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Produced by the NASA Center for Aerospace Information (CASI)
THE GENERALIZED POLE ASSIGNMENT PROBLEM*  
T. E. Djaferis and S. K. Mitter  
Department of Electrical Engineering and Computer Science  
Laboratory for Information and Decision Systems  
Massachusetts Institute of Technology  
Cambridge, Massachusetts 02139

Abstract

For some linear, strictly proper system given by its transfer function, two dynamic output feedback problems can be posed. The first one is that of using dynamic-output feedback to assign the closed-loop characteristic polynomial and the second that of assigning the closed-loop invariant factors. We are concerned with these problems and their inter-relationships. The formulation is done in the frequency domain and the investigation carried out from an algebraic point of view, in terms of linear equations over rings of polynomials. Using the notion of genericity, we express several necessary and sufficient conditions.

1. Introduction

Two of the central results of linear system theory are the following:

(A) Let A and B be matrices of dimension n x n and n x k, respectively. The pair (A,B) is controllable if and only if for every symmetric set A of n complex numbers, there is a matrix C such that A + BC has A for its set of eigenvalues;

(B) Let A and B be matrices of dimension n x n and n x k, respectively with (A,B) being a controllable pair. The input-state transfer function P is given by $P=(sI-A)^{-1}B$. If state feedback $y = Cx + v$ is used, the closed-loop transfer function G is given by $G = P(I+CP)^{-1}$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ be the controllability indices of P. Let $\phi_i$ be polynomials such that $\phi_i|\phi_{i-1}$ with $\sum \phi_i = n$.

Then, there exists a constant C such that the invariant polynomials of G are the $\phi_i$ if and only if

$$\sum_{i=1}^{k} \phi_i = \lambda_i \quad \text{at} \quad k = \lambda.$$

Subsequently there has been considerable work to generalize (A) to the case where static output feedback is allowed. For the most recent results on this topic, see Kilans and Hesselink [14] and Brockett and Byrnes [4]. Some recent work involving dynamic output feedback can be found in [2,3,7,14,15].

Generalization of problem (B) to the output feedback case has been investigated by Rosenbrock and Hayton [13]. They consider a transfer function given in Rosenbrock's system matrix form and present several interesting results. We consider the same problem in the following form.

...
The invariant factors are closely related with the transmission zeros of a plant as defined by Desoer and Schilman. Let \( P \) be an \( m \times k \) plant with Smith-McMillian form given by \( M_p \):

\[
M_p = \begin{bmatrix}
\phi_1 & \phi_2 & \cdots & \phi_k
\end{bmatrix}
\]

The \( \phi_i \) are the invariant factors of \( P \) and the transmission zeros of \( P \) are associated with the zeros of the polynomials \( \phi_i \). Suppose that \( \phi_i \neq 0 \). Then, [8,9], \( z \in \mathbb{C} \) is a zero of \( P \) of order \( n \) iff \( \phi_i(z) = 0 \). The significance of this order, roughly speaking, is that the system completely blocks the transmission of some input of the form \( \frac{1}{z^k} e^{tx} \) for \( k < n \).

For \( \sigma = 0,1, \ldots, k \). For \( \sigma = m \), there is an input of this form for which the output is non-zero and proportional to \( e^{tx} \). Therefore, if two systems \( P \) and \( \bar{P} \) have the same characteristic polynomial \( \chi = \chi \left( \phi_1^{(n)} \phi_2^{(n)} \phi_k^{(n)} \phi_{k+1}^{(n)} \phi_{k+2}^{(n)} \right) \) but different invariant factors (and zeros), the transmission-blocking properties of the two systems would be different.

This paper is divided into five sections. In section 2, we formulate the problem in an algebraic manner using the notion of matrix fraction representation. This, in a very natural way, will suggest a method of solution and in doing so, demonstrate the importance of the equation \( XD + YN = \phi \), where \( X, Y, D, N \) and \( \phi \) are all matrices with elements in \( \mathbb{R} [s] \). In section 3, we will study this equation as it pertains to our problem and will construct what we shall call 'acceptable' solutions. In sections 4 and 5, we discuss the characteristic polynomial problem and the invariant factor problem. From this it will be seen that the results are unsatisfactory in two ways. On the one hand, they are only sufficient conditions, and on the other, they apply in 'almost all' cases. In section 6 we show that by introducing the notion of genericity, more complete results can be formulated. Remaining questions are under continued investigation. Even though we do not specifically address ourselves to specific algorithms for solution, the procedures used are constructive and can be programmed on a digital computer.

2. Formulation and Method of Solution

Assume that we have the feedback system shown in Fig. 1 with \( P \) being a strictly proper \( m \times k \) input-output transfer function and \( C \) some \( k \times m \) proper dynamic compensator. Both \( P \) and \( C \) have elements in \( \mathbb{R} [s] \). The closed-loop transfer function \( G \) is given by

\[
G = P(1 + CP)^{-1}
\]

where we assume that \((1 + CP)^{-1}\) exists. Since \( P \) is a rational matrix, it can be factored [Desoer-Vidyasagar] as follows:

\[
P = BA^{-1} = D^{-1}N
\]

where \( B, A, D, N \) are polynomial matrices. We use the following notation:

\[
P = B_{RP} A_{RP}^{-1} \text{ some right representation of } P
\]

\[
= A_{LP}^{-1} B_{LP} \text{ some left representation of } P
\]

\[
= N_{RP} D_{RP}^{-1} \text{ some right coprime representation of } P
\]

\[
= D_{LP}^{-1} L_{LP} \text{ some left coprime representation of } P.
\]

