ON THE SPECTRUM

OF

INHOMOGENEOUS TURBULENCE

Prepared for NASA-LaRC

by

Dr. George Treviño

and

Del Mar College

Summer 1979

(NASA-CR-162137) ON THE SPECTRUM OF INHOMOGENEOUS TURBULENCE (Del Mar Coll.) 27 p HC A03/MF A01 CSCL 12A

N79-30987

Unclas

G3/64 31811

This report constitutes the results of research conducted for NASA-LaRC under NASA Grant No. 1615

1. Introduction and General Theory

Inhomogeneous turbulence is defined in the literature as turbulence whose statistics are functions of spatial position. The "statistic" of general interest in this investigation is the turbulence spectrum, and particularly how the shape of the spectrum varies, from point to point in space, as a consequence of well-defined spatial variations in the turbulence intensity and/or integral scale.

The intensity, $\sigma_i(\underline{x}), i = 1,2,3$, is defined as

$$\sigma_i(\underline{x}) = \langle u_i(\underline{x}) u_i(\underline{x}) \rangle^{\frac{1}{2}} \tag{1.1}$$

where $u_i(\underline{x})$ is a turbulence velocity component of zero mean value, $\langle\ \rangle$ denotes mean value, $\underline{x}$ is a three-dimensional vector coordinate defining spatial position, and the so-called "subscript summation convention" is not intended here. The integral scale, $\Lambda_{ijl}(\underline{x})$, $i,j,l = 1,2,3$, is defined as

$$\Lambda_{ijl}(\underline{x}) = \int_0^\infty Q_{ij}(\underline{x}, \underline{r}_{(l)})dr \tag{1.2}$$

where $Q_{ij}(\underline{x},\underline{r})$ is the "normalized" correlation function of the turbulence, and $\underline{r}_{(l)} = (r,0,0)$ for $l = 1$, $\underline{r}_{(l)} = (0,r,0)$ for $l = 2$, $\underline{r}_{(l)} = (0,0,r)$ for $l = 3$. The normalized correlation function itself is defined through the fundamental equation†

$$C_{ij}(\underline{x}_1,\underline{x}_2) = \langle u_i(\underline{x}_1) u_j(\underline{x}_2) \rangle = \sigma_i(\underline{x}_1)\sigma_j(\underline{x}_2) Q_{ij}(\underline{x}_1,\underline{x}_2) \tag{1.3}$$

---

†For a complete derivation of Equ. (1.3) see Appendix A.

which, by means of the coordinate transformation

$$\underline{x} = \tfrac{1}{2}(\underline{x}_1+\underline{x}_2)\ ,\ \underline{r} = \underline{x}_2-\underline{x}_1 \tag{1.4}$$

becomes

$$C_{ij}(\underline{x},\underline{r}) = \langle u_i(\underline{x}-r/2)u_j(\underline{x}+r/2)\rangle = \sigma_i\ (\underline{x}-\underline{r}/2)\sigma_j(\underline{x}+\underline{r}/2)Q_{ij}(\underline{x},\underline{r}) \tag{1.5}$$

Again, the subscript summation convention is not intended in Eqs. (1.2)-(1.5). It follows from Eqs. (1.1) and (1.2) that fully three-dimensional inhomogeneous turbulence has three distinct intensities and twenty-seven different scales.

In view of the definition of inhomogeneous turbulence, it can be shown through the continuity condition for an incompressible fluid,

$$\nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{1.6}$$

that, for fully three-dimensional inhomogeneous turbulence, changes in the intensity of the turbulence are not independent of changes in the integral scale. This circumstance exists by virtue of the fact that spatial changes in the function $C_{ij}(\underline{x},\underline{r})$ arise from spatial changes in the turbulence intensity as well as spatial changes in the function Q_{ij}. Indeed, the continuity condition yields two sets of partial differential equations (see Appendix B) which together govern the complete kinematical behavior of C_{ij}, and which clearly demonstrate the scale-intensity coupling. This characteristic of inhomogeneous turbulence is of paramount importance because it signifies that inhomogeneous turbulence is remarkably different from homogeneous turbulence, where the respective values of the scale and intensity are quite independent of one another. It should also be pointed out that inhomogeneous turbulence, by its very definition, is also nonisotropic, since even in its simplest form, viz. the case where there is only one direction of inhomogeneity, there will always exist a "preferred direction" to the turbulence, that direction being the direction of inhomogeneity.

The space-varying spectrum of inhomogeneous turbulence, defined as the multi-dimensional Fourier transform†

$$\psi_{ij}(\underline{x},\underline{k}) = \int C_{ij}(\underline{x},\underline{r})e^{-\iota\underline{k}\cdot\underline{r}}d\underline{r} \tag{1.7}$$

of the correlation function C_{ij}, is expressed as

$$\psi_{ij}(\underline{x},\underline{k}) = \left(\frac{1}{2\pi}\right)^3 \int \sigma_i(\underline{x}-\underline{r}/2)\sigma_j(\underline{x}+\underline{r}/2)\phi_{ij}(\underline{x},\underline{k}')e^{-\iota(\underline{k}-\underline{k}')\cdot\underline{r}}d\underline{r}d\underline{k}' \tag{1.8}$$

where

$$\phi_{ij}(\underline{x},\underline{k}) = \int Q_{ij}(\underline{x},\underline{r})e^{-\iota\underline{k}\cdot\underline{r}}d\underline{r} \tag{1.9}$$

$\underline{k}$ is a three dimensional wave-number vector, and ι ("iota") = $\sqrt{-1}$. Note that the turbulence spectrum, as given by Eq. (1.8), is not a simple product of the intensity and the energy spectrum of the normalized correlation function, but rather is a multi-dimensional convolution of the spectral distribution

$$S_{ij}(\underline{x},\underline{k}) = \int \sigma_i(\underline{x}-\underline{r}/2)\sigma_j(\underline{x}+\underline{r}/2)e^{-\iota\underline{k}\cdot\underline{r}}dr \tag{1.10}$$

and the tensor $\phi_{ij}(\underline{x},\underline{k})$. Indeed,

$$\psi_{ij}(\underline{x},\underline{k}) = \left(\frac{1}{2\pi}\right)^3 \int S_{ij}(\underline{x},\underline{k}-\underline{k}')\phi_{ij}(\underline{x},\underline{k}')d\underline{k}' \tag{1.11}$$

---

†Unless otherwise indicated the limits on all integrals are $-\infty$ to $+\infty$.

which indicates that the distribution of energy in inhomogeneous turbulence is dependent upon the spectral distribution of the turbulence intensity as well as the spectral distribution of $Q_{ij}(\underline{x},\underline{r})$.

2. Correlation Function and Spectrum for Simple Inhomogeneous Turbulence

In order to investigate and understand the essential nature of inhomogeneous turbulence, viz. varying scale and intensity and scale-intensity coupling, and the effects of this nature on the shape of the spectrum, a simple one-dimensional representation of inhomogeneous turbulence, with well-defined variations in both the scale and intensity, is introduced here. Since it will be necessary to compare the spectrum of inhomogeneous turbulence to the spectrum of homogeneous turbulence, the formulation suggested by Equ. (1.5) will be employed.

Inhomogeneous Turbulence with Varying Scale Only

Accordingly, the correlation function of interest, say the correlation between "upwash" components at two points along the x-axis, is written as

$$C_{33}(x,r) = \langle w(x-r/2)w(x+r/2) \rangle = \sigma_3(x-r/2)\sigma_3(x+r/2)Q_{33}(x,r) \qquad (2.1)$$

where x and r are simple scalar variables. In the following discussion the subscript "3" will be omitted for brevity from the notation, and it will be tacitly assumed that the turbulence component of interest is always the upwash component. The normalized correlation function appearing in Equ. (2.1) is now decomposed into the form

$$Q(x,r) = Q^{(1)}(r) + Q^{(2)}(x,r) \qquad (2.2)$$

where $Q^{(1)}$ is a function of "r" only and defines the "homogeneous" (i.e., constant) part of the turbulence scale, while $Q^{(2)}$ is a function of both x and r and defines the "inhomogeneous" (i.e., space-varying) part of the turbulence scale, In their simplest form the functions $Q^{(1)}$, $Q^{(2)}$, and Q have the shapes depicted respectively in Figs. 1.a, 1.b, and 1.c.

