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A MULTILOOP GENERALIZATION OF THE
CIRCLE CRITERION FOR STABILITY MARGIN ANALYSIS[*]

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ABSTRACT

In order to provide a theoretical tool well suited for use in characterizing the stability margins (e. g., gain and phase margins) of multiloop feedback systems, multiloop input-output stability results generalizing the circle stability criterion are considered. Generalized conic sectors with "centers" and "radii" determined by linear dynamical operators are employed to enable an engineer to specify the stability margins which he desires as a frequency-dependent convex set of modeling errors--including nonlinearities, gain variations and phase variations--which the system must be able to tolerate in each feedback loop without instability. The resulting stability criterion gives sufficient condit· ions for closed-loop stability in the presence of such frequency-dependent modeling errors, <u>even when the modeling errors occur simultaneously in all loops</u>; so, for example, stability is assured as loop gains and phases vary throughout a "set of non-zero measure" whose boundaries are frequency-dependent. The stability conditions yield an easily interpreted scalar measure of the amount by which a multiloop system exceeds, or falls short of, its stability margin specifications.

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I. INTRODUCTION

A key step in the synthesis of robustly stable feedback systems is the characterization of a set of feedback laws that are stabilizing for every element of the set of possible plant dynamics. This type of information is precisely what is provided for single-loop feedback systems by such input-output stability criteria as the Nyquist, Popov, and circle theorems. Indeed, the practical merit of classical feedback design procedures involving Nyquist loci, Bode plots, and Nichols charts is in a large measure directly attributable to the fact that these design procedures provide the designer with easily interpretable characterizations of such sets of robustly stable feedback laws. For single-loop feedback systems, these stability theorems enable engineers to meaningfully characterize the tolerable amount of gain and phase variation in the loop at each frequency, and even the tolerable amount of unmodeled nonlinearity. These tolerances of modeling error are in broad terms what we call stability margin, classical gain and phase margin being two familiar measures of stability margin for single feedback loops. Although multiloop generalizations of the Nyquist stability criterion have been developed (e. g. , [1] -[3]), it has been difficult to meaningfully relate the conditions of these multiloop criteria to tolerance of open-loop modeling error except in special cases such as diagonally dominant systems, normal systems, and systems in which feedback loop gains vary only over certain "sets of zero measure" [4]. The results of the present paper are intended to address the need for an improved method for characterizing the stability margins of multi-loop feedback systems.

In broad and imprecise terms what seems to be necessary to meaningfully characterize multiloop stability margins is a stability criterion that guarantees stability for every multiloop feedback operator within a given "frequency-dependent ball" in an appropriate space of input-output relations.

this ball being centered at the system's nominal "open-loop gain" operator.
It is important that the size of this ball be permitted to be frequency-
dependent so that one can account for frequency-dependent variations
in the precision of mathematical models such as result from such ubiquitous
effects as singular perturbations, hysteresis, imprecisely known time-
delays, or any sort of unmodeled dynamics. Also, since in general
one may expect modeling imprecision in certain feedback loops to be large
relative to other loops, it should be possible to specify that this ball be
somewhat egg-shaped, having different diameters in the various "directions"
corresponding to the "gains" of individual feedback loops. So, perhaps
the necessary ball of stable multiloop feedback operators could be better
described as a "frequency-dependent egg. " All of this is of course too
vague and imprecise to be of immediate use--what is needed is a stability
criterion dealing with a precise mathematical description of this frequency-
dependent ball (or, "egg") and of the space of operators in which is is
embedded,

       Stability results in this general spirit are provided by the Zames-
Sandberg input-output stability theory [5]-[6]. Sandberg's frequency-
domain stability criterion[6] for systems with multiple nonlinearities can
be interpreted as guaranteeing stability for a collection of nonlinear feed-
back operators inside a spherical (i. e., not "egg-shaped") non-frequency-
dependent ball centered at the identity operator times a scalar; the now
well-known circle stability criterion emerged in [7] as a special case of
this result. Zames' conic sector stability theorem [5, Theorem 2] is an
abstract generalization of Sandberg's criterion that makes the connection
with balls of stable multiloop feedback operators even more transparent:
the conditions of Zames' theorem involve conic sectors which it happens
are simply spherical balls, centered at the identity operator times a
scalar in an extended normed space of input-output relations.

In the past fifteen years frequency domain stability criteria based on the Zames-Sandberg theory have been improved and refined in many significant ways. Reference [8] provides a good overview of much of this work. Reference [9] develops similar results in a Lyapunov setting. References [10] - [12] and the additional references cited therein describe many stability results developed specifically for interconnected (i.e., multiloop) nonlinear systems. However, the previous literature in this area has focused primarily on <u>nonlinear</u> stability; though Zames [5] makes some key suggestive remarks about the broader implications conic sector results regarding imprecisely modeled systems. The stability margin implications of the results have not been stressed and no results based on the Zames-Sandberg theory have been published which address the need for a <u>frequency-dependent</u> characterization of multiloop stability margins.

The main objective of the present paper is to present a multiloop input-output stability criterion that is tailored to the task of multiloop feedback stability margin analysis. Our main result (Theorem 1) shows that multiloop stability margins--including tolerance of unmodeled non-linearity and of dynamical modeling errors of <u>frequency-dependent</u> magnitude--can be directly related to open-loop system frequency-response quantities. The results, expressed in terms of the "singular values" of certain matrices, are observed to yield an easily interpretable scalar measure of a systems "excess stability margin"--i.e., of the amount by which a multiloop feedback design exceeds its stability margin specifications. A related result described in [21] makes use of Theorem 1 in generating generalized conic sector bounds for characterizing the sensitivity of multiloop systems to large dynamical modeling errors of frequency-dependent magnitudes; in effect the result of [21] provides a nonlinear multiloop generalization of classical M-circle ideas.

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The role of singular values in connection with stability was first noted by Sandberg [1], though Sandberg does not specifically use the term singular value. Earlier versions of the results in present paper and their connections with stability margin analysis were first reported in [12] - [14]. Stressing the use of singular values, Doyle [15] establishes important geometric connections between multivariable Nyquist criteria and the stability margin results of [13], [14], and the present paper; additionally, Doyle [15] and Stein and Doyle [16] cite a number of illustrative examples that present a compelling case for the use of results of this type in the analysis of multivariable feedback stability margins. MacFarlane and Scott-Jones [25] discuss at length the relationships between the eigenvalues and the "principal gains" (i.e., singular values) of a multiloop-system's transfer matrix. Nuzman and Sandell [23] establish some inequalities relating these results to the singular values of the return difference matrix for multiloop systems and discuss the connection with the guaranteed stability margins of full-state feedback linear optimal regulators (see [24]). The paper by Sandell [17] discusses in broad and simple terms the role of singular values in coping with modeling imprecision in a wide range of engineering and numerical problems. Numerical aspects of singular value computation are surveyed in [18], wherein sophisticated and widely available computer routines for singular value computation are also referenced.

