STABILITY OF THE LAMINAR BOUNDARY LAYER

By Neal Tetervin

Langley Aeronautical Laboratory

This paper treats the stability theory for the laminar boundary layer and its applications. First, a short history of the theory similar to that in a paper by Pillow (reference 1), which contains a comprehensive list of references, is given; then, an outline of the theory for incompressible flow (reference 2) is presented. This is followed by a summary of the recent applications of the theory for incompressible flow. Finally, the results of investigations concerning the effect of compressibility and the effect of curvature on the stability of the laminar boundary layer is summarized.

In Prandtl's paper of 1904 (reference 3), which founded boundary-layer theory, the region of flow around a body was divided into two parts. One region includes almost the entire flow field and has the property that the viscosity of the fluid in this region has no effect on its motion. The other region is a narrow one next to the body where the fluid velocity rises rapidly from zero at the surface to a value which then changes slowly or not at all with further increase in distance from the surface. The narrow region in which the velocity changes so rapidly that the viscous forces are not negligible even in fluids of small viscosity is called the boundary layer. To this boundary layer can be traced the origin of the differences between the behavior of bodies in real and in nonviscous fluids.

Boundary layers are generally classified as either laminar or turbulent. Laminar flow is defined as one in which almost all of the interchange of momentum between adjacent layers of flowing fluid takes place by molecular diffusion (reference 4). At the stagnation point of a body and usually for some distance downstream, the flow in the boundary layer is laminar. Far enough from the stagnation point, however, the flow in the boundary layer changes from the smooth laminar flow to a violently fluctuating one – the turbulent flow. As shown in figure 1, the turbulent flow is associated with a different manner of increase of the average velocity with distance from the surface and with a higher skin friction. The skin friction for turbulent flow is usually several times the skin friction for laminar flow.

In normal flight attitudes the profile drag of wings and fuselages is almost directly proportional to their skin friction. It is thus possible to reduce greatly the drag of aircraft by so constructing them that extensive regions of laminar flow can exist. One method is to design shapes that are favorable for long lengths of laminar flow; another is to act directly upon the laminar boundary layer. The first method has led to the NACA 6-series airfoils (reference 5); the second which includes various types of suction and blowing is still in the relatively early stages of development (reference 6). The close connection between aircraft drag and the type of flow in the boundary layer thus makes it important to understand how the change from laminar to turbulent flow occurs. Such an understanding may eventually lead to aircraft with considerably lower drag (reference 7).

The change from laminar to turbulent flow is known as transition. Several causes of transition are (1) disturbances that originate in the outer stream such as those that occur when the free stream is turbulent, (2) disturbances introduced into the laminar boundary layer itself, for example, by surface roughness, and (3) a rising static pressure in the direction of flow that causes a complete reversal of the flow and eddies near the surface. Historically the subject first attracted attention because an apparently smooth flow would suddenly become turbulent.

It is interesting to note that the problem of the stability of laminar flow drew the attention of investigators years before modern aeronautics and boundary-layer theory began. The first recorded suggestion that the Navier-Stokes equation of motion might have unstable solutions was made by Stokes in 1843 (reference 8). Twenty-five years later Helmholtz (see reference 11 of reference 1) showed that, in a nonviscous fluid, surfaces across which there was a discontinuity in the velocity were inherently unstable. Rayleigh (reference 9) was the first to really attack the problem. He published his first paper on the subject of stability in 1879 and his last on the same subject thirty-five years later (references 10 and 11). Rayleigh investigated the stability of various hypothetical velocity distributions with the effect of viscosity on the disturbed motion neglected.

In 1883, Reynolds (see reference 48 of reference 1) published the results of his classic experiments on the transition from laminar to turbulent flow in pipes. Later, in 1895 (reference 12), he investigated the transition problem theoretically by seeking to determine the smallest Reynolds number above which an arbitrary disturbance would increase initially. The work was criticized by Sharpe in 1905 (see reference 46 of reference 1) and by Lorentz (see reference 29 of reference 1) in 1907 on the ground that the critical Reynolds number depended strongly on the form of the disturbance. Between 1907 and 1909, Orr (reference 13) improved Reynolds' method by using the calculus of variations to find the largest Reynolds number below which all disturbances decrease. Orr's work, however, has in turn been criticized because it allows all disturbances and, therefore, gives critical Reynolds numbers that are much smaller than those observed for quiet flows.

In 1908, a short time after Orr's work was published, Sommerfeld (see reference 26 of reference 1) independently set up the problem for the two-dimensional flow in which the velocity is parallel to the wall and is dependent only on the distance from the wall. Sommerfeld's and Orr's investigations formed the basis of the work leading up to the present theory of boundary-layer instability. During the following years, Von Mises (see references 27 and 28 of reference 1) and Hopf (reference 14), by making use of the work of Orr and Sommerfeld, found plane Couette flow, the flow which exists when two parallel planes separated by fluid slide past one another, to be stable for all the Reynolds numbers that were investigated. For the plane Couette flow the velocity varies directly with the distance from the wall.

Taylor, in 1923, (reference 15) investigated the Couette motion between rotating cylinders theoretically and checked the results experimentally. In contrast to most of the work on plane flows where the disturbances were assumed to be two dimensional, Taylor's theory was based on three-dimensional disturbances. For a number of years Taylor's work was a high-water mark in the understanding of the breakdown of laminar flow.

In 1924, Heisenberg (reference 16) successfully studied the stability of a variable continuous vorticity distribution by making use of the work of Orr and Sommerfeld. As an example he showed that plane Poiseuille flow, the flow under a uniform pressure gradient between fixed parallel planes, is unstable for sufficiently large Reynolds numbers. This flow has a parabolic velocity distribution. Heisenberg's theory was not generally accepted, perhaps, because his computations were incomplete and rough.

The first to investigate the stability of the boundary layer was Tietjens (reference 17) in 1925. He replaced the velocity profile by line segments and applied Rayleigh's theory, taking account of viscosity near the wall. Tietjens did not obtain a critical Reynolds number for the flat plate. The use of line segments to replace a velocity profile had already been shown to be invalid by Heisenberg. The next to investigate the stability of the boundary layer were Tollmien in 1929 (reference 18) and Schlichting in 1932 (reference 19). Both used what was essentially Heisenberg's theory and during the 1930's developed it sufficiently for use as a research tool (references 20 and 21). In 1945, Lin published his comprehensive work on the stability of two-dimensional parallel flows. (See reference 2.) This work made the theory more rigorous mathematically, provided a rapid approximate means of determining the minimum critical Reynolds number of a flow, and improved the physical picture of the instability. In addition it provided stability limits for the flow over a flat plate that agree better with experimental results than do the calculations of Tollmien and Schlichting.

The following is an outline of Lin's stability theory (reference 2). The purpose of the theory is to determine whether a particular flow is unstable for sufficiently large Reynolds numbers, to determine the minimum critical Reynolds number at which instability begins, and to understand the physical mechanism of the growth or decay of disturbances. The basic assumptions of the theory are that (1) the disturbances are small, (2) two-dimensional disturbances alone are considered, (3) the flow is essentially parallel to one direction (thus, the boundary-layer approximation that the derivative parallel to the surface of any quantity connected with the main flow is negligible compared with the derivative normal to the surface of the same quantity is applicable), (4) the velocity distribution normal to the surface is everywhere the same, and (5) the boundary conditions are everywhere the same.

The development of the theory is begun by writing the Navier-Stokes equation of motion for two-dimensional incompressible flow in a form that uses the vorticity ζ and thereby eliminates the pressure. The equation of motion then appears as:

$$\Delta\psi_t + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y = \nu\Delta\Delta\psi \qquad (1)$$

where x is the coordinate along the surface, y is the coordinate normal to the surface,

$$U = \psi_y = \frac{\partial\psi}{\partial y}$$

is the velocity parallel to the surface,

$$V = -\psi_x = -\frac{\partial\psi}{\partial x}$$

is the velocity normal to the surface,

$$\zeta = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = -\Delta\psi$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and ν is the kinematic viscosity.