In the form given by Equ. (2.2) it is possible to determine what effect changes in the integral scale have on the shape of the spectrum of inhomogeneous turbulence. To do so requires a Fourier transformation of the form

$$\phi(x,k) = \int Q(x,r)e^{-\iota kr}dr \tag{2.3}$$

$$= \phi^{(1)}(k) + \phi^{(2)}(x,k) \tag{2.4}$$

where

$$\phi^{(1)}(k) = \int Q^{(1)}(r)e^{-\iota kr}dr \tag{2.5}$$

and

$$\phi^{(2)}(k) = \int Q^{(2)}(x,r)e^{-\iota kr}dr \tag{2.6}$$

Choosing the functional form of $Q^{(1)}$ to be that given by the Dryden approach, i.e.

$$Q^{(1)}(r) = (1-r/4\Lambda_1)e^{-r/2\Lambda_1} \tag{2.7}$$

Λ_1 being the so-called "lateral" scale of the turbulence, and the functional form of $Q^{(2)}$ to be

$$Q^{(2)}(x,r) = \varepsilon(x)[1-\cos(\frac{2\pi r}{a\Lambda_1})], |r| \leq a\Lambda_1$$

$$= 0 \ , \ |r|>a\Lambda_1 \tag{2.8}$$

the resulting functions $\phi^{(1)}$ and $\phi^{(2)}$ are

$$\phi^{(1)}(\Omega) = \frac{2\Lambda_1(1+3\Omega^2)}{(1+\Omega^2)^2}, \ \Omega = \Lambda_1 k \tag{2.9}$$

and

$$\phi^{(2)}(x,\Omega) = 2\varepsilon(x)(\frac{2\pi}{a})^2 \frac{\Lambda_1 \sin a\Omega}{\Omega[(\frac{2\pi}{a})^2-\Omega^2]} \tag{2.10}$$

In Eqs. (2.8) and (2.10) the parameter "a" is a positive number that defines the domain of definition of $Q^{(2)}$ while $\varepsilon(x)$ is a function yet to be determined. In view of this, the integral scale, defined here as

$$\Lambda(x) = \int_0^\infty Q(x,r)dr = \int_0^\infty Q^{(1)}(r)dr + \int_0^\infty Q^{(2)}(x,r)dr \tag{2.11}$$

is written as

$$\Lambda(x) = \Lambda_1 + \Delta\Lambda(x) \tag{2.12}$$

where

$$\Lambda_1 = \int_0^\infty Q^{(1)}(r)dr \tag{2.13}$$

and

$$\Delta\Lambda(x) = \int_0^\infty Q^{(2)}(x,r)dr \tag{2.14}$$

Choosing the functional form of $\Lambda(x)$ to be

$$\Lambda(x) = \Lambda_1 + (\Lambda_2-\Lambda_1)U(x) \tag{2.15}$$

where $U(x)$ is the unit step function, it follows that

$$\varepsilon(x) = \left(\frac{\Lambda_2-\Lambda_1}{a\Lambda_1}\right)U(x) \tag{2.16}$$

Substituting Eqs. (2.9), (2.10), and (2.16), into Equ. (2.4), the complete spectrum for this type of inhomogeneous turbulence becomes

$$\psi(x,\Omega) = 2\Lambda_1\sigma^2\left\{\frac{1+3\Omega^2}{(1+\Omega^2)^2} + \left(\frac{\Lambda_2-\Lambda_1}{a\Lambda_1}\right)U(x)\left(\frac{2\pi}{a}\right)^2 \frac{\sin a\Omega}{\Omega[(\frac{2\pi}{a})^2-\Omega^2]}\right\} \tag{2.17}$$

where $2\Lambda_1\sigma^2 = \psi(x,0)$, $x<0$.

Inhomogeneous Turbulence with Varying Intensity Only

The effects of varying intensity on the shape of the spectrum will be investigated by considering the case where

$$\sigma(x) = \sigma_1+(\sigma_2-\sigma_1)\sin\alpha x \tag{2.18}$$

For this case (see Appendix C)

$$S(x,k) = [\sigma_1^2+(\sigma_2-\sigma_1)^2\sin^2\alpha x]2\pi\delta(k)+\sigma_1(\sigma_2-\sigma_1)\sin\alpha x[2\pi\delta(\alpha/2-k)+2\pi\delta(\alpha/2+k)] + \left(\frac{\sigma_2-\sigma_1}{2}\right)^2 [2\pi\delta(\alpha-k)-4\pi\delta(k)+2\pi\delta(\alpha+k)] \tag{2.19}$$

and

$$\psi(x,k) = [\sigma_1^2-2\left(\frac{\sigma_2-\sigma_1}{2}\right)^2+(\sigma_2-\sigma_1)^2\sin^2\alpha x]\phi^{(1)}(k)+\sigma_1(\sigma_2-\sigma_1)\sin\alpha x [\phi^{(1)}(k-\alpha/2)+\phi^{(1)}(k+\alpha/2)] + \left(\frac{\sigma_2-\sigma_1}{2}\right)^2 [\phi^{(1)}(k-\alpha)+\phi^{(1)}(k+\alpha)] \tag{2.20}$$

where $\phi^{(2)} \equiv 0$ since there is no scale variation. Note that even for the case where $(\sigma_2-\sigma_1)/\sigma_1 \ll 1$,

$$\psi(x,k) \approx \sigma_1^2\phi^{(1)}(k)+\sigma_1(\sigma_2-\sigma_1)\sin\alpha x[\phi^{(1)}(k-\alpha/2)+\phi^{(1)}(k+\alpha/2)] \tag{2.21}$$

Expressing α as

$$\alpha = \frac{2\pi}{c\Lambda_1} \tag{2.22}$$

where "c" is a positive number that effectively defines the wavelength of the oscillatory part of the turbulence, Equ. (2.21) reduces to

$$\psi(x,\Omega) \approx \sigma_1^2\phi^{(1)}(\Omega)+\sigma_1(\sigma_2-\sigma_1)\sin\alpha x[\phi^{(1)}(\Omega-\pi/c)+\phi^{(1)}(\Omega+\pi/c)] \tag{2.23}$$

where, recall, $\Omega = \Lambda_1 k$.

Inhomogeneous Turbulence with Both Varying Scale and Intensity

Combining the two previous analyses into one to determine the overall effects of varying scale and varying intensity results in

$$\psi(x,\Omega) \approx \sigma_1^2[\phi^{(1)}(\Omega)+\phi^{(2)}(x,\Omega)] + \sigma_1(\sigma_2-\sigma_1)\sin\alpha x[\phi^{(1)}(\Omega-\pi/c)$$

$$+\phi^{(1)}(\Omega+\pi/c)] + \sigma_1(\sigma_2-\sigma_1)\sin\alpha x[\phi^{(2)}(x,\Omega-\pi/c)+\phi^{(2)}(x,\Omega+\pi/c)] \qquad (2.24)$$

This expression can be written as

$$\psi(x,\Omega) \approx \psi_\Lambda(x,\Omega)+\psi_\sigma(x,\Omega)+\psi_{\sigma\Lambda}(x,\Omega) \qquad (2.25)$$

where $\psi_\Lambda(x,\Omega)$, given by Equ. (2.17), describes the effects of varying scale only, $\psi_\sigma(x,\Omega)$, given by the last term on the right hand side of Equ. (2.21), describes the effects of varying intensity only, and

$$\psi_{\sigma\Lambda}(x,\Omega) = \sigma_1(\sigma_2-\sigma_1)\sin\alpha x[\phi^{(2)}(x,\Omega-\pi/c)+\phi^{(2)}(x,\Omega+\pi/c)] \qquad (2.26)$$

describes the effects due to scale-intensity coupling. It is instructive to point out at this point one further reduction in the form of $\psi(x,\Omega)$ which results from the case where changes in the scale and changes in the intensity are both "small." For this circumstance

$$\psi(x,\Omega) \approx \psi_\Lambda(x,\Omega) + \psi_\sigma(x,\Omega) \qquad (2.27)$$

the scale-intensity coupling being of no importance.

