side is σ'_1 by definition and the first term on the right is its initial value σ'_{10}. So

$$\sigma'_1 = \sigma'_{10} + \left(\tau - \tau_o\right)$$
$$+ \frac{\varepsilon}{2}\left[f'_1(b' - 2/3) + f'_2(b' - 2/3) + f'b'_2 + f'b'_3\right]\left(\tau - \tau_o\right) \quad (29)$$

Now add the second and third equations of equation (27) to obtain

$$g' + h' = g'_o + h'_o + \frac{\varepsilon}{2}\nu\left(\tau - \tau_o\right)$$

where

$$\nu = f'_2\left(b' - 2/3\right) + f'\left(b'_2 + b'_3\right)$$

Taking the sine of both sides

$$\sin\left(g' + h'\right) = \sin\left(g'_o + h'_o\right)\cos\left[\frac{\varepsilon}{2}\nu\left(\tau - \tau_o\right)\right]$$
$$+ \cos\left(g'_o + h'_o\right)\sin\left[\frac{\varepsilon}{2}\nu\left(\tau - \tau_o\right)\right]$$

Now because $\Phi' = \Phi'_o$ = constant and $G' = G'_o$ = constant, multiply both sides by $\sqrt{2(\Phi' - G')}$ to obtain

$$\sqrt{2(\Phi' - G')}\,\sin\,(g' + h') = \sqrt{2\left(\Phi'_o - G'_o\right)}\,\sin\left(g'_o + h'_o\right)\,\cos\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right]$$

$$+\sqrt{2\left(\Phi'_o - G'_o\right)}\,\cos\left(g'_o + h'_o\right)\,\sin\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right]$$

using equations (17)

$$\sigma'_2 = -\sqrt{2\left(\Phi' - G'\right)}\,\sin\left(g' + h'\right)$$

$$\sigma'_{20} = -\sqrt{2\left(\Phi'_o - G'_o\right)}\,\sin\left(g'_o + h'_o\right)$$

$$\rho'_{20} = \sqrt{2\left(\Phi'_o - G'_o\right)}\,\cos\left(g'_o + h'_o\right)$$

Finally

$$\sigma'_2 = \sigma'_{20}\,\cos\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right] - \rho'_{20}\,\sin\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right] \tag{30}$$

By a similar procedure

$$\rho'_2 = \rho'_{20}\,\cos\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right] + \sigma'_{20}\,\sin\left[\frac{\varepsilon}{2}\,\nu\left(\tau - \tau_o\right)\right] \tag{31}$$

$$\sigma'_3 = \sigma'_{30}\,\cos\left[\frac{\varepsilon}{2}\,f'b'_3\left(\tau - \tau_o\right)\right] - \rho'_{30}\,\sin\left[\frac{\varepsilon}{2}\,f'b'_3\left(\tau - \tau_o\right)\right] \tag{32}$$

$$\rho'_3 = \rho'_{30}\,\cos\left[\frac{\varepsilon}{2}\,f'b'_3\left(\tau - \tau_o\right)\right] + \sigma'_{30}\,\sin\left[\frac{\varepsilon}{2}\,f'b'_3\left(\tau - \tau_o\right)\right] \tag{33}$$

Because from equation (17) $\sigma_4' = \ell'$, it follows from equation (27) that

$$\sigma_4' = \sigma_{40}' + \frac{\mu}{\left(2L_o'\right)^{3/2}}\left(\tau - \tau_o\right) + \frac{\varepsilon}{2} f_4'\left(b' - 2/3\right)\left(\tau - \tau_o\right) \tag{34}$$

And also from the fact that Φ' and L' are constant, from equation (17)

$$\rho_1' = \Phi' = \Phi_o' = \rho_{10}' \tag{35}$$

$$\rho_4' = L' = L_o' = \rho_{40}' \tag{36}$$

The first-order DSϕ solution transformed by substitution into the PSϕ solution equation (29) through equation (36) provides the means of propagating the initial perturbed values, ρ_{ko}' and σ_{ko}' forward to their values ρ_k' and σ_k' at a specified (or otherwise determined) value of the independent variable.