The stream function ψ is assumed to be the sum of the stream function of the steady flow $\bar{\psi}$ and of the stream function of the disturbance ψ'. The introduction of the stream functions makes both the mean and the disturbance velocities satisfy the equation of continuity. Thus, let

$$\psi = \bar{\psi}(x,y) + \psi'(x,y,t)$$

and substitute into equation (1). Then, because the disturbance is small, terms quadratic in ψ' and its derivatives can be neglected. Equation (1) then becomes

$$\Delta\psi'_t + \bar{\psi}_y\Delta\psi'_x - \bar{\psi}_x\Delta\psi'_y + \psi'_y\Delta\bar{\psi}_x - \psi'_x\Delta\bar{\psi}_y = \nu\Delta\Delta\psi' \qquad (2)$$

The flow is now assumed to be essentially parallel to the x-axis, thus making the boundary-layer approximations applicable. Therefore, it is permissible to neglect the x-derivative of any quantity connected with the main flow compared with the y-derivative of the same quantity. For the disturbance, however, the quantities ψ'_y and ψ'_x, which are the disturbance velocities u' and $-v'$ along the x- and y-axes, respectively, are of the same order of magnitude. After making the boundary-layer approximation, equation (2) becomes

$$\Delta\psi'_t + \Psi_y \Delta\psi'_x - \psi'_x \frac{\partial^3 \Psi}{\partial y^3} = \nu\Delta\Delta\psi' \tag{3}$$

The approximation that the velocity distribution normal to the wall is independent of x now makes it permissible to use the local values at a given value of x for $\bar{u} = \frac{\partial \Psi}{\partial y} = \Psi_y$ and for $\frac{\partial^2 \bar{u}}{\partial y^2} = \frac{\partial^3 \Psi}{\partial y^3}$. Equation (3) then becomes

$$\Delta\psi'_t + \bar{u}(y)\Delta\psi'_x - \frac{\partial^2 \bar{u}(y)}{\partial y^2} \psi'_x = \nu\Delta\Delta\psi' \tag{4}$$

A main flow with an arbitrary distribution of velocity $\bar{u}(y)$ is now assumed to exist between two parallel planes $y = y_1$ and $y = y_2$. Then the disturbance stream function $\psi'(x,y,t)$ must be made to satisfy both equation (4) and the conditions $u' = v' = 0$ at $y = y_1$ and $y = y_2$ where u' and v' are the disturbance velocities. The disturbance stream function is now assumed to be given by

$$\psi' = \phi(y)e^{i\alpha(x-ct)}$$

where ϕ disturbance amplitude function

$$\alpha = \frac{2\pi}{\text{Wave length of disturbance}}$$

x coordinate along the plate

t time

and c is complex; the real part of c, that is, c_r, is the velocity with which the disturbance moves downstream; and the imaginary part of c, that is, c_i, determines whether the disturbance dies out $(c_i < 0)$, does not change with time $(c_i = 0)$, or increases in amplitude with time $(c_i > 0)$

After all the velocities have been referred to a reference velocity U and all lengths, to a reference length l, a Reynolds number $R = \frac{Ul}{\nu}$ has been defined, and the equation for ψ' has been used, equation (4) becomes the linearized differential equation for $\phi(y)$ which is known as the Orr-Sommerfeld equation.

$$(\bar{u} - c)\left(\frac{\partial^2\phi}{\partial y^2} - \alpha^2\phi\right) - \frac{\partial^2\bar{u}}{\partial y^2}\,\phi = -\frac{i}{\alpha R}\left(\frac{\partial^4\phi}{\partial y^4} - 2\alpha^2\frac{\partial^2\phi}{\partial y^2} + \alpha^4\phi\right) \qquad (5)$$

Equation (5) is a homogoneous, linear, ordinary differential equation of the fourth order. Its solution is

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4 \qquad (6)$$

where the ϕ's are particular solutions and the C's are constants of integration.

The four boundary conditions which are independent of x and which must be satisfied are

$$\left.\begin{aligned} &\phi(y_1) = 0 \\ &\phi(y_2) = 0 \\ &\text{that is, } v' = 0 \text{ at } y = y_1 \text{ and } y = y_2, \text{ and} \\ &\left(\frac{d\phi}{dy}\right)_{y_1} = 0 \\ &\left(\frac{d\phi}{dy}\right)_{y_2} = 0 \\ &\text{that is, } u' = 0 \text{ at } y = y_1 \text{ and } y = y_2. \end{aligned}\right\} \qquad (7)$$

When these boundary conditions are used with equation (6), the result is the determinant

$$\begin{vmatrix} \phi_1(y_1) & \phi_2(y_1) & \phi_3(y_1) & \phi_4(y_1) \\ \phi_1(y_2) & \phi_2(y_2) & \phi_3(y_2) & \phi_4(y_2) \\ \left(\frac{d\phi_1}{dy}\right)_{y_1} & \left(\frac{d\phi_2}{dy}\right)_{y_1} & \left(\frac{d\phi_3}{dy}\right)_{y_1} & \left(\frac{d\phi_4}{dy}\right)_{y_1} \\ \left(\frac{d\phi_1}{dy}\right)_{y_2} & \left(\frac{d\phi_2}{dy}\right)_{y_2} & \left(\frac{d\phi_3}{dy}\right)_{y_2} & \left(\frac{d\phi_4}{dy}\right)_{y_2} \end{vmatrix} = 0 \qquad (8)$$

which involves the solution of equation (5). After the functions ϕ_1, ϕ_2, ϕ_3, and ϕ_4 which contain the parameters α, R, and c have been determined with sufficient precision, which is a very involved process, the determinant (8) is written out and the real and imaginary parts equated to zero. The result is two real equations involving the parameters α, R, c_r, and c_i. If c_i is made zero and c_r is eliminated between the two real equations, the result is a relation between α and R. This relation between α, a quantity inversely proportional to the wave length of the disturbance, and R, the Reynolds number, defines the neutral curve along which the disturbances are neither damped nor amplified. The curve divides the α,R-plane into a stable region and an unstable region. The smallest value of the Reynolds number for which amplification can occur is called the minimum critical Reynolds number. Above the minimum critical Reynolds number, disturbances in the correct frequency range are amplified and, if they grow large enough, cause transition to turbulent flow. Lin has found that all velocity distributions of the symmetrical type and of the boundary-layer type are unstable for sufficiently large, but finite, Reynolds numbers. In his paper, Lin has given a useful approximate rule for the determination of the minimum critical Reynolds number; the rule is

$$R_\delta = \frac{25\left(\frac{d\bar{u}}{dy}\right)_1}{c^4}$$

where

$$R_\delta = \frac{U_\delta \delta}{\nu}$$

and where c is equal to the value of $\bar{u}$ for which

$$-\pi\left(\frac{d\bar{u}}{dy}\right)_1 \left[3 - \frac{2\left(\frac{d\bar{u}}{dy}\right)_1 y}{\bar{u}}\right]\left[\frac{\bar{u}\left(\frac{d^2\bar{u}}{dy^2}\right)}{\left(\frac{d\bar{u}}{dy}\right)^3}\right] = 0.58$$

U_δ is the velocity at the edge of the boundary layer, δ is the thickness of the boundary layer, and subscript 1 denotes "at surface." The velocities are referred to the velocity at the boundary-layer edge and the lengths, to the distance from the wall to where $\bar{u} = 1$.

The physical interpretation of the instability process (references 2 and 22) is that the viscosity shifts the phase between the x- and y-components of the disturbance in such a way that energy is drawn from the main flow and builds up the amplitude of the disturbance.