The closed-loop transfer function \( G \) can then be expressed in the following ways:

\[
G = P(I + CP)^{-1}
\]

\[
= B_{RP} A_{RP}^{-1} + L_{RP} R_{RP}^{-1} A_{RP}^{-1}
\]

\[
= N_{RP} D_{RP}^{-1} + B_{LC} N_{LC}^{-1} A_{LC}^{-1}
\]

\[
= N_{RP} D_{RP}^{-1} + N_{LC}^{-1} D_{LC} + N_{RP} R_{RP}^{-1} D_{LC}
\]

\[
= N_{RP} + D_{LP}^{-1} L_{LP} \text{ (least order)},
\]

where \( N_{RP}, D_{LP}^{-1} L_{LP} \) are right coprime and \( N_{LC}, L_{LC} \) left coprime.

From [5,7] we have that \( \chi \), the characteristic polynomial of the closed-loop system, can be written as \( \chi = \alpha \det N \), a non-zero constant. If coprime representations for both the plant and compensator are not used, then

\[
\chi = \frac{\alpha \det (A_{LC} B_{RP} + B_{LC} A_{RP})}{\det K \cdot \det L}
\]

where \( \alpha \neq 0 \) is a constant, \( K \) a greatest common left divisor of \( A_{LC}, B_{LC} \) and \( L \) a greatest common right divisor of \( B_{RP}, A_{RP} \).

If \( M_G \) is the Smith-McMillan form of \( G \),

\[
M_G = \begin{bmatrix}
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k
\vdots & \varepsilon_1 & \cdots & \varepsilon_k
\end{bmatrix}
\]

we call \( \mathcal{P} = \left[ \begin{array}{c} \psi_1 \varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \\ 0 \varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \\ \vdots \varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \\ 0 \varepsilon_1 \varepsilon_2 \cdots \varepsilon_k \end{array} \right] \) the invariant factor matrix

of \( G \), \( \psi_i \) being the \( i \)th invariant factors. As shown in [9], we have that \( \mathcal{P} \) and \( \mathcal{C} \) are equivalent.

One way to proceed is to utilize the form in which the closed-loop transfer function has been expressed:

\[
G = N_{RP} A_{LC} D_{RP}^{-1} + B_{LC} N_{RP}^{-1} A_{LC}^{-1}
\]

\[
= N_{RP} D_{LP}^{-1} + N_{LC}^{-1} D_{LC} + N_{RP} R_{RP}^{-1} D_{LC}
\]

\[
= N_{RP} + D_{LP}^{-1} L_{LP} \text{ (least order).}
\]

Suppose we are investigating the characteristic polynomial problem. Let \( \phi \) be some \( k \times m \) matrix with \( \det \phi = \alpha \), where \( \alpha \) is some given polynomial. If a
polynomial solution \(X, Y\) to \(XD_{RP} + YN_{RP} = \phi\) exists with \(X^{-1}Y\) existing and proper, then \(C = X^{-1}Y\) is a proper common denominator making the closed-loop characteristic polynomial \(\chi\) equal to \(\chi = \phi/q\) where \(q = \det K\), \(K\) a greatest common left divisor of \(X\) and \(Y\). If, in addition, \(X, Y\) are left coprime, then \(\chi = \sigma\). In a similar manner, suppose we are looking at the invariant factor problem. Let \(\phi = (\phi)\) be an \(L \times L\) matrix in Smith form. If a polynomial solution \(X, Y\) to \(XD_{RP} + YN_{RP} = \phi\) exists with \(X^{-1}Y\) existing and proper, \(X, \phi\) being left coprime and \(N_{RP}, \phi\) being right coprime, then \(C = X^{-1}Y\) is a proper compensator which makes the closed-loop invariant factors \(\psi\) equal to \(\psi_1\).

It is clear from the above that equation \(XD + YN = \phi\) plays a very important role in our investigation. We devote the next section to the study of this equation. Before doing this we also formulate a third problem, the Denominator Matrix problem.

(3.1) The Denominator Matrix Problem

Let \(P = N_{RP}D_{RP}^{-1}\) be an \(m \times L\) strictly proper transfer function described by the right coprime representation \(N_{RP}, D_{RP}\). Let \(\phi\) be an \(L \times L\) matrix. What are necessary and sufficient conditions for the existence of a polynomial solution \(X, Y\) of \(XD_{RP} + YN_{RP} = \phi\) for which \(X^{-1}Y\) exists and is proper? A variant of this would be to require also that \(N_{RP}, \phi\) are right coprime and \(X, \phi\) left coprime.

Remark. The issue of coprineness has not been explicitly dealt with by Rosenbrock and Hayton [13].