3. Illustrative Examples and Discussion

The three specific cases of inhomogeneous turbulence analyzed in the previous section were each studied for selected values of the various parameters appearing in the respective formulations. Figure 2 is a plot of $[\psi(x,\Omega)/2\Lambda_1\sigma^2]$ for the set of values, $p = (\Lambda_2 - \Lambda_1)/\Lambda_1 = 0.10$, $a = 4\pi$, and $x \geq 0$ for the case of varying scale only. (The case for $x<0$ is not shown since that case is the well-known Dryden spectrum.) The normalized correlation function $Q(x,r)$ corresponding to these parameters is depicted in Fig. 3. In Fig. 2 note the slight "dip" in the shape of the spectrum immediately to the left of the so-called "knee"; this dip ultimately is due to the presence of the term $Q^{(2)}(x,r)$ in the normalized correlation function and as such can be attributed directly to the fact that there is a change in the magnitude of the scale of the turbulence at $x = 0$. Clearly, the level of the dip depends on the value of "p" while the location of it, on the Ω-axis, depends on the value of "a". Indeed, for the set of values $p = 0.20$, $a = 2\pi$, and $x \geq 0$ (see Fig. 4) the spectrum of inhomogeneous turbulence has the "mild knee" characteristic introduced and discussed relevant to Figure 19 of Reference 1. Here, however, the mild knee characteristic is due not only to the fact that a change in the scale of the turbulence has occurred at $x = 0$, but moreso, to the fact that this change has <u>necessarily</u> been accounted for whereas, in Reference 1, the mild knee characteristic is due to changes in the intensity of the turbulence, and results only when a "composite spectrum", i.e., a spectrum shape that does not depend upon x but rather has the x-variation of the intensity averaged out, is constructed.

Figure 5 is a plot of $[\psi(x,\Omega)/2\Lambda\sigma_1^2]$ for the case of varying intensity only for the values, $q = (\sigma_2-\sigma_1)/\sigma_1 = 0.10$ and $c = 1.00$. The values $\sin\alpha x = 0.00$ and $\sin\alpha x = 1.00$ were chosen since these are the cases for which the variations in the intensity respectively make their minimum contribution and maximum positive contribution to $\psi(x,\Omega)$. Note the presence of the "hump", in the shape of the spectrum for $\sin\alpha x = 1.00$, for the range of values of $\Omega, 2.0<\Omega<6.0$; this hump is due to the presence of intensity variations of the form

$$\Delta\sigma = \left(\frac{\sigma_2-\sigma_1}{\sigma_1}\right) \sin\left(\frac{2\pi x}{c\Lambda}\right) = 0.10 \sin\left(\frac{2\pi x}{\Lambda}\right) \tag{3.1}$$

Indeed, if c becomes $c = 0.50$ while q remains unchanged, the hump moves farther out on the Ω-axis and consequently becomes more pronounced (see Fig. 6). The mild knee characteristic for this case cannot be produced regardless of the choice of values of q and c; hence, it may be concluded that the mild knee characteristic for turbulence with varying intensity is at best a second-order effect.†

Figure 7 is a plot of $[\psi(x,\Omega)/2\sigma_1^2\Lambda_1]$ for the values $p = 0.10$, $q = 0.10$, $a = 4\pi$, $c = 1.00$, $\sin\alpha x = 1.00$, and $x \geq 0$ for the case of both varying scale and varying intensity. The presence of both the "dip" and the "hump" is as expected. For the chosen values of p and q the scale-intensity coupling is a second-order effect and therefore has no noticeable effect on the shape of the spectrum.

---

†A thorough analysis of the effects, on the shape of the spectrum, of "slow" variations in the intensity of turbulence is presented in Reference 2.

Second-Order Effects

Second-order effects come in two varieties, one due to scale-intensity coupling (see Equ. 2.26), and the other due to large variations in the intensity such that terms of the form $[(\sigma_2-\sigma_1)/\sigma_1]^2$ are not negligible. To ascertain which of these two varieties is more prominent the function $[\psi(x,\Omega)/2\sigma_1^2\Lambda_1]$ is plotted for values of p = 0.50, q = 0.50, a = 4π, and c = 1.00 in Fig. 8 for the case where second-order terms due to scale-intensity coupling are included while second-order terms due to large variations in the intensity are neglected. The functional form of $\psi(x,\Omega)$ for this case is given by Equ. (2.24) when $\sin\alpha x = 1.00$. Figure 9 represents the case where second-order terms due to large variations in the intensity are included while second-order terms due to scale-intensity coupling are neglected. The functional form of $\psi(x,\Omega)$ for this case is

$$\psi(x,\Omega)\approx\sigma_1^2[\phi^{(1)}(\Omega)+\phi^{(2)}(x,\Omega)] + \sigma_1(\sigma_2-\sigma_1)\ \sin\alpha x$$

$$\phi^{(1)}(\Omega-\pi/c)+\phi^{(1)}(\Omega+\pi/c)] +$$

$$[(\sigma_2-\sigma_1)^2\sin^2\alpha x - \tfrac{1}{2}(\sigma_2-\sigma_2)^2]\phi^{(1)}(\Omega) +$$

$$\left(\frac{\sigma_2-\sigma_1}{2}\right)^2 [\phi^{(1)}(\Omega-2\pi/c)+\phi^{(1)}(\Omega+2\pi/c)] \qquad (3.2)$$

The spectrum shape depicted in Figure 9 is that for $\sin\alpha x = 1.00$.

4. References

1. Houbolt, J. C., "Atmospheric Turbulence", AIAA Journal, Vol. 11, No. 4, April 1973, pp. 421-437.

2. Mark, W. D. and Fischer, R. W., "Investigation of the Effects of Nonhomogeneous (or Nonstationary) Behavior of the Spectra of Atmospheric Turbulence", NASA CR-2745, October 1976.