The validity of the assumption that for a parallel flow it is necessary to investigate only two-dimensional disturbances was confirmed by Squire (reference 23). In 1933 he showed that a two-dimensional disturbance produces instability at a smaller Reynolds number than a corresponding three-dimensional disturbance.

In 1941, Pretsch (reference 24) showed that the relations between the parameters α, R, and c are the same whether both the mean and the disturbance velocities in the boundary layer are functions of x and y or of y alone as assumed in the development of the theory. This important result means that the stability of the boundary layer at any value of x is dependent only on the local velocity distribution. The present theory can therefore be used when both the velocity distribution in the boundary layer and its thickness change along the surface.

It should be kept in mind that the theory is a small-disturbance theory. Therefore, conclusions drawn from it should not be applied to cases where finite disturbances are introduced into the boundary layer. Such disturbances are often introduced by roughness particles, which although small, may easily produce disturbances much greater than the vanishingly small disturbances allowed by the theory. It should also be noted that the theory merely predicts when infinitesimal disturbances will begin to grow. The disturbance cannot be traced by the theory to the stage where the disturbance has grown large enough to produce turbulent flow. The growth of the infinitesimal disturbance takes time; and, therefore, when transition develops from the growth of infinitesimal disturbances, the transition point lies some distance downstream of the instability point. The magnitude of the distance depends on the rate of amplification of the disturbance and therefore on the flow conditions.

Because of the many assumptions and because of the complexity of the mathematical development, the theory and its predictions were not taken seriously by many until fairly recently. In 1943 the results of the outstanding experimental work of Schubauer and Skramstad appeared (reference 25). The results showed that the laminar boundary-layer oscillations predicted by the stability theory of Tollmien and Schlichting not only were present but that the theory correctly predicted their characteristics. Figure 2 shows the neutral curve calculated by Lin, probably the most accurate calculation to date, and the experimental points obtained by Schubauer and Skramstad for flow over a flat plate. The circle symbols should lie on branch I; the cross symbols, on branch II.

In Germany during the war, the theory was used to calculate stability limits for flows in which there were small velocities through the surface. For these suction or blowing flows, the same stability theory was used as for impervious walls. This use is permissible because both the equations describing the motion and the boundary conditions that have to be satisfied by the disturbances are unchanged by small flows through the wall. The stability limits were computed for four exact solutions of the Prandtl boundary-layer equations. A boundary-layer velocity distribution must be known precisely before its stability limits can be determined accurately. The following results were taken from a paper by Ulrich (reference 26). The first case is the "Asymptotic Case." It applies to flow over a flat plate with a constant flow velocity into the plate and concerns only the region that is so far from the leading edge that no boundary-layer characteristic changes with a further increase in distance from the leading edge. For this case, the surface friction coefficient is independent of the viscosity and, for equal boundary-layer Reynolds numbers, is 1.75 times greater than the surface friction on the plate without suction. The minimum critical Reynolds number $\frac{U_o\delta^*}{\nu}$, where δ^* is the displacement thickness, is given by Pretsch as 55,200 (reference 27) in contrast to 575 obtained by Schlichting for the flat plate without suction. Other German investigators have obtained the value 70,000 for $R_{\delta}*$ (reference 28) instead of 55,200 so that there seems to be some differences caused by different calculating procedures. In order to keep the boundary-layer Reynolds number always less than the minimum critical Reynolds number, and thus to keep the boundary layer stable, making the suction ratio $-\frac{v_o}{U_o} > 1.8 \times 10^{-5}$ is sufficient when 55,200 is used for the value of $R_{\delta}*_{cr}$; v_o is the suction velocity and is negative when its direction is into the plate and U_o is the free-stream velocity.

The second case is the "Constant Suction" flow. Here also, there is a constant suction velocity through the surface of the plate, but the entire plate is treated and the velocity profiles are not similar to one another. Near the leading edge of the plate, the profile is the Blasius flat-plate profile (reference 29); but as the distance from the leading edge increases, the profile becomes more convex and finally approaches the asymptotic profile at large distances from the leading edge. When the suction ratio $\frac{-v_o}{U_o} > 1.2 \times 10^{-4}$, the flow is stable over the entire plate. This suction ratio, 1.2×10^{-4} is about seven times the ratio necessary for stability with the asymptotic profile. The greater suction is necessary because the velocity profiles near the leading edge of the plate are not as stable as the more convex asymptotic profile. Note, however, that the required suction ratio is still very small. The flow velocity through the plate is about 0.001 of the free-stream velocity.

Another case for which the stability computations based on exact solutions of the boundary-layer equations were made is the one in which the suction velocity varies inversely as $\sqrt{x}$ from the leading edge of a flat plate. The results are shown in figure 3. For this flow all the velocity profiles are similar to one another and change their form only when the suction coefficient C_Q is changed. The suction coefficient is defined by

$$C_Q = -\frac{Q}{lbU_o}$$

where

l	length of plate
b	width of plate
U_o	free-stream velocity
Q	total suction quantity

The value 575 for R_{δ^*} corresponds to the value 1.1×10^5 for R_x; the value 10^4 for R_{δ^*} corresponds to the value 8.3×10^7 for R_x, an increase in R_x of about 750 times. Figure 3 clearly shows that sucking, positive C_Q, increases the stability of the flow over that on an impervious flat plate and that blowing, negative C_Q, decreases the stability. In general, suction increases the stability of a boundary layer both because the boundary layer is kept thin and because the velocity profile is made more convex.

The fourth case for which stability computations based on exact solutions of the boundary-layer equation exist is that for the flow near the stagnation point of a two-dimensional body which has a constant suction or blowing velocity through its surface. The region considered is that region where the velocity at the edge of the boundary layer varies directly as the distance from the stagnation point. The results are shown in figure 4. In this region $U = u_1 x$, where U is the velocity at the edge of the boundary layer, u_1 is a constant, and x is the distance from the stagnation point measured along the surface. Here again, all the velocity profiles are similar to one another and change in shape only when C_o, the suction coefficient, is changed. The boundary-layer thickness is independent of x. It should be noticed that the flow near the stagnation point has a falling pressure in the direction of the flow; the previously mentioned flows were all for zero pressure gradient. The increased stability caused by the falling pressure is shown in figure 4. An amount of blowing corresponding to $C_o < -3$, where $C_o = \dfrac{-v_o}{\sqrt{u_1 \nu}}$, is necessary before stability is reduced from that for no flow through the surface to that for the impervious flat plate. When there is no flow through the surface, the boundary layer near a stagnation point has a critical Reynolds number of 12,300 in contrast to the value of 575 for the flat plate; the increase of about 20 times is caused by the falling pressure along the surface.

In figure 5 is shown the theoretically predicted drag reduction for two types of flow over flat plates with just enough suction to maintain stability; one is the "Constant Suction" case and the other is the case for which the suction velocity is inversely proportional to $\sqrt{x}$. The drag reduction is a large percentage of the drag of a plate with a completely turbulent boundary layer and, for R_x less than 10^8, a constant suction velocity is better than a suction velocity inversely proportional to $\sqrt{x}$.

The skin-friction values upon which the comparison in figure 5 is based are obtained from the velocity derivative at the surface. The sucked-in fluid remains at rest in the plate and the power required to suck the fluid into the plate is not considered. If, however, it is assumed that the sucked-in fluid is ejected with free-stream total head and that, in order to do this, total head is added to the fluid with an efficiency of unity, then the drag reduction shown is the true drag reduction if the total-head loss through the surface is equal to the free-stream dynamic pressure. If the total-head loss through the surface is greater than the free-stream dynamic pressure, then the drag reduction will be less than shown and vice versa. Because only small quantities of suction air are required to maintain laminar flow, the percentage drag reduction changes fairly slowly with a change of total-head loss through the plate.