II. NOTATION

The following notation is used: A^T and x^T denote respectively the transpose of the matrix A and the vector x; A^* and x^* denote the complex conjugate of the matrix A^T and the vector x^T respectively; the determinant of a matrix A is denoted $\det(A)$; the Euclidian norm of a vector x is $\|x\|_E \triangleq \sqrt{x^* x}$; R_+ denotes nonnegative real numbers; the functional norm $\|x\|_T$ and inner product $<x_1, x_2>_T$ are defined for functions x:

$R_+ \rightarrow R^n$ as

$$\| x \|_\tau \triangleq \sqrt{< x, x >_\tau} \tag{1}$$

where for any x_1 and x_2

$$< x_1, x_2 >_\tau \triangleq \int_0^\tau x_1^T(t) x_2(t) \, dt \; . \tag{2}$$

The space $L_{2e}(R_+, R^n)$ is defined as

$$L_{2e}(R_+, R^n) \triangleq \left\{ x : R_+ \rightarrow R^n \mid \| x \|_\tau < \infty \quad \forall \, \tau \in R_+ \right\}, \tag{3}$$

where the symbol $\forall$ means "for all." We define $L_{2e} \triangleq \underset{n}{\cup} L_{2e}(R_+, R^n)$. Laplace transforms are denoted by capital letters, e. g., $X(s)$ denotes the Laplace transform of $x(t)$.

Given any matrix A, the square-roots of the eigenvalues of A^*A are called the singular values of A. For any matrix A, we use the notation $\sigma_{max}(A)$ to denote the largest singular value of A and $\sigma_{min}(A)$ to denote the smallest singular value of A. Singular values are always nonnnegative real numbers since A^*A is always positive semidefinite.

A functional relation is a mapping of functions into sets of functions; for example, a dynamical system mapping inputs in L_{2e} into outputs in L_{2e} defines a relation (e. g., [5]). An operator is a special type of relation which maps each input function into exactly one output function, i. e., into a set with exactly one element. All functional relations considered in this paper are implicitly assumed to be mappings of L_{2e} into L_{2e}. A relation $\underset{\sim}{H}$ is said to be nonanticipative if for all t_0, the output $(\underset{\sim}{H} x)(t_0)$

does not depend on $x(t)$ for $t > t_0$. We say that a relation $\underset{\sim}{H}$ is L_{2e}-$\underline{stable}$[1] if there exists a constant $k < \infty$ such that for all $x \, \epsilon \, L_{2e}$ and all $\tau \, \epsilon \, R_+$

$$\| \underset{\sim}{H} x \|_\tau \leq k \| x \|_\tau \; . \tag{4}$$

Generalizing some of the L_{2e} conic sector conditions of Zames [5], we employ the following definitions which are a special case of the generalized sector conditions of [13], [19], [20]. Given an operator $\underset{\sim}{H}$, if there exist operators $\underset{\sim}{C}$, $\underset{\sim}{R}$, and $\underset{\sim}{S}$ such that

$$\| \underset{\sim}{S}(y - \underset{\sim}{C}x) \|_\tau^2 \leq \| \underset{\sim}{R} x \|_\tau^2 - \epsilon (\| x \|_\tau^2 + \| y \|_\tau^2) \tag{5}$$

for all $y = \underset{\sim}{H} x$, all x, all τ, and some $\epsilon > 0$, then we say "$\underset{\sim}{H}$ is strictly inside the L_{2e} conic sector with center C and radius $(\underset{\sim}{R}, \underset{\sim}{S})$"; equivalently, we write

$$\underset{\sim}{H} \text{ strictly inside } L_{2e}\text{-Cone}(\underset{\sim}{C}, \; \underset{\sim}{R}, \; \underset{\sim}{S}).[2] \tag{6}$$

Given a relation $\underset{\sim}{G}$, if

$$\| \underset{\sim}{S}(y - \underset{\sim}{C}x) \|_\tau^2 \geq \| \underset{\sim}{R} x \|_\tau^2 \tag{7}$$

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[1] For nonanticipative operators, L_{2e}-stability as defined here is equivalent to the usual notion of L_2-stability, e.g., [8].

[2] The "strictly inside" conic sector condition of [5] can be demonstrated to be a special case of (6); however, the term $\epsilon \| y \|_\tau^2$ is only implicit in [5].

for all $x = -\underset{\sim}{G}y$, all y, and all τ, then we say "the inverse relation of $-\underset{\sim}{G}$ is outside the L_{2e} conic sector with center $\underset{\sim}{C}$ and radius $(\underset{\sim}{R}, \underset{\sim}{S})$"; [3] equivalently, we write

$$(-\underset{\sim}{G})^I \quad \text{outside} \quad L_{2e} - \text{Cone} \; (\underset{\sim}{C}, \; \underset{\sim}{R}, \; \underset{\sim}{S}) \; . \tag{8}$$

The notation $\text{col}(x_1, \ldots, x_N)$ denotes the column vector

$$\text{col}(x_1, \ldots, x_N) \quad \triangleq \quad \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad . \tag{9}$$

The relation $\text{diag}(\underset{\sim}{H}_1, \ldots, \underset{\sim}{H}_N)$ is defined by

$$\text{diag}(\underset{\sim}{H}_1, \ldots, \underset{\sim}{H}_N) \cdot \text{col}(x_1, \ldots, x_N) \quad = \quad \text{col}(\underset{\sim}{H}_1 x_1, \ldots, \underset{\sim}{H}_N x_N) \; . \tag{10}$$

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[3] Following [13], [19], [20], the notation $(-\underset{\sim}{G})^I$ is used for the inverse of the relation $-\underset{\sim}{G}$; i.e., $(-\underset{\sim}{G})^I$ is the relation which maps each $y \epsilon L_{2e}$ into the <u>set</u> of functions $x \epsilon L_{2e}$ such that $y = -\underset{\sim}{G} x$. The inverse <u>relation</u> $(-\underset{\sim}{G})^I$ always exists even for operators $\underset{\sim}{G}$ for which the inverse operator, denoted $(-\underset{\sim}{G})^{-1}$, does not exist.

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III. PROBLEM FORMULATION

Our results concern the input-output stability of systems consisting of a dynamical <u>linear time-invariant</u> (LTI) interconnection of N imprecisely modeled components, including imprecisely modeled LTI components and <u>nonlinear time-varying</u> (NTV) components as well as dynamical nonlinear components comprised of interconnections of LTI and NTV subcomponents. The system equations thus take the following form (see Fig. 1):