3. The Equation \(X D_{RP} + Y N_{RP} = \phi\)

The importance of this equation in the problems at hand has been shown in the last section. It is nothing else but a set of linear equations over the field of rational functions \(R(s)\),

\[
\begin{bmatrix}
X & Y
\end{bmatrix}
\begin{bmatrix}
D_{RP} \\
N_{RP}
\end{bmatrix} = \phi
\]

(i.e., \(Z \neq \phi\)). As such all (rational) solutions can be written as \(Z = Z_1 + Z_2\), where \(Z_1\) is a particular solution and \(Z_2\) is such that \(Z_2\phi = 0\). We, though, are interested only in polynomial solutions, and as can be shown [9, 11]:

Proposition 3.1. Let \(U, V\) be a polynomial solution to \(UD_{RP} + VN_{RP} = I\). Then all polynomial solutions \((X, Y)\) of \(XD_{RP} + YN_{RP} = \phi\) can be expressed as:

\[
\begin{align*}
X &= \phi U - NV_{LP} \\
Y &= \phi V + ND_{LP}
\end{align*}
\]

where \(N\) is a polynomial matrix.

Now from [12], we know that since \(N_{RP}, D_{RP}\) are right coprime, a polynomial solution \(X, Y\) always exists for any \(\phi\). This is an algebraic condition. Emre, in a recent paper [10], gives a nice system theoretic interpretation of this, using module theory and the realization techniques suggested by Fuhrmann. He also suggests an alternate description of all polynomial solutions, which has a system theoretic flavor.

As we have noted, we are interested in solutions of \(XD_{RP} + YN_{RP} = \phi\), which are polynomial but which in addition have the property that (a) \(\det X \neq 0\) and (b) \(X^{-1}Y\) is proper. We call such solutions acceptable.

Satisfying the first requirement is easy, as we see from [9].

Proposition 3.2. Let \(\phi\) be an \(L \times L\) matrix with \(\det \phi \neq 0\). Then there exists a polynomial solution \(X, Y\) to equation (3.1) for which \(\det X \neq 0\).

This next result describes how both requirements are satisfied simultaneously, which consequently plays a crucial role in our investigation [9, 13].

Theorem 3.3. Let \(P = N_{RP}D_{RP}^{-1}\) be a strictly proper \(m \times L\) rational transfer function with \(D_{RP}^{-1}\) being column-proper with column degrees (controllability indices) \(\lambda_1 \geq \lambda_2 \geq \ldots \geq 0\). Let \(\phi\) be an \(L \times L\) non-singular matrix with \(q = \det \phi\) \(-\det D_{RP} \geq 0\). Let \(X, Y\) be a polynomial solution of \(XD_{RP} + YN_{RP} = \phi\). Then \(X^{-1}Y\) exists and is proper iff there exists a unimodular matrix \(N\) and indices \(d_1 \geq 0\) satisfying \(\sum_{i=1}^{d_1} q\) such that

\[
\text{diag}(s^{-d_1}) M[Y, \phi \text{diag}(s^{-d_1})] \text{ is proper}. \quad (3.2)
\]

Remark. The above theorem is clarified if we look at what happens in the single-input, single-output (siso) situation. Let \(x, y\) be a solution to \(xd_p + yu_p = \phi\), \(n_{d_p}, p\) coprime, \(\phi(\theta) < \theta(d_p) = \alpha\).

Necessity: If \(X^{-1}Y\) is proper, we must have

\[
\theta(x) + \alpha < \theta(\phi) = \alpha + q \quad \text{and} \quad q = \theta(x) \geq \theta(y).
\]

Therefore, we must have \(s^{-d_1}[\theta(x), \phi] s^{-d_1}\) proper.

Sufficiency: If \(s^{-d_1}[\theta(x), \phi] s^{-d_1}\) is proper, we have that \(\theta(y) \leq q\) and since \(\theta(\phi) = \alpha + q\), we must have \(\theta(x) = q\), otherwise \(\theta(x) + \theta(y) \neq \theta(\phi)\). This means that \(Y\) exists and is \(p\) proper.

After looking at Theorem 3.1, it is quite natural to attempt the construction of acceptable solutions by making sure that requirement (3.2) is satisfied. We know that all solutions to (3.1) are given by

\[
\begin{align*}
X &= \phi U - NV_{LP} \\
Y &= \phi V + ND_{LP}
\end{align*}
\]

where \(UD_{RP} + VN_{RP} = I\). The question is how to choose \(N\). Let us look at the sisoo situation for a moment. \(Y = \phi V + nd_{n_p}\), or written differently, \(\phi V = nd_{n_p} + y\). We know that whatever \(\phi V\) there exists an \(n_p\) such that \(\theta(y) < \theta(d_p)\). This is nothing else but division by \(d_p\). This as shown in [15] and holds in the matrix case where the column degrees of \(Y\) are strictly less than the column degrees of \(D_{LP}\). If we then let \(D_{LP}, N_{LP}\) be row-proper, the row degrees are the observability indices of \(P\) and we can construct a unique \(Y\) with the \(\theta(y) \leq \mu_1 - 1\), \(\mu_1\) the largest observability index. Therefore, \(\text{diag}(s^{-d_1}) M[Y, \phi \text{diag}(s^{-d_1})] \text{ is proper}\).

To fulfill requirement 3.2, care must be taken in choosing \(\phi\) that makes \(\theta(\phi) \leq \theta(d_p)\).

Theorem 3.1 provides a test for determining whether a specific polynomial solution is actually an acceptable one. It would be greatly desirable, though, if a particular solution could serve as a representative for all solutions. In the following situation, this can be done.

