VARIATIONAL DESCRIPTION OF THE POSITIVE COLUMN
WITH TWO-STEP IONIZATION

by

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ABSTRACT

The ionization balance in diffusion-dominated discharges may depend on both one- and two-step ionization processes. The Spenke diffusion equation ($D\nabla^2 n + \nu n + kn^2 = 0$) describing such conditions is solved in this paper by the Rayleigh-Ritz variational method. Simple analytic approximations to the density profile, and the similarity relation between ν, k, D and the discharge dimensions, are derived for planar and cylindrical geometry, and compared with exact computations for certain limiting cases.

1. INTRODUCTION

In 1924, Schottky[1] published a theory of the steady state positive column appropriate to neutral gas pressures high enough for ambipolar diffusion to control the loss of electron-positive ion pairs to the discharge wall. His diffusion equation,

$$D\nabla^2 n + \nu n = 0 \qquad \text{(Schottky Equation)}, \quad (1)$$

implies that the ambipolar diffusion coefficient, D , is spatially independent; $\nu(>0)$ represents the effective (one-step) electron-neutral ionizing collision frequency, and n is the charge number density in the quasineutral, weakly-ionized column.

Equation (1) is readily solved in planar and cylindrical geometry, subject to the boundary conditions $n = n_0$ and $dn/dx = 0$ (or $dn/dr = 0$) at $x = 0$ (or $r = 0$) , to obtain the familiar results,

$$n = n_0 \cos\left(\frac{\pi}{2}\frac{x}{a}\right) \quad , \quad \left(\frac{\pi}{2a}\right)^2 = \frac{\nu}{D} \qquad \text{(Planar geometry)},$$

$$n = n_0 J_0\left(\gamma_1 \frac{r}{a}\right) \quad , \quad \left(\frac{\gamma_1}{a}\right)^2 = \frac{\nu}{D} \qquad \text{(Cylindrical geometry)}, \quad (2)$$

where $\gamma_1(= 2{\cdot}405)$ is the first root of the J_0 Bessel function. The discharge wall is generally taken arbitrarily to be at $x = a$ (or $r = a$), and cannot exceed this dimension since $n \geq 0$.

In 1950, Spenke[2] extended the analysis to include two-step ionization processes via the equation,

$$D\nabla^2 n + \nu n + kn^2 = 0 \qquad \text{(Spenke Equation)}. \quad (3)$$

Unfortunately, analytical solutions as simple as those of Eq. (2) cannot generally be obtained, though the solution in planar geometry can at least be expressed in terms of elliptic functions.[2] For this case, Spenke plotted the density profiles for $kn_0/\nu = 0$, 0.5 and ∞ , and obtained an approximate relation between a , ν/D , and kn_0/D. An analogous approximate relation was obtained for cylindrical geometry, and density profiles were plotted for $kn_0/\nu = 0$ and ∞ .

A thorough numerical study of Eq. (3) has been undertaken only very recently, by Rogoff.[3-5] He has enumerated a variety of ionization, recombination and attachment processes described by the deceptively simple Spenke Equation,[4] and has clarified the similarity properties of axially-uniform $(\nabla_{\parallel} = 0)$ discharges obeying it by introducing the solution $n = n_0 g$, and integrating over the cross-sectional area, A , to obtain

$$\left(\frac{\nu}{D}\right) A + \left(\frac{k}{D}\right) N = - \int_A \frac{\nabla_\perp^2 g}{g} \, dA = S \quad \text{(Rogoff Equation)}, \qquad (4)$$

where N is the total number of electrons (or ions) per unit length of the column. The integral is a dimensionless number, S , depending only on the shape of the cross-section and the parameter kn_0/ν . To determine S , it is necessary to first solve Eq. (3). This has been done numerically by Rogoff for rectangular and cylindrical geometry.[5]

It would be very useful if good analytic approximations could be obtained for the density profile, g , and the Rogoff equation. A possible means of obtaining such approximations is suggested by the fact that the numerical solutions for g show little change in shape as kn_0/ν varies from 0 to ∞ : in this paper, we shall investigate by variational methods the perturbations on the solutions obtained with $kn_0/\nu = 0$ (= Eq. (2)) resulting from variation of kn_0/ν from -1 to ∞ , corresponding to the range investigated numerically by Rogoff.[5]

The plan of the paper is as follows. In Section 2 we examine the properties of the Spenke Equation for different combinations of positive and negative ν and k , and demonstrate the significance of the range $-1 \leq kn_0/\nu \leq \infty$. A variation principle suitable for use in this range is presented, and it is shown how it may be applied in conjunction with the Rayleigh-Ritz method to solution of the Spenke Equation. Section 3 carries out this procedure for planar geometry, and presents computations of the density profile in the column, a similarity relation between ν, k, D and a, and useful working analytic approximations to these results. Section 4 repeats the procedure for cylindrical geometry. Our similarity relations are not obtained in the Rogoff form of Eq. (4). It is shown in Section 5, by suitable rearrangement of the terms, that they are identical, however, and analytic approximations are derived for the Rogoff form. Section 6 closes the paper with a brief discussion.

2. BASIC EQUATIONS

We shall write Eq. (3) as

$$\nabla^2 g + \alpha g + \beta n_0 g^2 = 0 \quad , \quad \alpha = \frac{\nu}{D} \, , \; \beta = \frac{k}{D} \quad , \tag{5}$$

and first investigate some of its general properties in planar and cylindrical geometries. We shall then show how its solution can be approximated by the Rayleigh-Ritz method.

2.1. Planar Geometry

Equation (5) and its first integral are given by

$$\frac{d^2g}{dx^2} + \alpha g + \beta n_0 g^2 = 0 \quad , \quad \left(\frac{dg}{dx}\right)^2 = \alpha\left(1-g^2\right) + \frac{2\beta n_0}{3}\left(1-g^3\right) . \tag{6}$$

We require, then, for a physical solution

$$\left(\frac{dg}{dx}\right)^2 = \alpha(1-g)(1-g_+)(1-g_-) \geq 0 \quad , \; g_\pm = \pm\frac{1}{2}\left[\left(1 + \frac{3\alpha}{2\beta n_0}\right)\left(\frac{3\alpha}{2\beta n_0} - 3\right)\right]^{\frac{1}{2}} - \frac{1}{2}\left(1 + \frac{3\alpha}{2\beta n_0}\right) . \tag{7}$$

We note that $g_\pm$ are real for $-\frac{3}{2} \leq \frac{\beta n_0}{\alpha} \leq \frac{1}{2}$, and complex outside this range. The quantity $(dg/dx)^2/\alpha$ is plotted in Fig. 1.

Case 1: $\alpha, \beta \geq 0$: Figure 1(a) (or Eq. (7)) shows that $(dg/dx)^2/\alpha \geq 0$ for $0 \leq g \leq 1$. A simple special case is[2]

$$g = 1 - 3\tanh^2\left(0.658\,\frac{x}{a}\right) \; , \qquad \left(\frac{1.317}{a}\right)^2 = \alpha \qquad \left(\frac{\beta n_0}{\alpha} = \frac{1}{2}\right), \tag{8}$$

where the numerical factor $1.317 = \cosh^{-1}2$. For future reference, we note that the second relation can be written as,

$$\left(\frac{\pi}{2a}\right)^2 = \alpha + 0.845\,\beta n_0 \qquad \left(\frac{\beta n_0}{\alpha} = \frac{1}{2}\right). \tag{9}$$

Case 2: $\alpha \geq 0$, $-1 \leq \beta n_0/\alpha \leq 0$: As $\beta n_0/\alpha$ becomes more negative, there is a change of behavior at $\beta n_0/\alpha = -1$, at which point $(dg/dx)^2/\alpha$ is apparently positive for $g \gtrless 1$. Examination of Eq. (6) shows, however, that $d^2g/dx^2 = dg/dx = 0$ for $g = 1$, i.e. the column density is independent of x for the condition $\beta n_0/\alpha = -1$.