<u>components</u>

$$y_i = \underset{\sim}{H}_i x_i \quad , \qquad (i = 1, \dots, N) \tag{11}$$

<u>dynamical LTI interconnection</u>

$$\underline{X}(s) = -\underline{G}(s) (\underline{Y}(s) + \underline{V}(s)) + \underline{U}(s) \tag{12}$$

where

$$\underline{Y}(s) = \text{col} (Y_1(s), \dots, Y_N(s)) \tag{13}$$

$$\underline{X}(s) = \text{col} (X_1(s), \dots, X_N(s)) \tag{14}$$

$$\underline{U}(s) = \text{col} (U_1(s), \dots, U_N(s)) \tag{15}$$

$$\underline{V}(s) = \text{col} (V_1(s), \dots, V_N(s)) \tag{16}$$

$$\underline{G}(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1N}(s) \\ & \vdots & \\ & \vdots & \\ G_{N1}(s) & \dots & G_{NN}(s) \end{bmatrix} \tag{17}$$
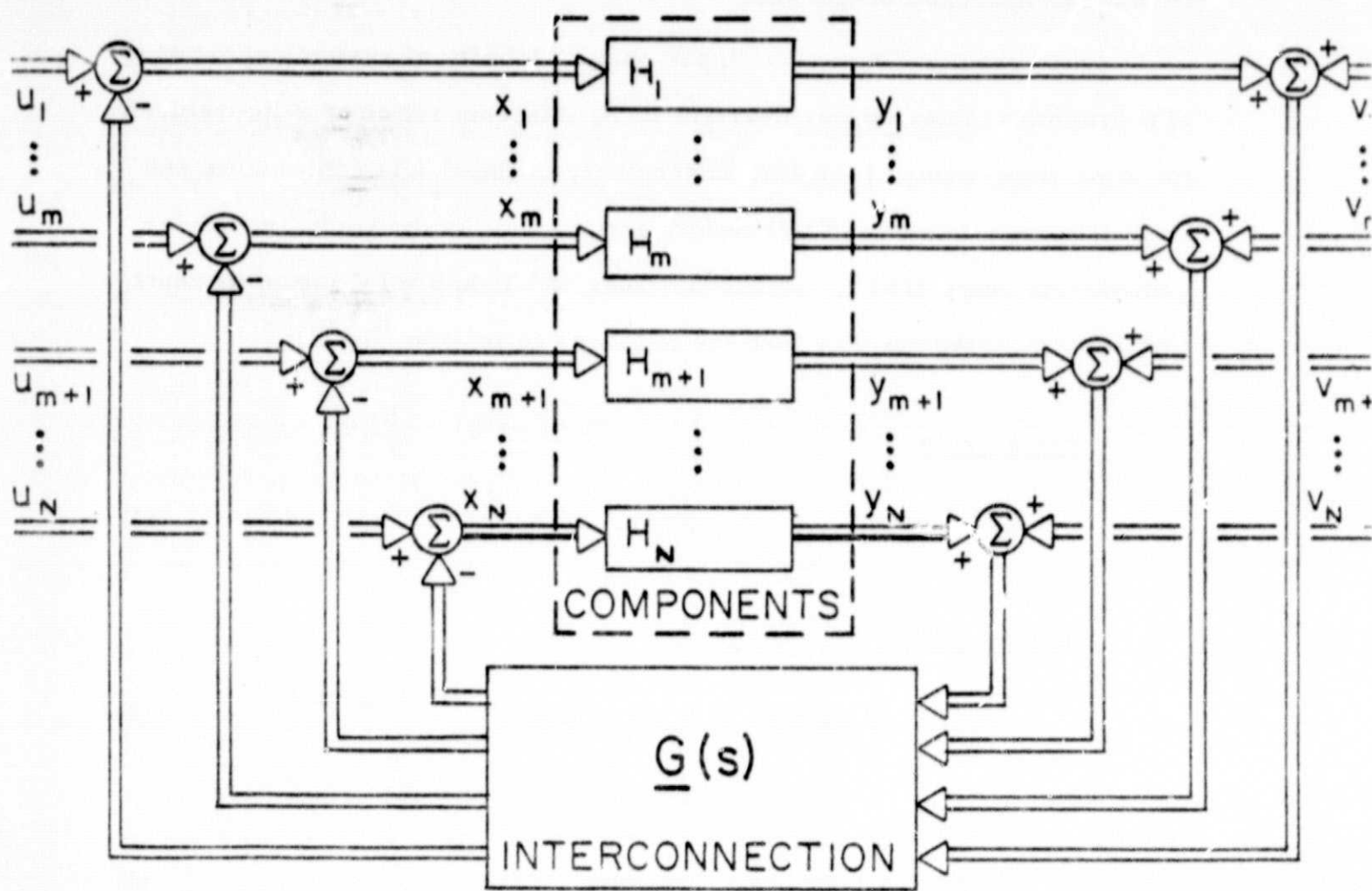
Fig. 1   The System

The endogenous variables $y_i(t)$ and $x_i(t)$ are the system "outputs" and the exogenous variables $u_i(t)$ and $v_i(t)$ are the system "inputs". Each of the "components" $\underset{\sim}{H}_i$ may itself be a multi-input-multi-output a (MIMO) system in general, though our results are most easily used and inperpreted when the components are single-input-single-output (SISO).

We assume that for each of the imprecisely modeled components $\underset{\sim}{H}_i$ we have a crude approximate LTI model $\underset{\sim}{C}_i$ and that LTI operators $\underset{\sim}{R}_i$ and $\underset{\sim}{S}_i$ can be found such that the modeling error in each $\underset{\sim}{H}_i$ is bounded by a generalized conic sector condition such as (6); i.e.,

$$\underset{\sim}{H}_i \text{ strictly inside } L_{2e}\text{- Cone}(\underset{\sim}{C}_i, \underset{\sim}{R}_i, \underset{\sim}{S}_i) \tag{18}$$

for $i = 1, \ldots, N$. For notational convenience we define

$$\underset{\sim}{C} \triangleq \text{diag}(\underset{\sim}{C}_1, \ldots, \underset{\sim}{C}_N) \tag{19}$$

$$\underset{\sim}{R} \triangleq \text{diag}(\underset{\sim}{R}_1, \ldots, \underset{\sim}{R}_N) \tag{20}$$

$$\underset{\sim}{S} \triangleq \text{diag}(\underset{\sim}{S}_1, \ldots, \underset{\sim}{S}_N) \tag{21}$$

Comments:

The generalized conic sector error bound (18), though somewhat abstract, is fairly easily related to meaningful quantities. Lemmas A4 and A5 in the Appendix relate condition (18) to simple Euclidean norm bounds for multi-input-multi-output (MIMO) NTV and LTI $\underset{\sim}{H}_i$. For example, if $\underset{\sim}{H}_i$ is a stable SISO LTI element, then it follows from Lemma A4 that the simple frequency domain condition

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$$|H_i(j\omega) - c_i(j\omega)|^2 \le |r_i(j\omega)|^2 - \varepsilon \qquad (22)$$

for some $\varepsilon > 0$ and all ω, (see Fig. 2b) implies that (18) is satisfied for any stable SISO LTI nonanticipative $\underset{\sim}{S}$ and $\underset{\sim}{R}$ satisfying