These results are the sum total of the known stability computations based on exact solutions of the laminar boundary-layer equations. The only case directly applicable to flow about an airfoil is the stagnation-point flow.

Before the stability boundaries for an airfoil can be computed, the velocity distributions through the boundary layer must be known. During the war, Schlichting developed an approximate method for the computation of the laminar boundary layer over an arbitrary two-dimensional body with an arbitrary distribution of suction along the surface (reference 30). The method is related to the Pohlhausen method which treats flows without suction. Schlichting's method uses the boundary-layer momentum equation for the case where there is flow through the surface and assumes a one-parameter family of curves for the boundary-layer velocity distributions. The parameter for the velocity distribution depends on the pressure distribution over the body and on the suction flow through the surface.

The critical Reynolds number of a velocity profile is sensitive to its shape. Therefore, the accuracy of an approximate method, such as Schlichting's, when the results are to be used for stability computations, can be tested only by comparing the critical Reynolds numbers with those from an accurate computation of the boundary layer.

The foregoing discussion was restricted to incompressible flow. The problem of the stability of the laminar boundary layer in a compressible gas has, however, not been neglected. The increase in flight speeds has given the problem practical, as well as purely scientific, importance.

The stability theory for compressible flow has been developed by Lees and Lin (references 31 and 32) to about the same state as the theory for incompressible flow. The development of the theory for compressible flow is similar to that for incompressible flow. In the theory for compressible flow, however, in contrast to the theory for incompressible flow, the heat energy is important and the physical properties of the gas are not fixed. Nevertheless, the main physical mechanism is not changed. The stability of a velocity distribution depends on the distribution of the product of density and vorticity and on the effect of the viscous forces but not directly on the heat conductivity. The expression $\frac{\partial}{\partial y}\left(\rho \frac{\partial u}{\partial y}\right)$ for compressible flow takes the place of the expression $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ for incompressible flow as an important factor in determining the stability. It is noted, however, that as yet for compressible flow there is no rigorous proof that the two-dimensional disturbances upon which the theory is based are more unstable than three-dimensional disturbances.

The main results of Lees' and Lins' work can be summed up in the following statements:

(1) When the free-stream velocity is subsonic, every laminar boundary-layer flow is unstable at sufficiently large Reynolds numbers.

(2) At all free-stream Mach numbers the flow is unstable at sufficiently large Reynolds numbers if the y derivative of $\rho \frac{\partial u}{\partial y}$ is zero for a value of $u > 1 - \frac{1}{M_o}$.

(3) An approximate expression for the minimum critical Reynolds number is obtained, similar to the expression obtained by Lin for incompressible flow.

(4) As shown in figure 6 the stability of the laminar boundary layer on an insulated surface decreases with increase in Mach number. At $M_o = 1$, $R_{x_{cr_{min}}}$ is less than half its value at $M_o = 0$.

(5) As shown in figure 7, the ratio of the surface temperature to the free-stream temperature has a large effect on the boundary-layer stability. Thus, at a Mach number of 0.7 the value of the boundary-layer Reynolds number R_θ, based on the momentum thickness as the length, at which the boundary layer first becomes unstable increases about 40 times when the surface temperature is changed from 110 percent of the free-stream temperature, the stagnation-temperature ratio for a Mach number of 0.7, to 70 percent of the free-stream temperature. On the other hand, an increase of surface temperature from 110 percent of the free-stream temperature to 125 percent of the free-stream temperature halves the Reynolds number at which the flow becomes unstable.

(6) At supersonic free-stream velocities, the boundary layer can be made stable at all Reynolds numbers by maintaining the surface temperature at a small enough fraction of the free-stream temperature. For $M_o > 3$ at 50,000 feet altitude and for $M_o > 2$ at 100,000 feet altitude, the radiation of heat from a surface can make the ratio of the surface temperature to the free-stream temperature small enough to ensure a stable boundary layer at all Reynolds numbers, in the absence of an adverse pressure gradient.

The stability theories for both the incompressible and the compressible laminar boundary layer, which have just been discussed, were developed for flows in which the effects of surface curvature were negligible. Because most aircraft components are curved, it was not clear whether the stability theory for flat surfaces was directly applicable. The effect of curvature on the stability of the incompressible boundary layer was investigated theoretically by Görtler about 1940 (references 33 to 35) and experimentally by Liepmann (references 36 and 37) in the following years.

Görtler found that the two-dimensional wavelike disturbances were hardly affected by wall curvature. When, however, the stability of the boundary layer on curved walls was considered by investigating the behavior of vortices with their axis parallel to the main flow, analogous to the Taylor vortices in flow between concentric rotating cylinders, an instability caused by these vortices was found to be possible only on concave walls. The effect was so large that the effect of the usual two-dimensional disturbances was completely overshadowed. Görtler's theory is, like the two-dimensional disturbance theory, a small-disturbance theory that assumes the main boundary-layer flow to be the same over the entire surface. Also, the boundary-layer thickness is assumed to be small compared with the radius r of the wall. It was found, as shown in figure 8, that the wall curvature and the Reynolds number occur in the combination $R_\theta\sqrt{\frac{\theta}{r}}$ and that instability occurs above a value of $R_\theta\sqrt{\frac{\theta}{r}}$ that depends on $\alpha\theta$, where α is inversely proportional to the wave length and θ is the boundary-layer momentum thickness. The neutral curve shown is for the Blasius velocity distribution. Görtler found that the instability region was only slightly affected by the shape of the velocity distribution through the boundary layer when the momentum thickness θ was used as the measure of the boundary-layer thickness.

In agreement with Görtler's theoretical work, Liepmann found experimentally that $R_\theta\sqrt{\frac{\theta}{r}}$ was the parameter defining the stability of the boundary layer on concave surfaces. Liepmann concluded that transition can be expected when the value of $R_\theta\sqrt{\frac{\theta}{r}}$ reaches about 9.0. It may be observed that Görtler found the minimum critical value of $R_\theta\sqrt{\frac{\theta}{r}}$ to be 0.58. It should be noted, however, that Liepmann's criterion concerns transition, whereas Görtler's concerns the stability of the boundary layer. Liepmann also found, in agreement with Görtler's work, that in contrast to flow over convex or plane surfaces, a pressure gradient along the wall had a negligible effect on the stability of the flow over concave walls. Thus, on convex and plane surfaces instability of the boundary layer is caused by the Tollmien-Schlichting waves; whereas the instability on concave walls is caused by three-dimensional disturbances. In figure 9 is shown the dependence of Reynolds number for transition $R_{\theta_{tr}}$ on the effective curvature θ/r. The value of $R_{\theta_{tr}}$ is practically independent of curvature for convex walls and is about equal to the value for the flat plate. The value of $R_{\theta_{tr}}$ for concave walls, however, decreases rapidly as the effective curvature increases. The data in figure 10 show that the experimentally determined stability limits for the boundary layer on a convex wall and the calculated stability limits for the boundary layer on a flat plate are about the same except at the lowest Reynolds numbers. The upright triangles should lie on the upper branch of the neutral curve; the inverted triangles, on the lower branch. The neutral curve for the experimental points for $r = 20$ feet and also the curve for the points for $r = 2\frac{1}{2}$ feet, not shown in the figure, have a slightly higher minimum critical Reynolds number than the neutral curve for the flat plate. The reason for the difference is not definitely known.

This paper has attempted to present a short history of the theory of the stability of laminar flow, an outline of the theory for incompressible plane flow, a summary of the applications of the theory in combination with suction flows, a resumé of the results of the theory for compressible plane flow, and a summary of the theoretical and experimental results for curved flows. The stability theory based on infinitesimal disturbances may be regarded as experimentally verified for incompressible flow over plane surfaces and, probably, also for curved surfaces. Experimental work remains to be done in verifying the stability theory for compressible flows. An extension of the stability theory to the realm of finite disturbances for the purpose of calculating transition points is desirable.