5. List of Figures

5. List of Figures

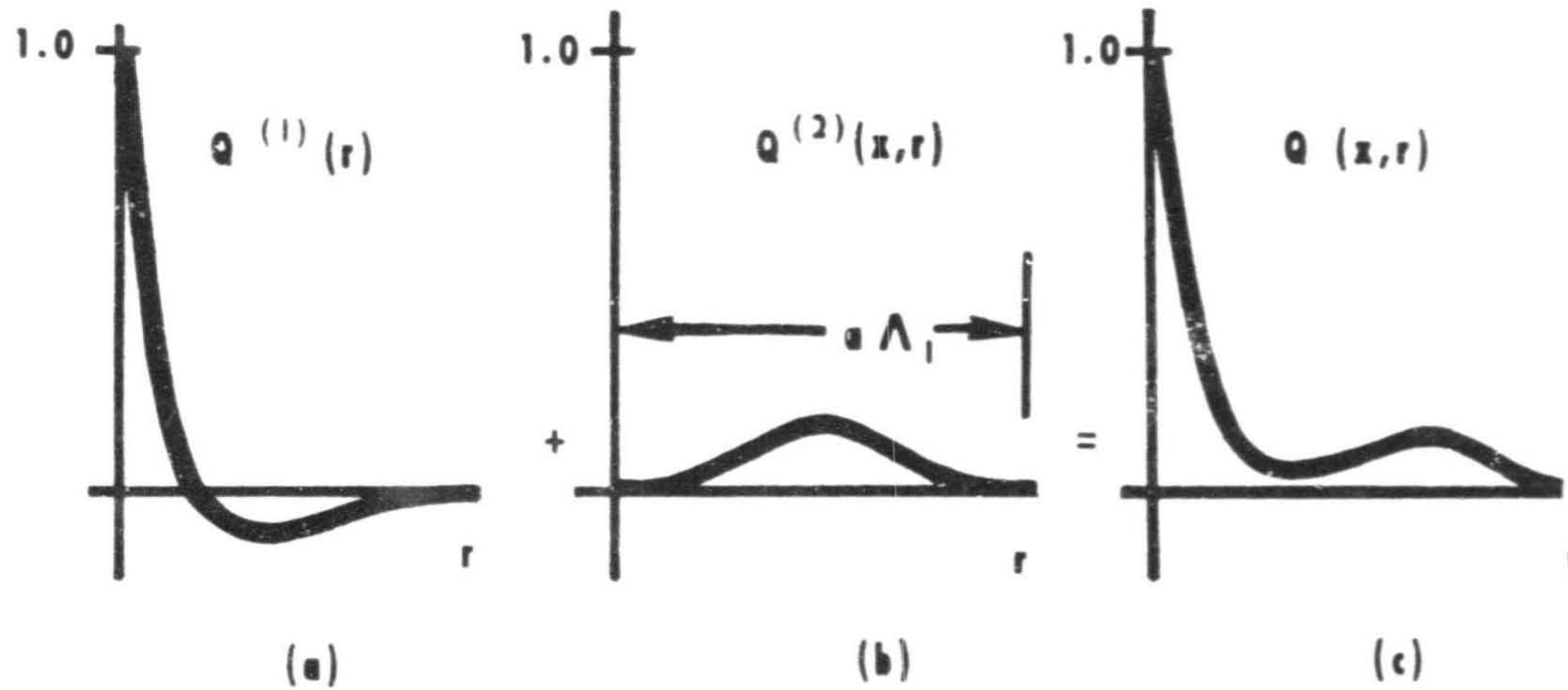
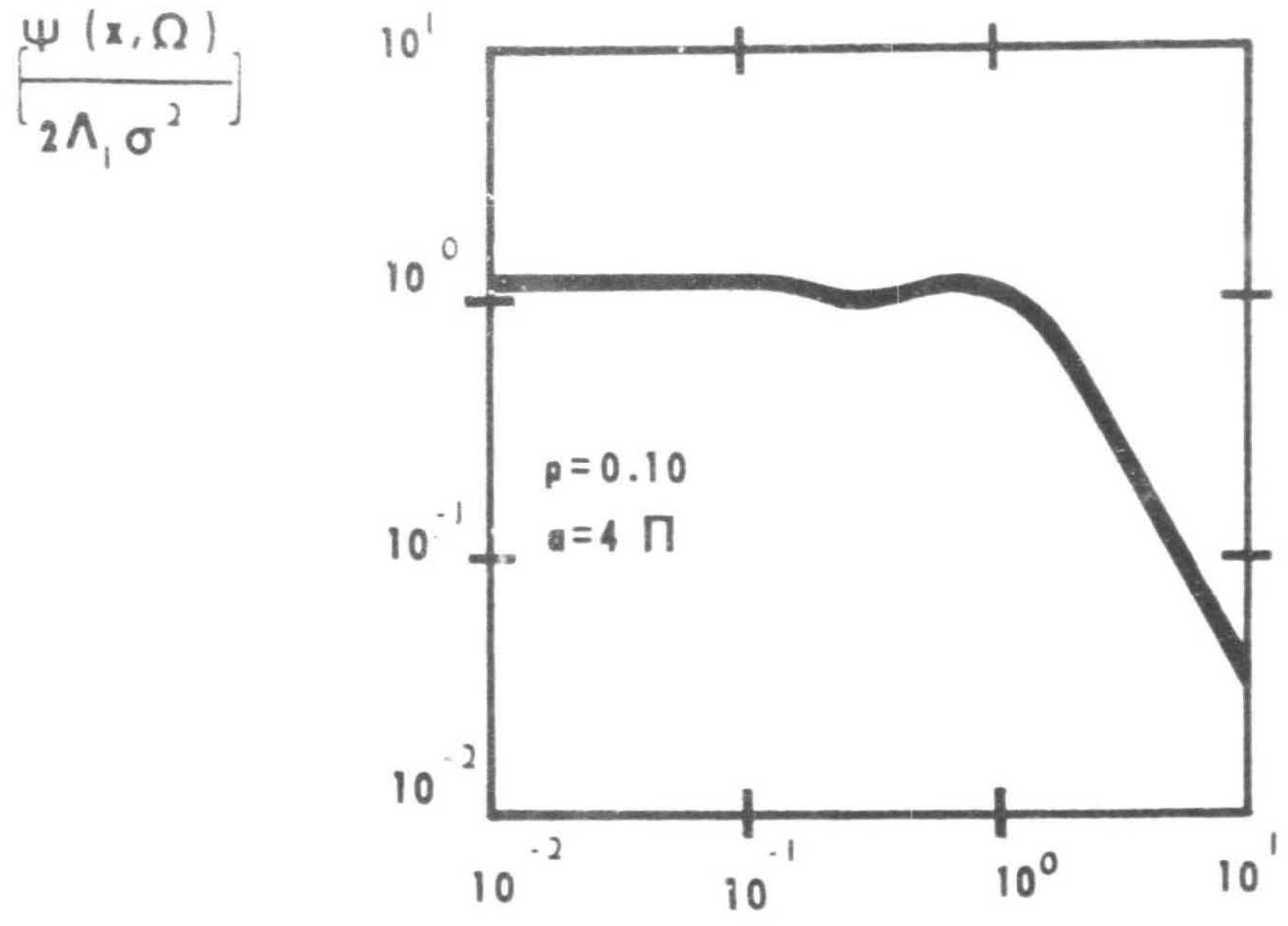


Figure 1

General form of $Q(x,r) = Q^{(1)}(r) + Q^{(2)}(x,r)$



Dimensionless wavenumber, Ω

Figure 2

Spectrum of inhomogeneous turbulence with varying scale only.