$$C_i(j\omega) = c_i(j\omega) \qquad (23)$$

$$\frac{R_i(j\omega)}{S_i(j\omega)} = r_i(j\omega) \ . \qquad (24)$$

If $\underset{\sim}{H}_i$ is a SISO memoryless NTV element defined by

$$y_i(t) = h_i(x_i(t), t) \qquad (25)$$

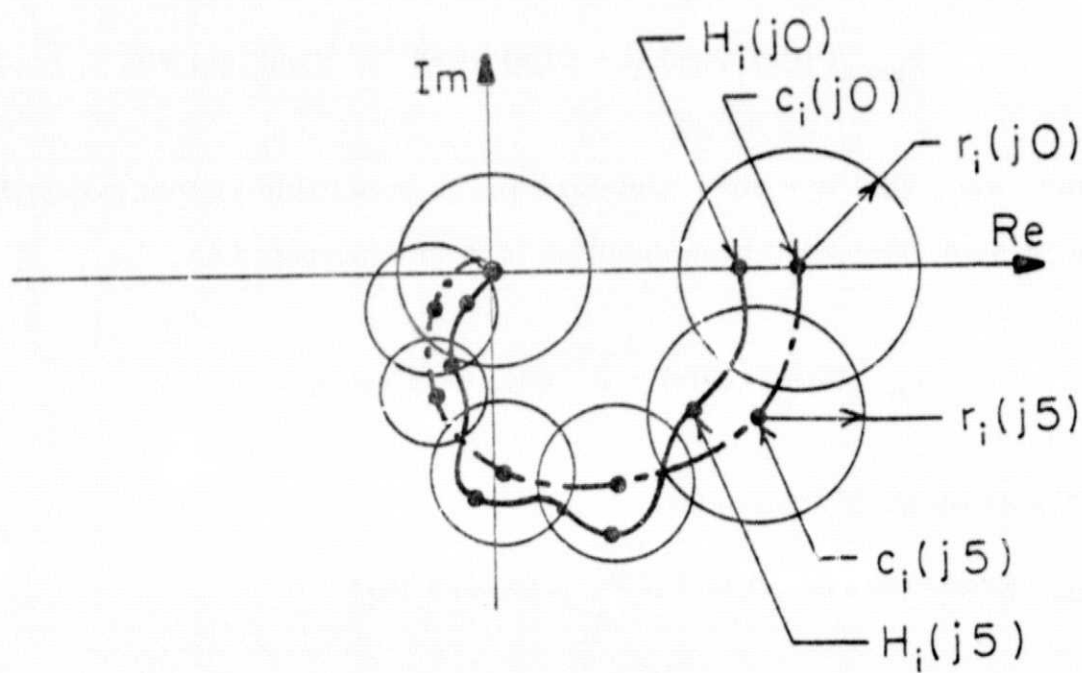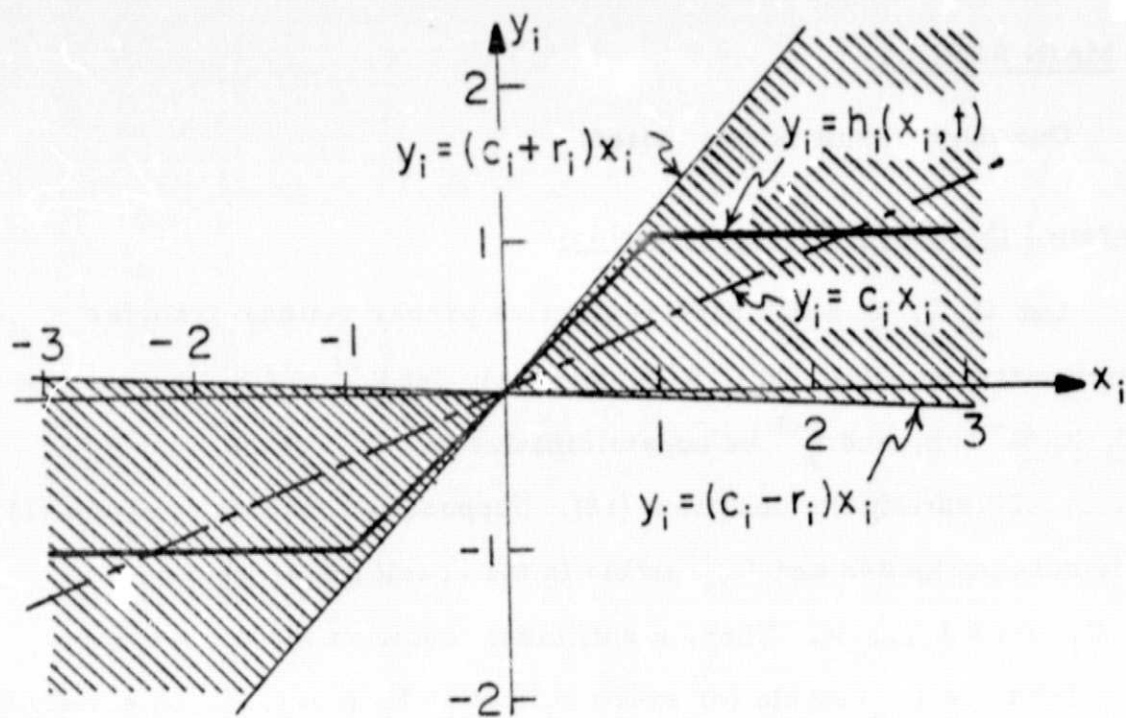and if for some c, r and some $\varepsilon > 0$

$$\left| \frac{h_i(\alpha, t) - c_i \alpha}{\alpha} \right|^2 < r_i^2 - \varepsilon \qquad \forall \ \alpha \ne 0 \ , \qquad (26)$$

then from Lemma A5 it follows that (18) is satisfied for any <u>constant</u> $C_i(s)$, $R_i(s)$, $S_i(s)$ satisfying (23) - (24); this is the usual sector nonlinearity condition (e. g., [5-8])--see Fig. 2a. In more complicated situations where a component $\underset{\sim}{H}_i$ consists of an interconnection of several LTI elements and/or memoryless NTV elements, the result of [21] frequently may be invoked to determine suitable $\underset{\sim}{C}_i$, $\underset{\sim}{R}_i$, and $\underset{\sim}{S}_i$.

(a) Nonlinear component satisfying (26)

(b) Nyquist locus of LTI component satisfying (22)

Fig. 2   SISO Components

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Top figure labels:
$y_i = (c_i + r_i)x_i$

$y_i = h_i(x_i, t)$

$y_i = c_i x_i$

$y_i = (c_i - r_i)x_i$

Bottom figure labels:
$H_i(j0)$

$c_i(j0)$

$r_i(j0)$

$r_i(j5)$

$c_i(j5)$

$H_i(j5)$

Im

Re

IV. MAIN RESULT

Our main result is now stated.

Theorem 1 (Multiloop Circle Criterion)

Let $\underset{\sim}{G}$, $\underset{\sim}{C}$, $\underset{\sim}{R}$, and S have respective proper rational transfer function matrices $G(s)$, $C(s)$, $R(s)$, and $S(s)$; let $\underset{\sim}{R}^{-1}$ and $\underset{\sim}{S}^{-1}$ exist; and let $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{R}^{-1}$, $\underset{\sim}{S}$, and $\underset{\sim}{S}^{-1}$ be nonanticipative and L_{2e}-stable; let $\underset{\sim}{H}_i$ ($i = 1, \ldots, N$) satisfy the condition (18). Suppose the feedback system (11) - (12) is nonanticipative and L_{2e}-stable in the special case where $\underset{\sim}{H}_i = \underset{\sim}{C}_i \; \forall \, i = 1, \ldots, N$. Then, a sufficient condition for the system (11) - (12) to be L_{2e}-stable for every collection H_i ($i = 1, \ldots, N$) satisfying (18) is

$$\sigma_{max}\left(R(j\omega) \; G(j\omega) \left(I + C(j\omega) \, G(j\omega)\right)^{-1} S^{-1}(j\omega)\right) \leq 1 \; \forall \, \omega. \qquad (27)$$