THE METHOD OF SOLUTION

The unprimed and primed PSϕ elements are related through the generating function $S(\rho',\bar{\sigma})$ by equation (24). If, for example, the initial unperturbed elements have been obtained from, say, Cartesian coordinates, then the initial perturbed elements may be computed from

$$\begin{aligned} \sigma_{ko}' &= \sigma_{ko} + \varepsilon \left.\frac{\partial S_1}{\partial \rho_k'}\right|_{\tau_o} \\ \rho_{ko}' &= \rho_{ko} - \varepsilon \left.\frac{\partial \bar{S}_1}{\partial \sigma_k}\right|_{\tau_o} \qquad k = 1,\ldots,4 \end{aligned} \tag{37}$$

Because the primed (perturbed) elements are related to the unprimed through order ε, either the primed or the unprimed values of ρ_k and σ_k may be used in evaluating the derivatives of $\bar{S}_1$ on the right hand side of equation (37). The resulting error will be of order ε^2, which is beyond the accuracy of the first-order solution.

Similarly, after a prescribed propagation interval, the unperturbed PSϕ elements may be computed again by using equation (24).

$$\sigma_k(\tau) = \sigma'_k(\tau) - \varepsilon \left. \frac{\partial \bar{S}_1}{\partial \rho'_k} \right|_\tau$$

$$\rho_k(\tau) = \rho'_k(\tau) + \varepsilon \left. \frac{\partial \bar{S}_1}{\partial \sigma_k} \right|_\tau \qquad k = 1,\ldots,4 \tag{38}$$

As in the initialization described in equation (37) either primed or unprimed values of ρ_k and σ_k may be used in numerically evaluating the derivatives of the generating function in equation (38).

The computation procedure can be visualized by referring to figure 1. The conversion from Cartesian coordinates to PSϕ elements and its inverse are given in the appendix.

SPECIFIED TIME

Because time is not the independent variable in the DSϕ and PSϕ theories, provision must be made for propagating forward to a specified time. The procedure for accomplishing this is iterative. The following equations are evaluated with the perturbed (primed) values of ρ_k and σ_k. The prime is omitted for convenience.

From the appendix, the time equation is

$$t(\tau) = \sigma_4 + \frac{\mu}{\left(2\rho_4\right)^{3/2}} \left(E - \phi - \frac{r}{p}\sqrt{1 - e^2}\, e \sin \phi \right) \tag{39}$$

A standard Newton iteration procedure is now used to solve for the value of τ, which corresponds to the specified value of time, T

$$\tau = \tau_1 - \frac{T - t\left(\tau_1\right)}{(dt/d\tau)_1} \tag{40}$$

The derivative is given by

$$\frac{dt}{d\tau} = r^2/q$$

which is known at τ_1. The iteration is continued until $\tau - \tau_1$ is less than a previously specified tolerance. Each time the tolerance is not met, τ_1 (the initial guess of τ) is set equal to the last computed value of τ from equation (40).

The first value of the initial guess τ_1, is chosen by referring to equation (27). To a good approximation

$$\tau_1 - \tau_o = \frac{T - \ell_o'}{\mu\left(2L'\right)^{3/2} + \frac{\varepsilon}{2} f_4'\left(b' - 2/3\right)} \tag{41}$$

The denominator of this equation is computed at the same time that the other quantities in the DSϕ solution are computed.

SOME NUMERICAL RESULTS

The solution for the J_2 problem in the singularity-free PSϕ variables has been programed on the Univac 1110 computer and is available for public use. This prototype program is called PSANS (for Poincaré-Similar analytical short-period eliminated solution).

Several typical trajectories for satellites orbiting the Earth were computed for times corresponding to 1, 10, 50, and 100 revolutions of the satellites for each case. The initial conditions (in terms of classical orbital elements) are given, and the results of comparisons between the analytical solution computed by this method (PSANS) and double-precision numerical integration are given in table I. It is seen that for a wide range of inclinations (0 to $\Pi/6$ radians (0 to 30 degrees)) and eccentricities (0 to 0.1) the maximum errors are less than 1 kilometer. The errors were computed from the formula

$$\text{Error} = \left| \underline{r}_{\text{PSANS}} - \underline{r}_{\text{INTEG}} \right|$$

at several points during each revolution. The maximum value of the error during a given revolution were output for the presentation in table I.

CONCLUDING REMARKS

From the numerical results presented, it can easily be seen that the analytical solution to the J_2 satellite problem in the PSϕ variables overcomes the nuisance of the singularities due to small eccentricities and inclinations that exist in the solution when Delaunay variables are used.