There is an additional range of values of $g > 1$ for which $(dg/dx)^2/\alpha > 0$ (for example, $g > g_A$ for $\beta n_0/\alpha = -0.75$ in Fig. 1(a)). It is not accessible, however, from $g = 1$ through real values of dg/dx.

Case 3: $\alpha \geq 0$, $\beta n_0/\alpha < -1$: As $\beta n_0/\alpha$ decreases further, solutions can only be found for $g \geq 1$, so the wall can no longer be arbitrarily located at $g = 0$, but must be assigned a non-zero value of g. Taking $\frac{\beta n_0}{\alpha} = -\frac{3}{2}$ as a simple special case, we have

$$g = \sec^2 \left(\frac{\pi x}{2a}\right) , \quad \left(\frac{\pi}{2a}\right)^2 = \alpha \quad \left(\frac{\beta n_0}{\alpha} = -\frac{3}{2}\right) \quad (10)$$

if the wall is located where $g \to \infty$.

For $-\frac{3}{2} < \beta n_0/\alpha < -1$ there is a range of solutions with $0 \leq g < 1$ for which $(dg/dx)^2/\alpha > 0$ (for example, $0 \leq g \leq g_B$ for $\beta n_0/\alpha = -1.25$ in Fig. 1(a)), but it is not accessible through real values of dg/dx.

Case 4: $\alpha, \beta < 0$: For $\alpha < 0$, Fig. 1(b) is appropriate. Solutions are obtainable for $1 \leq g \leq \infty$. The solution for $\beta n_0/\alpha = 1/2$, analogous to Eq. (8) is,

$$g = 1 + 3 \tan^2 \left(\frac{\pi x}{2a}\right) , \quad \left(\frac{\pi}{a}\right)^2 = -\alpha \quad \left(\frac{\beta n_0}{\alpha} = \frac{1}{2}\right), (11)$$

if the wall is located where $g \to \infty$.

Case 5: $\alpha < 0$, $-1 \leq \beta n_0/\alpha < 0$: In this range, solutions are possible for $1 \leq g < \infty$ (for example, $1 \leq g \leq g_A$ for $\beta n_0/\alpha = -0.75$ in Fig. 1(b)). As $\beta n_0/\alpha \to -1$, the upper density limit decreases towards $g = 1$, and we obtain again the uniform density solution of Case 2 at $\beta n_0/\alpha = -1$.

Case 6: $\alpha < 0$, $\beta n_0/\alpha < -1$: For $-\frac{3}{2} < \frac{\beta n_0}{\alpha} < -1$, solutions are possible with $0 < g \leq 1$ (for example, $g_B \leq g \leq 1$ for $\beta n_0/\alpha = -1.25$ in Fig. 1(b)), so the wall must be assigned a non-zero density. For $\frac{\beta n_0}{\alpha} \leq -\frac{3}{2}$, solutions with $0 \leq g \leq 1$ are obtainable, i.e. the wall can be arbitrarily located where $g = 0$. The simple special solution for $\frac{\beta n_0}{\alpha} = -\frac{3}{2}$, analogous to Eq. (10), is

$$g = \mathrm{sech}^2 \left(\frac{\pi x}{2a}\right) , \quad \left(\frac{\pi}{2a}\right)^2 = -\alpha \quad \left(\frac{\beta n_0}{\alpha} = -\frac{3}{2}\right). \quad (12)$$

Here, a is a scale length, since $g \to 0$ only as $x \to \infty$.

The obvious conclusion to be drawn from this brief survey of Cases 1-6 is that there is a very rich variety of solutions to the Spenke Equation. In this paper, we shall consider only those for which $0 \le g \le 1$. Such solutions are associated with the ranges $\alpha \ge 0$, $-1 \le \beta n_0/\alpha \le \infty$, (Case 1) and $\alpha < 0$, $\beta n_0/\alpha \le -\frac{3}{2}$ (Case 6); we shall limit our consideration to the first of these, for which Rogoff has obtained numerical solutions in rectangular and cylindrical geometry.[5]

2.2. Cylindrical Geometry

It is reasonable to suppose that the six cases just discussed will appear for cylindrical geometry with qualitative similarities and quantitative differences. We shall not pursue the question here, but simply note that in cylindrical geometry, Eq. (5) is

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dg}{dr}\right) + \alpha g + \beta n_0 g^2 = 0 \ , \tag{13}$$

and is satisfied by $g = 1$ when $\beta n_0/\alpha = -1$. The range of interest to us is consequently the same as for planar geometry: $\alpha \ge 0$, $-1 \le \beta n_0/\alpha \le \infty$.

It may be remarked, in passing, that the parameter $\beta n_0/\alpha (= k n_0/\nu)$, which will figure prominently throughout this paper,may be interpreted physically as the ratio of the number of charges produced at the center of the discharge by two-step $\left(k n_0^2\right)$ and one-step $\left(\nu n_0\right)$ processes.

2.3. Variational Solution

Equation (5) is the variational solution to (see Appendix)

$$I = \int_A \left[\left(\nabla_\perp g\right)^2 - \alpha g^2 - \frac{2\beta n_0}{3}\, g^3\right] dA \ , \tag{14}$$

where the column is axially uniform $\left(\nabla_\parallel = 0\right)$, and A is its cross-sectional area. This implies that if a trial solution differing by δg from the exact solution is substituted in Eq. (14) the error in I is $O\left(\delta g^2\right)$ In our application of the Rayleigh-Ritz method,[6] our trial function will be the first two terms of a Fourier expansion for g of the form

$$g = \frac{n_1}{n_0}\, g_1 + \frac{n_2}{n_0}\, g_2 \ . \tag{15}$$

where $g_1(a) = g_2(a) = 0$ at the wall, and $g_1'(0) = g_2'(0) = 0$. Here, a prime denotes a derivative with respect to x (or r).

Since I is an extremum with respect to the trial function, we require $dI/dn_1 = dI/dn_2 = 0$. Substituting Eq. (15) in Eq. (14), and carrying out the differentiations, leads to the two equations,

$$I_1 + \left(\frac{n_2}{n_1}\right) I_2 = \beta n_1 \left(I_3 + 2\left(\frac{n_2}{n_1}\right) I_4 + \left(\frac{n_2}{n_1}\right)^2 I_5 \right) ,$$

$$\left(\frac{n_2}{n_1}\right) I_6 + I_2 = \beta n_1 \left(I_4 + 2\left(\frac{n_2}{n_1}\right) I_5 + \left(\frac{n_2}{n_1}\right)^2 I_7 \right) , \qquad (16)$$

where $I_1 - I_7$ are defined by

$$I_1 = \int_A \left(g_1'^2 - \alpha g_1^2\right) dA \ , \quad I_2 = \int_A \left(g_1' g_2' - \alpha g_1 g_2\right) dA \ , \quad I_6 = \int_A \left(g_2'^2 - \alpha g_1^2\right) dA \ ,$$

$$I_3 = \int_A g_1^3 dA \ , \quad I_4 = \int_A g_1^2 g_2 \, dA \ , \quad I_5 = \int_A g_1 g_2^2 \, dA \ , \quad I_7 = \int_A g_2^3 \, dA \ . \qquad (17)$$

The use of Eq. (16) will be central to the following sections.