REFERENCES

1. Pillow, A. F.: A Review of Hydrodynamic Stability and Its Bearing on Transition to Turbulent Flow in the Boundary Layer. Council for Sci. and Ind. Res., Div. Aero., Commonwealth of Australia, Rep. A.35, 1945.

2. Lin, C. C.: On the Stability of Two-Dimensional Parallel Flows. Part I. Quarterly Appl. Math., vol. III, no. 2, July 1945, pp. 117-142; Part II, vol. III, no. 3, Oct. 1945, pp. 218-234; and Part III, vol. III, no. 4, Jan. 1946, pp. 277-301.

3. Prandtl, L.: Motion of Fluids with Very Little Viscosity. NACA TM No. 452, 1928.

4. Fluid Motion Panel of the Aeronautical Research Committee and Others: Modern Developments in Fluid Dynamics. Vols. I and II, S. Goldstein, ed., The Clarendon Press (Oxford), 1938.

5. Abbott, Ira H., von Doenhoff, Albert E., and Stivers, Louis S., Jr.: Summary of Airfoil Data. NACA Rep. No. 824, 1945.

6. Goldstein, Sydney: Low-Drag and Suction Airfoils. Jour. Aero. Sci., vol. 15, no. 4, April 1948, pp. 189-214.

7. Squire, H. B., and Young, A. D.: The Calculation of the Profile Drag of Aerofoils. R. & M. No. 1838, British A.R.C., 1938.

8. Stokes, George Gabriel: On Some Cases of Fluid Motion. Mathematical and Physical Papers, vol. I, Univ. Press (Cambridge), 1880, pp. 17-68.

9. Rayleigh, Lord: On the Instability of Jets. Scientific Papers, vol. I, no. 58, Univ. Press (Cambridge), 1879, pp. 361-371.

10. Rayleigh, Lord: On the Stability or Instability of Certain Fluid Motions. Scientific Papers: Part I, vol. I, no. 66, 1880, pp. 474-487. Part II, vol. III, no. 144, 1887, pp. 17-23. Part III, vol. IV, no. 216, 1895, pp. 203-210.

11. Rayleigh, Lord: Stability of the Laminar Motion of an Inviscid Fluid. Phil. Mag., no. 26, Dec. 1913, pp. 1001-1010.

12. Reynolds, Osborne: On the Dynamical Theory of Incompressible Viscous Fluids and the Determination of the Criterion. Phil. Trans. Roy. Soc. (London), ser. A, vol. 186, 1895, pp. 123-164.

13. Orr, William M'Fadden: The Stability or Instability of the Steady Motions of a Liquid. Proc. Roy. Irish Acad., vol. XXVII, ser. A, no. 3, 1907, pp. 9–138.

14. Hopf, L.: Decrement of Small Vibrations in the Flow of a Viscous Fluid. Ann. d. Physik, vol. 44, no. 4, April 28, 1914, pp. 1–60.

15. Taylor, G. I.: Stability of Viscous Liquid Contained between Two Rotating Cylinders. Phil. Trans. Roy. Soc. (London), vol. 223, Feb. 8, 1923, pp. 289–343.

16. Heisenberg, Werner: Über Stabilität und Turbulenz von Flüssigkeitsströmen. Ann. d. Phys., Vierte Folge, Bd. 74, no. 15, 1924, pp. 577–627.

17. Tietjens, O.: Beiträge zur Entstehung der Turbulenz. Z.f.a.M.M., Bd. 5, Heft 3, June 1925, pp. 200–217.

18. Tollmien, W.: The Production of Turbulence. NACA TM No. 609, 1931.

19. Schlichting, H.: Zur Entstehung der Turbulenz bei der Plattenströmung. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl., 1933, pp. 181–208.

20. Tollmien, W.: General Instability Criterion of Laminar Velocity Distributions. NACA TM No. 792, 1936.

21. Schlichting, H.: Amplitudenverteilung und Energiebilanz der kleinen Störungen bei der Plattenströmung. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl., Neue Folge, Bd. 1, Nr. 4, 1935, pp. 47–78.

22. Prandtl, L.: The Mechanics of Viscous Fluids. Vol. III of Aerodynamic Theory, div. G, W. F. Durand, ed., Julius Springer (Berlin), 1935, pp. 34–208.

23. Squire, H. B.: Stability for Three-Dimensional Disturbances of Viscous Fluid Flow between Parallel Walls. Proc. Roy. Soc. (London), ser. A, vol. 142, no. , Nov. 1, 1933, pp. 621–628.

24. Pretsch, J.: Die Stabilität einer ebenen Laminarströmung bei Druckgefälle und Druckanstieg. Jahrb. 1941 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 158 – I 175.

25. Schubauer, G. B., and Skramstad, H. K.: Laminar-Boundary-Layer Oscillations and Transition on a Flat Plate. NACA ACR, April 1943.

26. Ulrich, A.: Theoretical Investigation of Drag Reduction in Maintaining the Laminar Boundary Layer by Suction. NACA TM No. 1121, 1947.

27. Pretsch, J.: Umschlagbeginn und Absaugung. Jahrb. 1942 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 1 – I 7.

28. Bussmann, K., and Münz, H: Die Stabilität der laminaren Reibungsschicht mit Absaugung. Jahrb. 1942 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 36 – I 39.

29. Iglisch, Rudolf: Exakte Berechnung der laminaren Grenzschicht an der längsangeströmten ebenen Platte mit honogener Absaugung. Schriften der Deutsche Akademie der Luftfahrtforschung, Bd. 8B, Heft 1, 1944.

30. Schlichting, H.: Ein Näherungsverfahren zur Berechnung der laminaren Grenzschicht mit Absaugung bei beliebiger Körperform. Bericht 43/14, Aerodynamisches Institut der T. H. Braunschweig, June 12, 1943.

31. Lees, Lester, and Lin, Chia Chiao: Investigation of the Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA TN No. 1115, 1946.

32. Lees, Lester: The Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA TN No. 1360, 1947.

33. Görtler, H.: Über den Einfluss der Wandkrümmung auf die Entstehung der Turbulenz. Z.f.a.M.M., Bd. 20, Heft 3, June 1940, pp. 138–147.

34. Görtler, H.: Über eine dreidimensionale Instabilität laminarer Grenzschichten an konkaven Wanden. Göttinger Nachrichten Neue Folge, 2 Nr. 1.

35. Görtler, H.: Instability of Laminar Boundary Layers on Concave Walls against Certain Three-Dimensional Disturbances. R.T.P. Translation No. 1588, British Ministry of Aircraft Production. (From Z.f.a.M.M., Bd. 21, Heft 4, Aug. 1941, pp. 250–252.)

36. Liepmann, Hans W.: Investigations on Laminar Boundary-Layer Stability and Transition on Curved Boundaries. NACA ACR No. 3H30, 1943.