Let \(P\) be a strictly proper transfer function with all observability indices equal to \(\mu\). Let \(D_{LP}, N_{LP}\) be row proper, \(D_{LP} = I + s^{\mu} + \ldots + s^{\mu_d}\).
Let $U, V$ be such that $UD_{RP} + VN_{RP} = I$ with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column proper with controllability indices $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Using right division, there exist unique $-N$ and $\bar{Y}$ such that

$$\phi = -N D_{LP} + \bar{Y} \quad \theta(\bar{Y}) < u.$$ 

Let $\bar{x} = DU + \bar{N}L_P$.

**Proposition 3.3.** Let $P = N_{RP}D_{RP}^{-1} = D_{LP}N_{LP}$ be as above. Let $\phi$ be a diagonal matrix with $\theta(q_{yy}) = \lambda_1 + \gamma, \gamma \geq 0$. Then $X D_{RP} + Y N_{RP} = \hat{\phi}$ has an acceptable solution iff $\bar{x}, \bar{y}$ is an acceptable solution.

**Remark:** The above does include the siso case.

4. **The Characteristic Polynomial Problem**

We are now in a position to give a partial answer to the Characteristic Polynomial Problem as stated in section 1. I say 'partial' because it is only a sufficient condition.

**Theorem 4.1.** Let $P$ be an $m \times l$ strictly proper transfer function and $N_{RP}D_{RP}^{-1}$ a right coprime representation

where $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column proper with column degrees $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ (controllability indices).

Let $D_{LP}N_{LP}$ be a left coprime representation, with

$${\begin{bmatrix} D_{LP} \\ N_{LP} \end{bmatrix}}$$

row proper. Let $\phi$ be a polynomial of degree $t = t(q_{yy}) + Z(\mu_i - 1)$ ($\mu_i$ the largest observability index). Then there exists a proper compensator $C$ such that the characteristic polynomial $\chi$ of the closed-loop system is given by

$$X = \Phi_{XY}$$

where $\Phi_{XY}$ is a polynomial with $0 \leq \theta(q_{XY}) \leq \lambda(\mu_i - 1)$.

The proof is constructive [9], with the compensator being given by $C = X^{-1}Y$ where $X, Y$ is an acceptable solution to some equation of the form $X D_{RP} + Y N_{RP} = \hat{\phi}$. The polynomial $\Phi_{XY}$ is nothing else but the Kronecker product of $X^{-1}$ and $Y$.

**Remark.** It is clear that this Theorem can be used for purposes of stabilization. If $\phi$ is chosen to be a stable polynomial, then so will be $X$, the closed-loop characteristic polynomial. We also note that the compensator may or may not be stable. We will investigate this issue later in this section.

**Remark.** From a closer examination of the procedure we can see that, in general, one does not have prior knowledge of what $\Phi_{XY}$ is. The larger the degree of $\Phi_{XY}$ implies a smaller increase in the overall dynamics of the system. One can therefore use this in the design of compensators.

**Remark.** Using a different output feedback configuration, Brasch and Pearson [3] show that the characteristic polynomial of the closed-loop system can be assigned by only increasing the system dynamics by $\mu_{i-1}$. These results can be obtained using the approach outlined in this paper [9]. Even though more dynamics are added in our approach, it may be that the computations are less cumbersome. This issue warrants further investigation.

**Remark.** The approach taken in [15] suggests that compensation involves input as well as output dynamics, and it also differs from the present approach in that it requires stable, pole-zero cancellation and the presence of 'hidden' modes.

On the one hand, it is quite worthwhile to investigate compensation schemes that require as little added dynamics as possible. An equally worthwhile task is to investigate whether, by adding more dynamics than the least required, one can achieve other design objectives as well [2]. The following Lemma and Example deal with this issue.

**Lemma 4.1.** Let $\phi$ be a polynomial with $\theta(\phi) = 2n - 1 + k, k \geq 1$. Let $x_1, y_1$ be an acceptable solution of

$$\Phi_{LP} + \Phi_{RP} = \Phi,$$

strictly proper, $\theta(\Phi_{LP}) = n$. All acceptable solutions are of the form

$$x_2 = x_1 - mn_p \quad y_2 = y_1 + mn_p$$

where $\theta(\Phi) \leq k - 1$.

**Example.**

Let $p = \frac{1}{s^2 - 1}$, and suppose that we want to construct a proper and stable compensator which makes the characteristic polynomial of the closed-loop system equal to the stable polynomial $\phi = s^3 + s^2 + 3s + 1$.

The compensator $C = \frac{s^3 - 2s}{s^2 - 1} = \frac{Y_1}{X}$ does satisfy the requirements except that it is unstable. Now, all acceptable solutions are given by

$$x_2 = x_1 - mn_p \quad y_2 = y_1 + mn_p$$

where $n$ is a constant. Let $n = -2$. Then

$$x_2 = s^{2} + s + 1 \quad y_2 = 3s^{2} + 2s + 2$$

Clearly, $C_{2} = 3s^{2} + 2s + 2$ satisfies all the requirements.