3. PLANAR GEOMETRY

A suitable trial function is

$$g = \frac{n_1}{n_0}\cos\left(\frac{\pi x}{2a}\right) + \frac{n_2}{n_0}\cos\left(\frac{3\pi x}{2a}\right) . \qquad (18)$$

As $\beta n_0/\alpha \to 0$, we expect $n_2/n_0 \to 0$, $n_1/n_0 \to 1$, to recover the simple solution of Eq. (2). The integrals of Eq. (17) may be evaluated as

$$I_1 = \frac{a}{2}\left[\left(\frac{\pi}{2a}\right)^2 - \alpha\right] , \quad I_2 = 0 , \quad I_6 = \frac{a}{2}\left[\left(\frac{3\pi}{2a}\right)^2 - \alpha\right] ,$$

$$I_3 = \frac{4a}{3\pi} , \quad I_4 = \frac{4a}{15\pi} , \quad I_5 = \frac{36a}{35\pi} , \quad I_7 = -\frac{4a}{9\pi} , \qquad (19)$$

and substituted in Eq. (16) to yield

$$\left(\frac{\pi}{2a}\right)^2 - \alpha = \frac{8\beta n_1}{3\pi}\left[1 + \frac{2}{5}\left(\frac{n_2}{n_1}\right) + \frac{27}{35}\left(\frac{n_2}{n_1}\right)^2\right] ,$$

$$\left(\frac{n_2}{n_1}\right)\left[\left(\frac{3\pi}{2a}\right)^2 - \alpha\right] = \frac{8\beta n_1}{15\pi}\left[1 + \frac{54}{7}\left(\frac{n_2}{n_1}\right) - \frac{5}{3}\left(\frac{n_2}{n_1}\right)^2\right] . \qquad (20)$$

As $\beta \to 0$, these yield correctly $(\pi/2a)^2 \to \alpha$ and $n_2/n_1 \to 0$.

3.1. Determination of n_2/n_1

Eliminating $(\pi/2a)^2$ from Eq. (20), and replacing n_1 by $n_0/(1 + n_2/n_1)$ gives

$$\frac{\beta n_0}{\alpha} = \frac{15\pi\left(\frac{n_2}{n_1}\right)\left(1 + \frac{n_2}{n_1}\right)}{1 - \frac{261}{7}\left(\frac{n_2}{n_1}\right) - \frac{59}{3}\left(\frac{n_2}{n_1}\right)^2 - \frac{243}{7}\left(\frac{n_2}{n_1}\right)^3} . \qquad (21)$$

This relation is shown in Fig. 2(a). For $\beta n_0/\alpha > 0$, an asymptotic value of $n_2/n_1 = 0.0264$ is approached. For $-1 < \beta n_0/\alpha < 0$, n_2/n_1 is negative and increases in magnitude as $\beta n_0/\alpha \to -1$. Equation (21) actually predicts a turning point at $\beta n_0/\alpha = -0.959$, $n_2/n_1 = -0.157$ but the two-term expansion is unlikely to be accurate for $n_2/n_1 < 1/9$ for the following reason.

For $g \le 1$ everywhere, it is necessary that $g''(0) \le 0$, which implies $-\frac{1}{9} \le n_2/n_1$. For $g \ge 0$ everywhere, it follows from Eq. (18) that $n_2/n_1 < 1/3$, corresponding to $g'(a) \le 0$. We have, then, the limits

$$-\frac{1}{9} \le \frac{n_2}{n_1} \le \frac{1}{2} \quad . \tag{22}$$

For $\beta n_0/\alpha > 0$, Fig. 2(a) shows that n_2/n_1 is well below the upper limit. For $\beta n_0/\alpha < 0$, the lower limit is reached at $\beta n_0/\alpha = -0.941$.

3.2. Determination of Discharge Dimensions

The first expression in Eq. (20) may be written as

$$\left(\frac{\pi}{2a}\right)^2 = \alpha + p\,\beta n_0 \quad , \quad p = \frac{8}{3\pi}\left[\frac{1 + \frac{2}{5}\left(\frac{n_2}{n_1}\right) + \frac{27}{35}\left(\frac{n_2}{n_1}\right)^2}{1 + \frac{n_2}{n_1}}\right] , \tag{23}$$

where the factor p depends on $\beta n_0/\alpha$ via Eq. (21). It is plotted in Fig. 3(a). For $\beta n_0/\alpha > 0$, the asymptotic limit of $p = 0.8362$ is approached. This may be compared with the value of 0.8366 given by Spenke.[3] At $\beta n_0/\alpha = 0$, $p = 0.849$. For $\beta n_0/\alpha < 0$, p increases rapidly towards unity at $\beta n_0/\alpha = -1$, the dashed curve being extrapolated for $n_2/n_1 < -1/9$. The exact value of p for the special case of Eq. (9) is shown on the figure, and lies precisely on the curve.

3.3. Density Profile

We now test the quality of the approximation represented by Fig. 2(a) by comparing exact solutions of Eq. (6) with Eqs. (18) and (23). This has been done by solving numerically

$$\frac{d^2 g}{d\xi^2} + \left(\frac{\pi}{2}\right)^2 \left(\frac{\alpha g + \beta n_0 g^2}{\alpha + p\,\beta n_0}\right) = 0 \quad , \qquad \xi = \frac{x}{a} , \tag{24}$$

using the value of p appropriate to the value of βn_0 chosen. We then find $g = 0$ at $\xi = \xi_0$. If it turns out that $\xi_0 \ne 1$, then the correct value of p should have been

$$p' \approx \frac{p}{\xi_0^2} + \frac{\alpha}{\beta n_0}\left(\frac{1}{\xi_0^2} - 1\right) . \qquad (25)$$

The profile is then recalculated using p' in Eq. (24)

Figure 4(a) shows the exact and variational (dashed) solutions for the extreme cases of $\beta n_0/\alpha \to \infty$ ($n_2/n_1 = 0.0264$, $p = 0.836$) and $\beta n_0/\alpha = -0.941$ ($n_2/n_1 = -1/9$, $p = 0.922$). No difference is visible on the curve for $\beta n_0/\alpha \to \infty$; the error is generally much less than 1 per cent. For $\beta n_0/\alpha = -0.941$, the error is observable, but is always less than 5 per cent. It is characteristic of variational solutions that eigenvalues are obtained more accurately than eigenfunctions. In this case, $p' = 0.901$ for $\beta n_0/\alpha = -0.941$, so the variational solution ($p = 0.922$) is only in error by 2 per cent.

If we restrict use of the variational solution to $-0.050 \le n_2/n_1$, $-0.8 \le \beta n_0/\alpha$, further calculations show that the errors are much reduced. Figure 4(a) shows $\beta n_0/\alpha = -0.794$ ($n_2/n_1 = -0.050$, $p = 0.877$), for which p' is found to be 0.873 and the separation between the exact and variational solutions is generally 1 per cent or less.

3.4. Further Approximations

So far, we have obtained results without simplifying Eq. (20). Since n_2/n_1 is found to be small, we may write the first relation of Eq. (20) and Eq. (21) as

$$\left(\frac{\pi}{2a}\right)^2 - \alpha \approx \frac{8\beta n_0}{3\pi}\left(1 - \frac{3}{5}\,\frac{n_2}{n_1}\right), \qquad \frac{\beta n_0}{\alpha} \approx \frac{15\pi\left(\frac{n_2}{n_1}\right)}{1 - \frac{268}{7}\left(\frac{n_2}{n_1}\right)} . \qquad (26)$$

The second equation rearranges to give an approximate version of Eq. (21),

$$\frac{n_2}{n_1} \approx 0.0212\left(\frac{\beta n_0}{\alpha + 0.812\,\beta n_0}\right) , \qquad (27)$$

which may be substituted in the first equation to give an approximate value for p of Eq. (23),

$$p \approx 0.849\left(\frac{\alpha + 0.800\,\beta n_0}{\alpha + 0.812\,\beta n_0}\right) . \qquad (28)$$

The expressions of Eqs. (27) and (28) are adequate over the range $-0.80 \lesssim \beta n_0/\alpha \leq \infty$, though empirical adjustment of the constants might effect some improvement in their accuracy, particularly over any more restricted range.

In practice, the mean number density in the column, $\bar{n}$, may be a more useful parameter than the axial density, n_0 : it can be measured from outside the column, for example, by microwave cavity perturbation methods, or inferred from current density measurements and estimated axial electron drift velocities. For the trial function of Eq. (18), we have

$$\bar{n} = \frac{2n_0}{\pi}\left(\frac{1 - \dfrac{n_2}{3n_1}}{1 + \dfrac{n_1}{n_2}}\right) , \tag{29}$$

so that Eq. (21) becomes

$$\frac{\beta\bar{n}}{\alpha} = \frac{30\left(\dfrac{n_2}{n_1}\right)\left(1 - \dfrac{n_2}{3n_1}\right)}{1 - \dfrac{261}{7}\left(\dfrac{n_2}{n_1}\right) - \dfrac{59}{3}\left(\dfrac{n_2}{n_1}\right)^2 - \dfrac{243}{7}\left(\dfrac{n_2}{n_1}\right)^3} . \tag{30}$$

This equation has been used to obtain the dashed curve in Fig. 2(a).