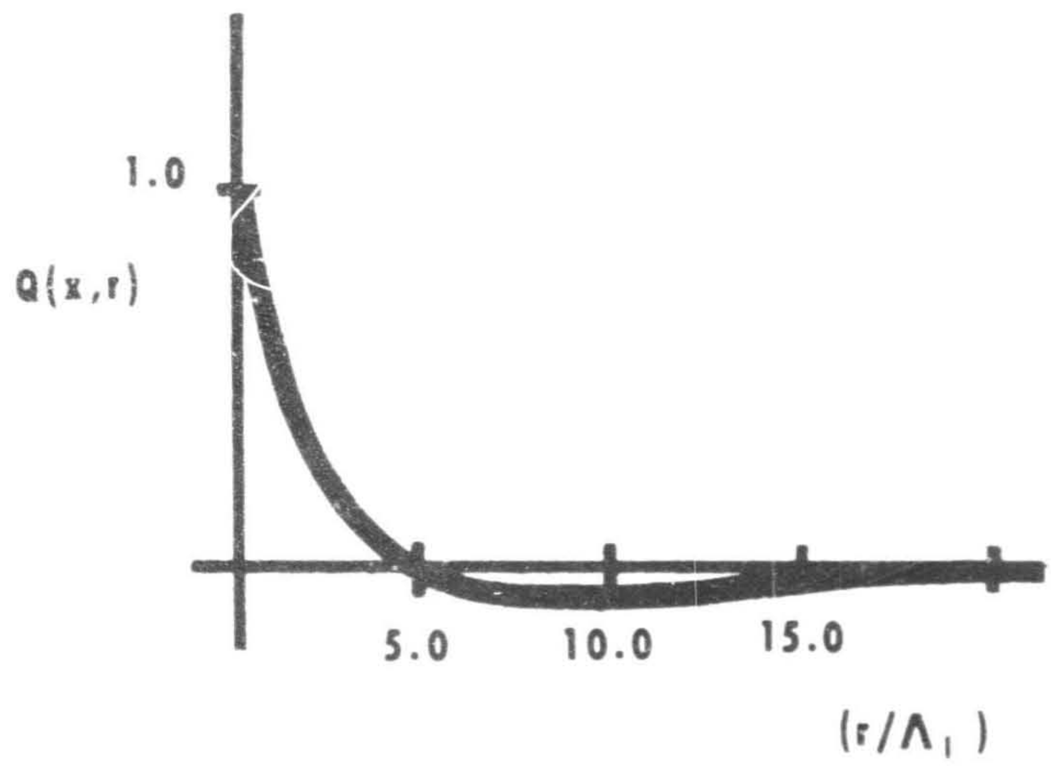


Figure 3

Normalized correlation function $Q(x,r)$, $(p=0.10, a=4\Gamma 1$, $x \geq 0)$, for inhomogeneous turbulence with varying scale only.

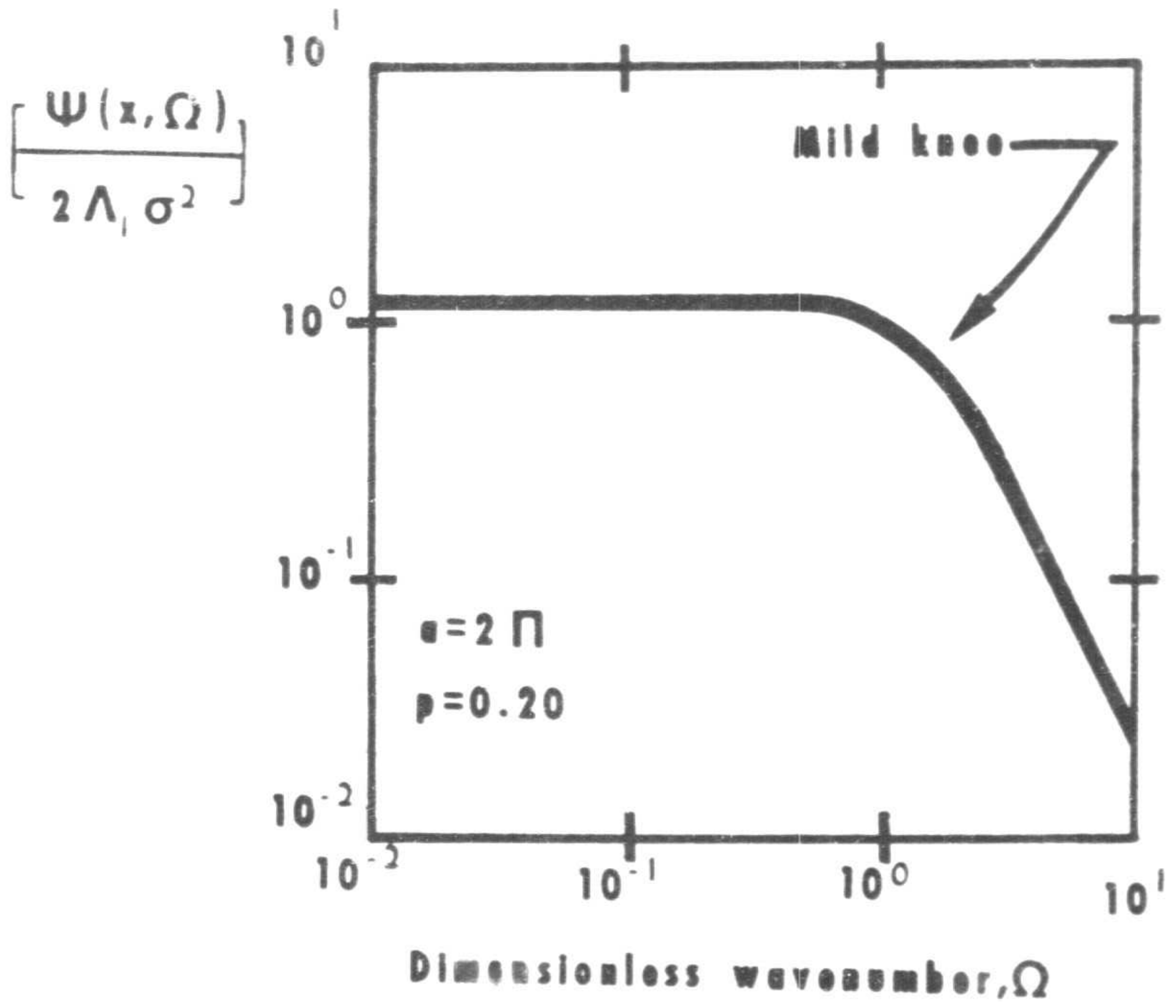


Figure 4

Spectrum of inhomogeneous turbulence with varying scale only $(x \geq 0)$.

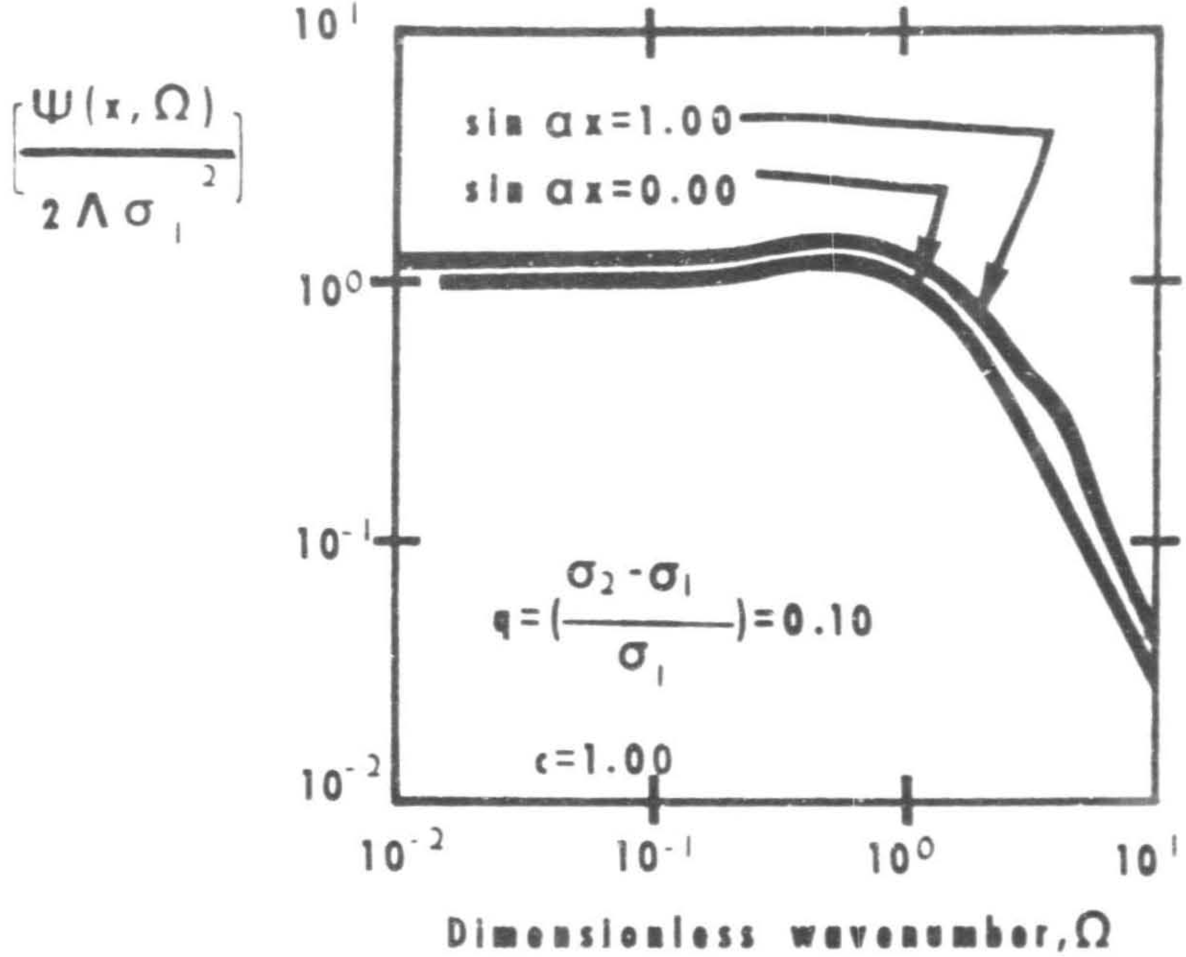


Figure 5

Spectrum of inhomogeneous turbulence with varying intensity only.