37. Liepmann, H. W.: Investigation of Boundary-Layer Transition on Concave Walls. NACA ACR No. 4J28, 1945.

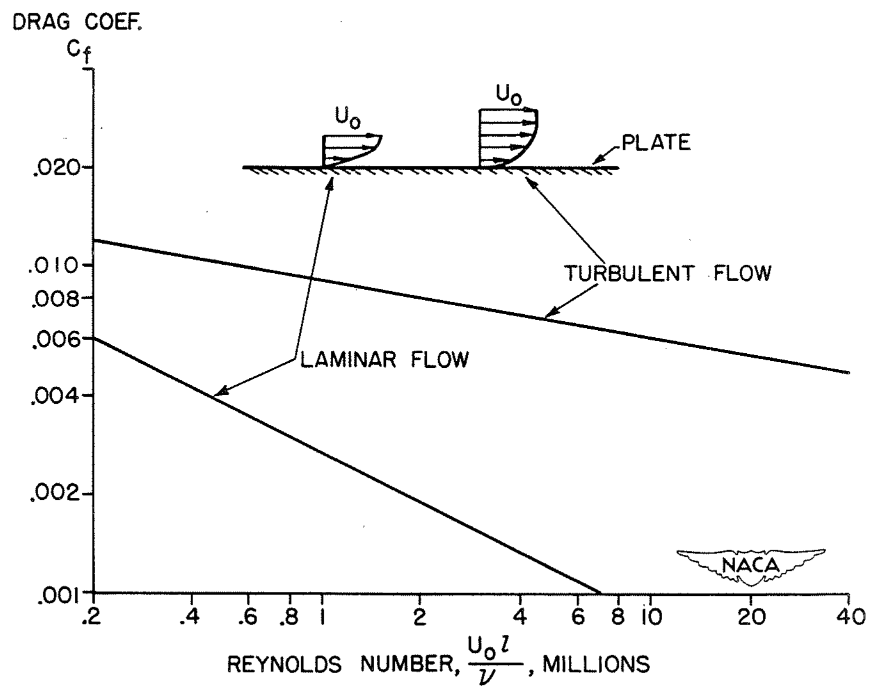

Figure 1.- Flat-plate drag coefficients for turbulent and laminar flow.

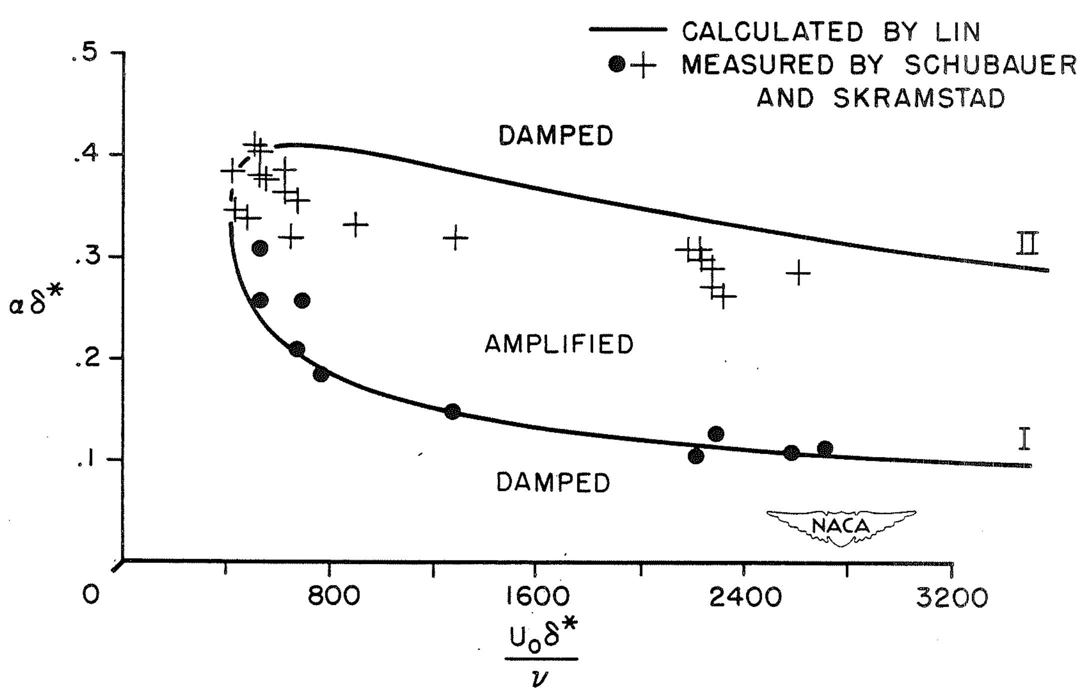

Figure 2.- Curve of neutral stability for Blasius profile.

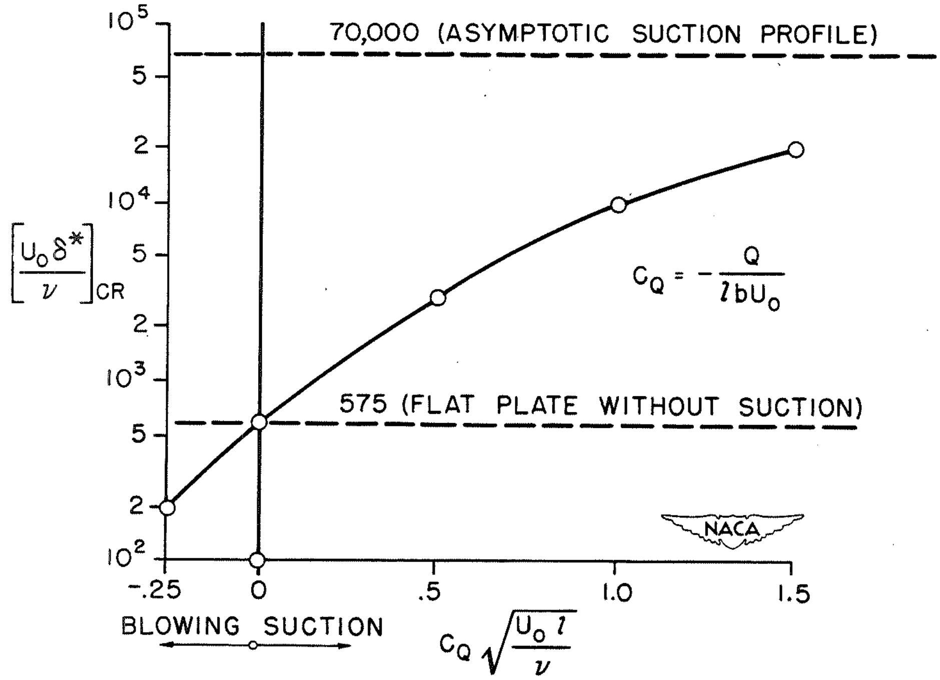

Figure 3.- Flow over a flat plate $\left(V_o \propto \frac{1}{\sqrt{x}}\right)$.

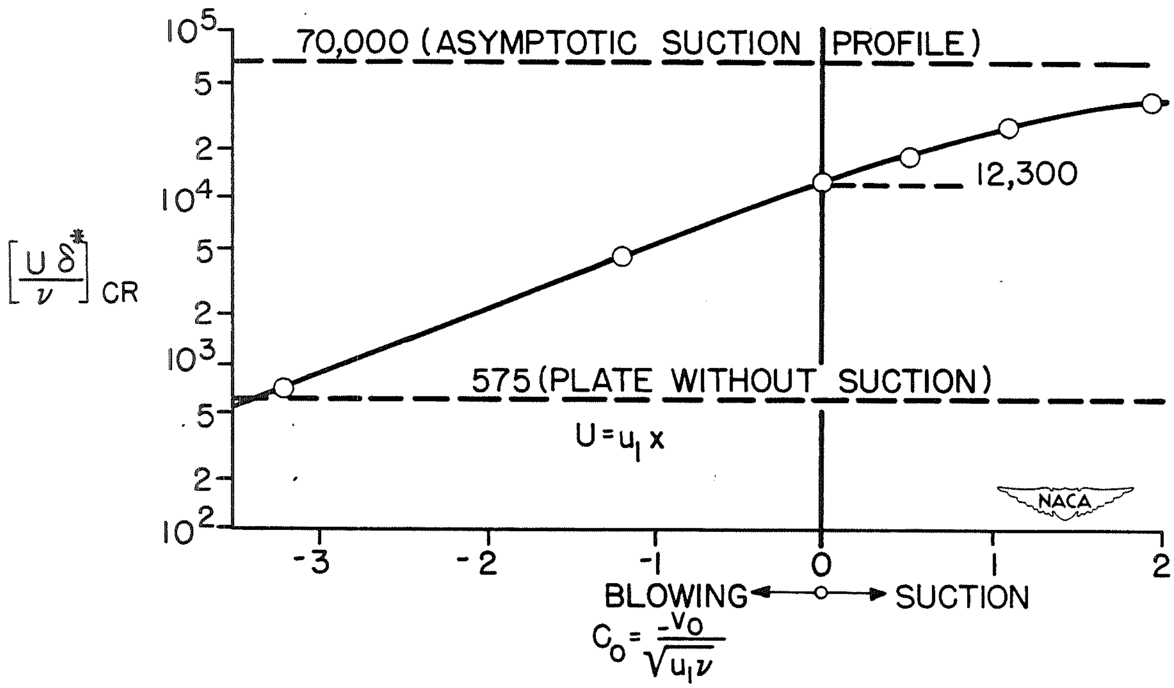

Figure 4.- Flow near a stagnation point.