The first expression in Eq. (20) becomes

$$\left(\frac{\pi}{2a}\right)^2 = \alpha + q\,\beta\bar{n} \quad , \quad q = \frac{4}{3}\left[\frac{1 + \dfrac{2}{5}\left(\dfrac{n_2}{n_1}\right) + \dfrac{27}{35}\left(\dfrac{n_2}{n_1}\right)^2}{1 - \dfrac{n_2}{3n_1}}\right] . \tag{31}$$

The quantity q has been plotted in Fig. 3(b). It depends on $\beta\bar{n}/\alpha$ via Eq. (30), and drops from an asymptotic value of 1.36 towards unity at $\beta\bar{n}/\alpha = -1$. We shall discuss Eq. (31) further in Section 5.

Equation (30) may be approximated for small n_2/n_1 , and rearranged to give,

$$\frac{\beta\bar{n}}{\alpha} \approx \frac{30\left(\frac{n_2}{n_1}\right)}{1 - \frac{776}{21}\left(\frac{n_2}{n_1}\right)} \quad , \quad \frac{n_2}{n_1} \approx \frac{1}{30}\left(\frac{\beta\bar{n}}{\alpha + 1.23 \quad \beta\bar{n}}\right) . \qquad (32)$$

Substituting in Eq. (31), and retaining only first-order terms, yields the useful expression,

$$q \approx \frac{4}{3}\left[\frac{\alpha + 1.26 \ \beta\bar{n}}{\alpha + 1.23 \ \beta\bar{n}}\right] . \qquad (33)$$

4. CYLINDRICAL GEOMETRY

A suitable trial function is

$$g = \frac{n_1}{n_0} J_0 \left(\gamma_1 \frac{r}{a}\right) + \frac{n_2}{n_0} J_0 \left(\gamma_2 \frac{r}{a}\right) , \tag{34}$$

where $\gamma_1 (= 2.405)$ and $\gamma_2 (= 5.520)$ are the first two zeros of the J_0 Bessel function. The simple solution of Eq. (2) will be recovered if $n_2/n_0 \to 0$ and $n_1/n_0 \to 1$, as $\beta n_0/\alpha \to 0$. The integrals of Eq. (17) may be evaluated as

$$I_1 = \left[\left(\frac{\gamma_1}{a}\right)^2 - \alpha\right] \frac{a^2}{2} J_1^2 (\gamma_1) \quad , \quad I_2 = 0 \quad , \quad I_6 = \left[\left(\frac{\gamma_2}{a}\right)^2 - \alpha\right] \frac{a^2}{2} J_1^2 (\gamma_2) ,$$

$$I_3 = 0.0975\ a^2 \quad , \quad I_4 = 0.0186\ a^2 \quad , \quad I_5 = 0.0366\ a^2 \quad , \quad I_7 = 0.00760\ a^2 , \tag{35}$$

the last four results being obtained numerically. Substituting in Eq. (16) yields

$$\left[\left(\frac{2.405}{a}\right)^2 - \alpha\right] = 0.723\ \beta n_1 \left[1 + 0.382 \left(\frac{n_2}{n_1}\right) + 0.376 \left(\frac{n_2}{n_1}\right)^2\right] ,$$

$$\left(\frac{n_2}{n_1}\right) \left[\left(\frac{5.520}{a}\right)^2 - \alpha\right] = 0.322\ \beta n_1 \left[1 + 3.93 \left(\frac{n_2}{n_1}\right) + 0.408 \left(\frac{n_2}{n_1}\right)^2\right] . \tag{36}$$

As $\beta \to 0$, these yield correctly $(2.405/a)^2 \to \alpha$ and $n_2/n_1 \to 0$.

4.1. Determination of n_2/n_1

Eliminating a^2 from Eq. (36), and replacing n_1 by $n_0/(1 + n_2/n_1)$ gives

$$\frac{\beta n_0}{\alpha} = \frac{13.3 \left(\frac{n_2}{n_1}\right)\left(1 + \frac{n_2}{n_1}\right)}{1 - 7.91 \left(\frac{n_2}{n_1}\right) - 4.12 \left(\frac{n_2}{n_1}\right)^2 - 4.46 \left(\frac{n_2}{n_1}\right)^3} . \tag{37}$$

This relation is shown in Fig. 2(b). A turning point at $\beta n_0/\alpha = -0.895$, $n_2/n_1 = -0.275$, indicates breakdown of the two-term variational approximation, so we should assess reasonable limits for n_2/n_1. For $g''(0) \le 0$, we require $-(\gamma_1/\gamma_2)^2 \le n_2/n_1$ and for $g'(a) \le 0$ we require $n_2/n_1 \le \gamma_1 J_1(\gamma_1)/\gamma_2 J_1(\gamma_2)$. We have, then, the limits

$$-0.190 \le n_2/n_1 \le 0.665. \tag{38}$$

For $\beta n_0/\alpha > 0$, Fig. 2(b) shows that n_2/n_1 is well below the upper limit. For $\beta n_0/\alpha < 0$, the lower limit is reached at $\beta n_0/\alpha = -0.856$.

4.2. Determination of Discharge Dimensions

The first expression in Eq. (36) may be written as

$$\left(\frac{2.405}{a}\right)^2 = \alpha + p\,\beta n_0 \quad , \quad p = 0.723\left[\frac{1 + 0.382\left(\frac{n_2}{n_1}\right) + 0.376\left(\frac{n_2}{n_1}\right)^2}{1 + \frac{n_2}{n_1}}\right], \tag{39}$$

where the factor p depends on βn_0 via Eq. (37). It is plotted in Fig. 3(a), and rises rapidly from its asymptotic limit of $p = 0.679$ towards unity as $\beta n_0/\alpha$ decreases below zero towards -1. It is interesting to note that Spenke[2] obtained $p \to 0.724$ as $\beta n_0/\alpha \to 0$, and $p \to 0.675$ as $\beta n_0/\alpha \to \infty$, by approximate methods

4.3. Density Profile

We have obtained some numerical solutions of Eq. (5), in the form

$$\frac{d^2 g}{d\xi^2} + \frac{1}{\xi}\frac{dg}{d\xi} + \gamma_1^2\left(\frac{\alpha g + \beta n_0\, g^2}{\alpha g + p\,\beta n_0}\right) = 0 , \tag{40}$$

for comparison with our variational solutions. Figure 4(b) shows the exact and variational (dashed) solutions for the extreme cases of $\beta n_0/\alpha \to \infty$ ($n_2/n_1 = 0.118$, $p = 0.679$) and $\beta n_0/\alpha = -0.856$ ($n_2/n_1 = -0.190$, $p = 0.840$). Agreement for $\beta n_0/\alpha \to \infty$ is generally to well within 1 per cent up to $r/a \sim 0.9$, and then to within 2 per cent; p' is found to be 0.676. For $\beta n_0/\alpha = 0.856$, the discrepancies are much larger, though p' is found to be 0.798, which differs from the approximate solution ($p = 0.840$) by only 5 per cent.

If we restrict use of the variational solution to $-0.125 \leq n_2/n_1$; $-0.751 \leq \beta n_0/\alpha$, the errors are considerably reduced, as shown in Fig. 4(b) for $\beta n_0/\alpha = -0.751$ ($n_2/n_1 = -0.125$, $p = 0.792$). For this case, $p' = 0.775$, which agrees with the variational result to within about 2 per cent.