Further, when $G(s)$ is a square matrix and is invertible almost everywhere on the $j\omega$-axis, then the above condition can be expressed as

$$\sigma_{min}(S(j\omega) \, (C(j\omega) + G^{-1}(j\omega)) \, R^{-1}(j\omega)) \geq 1 \qquad (28)$$

for all ω at which $G^{-1}(j\omega)$ exists. $\qquad \Box$

Proof: From Lemma A3 and (27), it follows that

$$\underset{\sim}{H} \overset{\Delta}{=} \; \text{diag}(H_1, \ldots, H_N) \; \underline{\text{strictly inside}} \; L_{2e} - \text{Cone}(\underset{\sim}{C}, \; \underset{\sim}{R}, \; \underset{\sim}{S}). \qquad (29)$$

From Lemma A2, it follows that (27) and (28) are equivalent when $G^{-1}(j\omega)$ exists almost everywhere and that

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$$(-\underset{\sim}{G})^{I} \quad \text{outside} \quad L_{2e} - \text{Cone}(\underset{\sim}{C}, \ \underset{\sim}{R}, \ \underset{\sim}{S}) \ . \tag{30}$$

L_{2e}-stability of (11)-(12) follows from the conic sector stability theorem, Theorem A4.

□

Remarks

It is also possible to prove Theorem 1 by applying Parseval's theorem to verify that the conditions of the well-known "small gain theorem" (e.g., [8]) are satisfied by the transformed system defined by Eqns. (A5)-(A10) of the Appendix. We consider the present proof more appealing because it stresses the direct connection between the conditions of Theorem 1 and the simple conic sector conditions of Theorem A4, just as Zames' proof in [22] of the well-known circle criterion stresses the direct connection between circle theorem conditions and the simple, but less general, conic sector conditions of [5, Theorem 2a].

V. DISCUSSION

There are essentially two main conditions which must be satisfied to conclude stability from Theorem 1: (i) The system must be stable when the uncertain components $\underset{\sim}{H}_i$ are replaced by the respective LTI approximations $\underset{\sim}{C}_i$; and, (ii) the frequency-domain condition (27)(or (28)) must be satisfied. The former condition can be verified a variety of ways; for example, one may check that the roots of the characteristic equation

$$\det (I + C(s) \ G(s)) = 0 \tag{31}$$

all have negative real parts;[4] alternatively, one may apply the multivariable Nyquist criterion, checking that the polar plot of the locus of $\det(C(j\omega)\,G(j\omega))$ encircles the point $-1 + j0$ exactly once counterclockwise for each unstable open-loop pole of $\underline{C}(s)\,\underline{G}(s)$ (multiplicities counted) [3]. The latter condition (27) (or (28)) requires that one plot the variable $\sigma_{max}(\cdot)$ (or $\sigma_{min}(\cdot)$) verses ω and verify that the appropriate inequality holds for all ω.

In the special case in which there is a single SISO nonlinearity $h(x, t)$ both of the conditions of Theorem 1 can be verified by inspection of the polar plot of $G(j\omega)\ \forall\omega$. In this case the conditions of Theorem 1 become precisely the conditions of the well-known circle stability criterion (cf. [7]-[8]). It is this which motivates us to refer to Theorem 1 as a "multiloop circle stability criterion"--despite the fact that in general no circles are employed in verifying its conditions.

One can interpret the uncertainty bounds (R_i, S_i) as specifications for the gain margins and phase margins of the system (11)-(12). For example, if the $\underline{H}_i$ are LTI and SISO and if $H_i(s) \equiv C_i(s)$ $(i = 1, \ldots, N)$, then it follows from (22) that under the conditions of Theorem 1 the system will remain stable despite variations in the individual component gains of magnitudes as great as $|r_i(j\omega)| \overset{\Delta}{=} |R_i(j\omega)/S_i(j\omega)|$, <u>even when the variations occur simultaneously in all components.</u> So, for example, the system can tolerate simultaneous gain variations or phase variations of at least

$$G_{M_i} \overset{\Delta}{=} \inf_{\omega}\ 20\ \log\ \left|\ \frac{r_i(j\omega)}{C_i(j\omega)}\ \right|, \quad db \tag{32}$$

---

[4] If $C(s)\,G(s)$ has any "decoupling zeroes" (i.e., uncontrollable or unobservable poles, then these will not be roots of (31) and one must check separately that these poles have negative real parts--cf. [3].

or

$$\Theta_{M_i} \triangleq 2 \inf_{\omega} \arcsin \left| \frac{r_i(j\omega)}{2C_i(j\omega)} \right| \tag{33}$$

in each of the respective component feedback loops; i.e., the system has gain margins of <u>at least</u> G_{M_i} and phase margins of <u>at least</u> Θ_{M_i} at the inputs to the respective components $C_i(s)$ $(i = 1, \ldots, N)$. The quantity

$$k_m(j\omega) \triangleq \sigma_{min}(S(j\omega) \, (C(j\omega) + G^{-1}(j\omega)) \, R^{-1}(j\omega)) \tag{34}$$

is the amount by which the uncertainty bounding matrices $R_i(j\omega)$ can be simultaneously increased without violating the stability conditions of Theorem 1--$k_m(j\omega)$ can be viewed as a lower bound on the amount by which the system (11)-(12) <u>exceeds</u> the stability margin specifications (18) at each frequency ω.

In general, the stability conditions of Theorem 1--and the estimate (34) of excess stability margin k_m--will be conservative. This conservativeness can usually be reduced by substituting weighted uncertainty bounding matrices $(\alpha_i R_i(s), \alpha_i S_i(s))$ for the original matrices $(R_i(s), S_i(s))$. Further, if it happens that $\underset{\sim}{H}_i$ is linear time-invariant for some i, then as a consequence of Lemma A4, the corresponding α_i may be replaced by a frequency-dependent $\alpha_i(j\omega)$ (provided that for some $k < \infty$ and some $\varepsilon > 0$, $k > |\alpha_i(j\omega)|^2 > \varepsilon$ for all ω). Iterative numerical methods would be required to enable one to efficiently compute the "optimal" weightings (i.e., the weightings leading to the least conservative stability conclusions). We hasten to add that the idea of using constant weightings to reduce conservativeness in multiloop nonlinear input-output stability results is not new: M-matrix tests provide a simple but conservative method to implicitly ensure the existence of constant

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weightings (see [10], [11]); other results have been stated in which constant weightings appear explicitly (e. g. , Moylan and Hill [12, Thms. 5 and 6]).