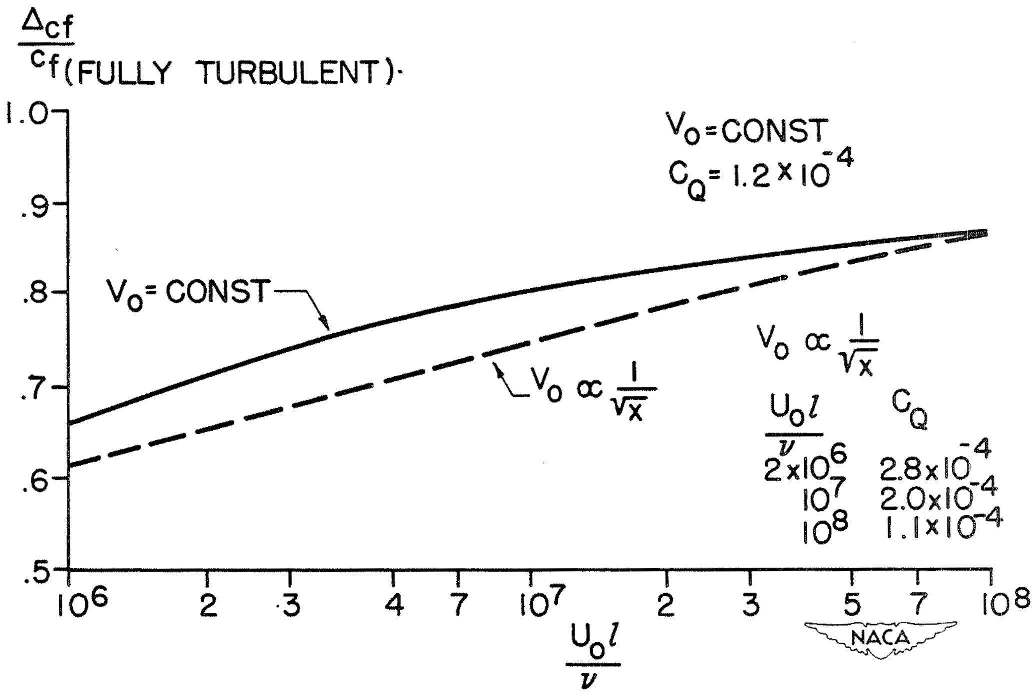

Figure 5.- Relative drag reduction for V_o = Constant and $V_o \propto \frac{1}{\sqrt{x}}$.

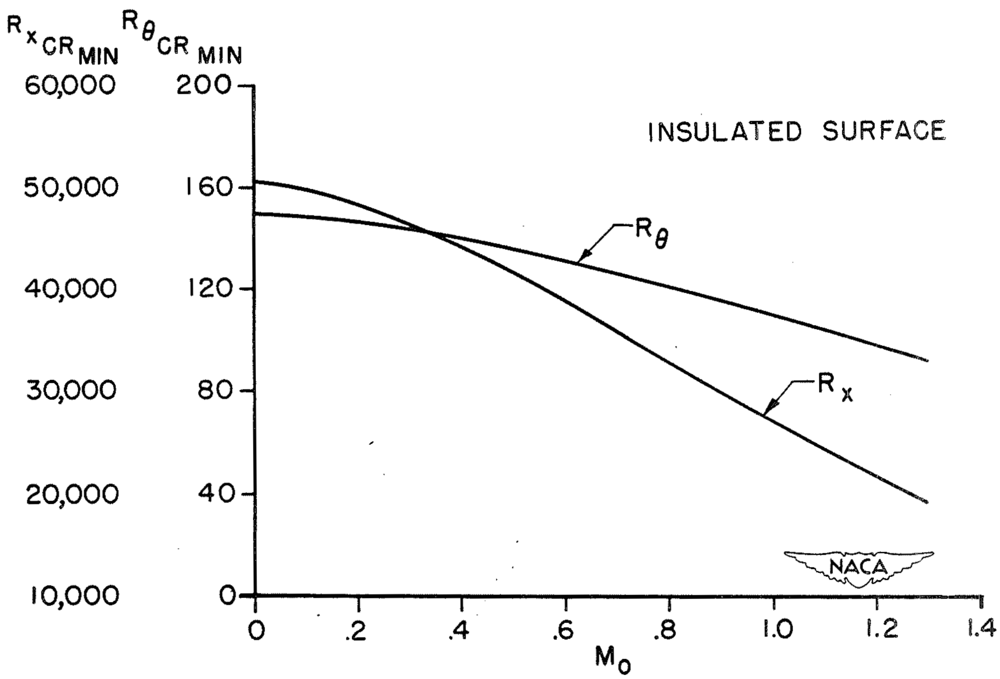

Figure 6.- Critical Reynolds number against Mach number.

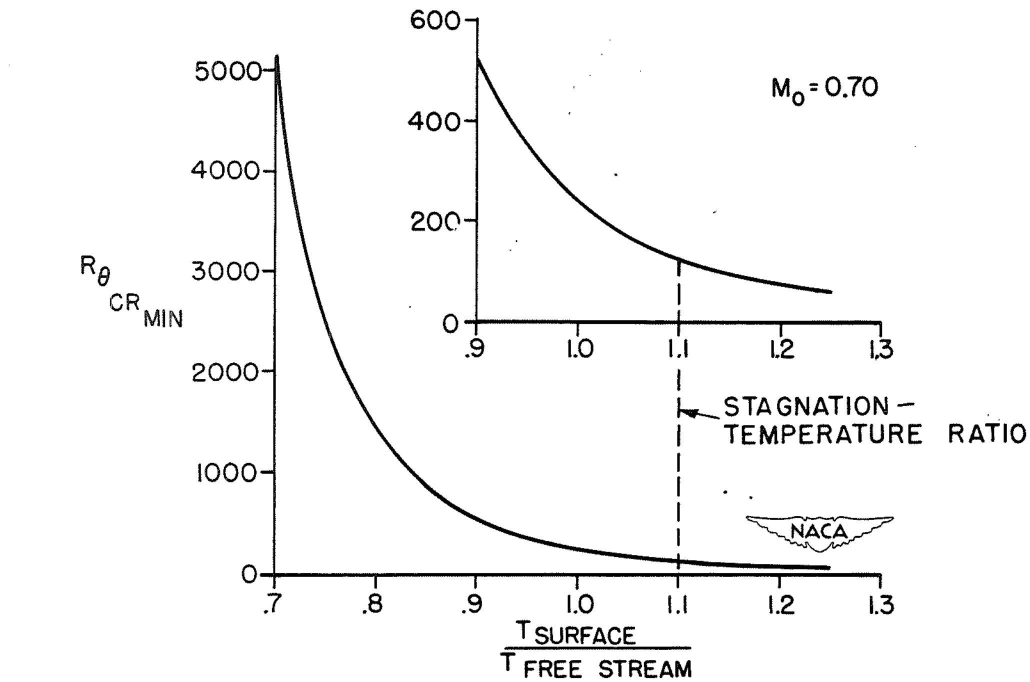

Figure 7.- Critical Reynolds number against surface temperature ratio.