The problem that remains is to systematically extend the solution to include the cases (1) where both short- and long-period terms are included in the solution, (2) where the effects of the higher order zonal terms are included, and (3) where the effects of atmospheric drag are included. All of these new extensions are currently being developed. The present version (PSANS), however, has demonstrated that it has a remarkable accuracy for medium lifetime satellites (less than 1 kilometer error in approximately 7 days).

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TABLE I.- ANALYTICAL SOLUTION AND NUMERICALLY INTEGRATED SOLUTION

COMPARING SIX TEST CASES

(a) Initial conditions

	Orbital elements					
Case	a	e	I	ω	Ω	M
1	1.050486	0.015	0	π/9	π/9	π/9
2	1.050486	0	π/6	π/9	π/9	π/9
3	1.050486	.001	π/6	π/9	π/9	π/9
4	1.160238	.1	π/6	π/9	π/9	π/9
5	1.070560	0	0	0	0	0
6	1.070560	0	π/6	0	0	0

[a]The orbital elements are defined as folows:

a = semimajor axis in Earth radii
e = eccentricity
I = inclination in radians
ω(or g) = argument of perigee in radians
Ω(or h) = argument of ascending node in radians
M = mean anomaly in radians

TABLE I.- Continued

(b) Revolution number, time function, maximum errors

Rev	Time, days	Max error, m
	Case 1	
1	0.06	8
10	.63	50
50	3.15	254
100	6.30	511
	Case 2	
1	0.06	11
10	.63	100
50	3.15	498
100	6.30	995
	Case 3	
1	0.06	11
10	.63	100
50	3.15	498
100	6.30	995
	Case 4	
1	0.07	10
10	.73	84
50	3.66	432
100	7.32	872
	Case 5	
1	0.06	8
10	.65	48
50	3.24	239
100	6.48	479