4.4. Further Approximations

Since n_2/n_1 is typically several times greater than for the planar case, for a given value of the parameter $\beta n_0/\alpha$, quadratic terms in Eq. (37) should be retained. The equations analogous to Eq. (26) for planar geometry are then

$$\left(\frac{2.405}{a}\right)^2 - \alpha \approx 0.723\ \beta n_0 \left[1 - 0.618\left(\frac{n_2}{n_1}\right) + 0.994\left(\frac{n_2}{n_1}\right)^2\right],$$

$$\frac{\beta n_0}{\alpha} \approx \frac{13.3\left(\frac{n_2}{n_1}\right)}{1 - 8.91\left(\frac{n_2}{n_1}\right) + 4.79\left(\frac{n_2}{n_1}\right)^2}. \qquad (41)$$

Solving for n_2/n_1 , and eliminating between these two expressions leads to

$$\frac{n_2}{n_1} \approx \frac{0.0754\ \beta n_0\ (\alpha + 0.671\ \beta n_0)}{(\alpha + 0.507\ \beta n_0)\ (\alpha + 0.836\ \beta n_0)},$$

$$p \approx 0.723 \left[\frac{1 + 1.30\left(\frac{\beta n_0}{\alpha}\right) + 0.398\left(\frac{\beta n_0}{\alpha}\right)^2}{1 + 1.34\left(\frac{\beta n_0}{\alpha}\right) + 0.424\left(\frac{\beta n_0}{\alpha}\right)^2}\right], \qquad (42)$$

which are adequate approximations over the range $-0.75 \lesssim \beta n_0/\alpha \leq \infty$, but are probably subject to improvement by empirical adjustment of the constants to fit the exact results.

If we wish to work in terms of the mean number density,

$$\bar{n} = 2\left[n_1 \frac{J_1(\gamma_1)}{\gamma_1} + n_2 \frac{J_1(\gamma_2)}{\gamma_2}\right] = 0.432\, n_0 \left[\frac{1 - 0.286\left(\frac{n_2}{n_1}\right)}{1 + \frac{n_2}{n_1}}\right], \quad (43)$$

then Eq. (37) becomes

$$\frac{\beta\bar{n}}{\alpha} = \frac{5.73\left(\frac{n_2}{n_1}\right)\left(1 - 0.286\left(\frac{n_2}{n_1}\right)\right)}{1 - 7.91\left(\frac{n_2}{n_1}\right) - 4.12\left(\frac{n_2}{n_1}\right)^2 - 4.45\left(\frac{n_1}{n_2}\right)^3} \quad . \quad (44)$$

This equation has been used to obtain the dashed curve of Fig. 2(b).

The first expression in Eq. (36) becomes

$$\frac{2.405}{a}^2 = \alpha + q\,\beta\bar{n}\ , \quad q = 1.68\left[\frac{1 + 0.382\left(\frac{n_2}{n_1}\right) + 0.376\left(\frac{n_2}{n_1}\right)^2}{1 - 0.286\left(\frac{n_2}{n_1}\right)}\right] . \quad (45)$$

The quantity q has been plotted in Fig. 3(b). It depends on $\beta\bar{n}/\alpha$ via Eq. (44), and drops from an asymptotic value of 1.82 towards unity at $\beta\bar{n}/\alpha = -1$. We shall discuss Eq. (45) further in Section 5.

Equation (44) may be approximated for small n_2/n_1 by dropping terms in $(n_2/n_1)^3$ and rearranging to give

$$\frac{\beta\bar{n}}{\alpha} \approx \frac{5.73\left(\frac{n_2}{n_1}\right)}{1 - 7.62\left(\frac{n_2}{n_1}\right) - 6.30\left(\frac{n_2}{n_1}\right)^2} \ , \quad \frac{n_2}{n_1} \approx \frac{0.175\left(\frac{\beta\bar{n}}{\alpha}\right)\left(1 + 1.33\,\frac{\beta\bar{n}}{\alpha}\right)}{1 + 2.66\left(\frac{\beta\bar{n}}{\alpha}\right) + 1.96\left(\frac{\beta\bar{n}}{\alpha}\right)^2} \ . \quad (46)$$

Substituting in Eq. (45) yields the useful expression

$$q \approx 1.68\left(\frac{1 + 2.78\left(\frac{\beta\bar{n}}{\alpha}\right) + 2.14\left(\frac{\beta\bar{n}}{\alpha}\right)^2}{1 + 2.66\left(\frac{\beta\bar{n}}{\alpha}\right) + 1.96\left(\frac{\beta\bar{n}}{\alpha}\right)^2}\right) . \quad (47)$$

5. THE ROGOFF EQUATION

We may write the Rogoff Equation (Eq. (4)) in the form

$$\alpha + \beta\bar{n} = \frac{S}{A} . \tag{48}$$

Comparison with Eqs. (31) and (45) indicates that it differs from them in form in that the effect of the function $q\left(\frac{\beta\bar{n}}{\alpha}\right)$ in those equations is exerted by the function $S\left(\frac{\beta\bar{n}}{\alpha}\right)$ $\left[\text{or } S\left(\frac{\beta n_0}{\alpha}\right)\right]$ in the Rogoff Equation. In what follows, we shall determine appropriate values of S using the variational results of Sections 3 and 4.

5.1. Planar Geometry

For the trial function of Eq. (18), we obtain directly

$$S = \int_A \frac{d^2g}{g\,dx^2}\, dA = \pi^2 \left(9 - \frac{8}{\left[\left(1 + \frac{n_2}{n_1}\right)\left(1 - \frac{3n_2}{n_1}\right)\right]^{\frac{1}{2}}}\right), \tag{49}$$

where we take $A = 4a^2$. In this open geometry, there is some arbitrariness in defining the width of the area under consideration, and hence the value of S . Our choice implies a width $2a$ equal to the separation between the planar walls. With this choice, when $\beta\bar{n} \to 0$ and $n_2/n_1 \to 0$, we retrieve the simple result of Eq. (2) with $S = \pi^2$.

Figure 5(a) shows $S\left(\frac{\beta n_0}{\alpha}\right)$ and $S\left(\frac{\beta\bar{n}}{\alpha}\right)$ obtained by using Eqs. (21) and (30). In general, $\bar{n}/n_0 < 1$, but Fig. 5(b) confirms the prediction of Section 2 that the column becomes uniform ($g \to 1$) as $\beta n_0/\alpha \to -1$, i.e. $\bar{n}/n_0 \to 1$. Since we are simply rearranging the same data, we should expect results for $n_2/n_1 < -0.05$ to be subject to significant error, as discussed in Section 3.3.

Approximate forms of Eq. (49) can be obtained by expanding for n_2/n_1 small,

$$S \approx \pi^2 \left[1 - \frac{8n_2}{n_1}\right] , \tag{50}$$

and substituting from Eqs. (27) and (32),

$$S \approx \pi^2 \left[\frac{\alpha + 0.642\ \beta n_0}{\alpha + 0.812\ \beta n_0}\right] \approx \pi^2 \left[\frac{\alpha + 0.965\ \beta\bar{n}}{\alpha + 1.23\ \ \beta\bar{n}}\right] . \tag{51}$$

5.2. Cylindrical Geometry

For the trial function of Eq. (34) we have

$$S = \pi\gamma_1^2 \int_0^1 \left[\frac{J_0(\gamma_1\xi) + \left(\frac{n_2}{n_1}\right)\left(\frac{\gamma_2}{\gamma_1}\right)^2 J_0(\gamma_2\xi)}{J_0(\gamma_1\xi) + \left(\frac{n_2}{n_1}\right) J_0\ (\gamma_2\xi)}\right] 2\xi\ d\xi$$

$$= \pi\ \gamma_1^2 \left[1 - 2\left(\left(\frac{\gamma_2}{\gamma_1}\right)^2 - 1\right)\int_0^1 \frac{\left(\frac{n_2}{n_1}\right) J_0(\gamma_2\xi)}{J_0(\gamma_1\xi) + \left(\frac{n_2}{n_1}\right) J_0(\gamma_2\xi)}\right]\xi d\xi . \tag{52}$$