**Remark.** This idea can certainly be extended to the multiple-input, multiple-output situation.

5. **The Invariant Factor Problem**

Let $P$ be an $m \times l$ strictly proper transfer function and $\Phi = \Phi_{LP}$ an $l \times k$ diagonal matrix in Smith form. If $P = N_{RP}D_{RP}^{-1}$ is some right coprime representation for which there exists an acceptable solution to the equation $X D_{RP} + Y N_{RP} = \Phi$, where $\Phi$ and $\Phi$ are equivalent, with (a) $\Phi$ left coprime and (b) $N_{RP}, \Phi$ right coprime, then $C = X^{-1}Y$ is a proper compensator making the invariant factor matrix $Y = \Phi_{LP}$ of $G$ equal to $\Phi$. If conditions (a) and (b) are not met, then (c) we have $\Phi_{LP}$, we also know [13] that if $P$ with controllability indices $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 0$ and observability indices $\mu_1 \geq \mu_2 \geq \cdots \mu_k \geq 0$, then also satisfies

$$k \sum_{i=1}^{k} \theta(q_{ai}) > \sum_{i=1}^{k} (\lambda_i + \mu_i - 1) \quad k = 1, 2, \ldots, \text{with}\text{ equality at } k = l,$$

then there exists a matrix $\Phi$ equivalent to $\Phi$ such that

$$\lim_{s \to \infty} [\text{diag}(s^{2a_i - 1}) \text{diag}(s^{2ai})] = I.$$

**Theorem 5.1.** Let $P, \Phi, \phi$ be as above, with the $\phi_i$ satisfying

$$k \sum_{i=1}^{k} \theta(q_{ai}) \geq \sum_{i=1}^{k} (\lambda_i + \mu_i - 1) \quad k = 1, 2, \ldots, \text{with}\text{ equality at } k = l.$$

Then there exists a proper compensator $C = X^{-1}Y$ with
\( X D_{RP} + Y N_{RP} = \phi \) and such that if \( \Psi \) is the closed-loop invariant factor matrix, then \( \Psi_1 \Phi_1 \).

We will only have \( \Psi_1 \Phi_1 \) if \( X \) and \( \Phi_1 \) are left coprime and \( N_{RP} \) and \( \Phi_1 \) are right coprime.

Remark. In earlier work Rosenbrock [12] gave a necessary and sufficient condition in the case of state feedback. That result can be obtained using the theory developed in this approach [9]. In that situation, the invariant factors are assigned exactly for all cases. Here, as we see, the conditions are merely sufficient and apply to 'some' cases; this warrants further clarification.

Remark. The fact that in the state feedback the system transfer function can take the form \( P = (sI-A)^{-1}B \) simplifies the problem, and using the procedure suggested here, Rosenbrock's earlier result can be proved.

The results we have discussed so far are unsatisfactory in two respects. We have seen that degree constraints on \( \psi_1 \) or \( \phi_1 \) are not enough to ensure that the closed-loop transfer function \( G \) will have the desired characteristics. If \( X, \phi_1 \), and left coprime and \( N_{RP} \Phi_1 \) are right coprime, this will be true. However, they are only sufficient conditions. It is therefore quite natural to ask whether degree constraints are solely sufficient in 'almost all' cases and whether these are necessary in 'almost all' situations as well. We will show in the next section that in some cases this is indeed true.

6. Generic Results

Let \( q \) be some positive integer. We define the Zariski topology on \( R^n \) this way. Let \( u \) be an ideal in \( R[x_1, \ldots, x_n] \). All points \( x = (x_1, \ldots, x_n) \) in \( u \) such that \( f(x) = 0 \) for all \( f \) in \( u \) form the variety of \( u \). If closed sets in \( \mathbb{C}^n \) are defined to be the varieties of \( \mathbb{C}^n \) [16], then \( \mathbb{C}^n \) becomes a topological space with the Zariski topology. Let \( R^n \) have the subspace topology.

Definition. A set \( S \subset R^n \) is called 'generic' if it contains a non-empty Zariski open set of \( R^n \). Roughly speaking, a set is generic if it contains almost all of \( R^n \), its complement is contained in a set of Lebesque measure zero. The way in which we use the notion of genericity is to first take a set of \( R^n \) and then define a property which is valid for all points in \( S \subset R^n \). We then attempt to show that \( S \) is generic. This means, in effect, that the property is valid on almost all of \( R^n \). We now give explicit definitions.

Definition. An \( nx \times \) strictly proper transfer function \( P \) of order \( n \), given by \( P = N_{RP}D_{RP} \), has the generic characteristic polynomial assignability property if the monic polynomials \( \lambda^k \Phi_1 \) for which there exists a proper compensator \( C \) making the closed-loop characteristic polynomial equal to \( \phi \) is a generic subset of \( R^n \).

Definition. An \( n \times \lambda \) strictly proper transfer function \( P \) of order \( n \), given by \( P = N_{RP}D_{RP} \), has the generic denominator matrix assignability property if the matrices \( \Phi_1 \) for which there exists an acceptable solution \( X, X, Y \) to \( X D_{RP} + Y N_{RP} = \phi \) with \( N_{RP}, \phi \) right coprime and \( X, \phi \) left coprime is a generic subset of \( R^n \).

In what follows, we find that looking at the equation \( X D_{RP} + Y N_{RP} = \phi \) as an operator is greatly advantageous. If

\[
\begin{align*}
X &= X_{k-1} + X_{k-2} + \cdots + X_0 \\
Y &= Y_{k-1} + Y_{k-2} + \cdots + Y_0 \\
D_{RP} &= D_{k-1} + \cdots + D_0 \\
N_{RP} &= N_{k-1} + \cdots + N_0
\end{align*}
\]

then

\[
[S_{D_{RP}}(D_{RP}N_{RP})]_{k} = [\delta_{k=t-1} \delta_{k+t-1} \cdots \delta_{k+1}]
\]

where

\[
\begin{bmatrix}
[1, X_{k-1} Y_{k-1} \cdots X_0 Y_0] & S(D_{RP}N_{RP}) \\
& \begin{pmatrix}
D_0 & D_{t-1} & 0 & \cdots & 0 \\
N_0 & 0 & 0 & \cdots & 0 \\
0 & N_{t-1} & 0 & \cdots & 0 \\
0 & 0 & N_{t-1} & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & & & 0 \\
& & & & N_0
\end{pmatrix}
\end{bmatrix}
\]

This [1] we immediately recognize as the generalized Sylvester Resultant of \( D_{RP} \) and \( N_{RP} \) of order \( k \) [it is a \( k(n-2)x(k+t) \) matrix with real entries].