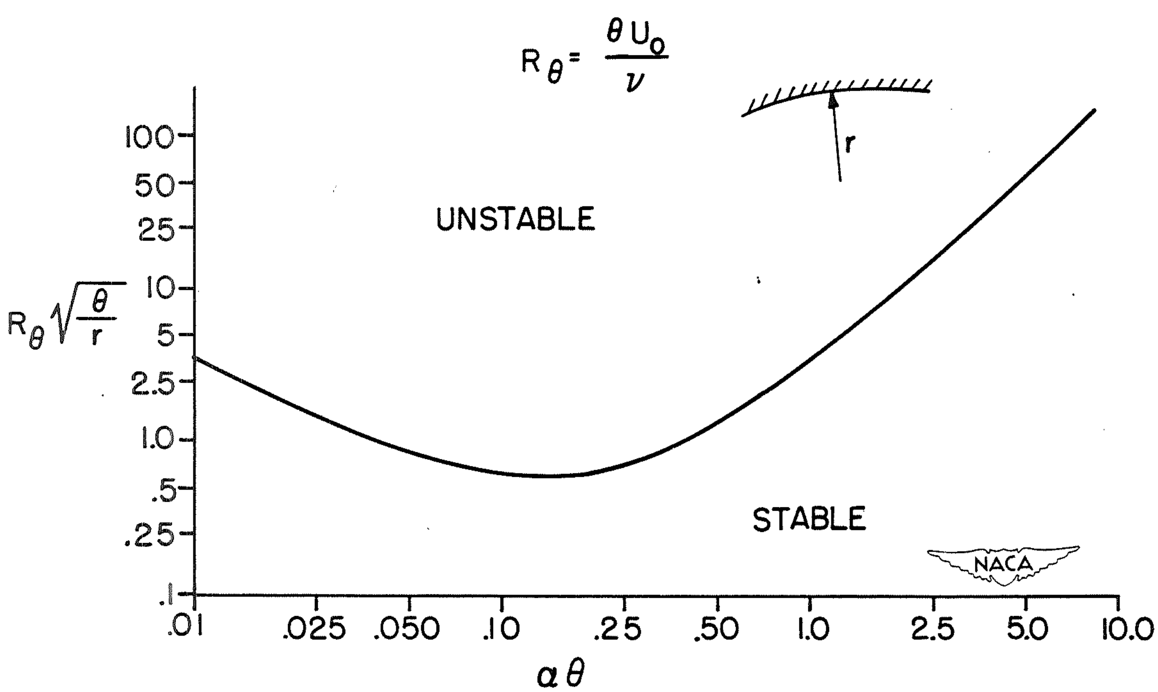

Figure 8.- Neutral curve for boundary layer on concave wall.

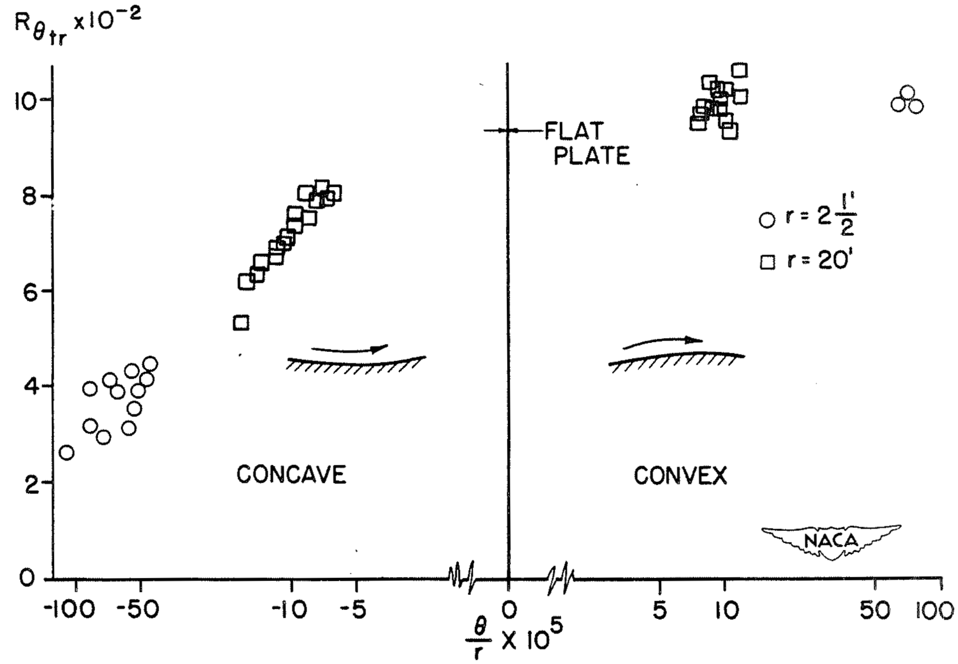

Figure 9.- Curvature effect on transition Reynolds number.

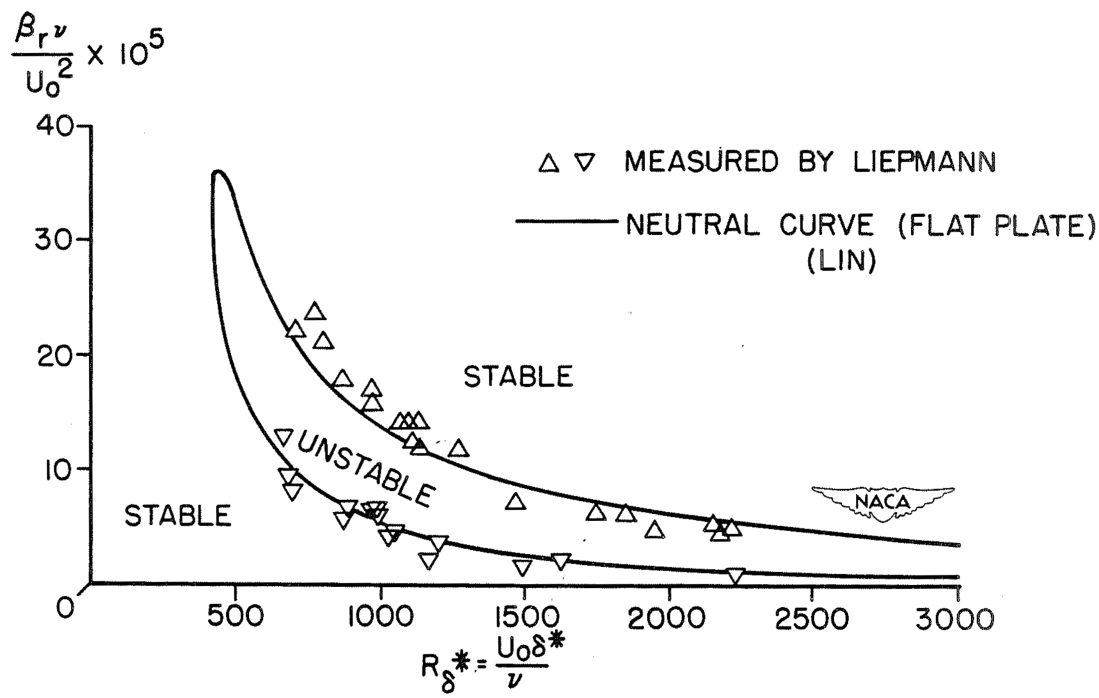

Figure 10.- Stability data for convex plate (r = 20 ft). β_r is the frequency of the oscillation.