Unlike the planar case, this cannot be integrated directly, nor expanded easily for n_2/n_1 small. We consequently take a different approach, starting from Eq. (36). The first expression may be written as

$$\alpha + \beta\bar{n} = \left(\frac{2.405}{a}\right)^2 + \left(\beta\bar{n} - 0.723\ \beta n_1\left(1 + 0.382\left(\frac{n_2}{n_1}\right) + 0.376\left(\frac{n_2}{n_1}\right)^2\right)\right)$$

$$= \left(\frac{2.405}{a}\right)^2 - \beta n_1\left(0.291 + 0.400\left(\frac{n_2}{n_1}\right) + 0.272\left(\frac{n_2}{n_1}\right)^2\right) \tag{53}$$

where we have used Eq. (33) Next, we eliminate α from Eq. (36) to obtain a relation between βn_1 and a^2,

$$\beta n_1 = \frac{76.7}{a^2}\left[\frac{\frac{n_2}{n_1}}{1 + 1.69\left(\frac{n_2}{n_1}\right) - 0.451\left(\frac{n_2}{n_1}\right)^2 - 0.854\left(\frac{n_2}{n_1}\right)^3}\right] . \tag{54}$$

Since the right-hand side of Eq. (53) is $S/\pi a^2$, we obtain finally from Eqs. (53) and (54),

$$S = \pi(2.405)^2 \left[\frac{1 - 2.18\left(\frac{n_2}{n_1}\right) - 5.76\left(\frac{n_2}{n_1}\right)^2 - 4.46\left(\frac{n_2}{n_1}\right)^3}{1 + 1.69\left(\frac{n_2}{n_1}\right) - 0.451\left(\frac{n_2}{n_1}\right)^2 - 0.854\left(\frac{n_2}{n_1}\right)^3} \right] . \qquad (55)$$

We have made no approximations in the algebra to this point. Though more complicated in form, Eq. (55) is analogous to Eq. (49) for planar geometry.

Figure 5(c) shows $S\left(\frac{\beta n_0}{\alpha}\right)$ and $S\left(\frac{\beta \bar{n}}{\alpha}\right)$ obtained by using Eqs. (37), (44) and (55). It can be compared with exact calculations published by Rogoff[3] for $0 \le \beta n_0/\alpha \le \infty$. There is, of course, exact agreement for $\beta n_0/\alpha = 0$. As nearly as can be determined from the published curves, $S = 15$ at $\beta n_0/\alpha = 1$, and $S \to 10$ as $\beta n_0/\alpha \to \infty$. Our variational analysis gives $S = 15.0$ and 10.0.

Approximate forms of Eq. (55) can be obtained by expansion for small n_2/n_1 ,

$$S \approx (2.405)^2 \left[1 - 3.87\left(\frac{n_2}{n_1}\right) + 1.22\left(\frac{n_2}{n_1}\right)^2\right] . \qquad (56)$$

and elimination of n_2/n_1 by use of the first expression in Eq. (39). This yields the alternative relations

$$S \approx \pi(2.405)^2 \left[\frac{1 + 1.05\left(\frac{\beta n_0}{\alpha}\right) + 0.235\left(\frac{\beta n_0}{\alpha}\right)^2}{1 + 1.34\left(\frac{\beta n_0}{\alpha}\right) + 0.424\left(\frac{\beta n_0}{\alpha}\right)^2} \right]$$

$$\approx \pi(2.405)^2 \left[\frac{1 + 1.99\left(\frac{\beta \bar{n}}{\alpha}\right) + 1.10\left(\frac{\beta \bar{n}}{\alpha}\right)^2}{1 + 2.66\left(\frac{\beta \bar{n}}{\alpha}\right) + 1.96\left(\frac{\beta \bar{n}}{\alpha}\right)^2} \right] . \qquad (57)$$

Of these, the second should be the more useful in practice, since the Rogoff Equation is expressed most conveniently in terms of $\bar{n}$ (Eq. (48)). We have given both forms throughout this section to facilitate comparison with Rogoff's calculations made in terms of $\beta n_0/\alpha$.[3]

6. DISCUSSION

The most significant new results of this paper are Eqs. (27) and (28) (or their alternative forms, Eqs. (32) and (33)), for planar geometry, and Eq. (42) (or the alternative forms, Eqs. (46) and (47)), for cylindrical geometry. These provide good working analytic approximations to the density profiles and similarity relations over the range $-0.75 \lesssim \beta n_0/\alpha \leq \infty$. They could probably be improved somewhat by empirical adjustment of the constants in them for best fit with exact calculations. If the Rogoff form of the similarity relations is preferred, it is given approximately by Eqs. (51) and (57).

For the range $-1 \leq \beta n_0/\alpha \lesssim -0.75$, the variational method breaks down. This is because the density profile is rapidly becoming flat everywhere ($g \to 1$ as $\beta n_0/\alpha \to -1$, see Section 2), and cannot be accurately approximated by a two-term Fourier (- Bessel) expansion. This analytical shortcoming is likely to be mitigated in experimental practice by the fact that discharge operation in this range would be difficult: if a is to remain finite as $\beta n_0/\alpha \to -1$, then we require $\alpha \to \infty$ and $\beta n_0 \to -\infty$.

Given the importance of two-step ionization processes in common discharges, it is surprising that the Spenke Equation has received so little attention over the last thirty years. For example, two-step processes are important in the mercury-vapor positive column[7] - it is estimated that at pressures above a few mTorr, and electron number densities above about 5×10^{10} cm^{-3}, more than half of the total ionization rate is contributed by two-step processes[8,9] - yet the author is only aware of the experiments of Howe[9] directed towards verifying the applicability of the Spenke Equation in such parameter ranges.

Future theoretical and experimental work should be directed towards exploring and verifying the solutions of the Spenke Equation for $\alpha(=\nu/D)$ and $\beta(=k/D)$ positive and negative. All four combinations are possible in practice[3]: α may contain positive contributions from one-step electron-neutral impact ionization, and two-step electron impact, excited state-ground state collisions and excited state-excited state collisions; these contributions may be offset by attachment. Similarly, β may contain positive contributions from two-step electron-neutral impact and excited state - excited state collisions; these contributions may be offset by electron-ion volume recombination.

Some parameter ranges do not describe self-sustained discharges, of course, e.g. $\alpha, \beta < 0$. They may still be of considerable practical interest, however, since they may apply to a plasma diffusing from a source region, taken as the "wall", into a region where it decays. A point which will require special care for $\alpha < 0$ is the modifying effect on D of the negative ions produced by attachment.[10]

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APPENDIX: FORMULATION OF THE VARIATION PRINCIPLE

Consider the expression,

$$I = \int_A \left[(\nabla_\perp g)^2 + \alpha_0 g^2 + \beta n_0 g^3\right] dA \;, \qquad (A.1)$$

and let $g = G + \delta G$, where G is the exact solution to Eq. (5) and δG is a small space-dependent discrepancy from it. We have

$$I + \delta I = \int_A \left[\left(\nabla_\perp (G + \delta G)\right)^2 + \alpha_0 \,(G + \delta G)^2 + \beta n_0 \,(G + \delta G)^3\right] dA, \qquad (A.2)$$

which separates into

$$I = \int_A \left[(\nabla_\perp G)^2 + \alpha_0 G^2 + \beta n_0 G^3\right] dA \;,$$

$$\delta I = \int_A \left[2\nabla_\perp G\cdot\nabla\delta G + \delta G\left(2\alpha_0 G + 3\beta n_0 G^2\right)\right] dA + 0\left(\delta G^2\right) \;. \qquad (A.3)$$

By use of the relations

$$\int_A (\nabla_\perp G)^2 dA = \int_A \nabla_\perp\cdot\left(G\nabla_\perp G\right) dA - \int_A G\nabla_\perp^2 G dA = - \int_A G\nabla_\perp^2 G \; dA \;,$$

$$\int_A \nabla_\perp G\cdot\nabla\delta G dA = \int_A \nabla_\perp\cdot\left(\delta G\nabla_\perp G\right) dA - \int_A \delta G\nabla_\perp^2 dA = - \int_A \delta G\nabla_\perp^2 G \; dA \;, \qquad (A.4)$$

which assume that $G(a) = \delta G(a) = 0$, Eq. (A.3) reduces to

$$I = \int_A \left[- \nabla_\perp^2 G + \alpha_0 G + \beta n_0 G^2 \right] G dA ,$$

$$\delta I = 2 \int_A \left[-\nabla_\perp^2 G + \alpha_0 G + \frac{3\beta_0 n_0}{2} G^2 \right] \delta G . dA + 0(\delta G^2) .$$

From comparison with Eq. (5), we see that $\delta I = 0(\delta G^2)$ if $\alpha_0 = -\alpha$ and $\beta_0 = -\frac{2}{3}\beta$. This establishes the variation principle in the form of Eq. (14).