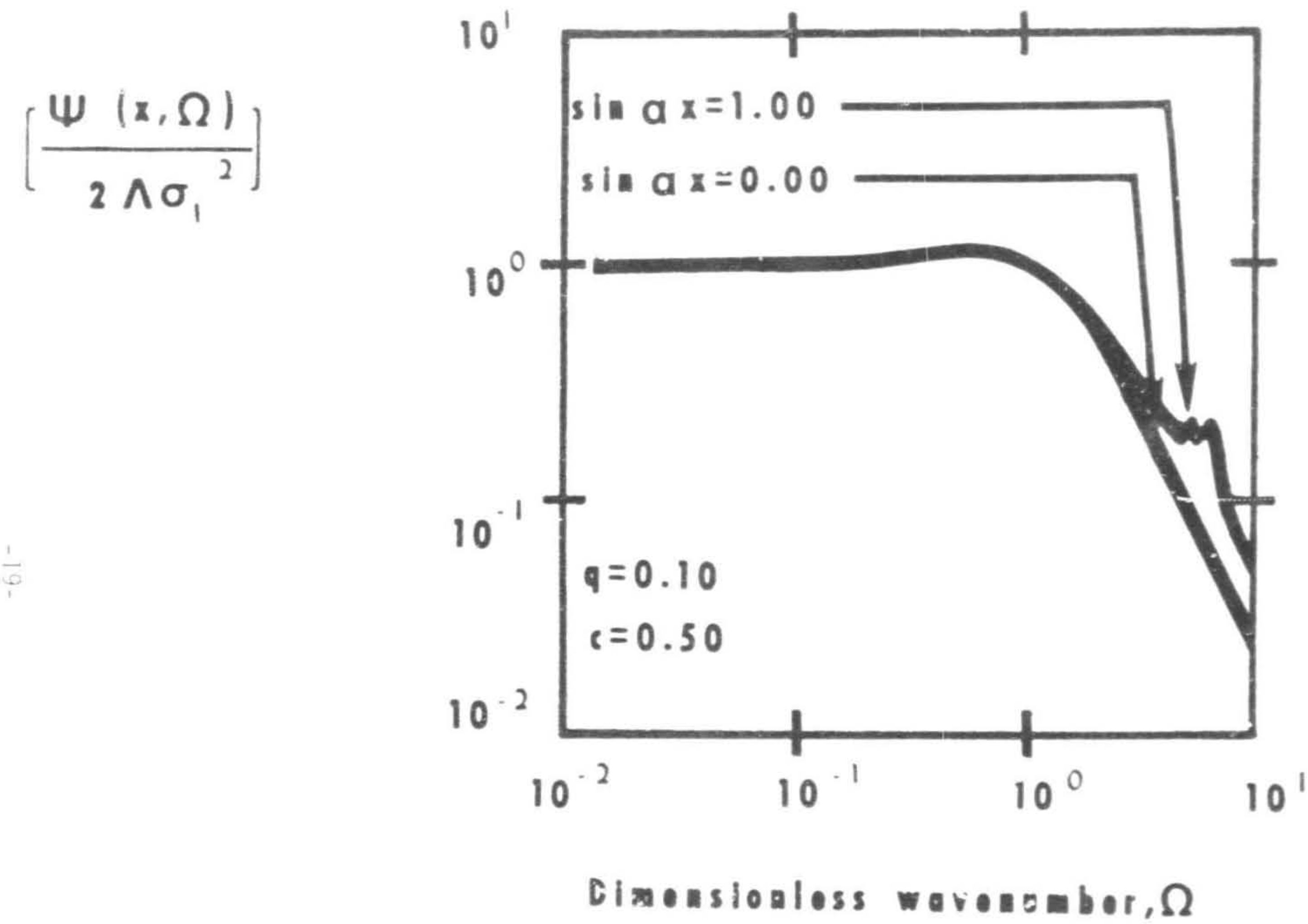


Figure 6

Spectrum of inhomogeneous turbulence with varying intensity only

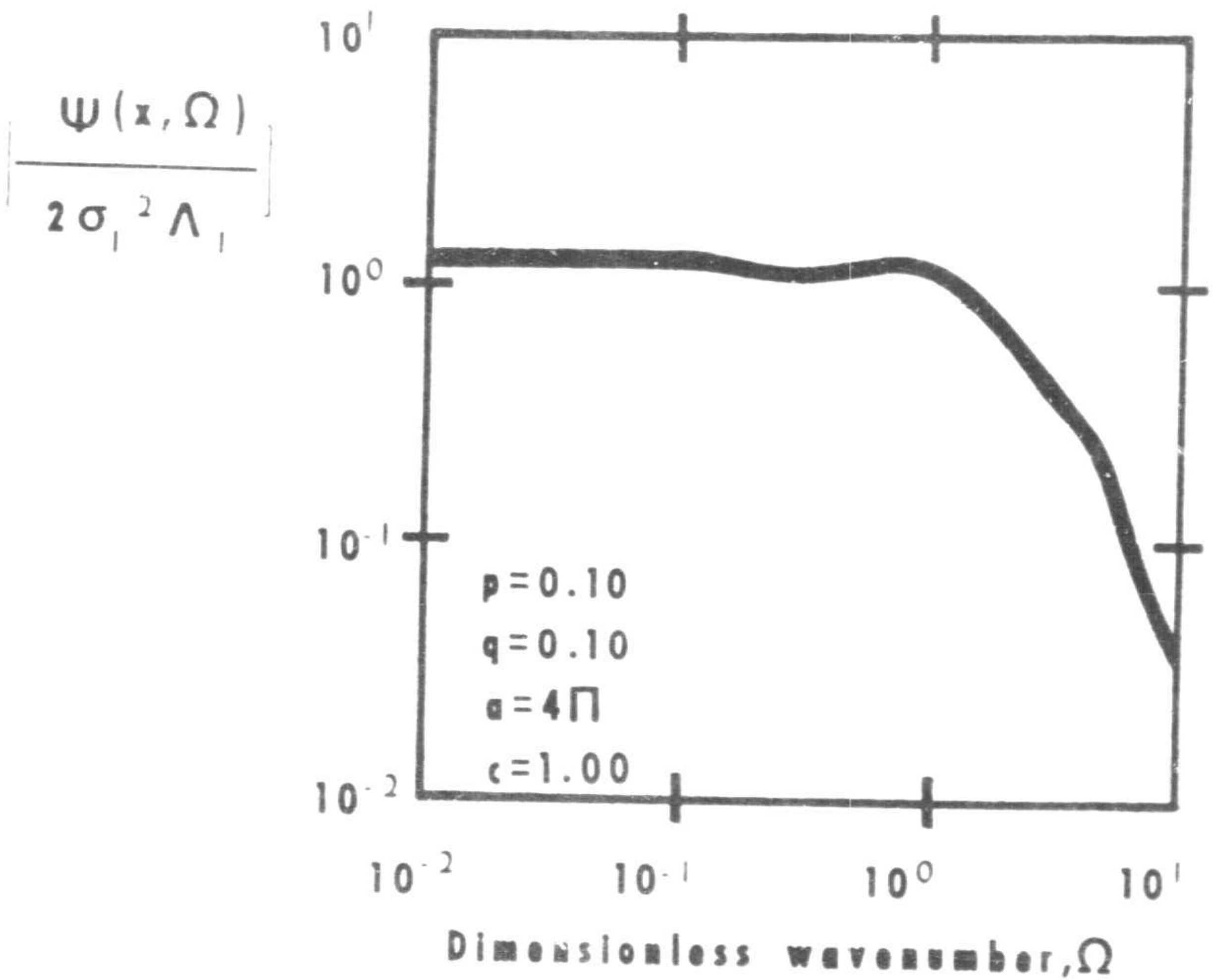


Figure 7

Spectrum of inhomogeneous turbulence with both varying scale and intensity ($\sin \alpha x = 1.00, x \geq 0$).