The following two lemmata taken from [1] give the rank of \( S(D,N) \) for some transfer function ND-1 in terms of the dual dynamical indices (observability indices if ND-1 proper) of ND-1 and relate coprimeness of \( N, D \) to the rank of some \( S_k(D,N) \). These are generalizations of siso results.

Lemma 6.1. Let \( ND^{-1} m \times k \) be proper with \( \lambda_1 \) observability indices of ND-1. Then

\[
\text{rank } S_k(D,N) = (n + m)k - \sum_{i=1}^{m} (k - \mu_i) < k
\]

Lemma 6.2. Let \( ND^{-1} m \times k \) be proper and \( q \) the least integer for which rank \( S_q(D,N) = \text{rank } S(D,N) - k \). Then, for \( n > q \), \( N, D \) are right coprime iff \( S_n(D,N) = kn - k \det D \).

A consequence of viewing the equation \( X D_{RP} + Y N_{RP} = \phi \) as an operator is:

Proposition 6.3. Let \( P = N_{RP}D_{RP} \) be an \( n \times \lambda \) strictly proper transfer function with controllability indices \( \lambda_1 = \lambda_2 = \cdots = \lambda_n \) and observability indices \( \mu_1 = \mu_2 = \cdots = \mu_n \) and \( D_{RP}, N_{RP} \) of the form \( D_{RP} = \text{Is}^{\lambda_1} + \text{Ds}^{\lambda_1} + \cdots + \text{Ds}^{\lambda_{n+1}} + \cdots + \text{Ds} + N_{RP} = \text{Ns}^{\mu_1} + \cdots + \text{Ns}^{\mu_{n+1}} + \cdots + N_0 \). Let \( R^n \) be the set of \( \lambda \times k \) matrices of the form

\[
\phi = \text{Is}^{\lambda_1} + \text{Ds}^{\lambda_1} + \cdots + \text{Ds}^{\lambda_{n+1}} + \cdots + \text{Ds} + \text{Ns}^{\mu_1} + \cdots + \text{Ns}^{\mu_{n+1}} + \cdots + N_0
\]

Let \( Q = \{(X,Y)| X = \text{Is}^{\lambda_1} + \text{X_{q-1}^{\mu_1-1}} + \text{X_0}, Y = \text{Y}^{\mu_1-1} + \cdots + Y_0 \} \)

A necessary and sufficient condition for the existence of a solution to \( X D_{RP} + Y N_{RP} = \phi \) in the class \( Q \) for generic \( \phi \) is \( q \geq k - 1 \).

Proof:

(necessity).

Equation \( X D_{RP} + Y N_{RP} = \phi \) with the conditions imposed can be written as

\[
[1, Y_q X_{q-1} Y_{q-1} \cdots X_0 Y_0] S_{q+1} = [1 \Phi_q, \lambda-1, \ldots, \Phi_0]
\]

\( S_{q+1} \) can be thought of as a function

\[
S_{q+1}: R^{(k+m)(q+1)} \rightarrow R^{(\lambda+q+1)k}
\]

From Lemma 6.1 we have that \( S_k \) [a \( k(n-2)x(k+t) \) matrix] has rank

\[
\text{rank } S_k = (k+m)k - \sum_{i=1}^{m} (k - \mu_i)
\]

which, under the special circumstances, becomes

\[
\text{rank } S_k = (k+m)k - m(k - \mu) \quad \text{if } \mu < k.
\]

By observing dimensions, we see that:

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a) $S_1, S_2, \ldots S_{\mu-1}$ are not onto

b) $S_\mu$ is both one-one and onto

c) $S_{\mu+1}, S_{\mu+2}, \ldots$ are onto.

Assume now that $q < \mu + 1$ and that $X D_{RP} + Y N_{RP} = \phi$ has a solution in $Q$ for generic $\phi$. Show a contradiction.

If we think of $(X,Y)$ as an element in $R^2(\ell + m)\ell + m$ and $\phi$ as an element in $R^2(\ell + m)\ell$, we see that the $\phi$ that can be reached from elements in $Q$ are a set of dimensions less than $\ell(q+\lambda)\ell$, which implies that the set of $\Phi$ which can be reached does not contain a non-empty Zariski open set. This is a contradiction; therefore, $q \geq \mu - 1$.