The results of [12] involving explicit weightings may be viewed as a special case in which the matrices $(C(j\omega),\ R(j\omega),\ S(j\omega))$, the interconnection matrix $G(j\omega)$, and the weightings $\alpha_i(j\omega)$ are not permitted to be frequency-dependent. We emphasize that the advantages offered by frequency-dependent $C(j\omega)$, $R(j\omega)$, $S(j\omega)$ and $\alpha_i(j\omega)$ are crucial in stability margin analysis where it usually is necessary to be able to characterize tolerance of dynamical modeling errors of frequency-dependent magnitude. Allowing the matrix $G(j\omega)$ to be frequency dependent eliminates the need for incorporating the dynamics of $\underset{\sim}{G}$ in additional dynamical $\underset{\sim}{H}_i$'s, thereby reducing the dimension N of $(\underset{\sim}{G},\ \underset{\sim}{C},\ \underset{\sim}{R},\ \underset{\sim}{S})$; this in general leads to less conservative stability conclusions from Theorem 1 and also broadens its scope of applicability (since $\underset{\sim}{G}$ need not be L_{2e}-stable under the conditions of Theorem 1 whereas each $\underset{\sim}{H}_i$ must).

We note that Theorem 1 is fairly broad in its scope of applicability. The transfer matrix $G(s)$ may be non-square and need not be open-loop stable (though the $\underset{\sim}{H}_i$ must be). The $\underset{\sim}{H}_i$ operators may be multi-input-multi-output and need not have equal numbers of inputs and outputs. Unlike some previous interconnected system results, no condition is imposed requiring either $c_i^2 - r_i^2 > 0$ $\forall i$ or $c_i^2 - r_i^2 < 0$ $\forall i$. The operators $(\underset{\sim}{C}_i,\ \underset{\sim}{R}_i,\ \underset{\sim}{S}_i)$ defining the conic sector condition (18) may be dynamical. Further, with the aid of Lemmas A4 and A5 and the aid of results such as in [21], it is practical to verify the conic sector condition (18), even for multi-input-multi-output dynamical nonlinearities.

VI. CONCLUSIONS

With a view towards developing a stability criterion well suited to the problem of multiloop feedback stability margin analysis, nonlinear input-output stability techniques generalizing the circle criterion have been re-examined. The stability margin implications of existing results have been stressed and an improved result has been generated allowing one to take account of the frequency-dependence of the magnitude of system modeling errors which commonly occurs in situations involving imprecisely modeled dynamics--e.g., singular perturbations, hysteresis, etc. Theorem 1 together with the related Lemmas A1 and A2 provide verifiable sufficient conditions for the stability of multiloop feedback systems using only crude conic sector bounds on system parameters, subsystem frequency responses, and nonlinearities. Potential applications include the testing of system integrity in the presence of actuator and/or sensor failures (cf. [2]) and the characterization of frequency-dependent gain and phase margins for multiloop feedback designs subject to multiple singular perturbations and dynamical nonlinearities leading to simultaneous frequency-dependent variations in gains and phases in the feedback loops.

The main result, Theorem 1, also plays a key role in a result described in [21] for generating conic sector bounds to characterize the sensitivity of multiloop systems to large dynamical modeling errors of frequency-dependent magnitudes in a manner similar to the way classical M-circle and Nichols diagram techniques enable one to quantitatively gauge the effect of open-loop gain variations in single-loop, unity-feedback systems. The result of [21] also can be useful in determining the conic sector bounds (C, R, S) required by Theorem 1.

VII. APPENDIX

In this appendix several results are stated which are needed in connection with Theorem 1. Theorem A1 and Lemmas A2 and A3 are used in the proof of Theorem 1. Lemmas A4 and A5 are useful in verifying the conic sector conditions (18) for memoryless nonlinear $\underset{\sim}{H}_i$ and for linear time-invariant dynamical $\underset{\sim}{H}_i$.

We note that while results similar to Lemmas A2, A4, and A3 have been presented in various forms elsewhere (e.g., [6]-[8], [13], [19], [20]), the very general case considered here (admitting, for example, dynamical and multi-input-multi-output $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$) is new, as is the explicit appearance of the term $\epsilon \|y\|_\tau^2$ in the "strict" conicity condition (5)-(6). The differences are sufficient to mandate the inclusion here of proofs for these Lemmas.

Theorem A1 (Conic Sector Stability Theorem)

Consider the feedback system

$$y = \underset{\sim}{H} x \tag{A1}$$

$$x = -\underset{\sim}{G}(y + v) + u \tag{A2}$$

where $x, y, u, v \in L_{2e}$ and $\underset{\sim}{G}, \underset{\sim}{H}: L_{2e} \to L_{2e}$; (u, v) is the "input" and (x, y) is the "output". If L_{2e}-stable linear operators $\underset{\sim}{C}$, $\underset{\sim}{R}$, and $\underset{\sim}{S}$ can be found such that

$$\underset{\sim}{H} \underline{\text{strictly inside }} L_{2e}\text{-Cone } (\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}) \tag{A3}$$

and

$$(-\underset{\sim}{G})^I \underline{\text{outside }} L_{2e}\text{-Cone } (\underset{\sim}{C}, \underset{\sim}{R}, \underset{\sim}{S}), \tag{A4}$$

then the feedback system (A1)-(A2) is L_{2e}-stable.

Proof: This result is a special case of the "sector stability theorem" ([13, p. 65], [19, Thm. 6.1]).

Remark

Theorem A4 also can be proved by applying the small gain theorem (e.g., [8]) to the "transformed" system

$$\tilde{y} = \underset{\sim}{S} (\underset{\sim}{H} - \underset{\sim}{C}) \underset{\sim}{R}^{-1} \tilde{x} + \tilde{v} \tag{A5}$$

$$\tilde{x} = -\underset{\sim}{R} \underset{\sim}{G} (\underset{\sim}{I} + \underset{\sim}{C} \underset{\sim}{G})^{-1} \underset{\sim}{S}^{-1} \tilde{y} + \tilde{u} \tag{A6}$$

where

$$\tilde{y} = \underset{\sim}{S} (y - \underset{\sim}{C}(x-u) + v) \tag{A7}$$

$$\tilde{x} = \underset{\sim}{R} x \tag{A8}$$

$$\tilde{u} = \underset{\sim}{R} u \tag{A9}$$

$$\tilde{v} = \underset{\sim}{S} (v + \underset{\sim}{C} u) , \tag{A10}$$

provided $\underset{\sim}{S}^{-1}$, $\underset{\sim}{G}(\underset{\sim}{I} + \underset{\sim}{C} \underset{\sim}{G})^{-1}$, and $\underset{\sim}{R}^{-1}$ exist and are L_{2e}-stable and non-anticipative. It can be shown that under the conditions of Theorem A4, the two operators $\underset{\sim}{S} (\underset{\sim}{H} - \underset{\sim}{C}) \underset{\sim}{R}^{-1}$ and $\underset{\sim}{R} \underset{\sim}{G} (\underset{\sim}{I} + \underset{\sim}{C} \underset{\sim}{G})^{-1} \underset{\sim}{S}^{-1}$ each have L_{2e}-gain less than one. Thus, Theorem A4 may be viewed as a characterization of the improvements on the small gain theorem obtainable by use of the nonlinear dynamical transformations (A5)-(A10); such transformations have been described as "loop-shifting" and "multiplier" transformations (e.g., [8]).