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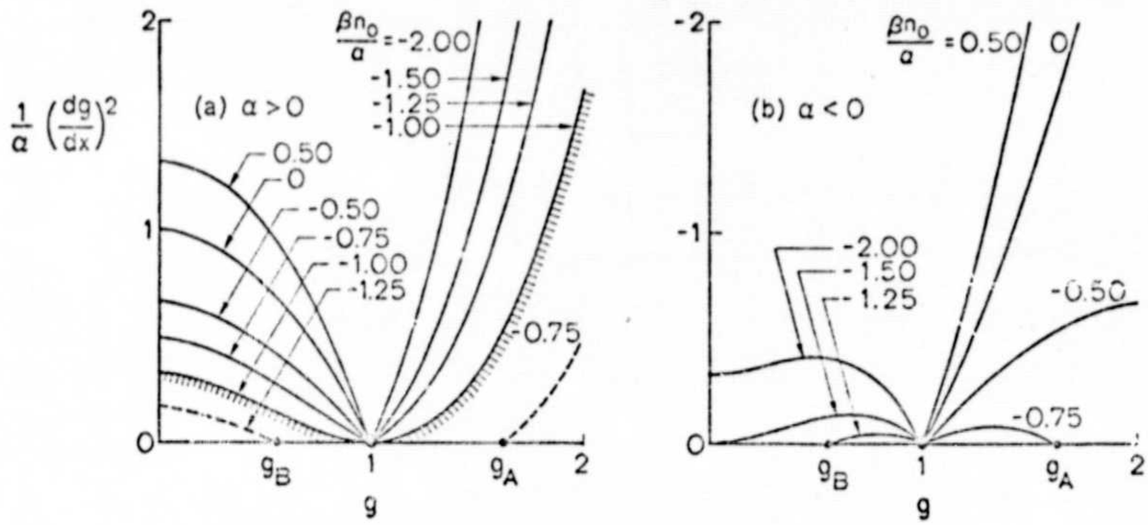

FIG. 1. Planar Geometry. Effects of $\beta n_0/\alpha$ on range of g for which real solutions ($(dg/dx)^2/\alpha \geq 0$) can be obtained.

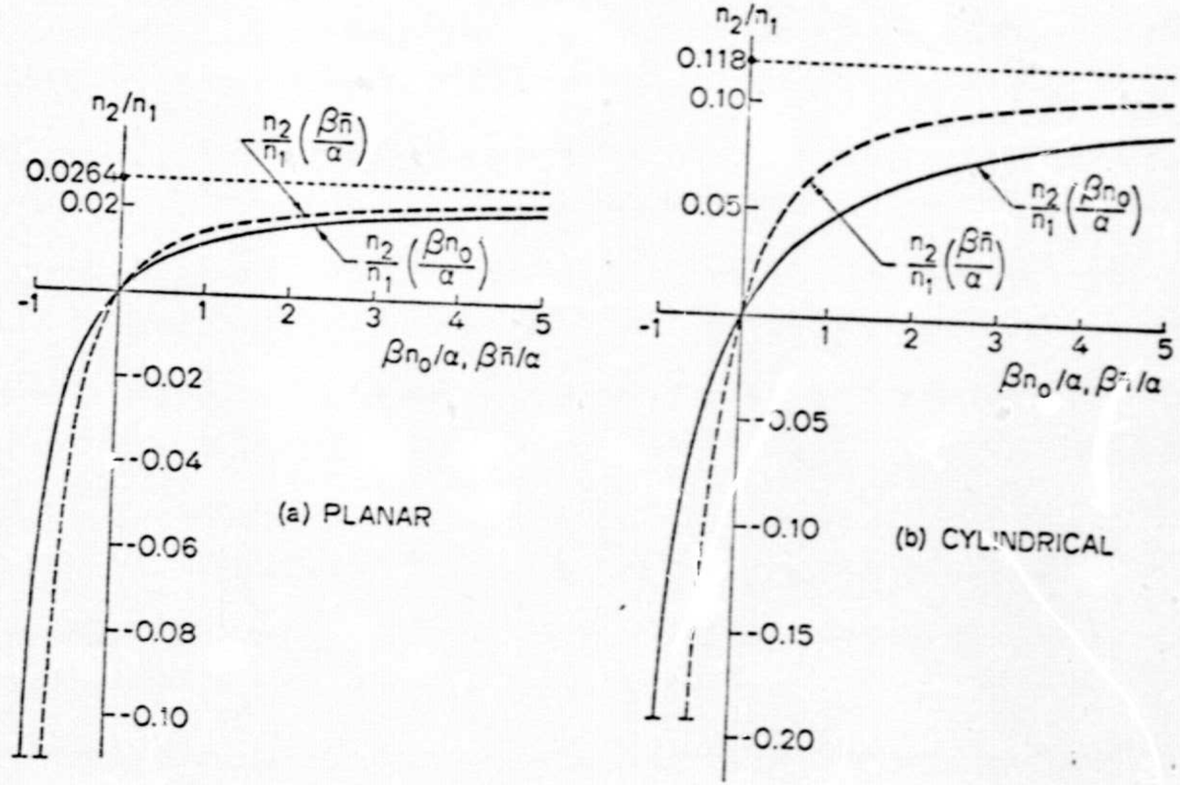

FIG. 2. Variation of n_2/n_1 with $\beta n_0/\alpha$ and $\beta\bar{n}/\alpha$ (Curves extend to $n_2/n_1 = -1/9$ (planar) and -0.190 (cylindrical)).