(Sufficiency)

Suppose that $q \geq \mu - 1$ (or equivalently, $q = \mu - 1 + k$, $k \geq 0$). We want to show that the set $\phi \in R^2$,

$$[\ell = \ell(\lambda + \mu + k)\ell - \ell]$$

for which a solution in $Q$ exists, is a generic subset of $R^2$. We already know that $S_{\mu k}$

is an $(\ell + m)(\ell + m)\ell + m$ matrix with

$$\text{rank}\ S_{\mu k} = (\ell + m)\ell + m \Delta k$$

This means that the operator

$$S_{\mu k} : R^2(\ell + m)\ell + m \rightarrow R^2(\ell + m)\ell + m \Delta k$$

is onto. We want to show that $S_{\mu k}(Q) = R^2$. For $\phi \in R^2$ there exists some $(X,Y)$ of the form

$$X = X_{\mu k-1} S_{\mu k-1}^{\mu k-1} + \ldots + X_0$$

$$Y = Y_{\mu k-1} S_{\mu k-1}^{\mu k-1} + \ldots + Y_0$$

such that

$$[X_{\mu k-1}, Y_{\mu k-1}, X_0, Y_0] S_{\mu k} = [I \Phi_{\lambda + \mu k - 1} \ldots \phi_\ell].$$

For this we must have $X_{\mu k-1} = 1$, which implies that $(X,Y) \in Q$. This completes the proof.

We are now in a position to give two results concerning the generic characteristic polynomial assignability property.

Theorem 6.4. Let $P = n d^{-1}$ be a iso, strictly proper transfer function of order $n$. Let $\phi$ be a monic polynomial with $\theta(\phi) = n + q (\phi \in R^{n+q})$. A necessary and sufficient condition for generic characteristic polynomial assignability is $q \geq n-1$.

Proof.

Since $\phi \in R^{n+q}$ is to be the characteristic polynomial of the closed system, the compensator accomplishing this must be of order $q$. From Proposition 6.5 we then have that a necessary condition is $q \geq \mu - 1$.

For sufficiency, assume that $q \geq \mu - 1$. We need to show that $S$ contains a non-empty Zariski open set (i.e., it is generic). Since $q \geq \mu - 1$ the Theorem 3.3 can be used to show that $S$ contains a non-empty Zariski open set. Since $q \geq \mu - 1$, $\phi_\ell$ monic with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell-1} \geq 0$. We have $S_{\mu k}(Q) = R^2$ for $\phi_\ell$ monic with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell-1} \geq 0$. We have $S_{\mu k}(Q) = R^2$ for $\phi_\ell$ monic with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell-1} \geq 0$.

Theorem 6.5. Let $P = n d^{-1}$ be a iso proper transfer function of order $n$. Let $\phi$ be a monic polynomial with $\theta(\phi) = n + q (\phi \in R^{n+q})$. A sufficient condition for generic characteristic polynomial assignability is $q \geq n-1$, the largest observability index of $P$. In the event that all observability indices are equal to $\mu$, then this condition is necessary as well.

Remark. In proving these results, we make use of the generalized Sylvester resultants. The results are confined to the case when the denominator matrix $(X D_{RP} + Y N_{RP} = \phi)$ is just a polynomial. For the general matrix case, a closer examination of the structure of the resultant matrices is needed.

Remark. Results similar to these proved in a different way can also be found in a recent paper of Willems and Hesselink [14].

The generalized Sylvester resultants can be used more effectively to treat the generic denominator matrix assignability problem. As expected for the single-input, single-output case, we have

Theorem 6.6. Let $P = n d^{-1}$ be a iso strictly proper transfer function of order $n$. Let $\phi$ be a monic polynomial with $\theta(\phi) = n + q (\phi \in R^{n+q})$. A necessary and sufficient condition for generic assignability is $q \geq n-1$.

The proof proceeds in a similar manner as that of Theorem 6.4. The multiple-input, multiple-output situation is much more challenging. For this, we interpret Lemma 6.2 in the following way: The matrices $N, D$ are right coprime iff at least one $\phi_\ell = \theta(\det D) \times \theta(\det D) \times \theta(\det D)$ minor of $S_{\mu 1}(0,N)$ is not zero. Denote these minors by $m_{i1}(0,N)$. By symmetry, the argument can also be made for left coprimeness. We can now state [9].

Proposition 6.7. Let $P = N_{RP}D_{RP}$ be an $m \times \ell$ strictly proper transfer function with $N_{RP}$ column proper, and column degrees $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell-1} \geq 0$. Let $N_{RP}D_{RP}$ be such that $\theta(\det D) = 1$. The largest observability index.

Let $R^\ell$ denote the set of $\ell \times \ell$ diagonal matrices $\phi_\ell = (\phi_\ell)$.

Let $N_{RP}D_{RP}$ be an $m \times \ell$ strictly proper transfer function with $N_{RP}$ column proper, and column degrees $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell-1} \geq 0$. Let $N_{RP}D_{RP}$ be such that $\theta(\det D) = 1$, the largest observability index. Denote these minors by $m_{i1}(0,N)$. By symmetry, the argument can also be made for left coprimeness. We can now state [9].

Remarks. For this result as well we see that degree constraints are not enough and that 'undesirable' additional conditions are present.

On the other hand, this is merely a sufficient condition. For the special case of diagonal systems, we have [9].
1. \[ P = \begin{bmatrix} n_1 & \cdots & n_k \\ d_1 & \cdots & d_2 \end{bmatrix} \]

with \( n_i, d_i \) coprime, \( d_i \) monic (this means that the controllability indices \( \lambda_i \) are equal to \( 0(\phi) \leq i \leq k \) and the observability indices \( \mu_i \) are equal to \( 0(d_i) \) \( 1 \leq i \leq k \) with \( \mu_{k+1} = \ldots = \mu_m = 0 \).)