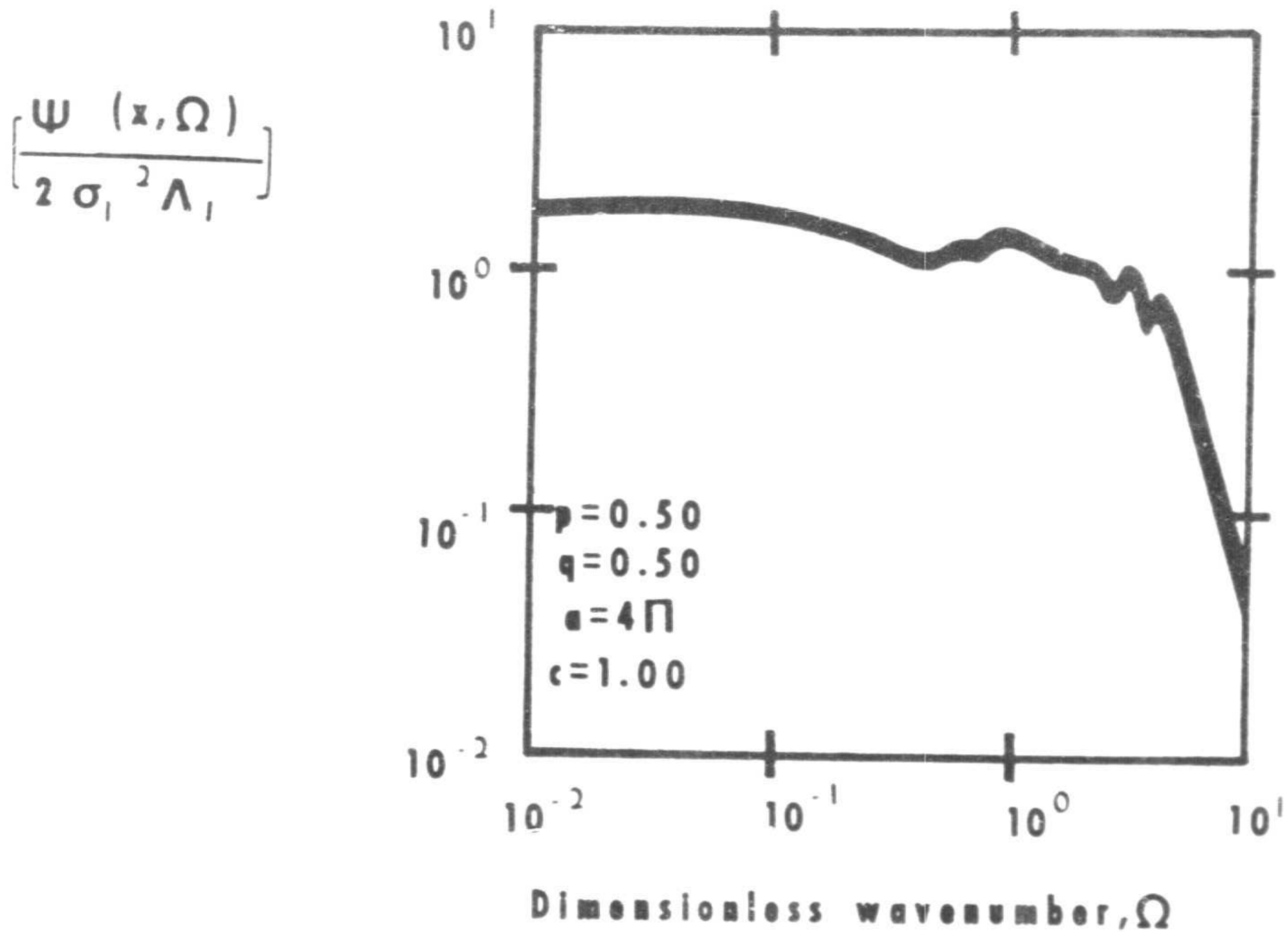


Figure 8

Spectrum of inhomogeneous turbulence with both varying scale and intensity ($\sin \alpha x = 1.00, x \geq 0$), including second-order effects of scale-intensity coupling only.