Lemma A2 (LTI Outside Conicity)

Let $\underset{\sim}{G}$, $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$ be linear time-invariant operators with respective

proper rational transfer functions $G(s)$, $C(s)$, $R(s)$, $S(s)$. Suppose that $S^{-1}(s)$ exists and has a proper rational transfer function matrix with no poles in $\text{Re}(s) \geq 0$. Suppose that R, $G(I + C\ G)^{-1}$ and S^{-1} are L_{2e} stable and non-anticipative. Then,

$$(-G)^{I} \quad \text{outside} \quad L_{2e}\text{-Cone} \ (C,\ R,\ S) \tag{A11}$$

if and only if the following condition holds for all real ω

$$\sigma_{\max}\!\left(R(j\omega)\ G(j\omega) \left(I + C(j\omega)\ G(j\omega)\right)^{-1} S^{-1}(j\omega)\right) \leq 1 \tag{A12}$$

almost everywhere.

When $G^{-1}(s)$ and $R^{-1}(s)$ exist almost everywhere on the $j\omega$-axis, then condition (A12) is equivalent to

$$\sigma_{\min}\!\left(S(j\omega)\big(C(j\omega) + G^{-1}(j\omega)\big)R^{-1}(j\omega)\right) \geq 1 \tag{A13}$$

almost everywhere.

Proof:

It is trivial to see that (A12) and (A13) are equivalent when $G^{-1}(j\omega)$ and $R^{-1}(j\omega)$ exist, since for any invertible matrix A

$$\sigma_{\min}(A^{-1}) = 1/\sigma_{\max}(A) \ .$$

Suppose that (A12) holds. Let (x, y) by any input-output pair satisfying $x = -G\,y$; let

$$\tilde{y} = S\ (y - C\ x) \tag{A14}$$

and let

$$\tilde{y}_\tau = \begin{cases} \tilde{y}(t) & \text{, if } 0 \leq t \leq \tau \\ 0 & \text{, otherwise} \end{cases} \quad . \tag{A15}$$

Let $\tilde{Y}_\tau(j\omega)$ denote the Fourier transform of $\tilde{y}_\tau$. Note that from (A12) it follows that for all $\tilde{Y}_\tau(j\omega)$

$$\| \tilde{Y}_\tau(j\omega) \|^2 - \| R(j\omega)\, G(j\omega)\, (I + C(j\omega)\, G(j\omega))^{-1}\, S^{-1}(j\omega)$$

$$\tilde{Y}_\tau(j\omega) \|^2 \geq 0 \quad . \tag{A16}$$

Now,

$$\| R\, x \|_\tau^2 = \| R\, G(I + C\, G)^{-1}\, S^{-1}\, \tilde{y} \|_\tau^2$$

$$= \| R\, G(I + C\, G)^{-1}\, S^{-1}\, \tilde{y}_\tau \|_\tau^2$$

$$\left(\begin{array}{c} \text{by the nonanticipativeness of} \\ R,\ G\, (I + C\, G)^{-1},\ S^{-1} \end{array} \right)$$

$$\leq \int_0^\infty \| R\, G\, (I + C\, G)^{-1}\, S^{-1}\, \tilde{Y}_\tau)\, (t) \|^2\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \| R(j\omega)\, G(j\omega)\, (I + C(j\omega)\, G(j\omega))^{-1}\, S^{-1}(j\omega)\, \tilde{Y}_\tau(j\omega) \|^2\, d\omega$$

$$\left(\begin{array}{c} \text{by Parseval's Theorem and the hypotheses that} \\ R,\ G(I + C\, G)^{-1},\ \text{and } S^{-1} \text{ are } L_{2e}\text{-stable.} \end{array} \right)$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \tilde{Y}_\tau(j\omega) \|^2 \, d\omega$$

$$= \| \tilde{y}_\tau \|_\tau^2 = \| \tilde{y} \|_\tau^2$$

$$= \| \underset{\sim}{S}(y - \underset{\sim}{C}x) \|_\tau^2 \quad , \tag{A.17}$$

which proves (A11) is implied by (A12).

Conversely, suppose that (A11) holds. Let Y_0 and ω_0 be arbitrary. Consider the L_{2e}-stable feedback system

$$x = -\underset{\sim}{G} \, y \tag{A.18}$$

$$y = \underset{\sim}{C} x + \underset{\sim}{S}^{-1} \, \tilde{y} \quad . \tag{A.19}$$

Let $\tilde{y}(t) \to Y_0 \, e^{j\omega_0 t}$. Then (letting $\tau \to \infty$ in (7)), it follows from (A11) that

$$\| R(j\omega) \, G(j\omega) \, (I + C(j\omega) \, G(j\omega))^{-1} \, S^{-1}(j\omega) \, Y_0 \|_E^2$$

$$\leq \| Y_0 \|_E^2 \tag{A.20}$$

and hence (A12) holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ☐

Lemma A3 (Composite Operator Conicity)

Let

$$\underset{\sim}{H} = \text{diag} \, (\underset{\sim}{H}_1, \dots, \underset{\sim}{H}_N) \tag{A21}$$

$$\underset{\sim}{C} = \text{diag} \, (\underset{\sim}{C}_1, \dots, \underset{\sim}{C}_N) \tag{A22}$$

$$R = \text{diag}\,(R_1, \ldots, R_N) \tag{A23}$$

$$S = \text{diag}\,(S_1, \ldots, S_N)\,. \tag{A24}$$

If for all $i = 1, \ldots, N$

$$H_i \text{ strictly inside } L_{2e}\text{-Cone}\,(C_i,\ R_i,\ S_i)\,, \tag{A25}$$

then

$$H \text{ strictly inside } L_{2e}\text{-Cone}\,(C,\ R,\ S)\,. \tag{A26}$$

Proof: This is a special case of the results in [13, p. 70] and [19, Lemma 6.2 (vi)]. □

Lemma A4 (LTI Conicity)

Let H, C, R, S be nonanticipative L_{2e}-stable linear-time-invariant operators with respective rational transfer function matrices $H(s)$, $C(s)$, $R(s)$, $S(s)$. Suppose that $R^{-1}(s)$ exists and has no poles in $\text{Re}(s) \geq 0$.

Then

$$H \text{ strictly inside } L_{2e}\text{-Cone}\,(C,\ R,\ S) \tag{A27}$$

if and only if

$$\|\,S(j\omega)\,\big(H(j\omega) - C(j\omega)\big)X(j\omega)\,\|_E^2 \;\leq\; \|\,R(j\omega)\,X(j\omega)\,\|_E^2$$

$$-\,\varepsilon\,\|\,X(j\omega)\,\|_E^2 \tag{A28}$$

for all $X(j\omega)$, all ω, and some $\varepsilon > 0$.