Let \( R^t \) denote the set of \( k \times k \) diagonal matrices \( \phi = (\phi_i), \phi_1 \) monic with \( \theta(\phi_i) = \lambda_i + t_i \),
\[ t = \sum_{i=1}^{\mu} (\lambda_i + t_i) \]. A sufficient condition for generic denominator matrix assignability is \( t \geq \mu - 1 \).

In the event that \( m = 0 \) and \( \lambda_1 = \lambda_2 = \ldots = \lambda_k = \lambda \) and \( \mu_1 = \mu_2 = \ldots = \mu_k = \mu = \mu \), then \( t \geq \mu - 1 \) is necessary condition as well.

Remark: Under the assumptions of Proposition 6.7, we have that a sufficient condition for generic denominator matrix assignability is
\[ a) \ t \geq \mu - 1 \]
\[ b) \ at least one \ m = 0, N \neq 0 \]
\[ c) \ at least one \ n = 0, X \neq 0 \]

It is desirable to eliminate condition (b). To accomplish this it is to show that for some \( \phi \in R^t \) we have \( m_1(\phi, N) \neq 0 \) and \( n_1(\phi, X) \neq 0 \).

Proposition 6.8 suggests a way in which this may be achieved. Instead of looking at some specific system and some space of \( \phi \), look at the space \( T \times \phi \), where \( T \) is an appropriate space of systems (which includes diagonal systems). Then attempt to show that for some \( \phi \) (a diagonal system) and some \( \phi \neq \phi \), \( m_1(N, \phi) \neq 0 \) and \( n_1(\phi, X) \neq 0 \). This way we will, in effect, have proved that 'almost all' systems in \( T \) have the generic denominator matrix assignability property, if \( t \geq \mu - 1 \).

Theorem 6.9

Let \( N, D \) be \( k \times k \) matrices and define \( W, Z, S, \phi \) as follows:
\[ W = \{ (N, D, \phi) \in R^{(k \lambda_1) \times 1} \ | \ N = D = D_0 \} \]
\[ Z = \{ \phi \in \phi \ | \ \phi = 15 \lambda^q \} \}
\[ S = \{ (N, D, \phi) \in R^{(k \lambda_1) \times 1} \mid \text{for which there exists an acceptable solution } X, Y \}

A necessary condition for \( S \) to be a generic subset of \( R^{(k \lambda_1) \times 1} \) is \( q \geq \lambda - 1 \).

Proof:

Suppose that \( S \) is generic (i.e., it contains a non-empty Zariski open set) and let \( q < \lambda - 1 \). Show a contradiction.

Let \( N, D \) be the subset of \( R^{(k \lambda_1) \times 1} \) for which \( N, D \) are right coprime and \( ND^{-1} \) has observability indices equal to \( \mu = \mu \). (If \( N, D \) are right coprime, the controllability indices of \( ND^{-1} \) are all equal to \( \lambda \).)

We have that \( S \) is generic because of the following:

The set \( F \in R^{(k \lambda_1) \times 1} \) for which \( \text{rank } X = 2k \) \( 1 \leq k \leq \lambda \) is generic. This means that for every \( (N, D, \phi) \in S \) we have that:

1) \( N, D \) are right coprime (Lemma 6.2)
2) Since \( ND^{-1} \) is proper, the observability indices of \( \lambda - 1 \) are all equal to \( \lambda \) (Lemma 6.1).

This implies that \( (F \cap S) \) is generic.

Since we have assumed \( S \) to be generic, we must have that \( S \cap N \) is non-empty. Let \( (N_0, D, \phi) \in S \cap N \). This means that for \( N_0, D \) and almost all \( \phi \in F \) there exists a symmetric solution \( X, Y \) of \( Z \) such that \( \phi = \phi \). This solution \( X, Y \) is an acceptable solution. It is clear that \( N, \phi \) are right coprime for almost all \( (N, D, \phi) \). The question then remains as to whether \( X, \phi \) are left coprime. (i.e., \( \phi, \phi \) 'right coprime').

From Proposition 6.8 we already know that there exists some diagonal system \( N D^{-1} \) and some diagonal \( \phi \in Z \) for which \( X, \phi \) are coprime [call the point \( (N, D, \phi) \in F \cap S) \). This means (Lemma 6.1) that
\[ \text{rank } X = \phi, X = 1 \leq i \leq \lambda - 1 \] and \( \text{rank } X = \phi, X = \lambda - 1 \). \( \phi \).

This implies that the above also hold for generic \( \phi \). Using Lemmas 6.2 we then have that \( X, \phi \) are left coprime for generic \( \phi \). This means that \( S \) is a generic subset of \( R^{(k \lambda_1) \times 1} \).

Remark: In Theorem 6.9 we see that \( q \geq \lambda - 1 \) is a necessary condition so that for almost all systems of order, \( k \lambda_1 \) and equal observability indices \( \lambda \), there exists an acceptable solution \( X, Y \) of \( X + Y \phi = \phi \) with \( X, \phi \) right coprime, \( \phi \) left coprime for almost all \( \phi \in Z \). In Proposition 6.8.2 we have that \( q \geq \lambda - 1 \) is a sufficient condition. We conjecture that \( q = \lambda - 1 \) is actually a sufficient condition, thus completing Theorem 6.9.

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References