TABLE I.- Concluded

(b) Concluded

Rev	Time, day	Max error, m
	Case 6	
1	0.06	12
10	.65	94
50	3.24	470
100	6.48	940

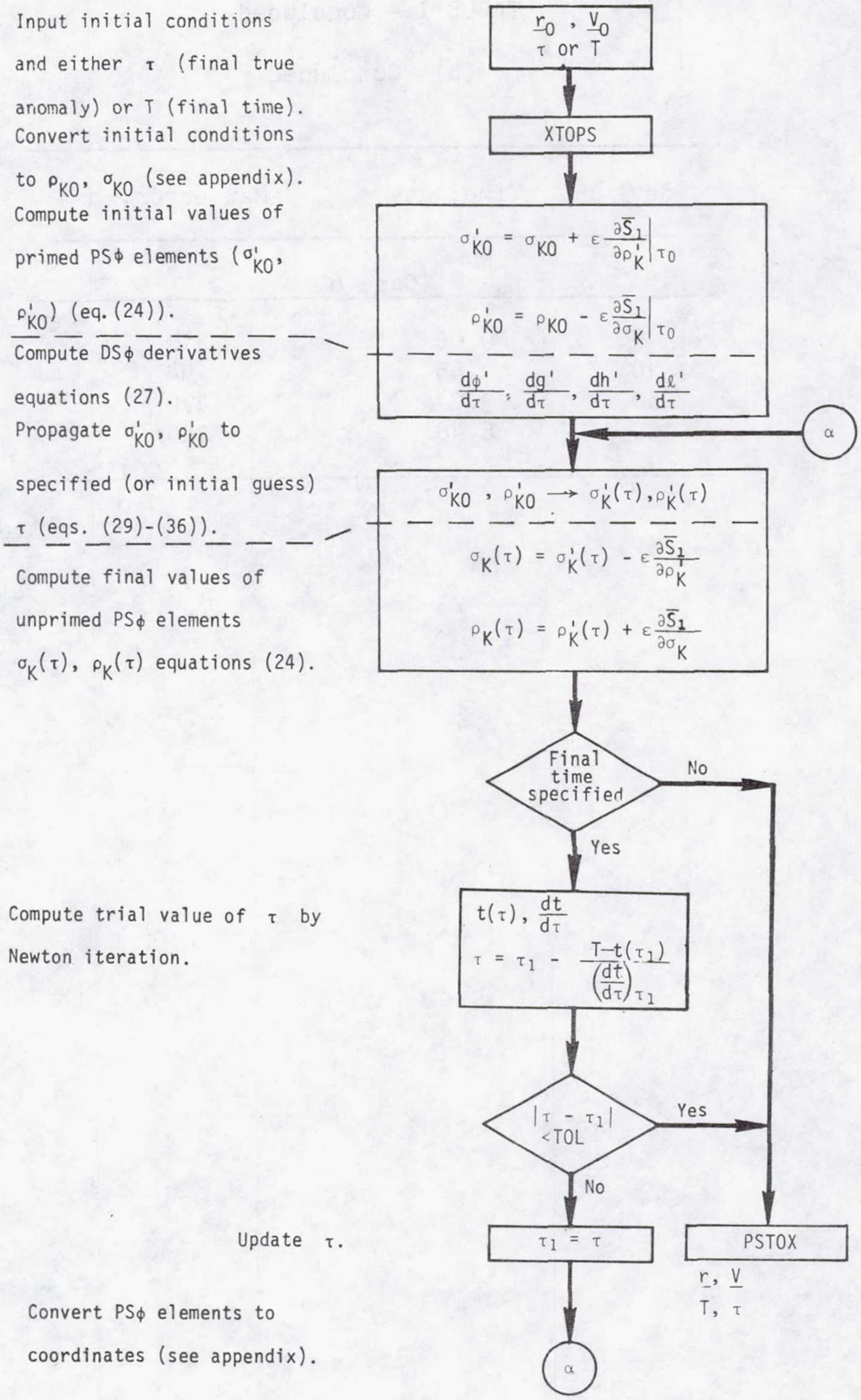

Figure 1.- Computation procedure for PSANS.

APPENDIX

COMPUTATIONAL PROCEDURE

The equations that transform from position and velocity in Cartesian coordinates and the time to the PSϕ elements and the inverse transformation are listed below.

1. From $\underline{r}$, $\underline{v}$, $\underline{t}$, to $\underline{\rho}$, $\underline{\sigma}$, (XTOPS).
   compute the perturbing potential V,

$$\underline{G} = \underline{r} \times \underline{v} \qquad \text{(the angular momentum vector)}$$

$$G = |\underline{G}|$$

$$H = G_3$$

$$r = |\underline{r}|$$

$$L = -\frac{1}{2}\,\underline{v}\cdot\underline{v} + \frac{\mu}{r} - V$$

$$\rho_4 = L$$

$$\sigma_3 = -2G_1/\sqrt{2(G + H)}$$

$$\rho_3 = -2G_2/\sqrt{2(G + H)}$$

$$\rho_1 = G + \frac{\mu}{\sqrt{2L}} - \sqrt{G^2 + 2r^2V}$$

$$R = \frac{x_3}{r}\sqrt{G}\,\frac{1}{\sqrt{1 - \frac{1}{4G}\left(\sigma_3{}^2 + \rho_3{}^2\right)}}$$

$$\sigma_1 = \tan^{-1}\left(\frac{\frac{x_2}{r} + \frac{R}{2G}\,\rho_3}{\frac{x_1}{r} + \frac{R}{2G}\,\sigma_3}\right)$$

$$q = -\left(\rho_1 - G\right) + \frac{\mu}{2} \frac{1}{\sqrt{2L}}$$

$$p = \frac{1}{\mu}\left[-\left(\rho_1 - G\right) + \mu/\sqrt{2L}\right]^2$$

$$Q = \frac{\sqrt{L}}{\mu}\left[\frac{2\mu}{\sqrt{L}} - \left(\rho_1 - G\right)\right]^{1/2}$$

$$Z_1 = \frac{1}{Q}\left(\frac{p}{r} - 1\right)$$

$$Z_2 = \frac{\underline{r}\cdot\underline{v}}{r} \frac{p}{Q\left(V\dfrac{r^2}{q} + G\right)}$$

$$\sigma_2 = Z_2 \cos\sigma_1 - Z_1 \sin\sigma_1$$

$$\rho_2 = Z_2 \sin\sigma_1 + Z_1 \cos\sigma_1$$

$$e\cos\phi = Q\left(\rho_2 \cos\sigma_1 - \sigma_2 \sin\sigma_1\right)$$

$$e\sin\phi = Q\left(\rho_2 \sin\sigma_1 + \sigma_2 \cos\sigma_1\right)$$

$$e^2 = (e\cos\phi)^2 + (e\sin\phi)^2$$

$$E - \phi = -2\tan^{-1}\left(\frac{e\sin\phi}{1 + \sqrt{1 - e^2} + e\cos\phi}\right)$$

$$\sigma_4 = t - \frac{\mu}{(2L)^{3/2}}\left(E - \phi - \frac{r}{p}\sqrt{1 - e^2}\, e\sin\phi\right)$$