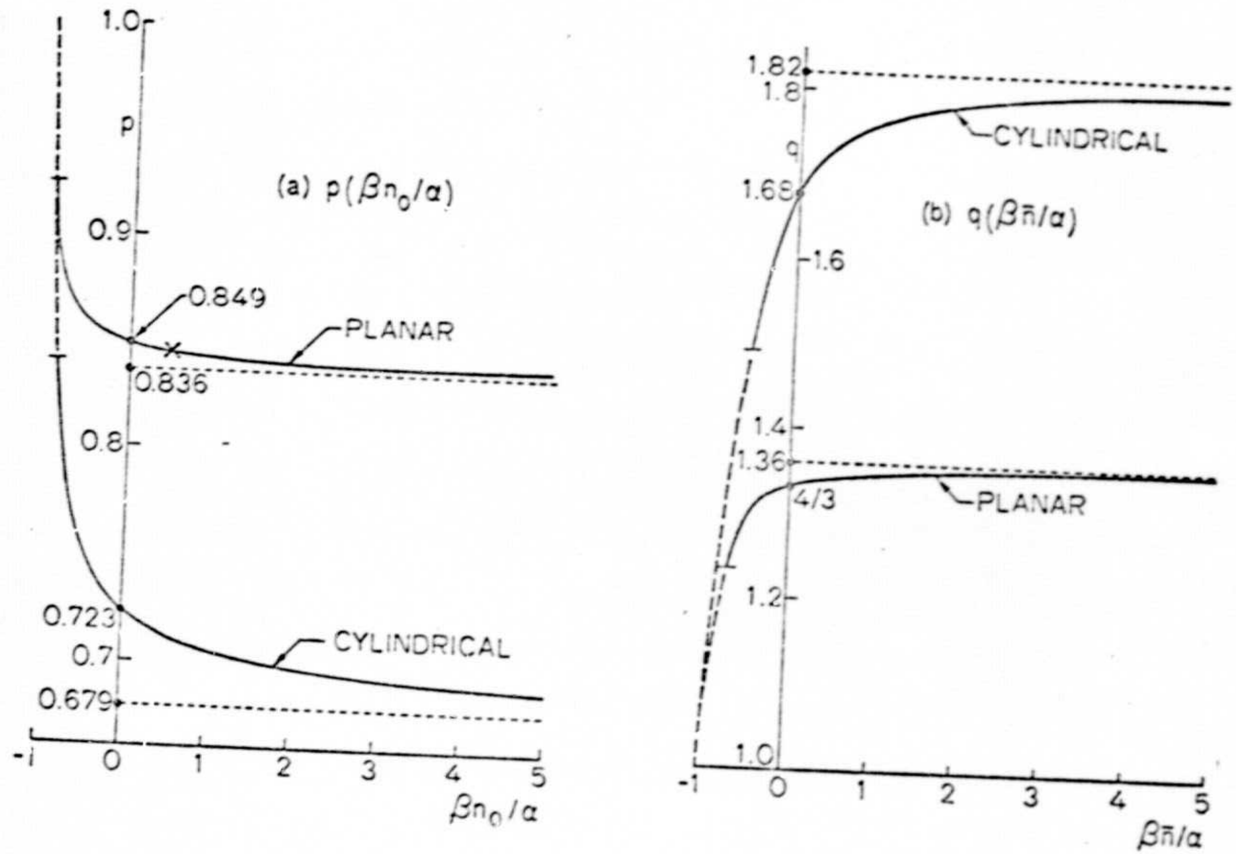

FIG. 3. Variation of $p(\beta n_0/\alpha)$ and $q(\beta\bar{n}/\alpha)$ (Curves extend to n_2/n_1 $n_2/n_1 = -1/9$ (planar) and -0.190 (cylindrical). The dashed extrapolations to unity are schematic; they have not been calculated).

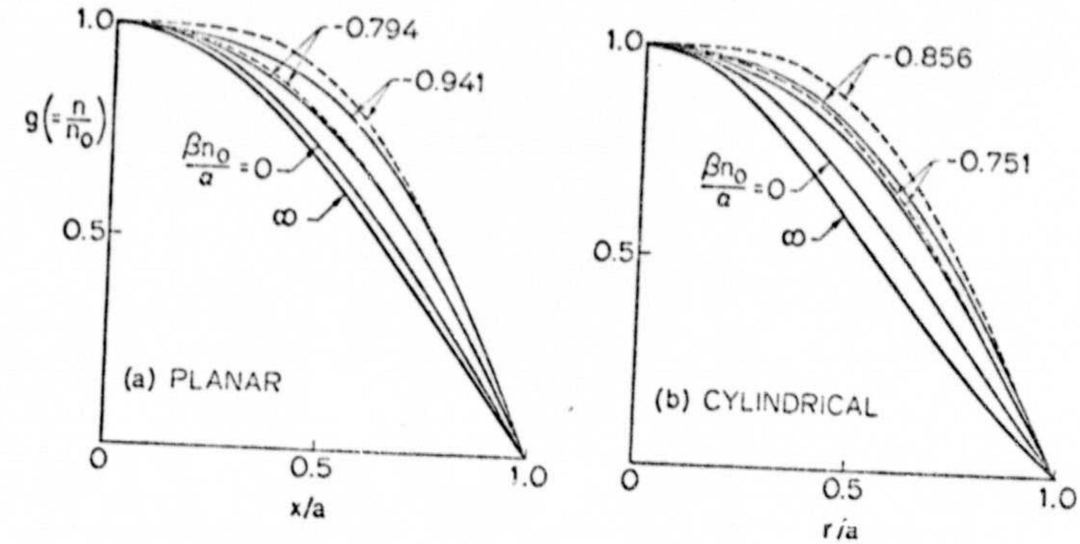

FIG. 4. Density profiles ($\beta n_0/\alpha$ = - 0.941 (planar) and - 0.856 (cylindrical) correspond to n_2/n_1 = - 1/9 and - 0.190; $\beta n_0/\alpha$ = - 0.794 (planar) and - 0.751 (cylindrical) correspond to n_2/n_1 = - 0.050 and - 0.125, which may be taken to be the limits for which the variational theory is adequate.)

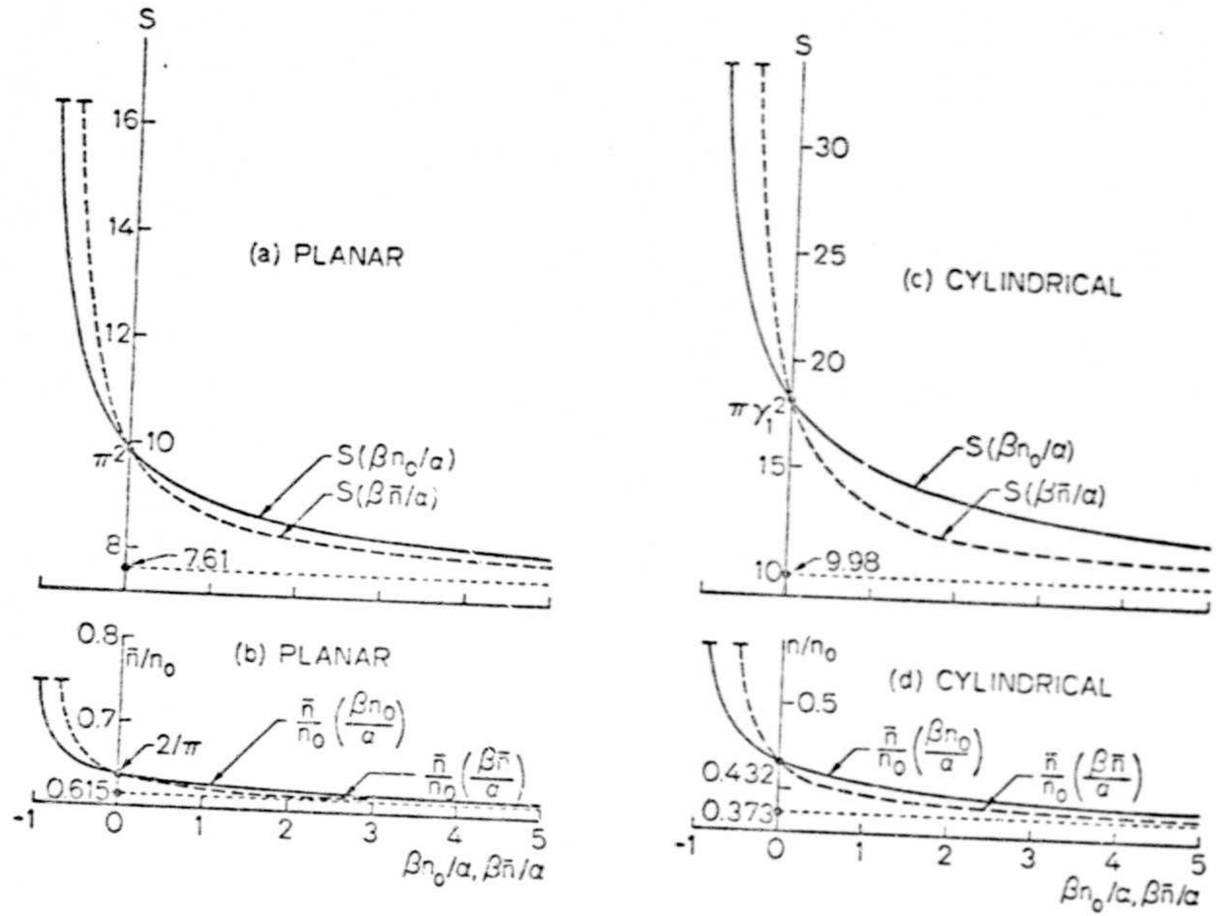

FIG. 5. Variation of Rogoff's S-parameter and mean density, $\bar{n}/n_0$, with $\beta n_0/\alpha$ and $\beta \bar{n}/\alpha$ (Curves extend to $n_2/n_1 = -1/9$ (planar) and -0.190 (cylindrical)).