<u>Proof:</u>   Let $\underset{\sim}{R}^{-1}$ denote the stable nonanticipative LTI operator having transfer function matrix $R^{-1}(s)$. Suppose that (A28) holds and let

$$\underset{\sim}{x}_\tau(t) \;=\; \begin{cases} (\underset{\sim}{R}\,x)(t) \;, & \text{if } t \le \tau \\[2mm] 0 & \text{, if } t > \tau \end{cases} \tag{A29}$$

and let $\tilde{X}_\tau(j\omega)$ denote the Fourier transform of $\tilde{x}_\tau(t)$. Then, for all $y = \underset{\sim}{H}x$ we have

$$\| \underset{\sim}{S}(y - \underset{\sim}{C}x) \|_\tau^2 \;=\; \| \underset{\sim}{S}(\underset{\sim}{H}x - \underset{\sim}{C}x) \|_\tau^2$$

$$=\; \| \underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\,x \|_\tau^2$$

(by linearity)

$$=\; \| \underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\,\underset{\sim}{R}^{-1}\underset{\sim}{R}\,x \|_\tau^2$$

(since $\underset{\sim}{R}^{-1}$ exists)

$$=\; \| \underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\,\underset{\sim}{R}^{-1}\,\tilde{x}_\tau \|_\tau^2$$

(by nonanticipativeness)

$$\le\; \int_0^\infty \| (\underset{\sim}{S}(\underset{\sim}{H} - \underset{\sim}{C})\,\underset{\sim}{R}^{-1}\,x_\tau)(t) \|_E^2 \, dt$$

(the integral exists since $\underset{\sim}{S}$, $\underset{\sim}{H}$, $\underset{\sim}{C}$, and $\underset{\sim}{R}^{-1}$ are stable)

$$=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \| S(j\omega)\big(H(j\omega) - C(j\omega)\big)R^{-1}(j\omega)$$

$$\cdot\, \tilde{X}_\tau(j\omega) \|_E^2 \, d\omega$$

(by Parseval's Theorem)

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \tilde{X}_T(j\omega) \|_E^2 - \varepsilon \| R^{-1}(j\omega) \tilde{x}_T(j\omega) \|_E^2 \ d\omega$$

(By A(?))

$$= \int_0^{\infty} \| \tilde{x}_T(t) \|_E^2 \ dt - \varepsilon \int_0^{\infty} \| (\underset{\sim}{R}^{-1} \tilde{x}_T)(t) \|_E^2 \ dt$$

(by Parseval's Theorem)

$$\leq \int_0^T \| \tilde{x}_T(t) \|_E^2 \ dt - \varepsilon \int_0^T \| (\underset{\sim}{R}^{-1} \tilde{x}_T)(t) \|_E^2 \ dt$$

$$= \| \tilde{x}_T \|_T^2 - \varepsilon \| \underset{\sim}{R}^{-1} \tilde{x}_T \|_T^2$$

$$= \| \underset{\sim}{R} x \|_T^2 - \varepsilon \| \underset{\sim}{R}^{-1} \underset{\sim}{R} x \|_T^2$$

(by nonanticipativeness of R^{-1}

$$= \| \underset{\sim}{R} x \|_T^2 - \varepsilon \| x \|_T^2$$

$$\leq \| \underset{\sim}{R} x \|_T^2 - \varepsilon'(\| x \|_T^2 + \| \underset{\sim}{H} x \|_T^2) \ . \tag{A30}$$

where

$$\varepsilon' = \frac{\varepsilon}{1 + \alpha^2} \tag{A31}$$

and

$$\alpha = \sup_{x, \ T} \left(\frac{\| \underset{\sim}{H} x \|_T}{\| x \|_T} \right) < \infty \tag{A32}$$

(since $\underset{\sim}{H}$ is stable).

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Conversely suppose (A27) holds. Let X_0 and ω_0 be arbitrary.
Then, letting $x(t) \to X_0 \cos(\omega_0 t + \phi)$ and $\tau \to \infty$, it follows that

$$\| S(j\omega_0) (H(j\omega_0) - C(j\omega_0)) X_0 \|$$

$$\leq \| R(j\omega_0) X_0 \|_E^2 - \varepsilon (\| X_0 \|_E^2 + \| H(j\omega_0) X_0 \|_E^2)$$

$$\leq \| R(j\omega_0) X_0 \|_E^2 - \varepsilon \| X_0 \|_E^2 \quad . \tag{A33}$$

$\square$

Lemma A5 (Nonlinear Time-Varying Conicity)

Let $h(x(t), t)$ be any function of $x(t)$ and t and let $\underset{\sim}{H}$ be given by

$$(\underset{\sim}{H} x)(t) = h(x(t), t) \quad . \tag{A34}$$

Let C, R, and S be constant matrices and let $\underset{\sim}{C}$, $\underset{\sim}{R}$, $\underset{\sim}{S}$ be the operators defined by

$$(\underset{\sim}{C} x)(t) = C\, x(t) \quad \forall\, x \tag{A35}$$

$$(\underset{\sim}{R} x)(t) = R\, x(t) \quad \forall\, x \tag{A36}$$

$$(\underset{\sim}{S} y)(t) = S\, y(t) \, . \quad \forall\, y \quad . \tag{A37}$$

Suppose S^{-1} exists, then

$$\underset{\sim}{H} \text{ strictly inside } L_{2e}\text{-Cone}\,(\underset{\sim}{C},\ \underset{\sim}{R},\ \underset{\sim}{S}) \tag{A38}$$

if and only if

$$\| S(h(x(t), t) - Cx(t)) \|_E^2 \leq \| Rx(t) \|_E^2 - \epsilon \| x(t) \|_E^2 \qquad \forall \, x(t) . \quad \text{(A39)}$$

□

Proof: Let $y(t) = h(x(t), t)$.

Suppose (A39) holds. Then,

$$\| y(t) \| \leq \alpha \| x(t) \| \qquad \qquad \text{(A40)}$$

where

$$\alpha = \frac{\sigma_{max}(R)}{\sigma_{min}(S)} + \sigma_{max}(C) . \qquad \text{(A41)}$$

Thus, taking

$$\epsilon' = \frac{\epsilon}{1 + \alpha^2} \qquad \qquad \text{(A42)}$$

we have that

$$\| \underset{\sim}{S}(y - \underset{\sim}{C} x) \|_\tau^2$$

$$= \int_0^\tau \| S(h(x(t), t) - Cx(t)) \|_E^2 \, dt$$

$$\leq \int_0^\tau \| R \, x(t) \|_E^2 - \epsilon \| x(t) \|_E^2 \, dt$$

$$\leq \int_0^\tau \| Rx(t) \|_E^2 - \epsilon'(\| x(t) \|_E^2 + \| y(t) \|_E^2 \, dt$$

$$= \| \underset{\sim}{R}x \|_\tau^2 - \epsilon \, (\| x \|_\tau^2 + \| y \|_\tau^2) . \qquad \qquad . \quad \text{(A43)}$$

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Conversely, when (A38) holds, then taking $x(t)$ to be the constant function $x(t) \equiv x_0$ we have that for some $\varepsilon > 0$

$$\| S(h(x_0, t) - C x_o) \|_E^2$$

$$= \frac{1}{\tau} \cdot \| \underset{\sim}{S} (\underset{\sim}{H} x - \underset{\sim}{C} x) \|_\tau^2$$

$$\leq \frac{1}{\tau} \left(\| \underset{\sim}{R} x \|_\tau^2 - \varepsilon \left(\| x \|_\tau^2 + \| y \|_\tau^2 \right) \right)$$

$$\leq \frac{1}{\tau} \left(\| \underset{\sim}{R} x \|_\tau^2 - \varepsilon \| x \|_\tau^2 \right)$$

$$= \| R x_0 \|_E^2 - \varepsilon \| x_0 \|_E^2 \ . \tag{A44}$$

$\square$

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