2. From $\underline{\sigma}$, $\underline{\rho}$ to $\underline{r}$, $\underline{v}$, $\underline{t}$ (PSTOX)

$$Q = \frac{\sqrt{\rho_4}}{\mu} \sqrt{\frac{2\mu}{\sqrt{2\rho_4}} - \frac{1}{2}\left(\sigma_2{}^2 + \rho_2{}^2\right)}$$

$$R = \rho_3 \sin \sigma_1 + \sigma_3 \cos \sigma_1$$

$$p = \frac{1}{\mu}\left[-\frac{1}{2}\left(\sigma_2{}^2 + \rho_2{}^2\right) + \frac{\mu}{\sqrt{2\rho_4}}\right]^2$$

$$q = -\frac{1}{2}\left(\sigma_2{}^2 + \rho_2{}^2\right) + \frac{1}{2}\rho_1 + \frac{\mu}{2\sqrt{2\rho_4}}$$

$$e \cos \phi = Q\left(\rho_2 \cos \sigma_1 - \sigma_2 \sin \sigma_1\right)$$

$$e \sin \phi = Q\left(\rho_2 \sin \sigma_1 + \sigma_2 \cos \sigma_1\right)$$

$$r = p/(1 + e \cos \phi)$$

$$x_1 = r\left(\cos \sigma_1 - \sigma_3 R/2G\right)$$

$$x_2 = r\left(\sin \sigma_1 - \rho_3 R/2G\right)$$

$$x_3 = \frac{rR}{2\sqrt{G}} \sqrt{(G + H)}$$

$$G = \rho_1 - \frac{1}{2}\left(\rho_2{}^2 + \sigma_2{}^2\right)$$

$$H = G - \frac{1}{2}\left(\rho_3{}^2 + \sigma_3{}^2\right)$$

$$\dot{R} = \frac{G}{r^2}\left(\rho_3 \cos \sigma_1 - \sigma_3 \sin \sigma_1\right)$$

$$\dot{r} = \frac{Q}{p}(2q - G)\left(\sigma_2 \cos \sigma_1 + \rho_2 \sin \sigma_1\right)$$

$$\dot{x}_1 = \dot{r}\,\frac{x_1}{r} - r\left(\frac{G}{r^2} \sin \sigma_1 + \frac{\sigma_3 \dot{R}}{2G}\right)$$

$$\dot{x}_2 = \dot{r}\,\frac{x_2}{r} + r\left(\frac{G}{r^2}\cos\sigma_1 - \frac{\rho_3\dot{R}}{2G}\right)$$

$$\dot{x}_3 = (\dot{r}R + r\dot{R})$$

$$e^2 = (e\cos\phi)^2 + (e\sin\phi)^2$$

$$E - \phi = -2\tan^{-1}\left(\frac{e\sin\phi}{1 + \sqrt{1 - e^2} + e\cos\phi}\right)$$

$$t = \sigma_4 + \frac{\mu}{\left(2\rho_4\right)^{3/2}}\left(E - \phi - \frac{r}{p}\sqrt{1 - e^2}\;e\sin\phi\right)$$

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-58221		

4. Title and Subtitle	5. Report Date
AN ANALYTICAL SINGULARITY-FREE SOLUTION TO THE J_2 PERTURBATION PROBLEM	July 1979
	6. Performing Organization Code JSC-13128
7. Author(s) Victor R. Bond	8. Performing Organization Report No.
	10. Work Unit No. 910-44-LH-00-72
9. Performing Organization Name and Address Lyndon B. Johnson Space Center Houston, Texas 77058	11. Contract or Grant No.
	13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The development of a singularity-free solution of the J_2 problem in satellite theory is presented. The procedure resembles that of Lyndane who rederives Brouwer's satellite theory using Poincaré elements. A comparable procedure is used in this report in which the satellite theory of Scheifele, who used elements similar to the Delaunay elements but in the extended phase space, is rederived using Poincaré elements also in the extended phase space. Only the short-period effects due to J_2 are included.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Satellite theory Analytical theory Perturbation theory Canonical variables	STAR Subject Category: 13 (Astrodynamics)

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	29	$4.00

*For sale by the National Technical Information Service, Springfield, Virginia 22161