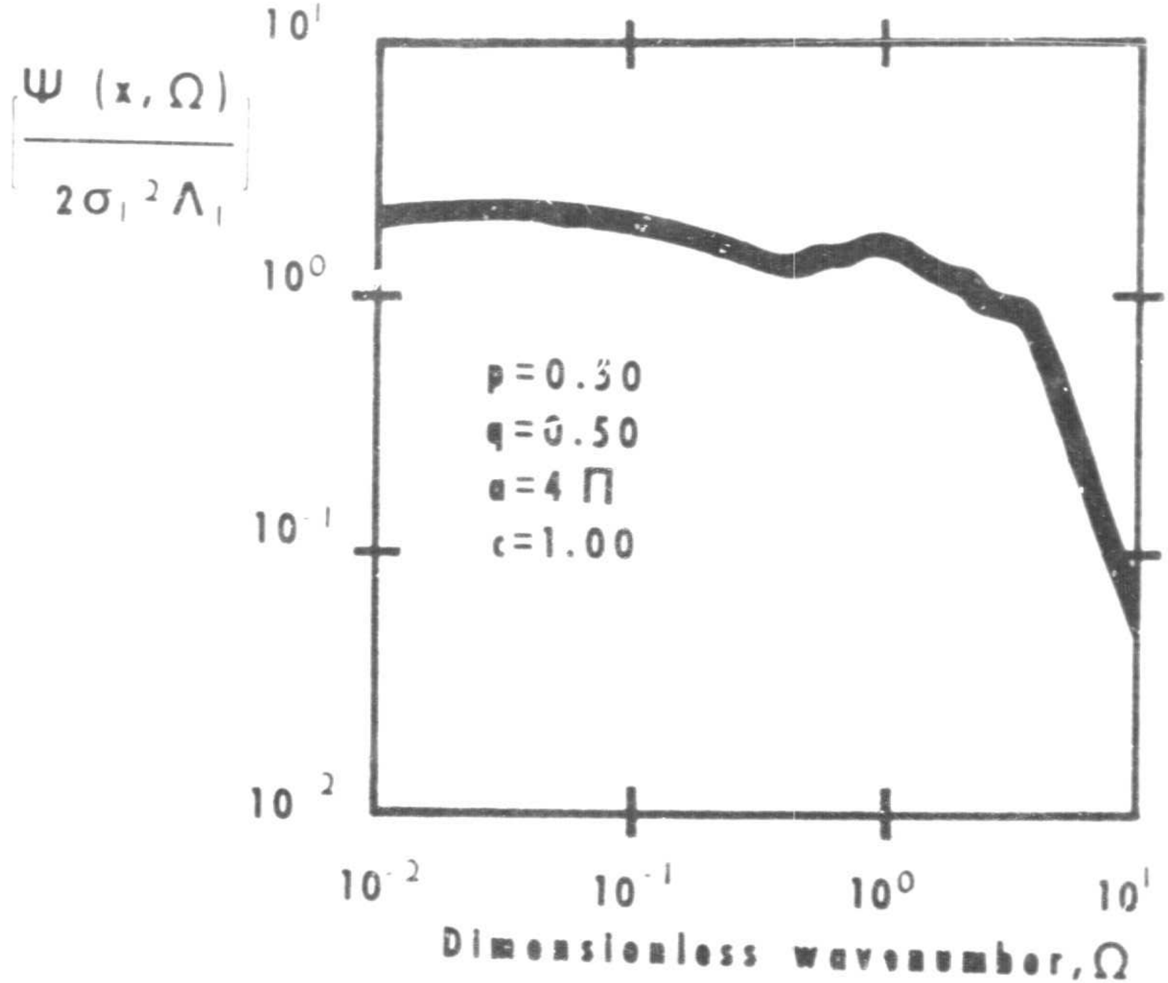


Figure 9

Spectrum of inhomogeneous turbulence with both varying scale and intensity ($\sin\alpha x = 1.00$, $x \geq 0$), including second-order effects of large intensity variation only.

Appendix A: Fundamental Form of Velocity Correlation Function

The correlation function C_{ij} is defined as the mean value

$$C_{ij}(\underline{x}_1,\underline{x}_2) = \langle u_i(\underline{x}_1)u_j(\underline{x}_2)\rangle \tag{A.1}$$

of the two-point velocity product $u_i(\underline{x}_1)u_j(\underline{x}_2)$. For three-dimensional inhomogeneous turbulence, this quantity is a second-order tensor with nine (9) distinct components, each component being formed by one of the various combinations of $i,j = 1,2,3$.

The Schwartz inequality

$$\langle[Au_i(\underline{x}_1) + u_j(\underline{x}_2)]^2\rangle \geq 0 \tag{A.2}$$

for inhomogeneous turbulence demands that

$$C_{ij}(\underline{x}_1,\underline{x}_2) \leq [C_{ii}(\underline{x}_1,\underline{x}_1)C_{jj}(\underline{x}_2,\underline{x}_2)]^{\frac{1}{2}} \tag{A.3}$$

and since

$$[C_{ii}(\underline{x}_1,\underline{x}_1)]^{\frac{1}{2}} = [\langle u_i(\underline{x}_1)u_i(x_1)\rangle]^{\frac{1}{2}} = \sigma_i(\underline{x}_1) \tag{A.4}$$

it follows that

$$\frac{C_{ij}(\underline{x}_1,\underline{x}_2)}{\sigma_i(\underline{x}_1)\sigma_j(\underline{x}_2)} \leq 1 \tag{A.5}$$

Equation (A.5) represents nine (9) expressions (the summation convention is not in effect!), and each one can be written as

$$\frac{C_{ij}(\underline{x}_1,\underline{x}_2)}{\sigma_i(\underline{x}_1)\sigma_j(\underline{x}_2)} = Q_{ij}(\underline{x}_1,\underline{x}_2) \leq 1 \tag{A.6}$$

From Equ. (A.6) it follows that

$$C_{ij}(\underline{x}_1,\underline{x}_2) = \sigma_i(\underline{x}_1)\sigma_j(\underline{x}_2)Q_{ij}(\underline{x}_1,\underline{x}_2) \tag{A.7}$$

where $Q_{ij}(\underline{x}_1,\underline{x}_2)$ is the "normalized" correlation function.

Appendix B: Derivation of Partial Differential Equations Governing Kinematical Behavior of C_{ij}

Beginning with the continuity condition

$$\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i} = 0 \qquad (B.1)$$

where the subscript summation convention is in effect, we have

$$\langle u_i(\underline{x}_1) \frac{\partial u_j(\underline{x}_2)}{\partial x_{2_j}} \rangle = 0 = \frac{\partial \langle u_i(\underline{x}_1) u_j(\underline{x}_2) \rangle}{\partial x_{2_j}} \qquad (B.2)$$

and

$$\langle \frac{\partial u_i(\underline{x}_1)}{\partial x_{1_i}} u_j(\underline{x}_2) \rangle = 0 = \frac{\partial \langle u_i(\underline{x}_1) u_j(\underline{x}_2) \rangle}{\partial x_{1_i}} \qquad (B.3)$$

With the transformation

$$\underline{x}_1 = \underline{x} - \underline{r}/2 \quad \text{and} \quad \underline{x}_2 = \underline{x} + \underline{r}/2, \qquad (B.4)$$

the partial derivatives in Eqs. (B.2) and (B.3) become

$$\frac{\partial}{\partial x_{1_i}} = \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x_{1_i}} + \frac{\partial}{\partial r_j} \frac{\partial r_j}{\partial x_{1_i}} = \frac{1}{2} \frac{\partial}{\partial x_i} - \frac{\partial}{\partial r_i} \qquad (B.5)$$

$$\frac{\partial}{\partial x_{2_j}} = \frac{\partial}{\partial x_i} \frac{\partial x_i}{\partial x_{2_j}} + \frac{\partial}{\partial r_i} \frac{\partial r_i}{\partial x_{2_j}} = \frac{1}{2} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial r_j} \qquad (B.6)$$

and the partial differential equations that govern the behavior of $C_{ij}(\underline{x}_1, \underline{x}_2)$ are:

$$\frac{1}{2} \frac{\partial C_{ij}}{\partial x_i} - \frac{\partial C_{ij}}{\partial r_i} = 0 \qquad (B.7)$$

$$\frac{1}{2} \frac{\partial C_{ij}}{\partial x_j} + \frac{\partial C_{ij}}{\partial r_j} = 0 \qquad (B.8)$$

Appendix C: Derivation of Equation [illegible]

Since

$$\sigma(x) = \sigma_1 + (\sigma_2-\sigma_1)\sin\alpha x \tag{C.1}$$

it follows that

$$\sigma(x-r/2)\sigma(x+r/2) = \sigma_1^2 + \sigma_1(\sigma_2-\sigma_1)\ [\sin\alpha(x-r/2) + \sin\alpha(x+r/2)] +$$

$$(\sigma_2-\sigma_1)^2\sin\alpha(x-r/2)\sin\alpha(x+r/2) \tag{C.2}$$

which reduces to

$$\sigma(x-r/2)\sigma(x+r/2) = \sigma_1^2 + 2\sigma_1(\sigma_2-\sigma_1)\sin\alpha x\cos(\alpha r/2) +$$

$$(\sigma_2-\sigma_1)^2\ [\sin^2\alpha x-\sin^2(\alpha r/2)] \tag{C.3}$$

With the definition

$$S(x,k) = \int \sigma(x-r/2)\sigma(x+r/2)e^{-\iota kr}dr \tag{C.4}$$

Equ. (C.3) transforms to

$$S(x,k) = [\sigma_1^2+(\sigma_2-\sigma_1)^2\sin^2\alpha x]2\pi\delta(k)+\sigma_1(\sigma_2-\sigma_1)\sin\alpha x$$

$$[2\pi\delta(\alpha/2-k)+2\pi\delta(\alpha/2+k)] + (\frac{\sigma_2-\sigma_1}{2})^2\ [2\pi\delta(\alpha-k)-4\pi\delta(k) + 2\pi\delta(\alpha+k)] \tag{C.5}$$

where δ is the Dirac function, and $\sin(\alpha r/2)$ is written as

$$\sin(\alpha r/2) = \frac{e^{\iota\alpha r/2}-e^{-\iota\alpha r/2}}{2\iota} \tag{C.6}$$